Computability of families of superatomic boolean algebras

Pavel E. Alaev, Novosibirsk

All considered boolean algebras are countable. Let S be a family of superatomic boolean algebras (SBA).

Question: when has S a computable numbering, i.e. a computable numbering $\{B_x\}_{x\in\omega}$ of boolean algebras s.t. $\{B_x \mid x \in \omega\} = S$?

Each SBA $A \neq \{0\}$ can be uniquely described by pair $\langle \alpha, n\rangle = \text{type}(A)$, where α is a countable ordinal, $n \in \omega$, $n \geqslant 1$ (by the well known way). We say that $\langle \alpha, n\rangle \leqslant \langle \beta, m\rangle$ if $\alpha < \beta$ or ($\alpha = \beta$ and $n \leqslant m$). It is easy to check (using some known facts) that if S is computable then there exists a computable ordinal α such that $\text{type}(A) \leqslant \langle \alpha, 1\rangle$ for all $A \in S$.

Let $I(S) = \{\text{type}(B) \mid B \in S\}$, $I_\beta(S) = \{n \in \omega \mid \langle \beta, n\rangle \in I(S)\}$, $I^*(S) = \{\beta \mid I_\beta(S) \neq \emptyset\}$. It is clear that S is completly described by $I(S)$.

Theorem 1. Let S be a family of SBA, $\alpha = \sup I^*(S)$ is a computable ordinal. Then the following are equivalent:

(1) S has a computable numbering

(2) $I_\beta(S) \in \Sigma^0_{2\beta+2}$ uniformly with respect to $\beta \leqslant \alpha$ and S is majorizable (see the definition below).

A family S is majorizable if, having pair $\langle \beta, n\rangle$ such that S has an algebra with a greater type, we can effectively find an index of BA from S with a greater type.

If $\alpha = \sup I^*(S)$, $I_\alpha(S) \neq \emptyset$, then the property of majorizable can be replaced by a more simple one:

Corollary. Let S be a family of SBA, $\alpha = \sup I^*(S)$ is a computable ordinal, $I_\alpha(S) \neq \emptyset$. Then the following are equivalent:

(1) S has a computable numbering

(2) $I_\beta(S) \in \Sigma^0_{2\beta+1}$ uniformly with respect to $\beta \leqslant \alpha$ and either $I_\alpha(S)$ if infinite and $\Sigma^0_{2\beta+2}$-majorizable (see the definition below) or is finite.

We say that $I \in \Sigma^0_2(X)$ is $\Sigma^0_2(X)$-majorizable, if there exists a X-computable function $f(n,t)$ such that:

(a) $f(n, 0) = n$

(b) $f(n,t) \leqslant f(n, t+1)$

(c) there is $\lim_{t\to\infty} f(n,t) \in I$.

By F_α we denote the Frechet ideal of level α.

Theorem 2. Let S be a family of SBA, α be a computable ordinal, $S/F_\alpha \neq \{0\}$. Then the following are equivalent:

(1) S has a computable numbering

(2) $\{B/F_\alpha \mid B \in S\}$ has a $\Delta^0_{2\alpha+1}$-numbering and $I_\beta(S) \in \Sigma^0_{2\beta+2}$ uniformly with respect to $\beta < \alpha$.

The property of majorizable in Theorem 1 can not be dropped neither in the case $\sup I^*(S) \in I^*(S)$ nor in the case $\sup I^*(S) \notin I^*(S)$.