ON AN EXTREMAL PROPERTY OF CLASSICAL ORTHOGONAL POLYNOMIALS

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Let $\Delta \subseteq \mathbb{R} := (-\infty, \infty)$ be either the interval $(-1,1)$ or $\mathbb{R}$, and $w(x) > 0$, $x \in \Delta$ is a weight such, that $\int_\Delta |p(x)|^2 \omega(x) dx < \infty$ for all polynomials $p(x)$. Given $G \in L^2(\omega)$ denote

$$\mathcal{D}_{k,\omega}(G) = [\mathrm{dist}_{L^2(\omega)}(G, \mathcal{P}_{k-1})]^2 := \inf_{c_0,\dots,c_{k-1}} \int_I \left| G(x) - \sum_{j=0}^{k-1} c_j x^j \right|^2 \omega(x) dx,$$

where $\mathcal{P}_k$ is the set of all polynomials of degree $\le k$. Let $AC^{(k)}(I)$ be the space of all functions on interval Δ with absolutely continuous $(k-1)$-st derivative. We study the problem under which conditions on weight $\omega(x)$ there exists a locally integrable function $\nu_k(x) \ge 0$ such, that for all $G \in AC^{(k)}(I)$ satisfying $G^{(k)} \in L^2(\nu_k)$ the two-sided inequality

$$\gamma_k \left| \int_\Delta G^{(k)}(x) \nu_k(x)\, dx \right|^2 \le \mathcal{D}_{k,\omega}(G) \le \gamma_k \int_\Delta |G^{(k)}(x)|^2 \nu_k(x)\, dx \qquad (1)$$

holds with some unimprovable constant $\gamma_k > 0$? An additional question is the existence of extremal functions for which both inequalities (1) become the equalities.

The problem is motivated by the well-known two-sided inequality of Probability theory (*Chernoff inequality*) of the form

$$[\mathbf{E}G'(X)]^2 \le \mathbf{D}[G(X)] \le \mathbf{E}[G'(X)]^2, \qquad (2)$$

valid for all $G \in AC^1(\mathbb{R})$ and a standard random variable X, where $\mathbf{E}$ and $\mathbf{D}$ denote the expectation and the variance of X and turning into equalities for $G(x) = const \cdot x$.

Let $\mathbb{H} := L^2(dF)$ be the Hilbert space with the inner product $(f,g)_\mathbb{H} := \int_\mathbb{R} fg\, dF$, and the norm $\|f\|_\mathbb{H} = (f,g)_\mathbb{H}^{1/2}$, where $dF(x) := \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$.

Denote $\{\psi_k\}$, the orthogonal system of Chebyshev-Hermite polynomials

$$\psi_k(x) = (-1)^k e^{x^2/2} \frac{d^k}{dx^k}(e^{-x^2/2}), \quad k \in \mathbb{N}_0.$$

Theorem. *Let $k \in \mathbb{N}$. Then for all $G \in AC^{(k)}(\mathbb{R})$ such, that $G^{(k)} \in \mathbb{H}$, the inequalities*

$$\frac{1}{k!} \left[\int_\mathbb{R} G^{(k)} dF \right]^2 \le \mathcal{D}_{k,F}(G) \le \frac{1}{k!} \int_\mathbb{R} |G^{(k)}|^2 dF,$$

hold becoming equalities for $G = \mathrm{const} \cdot \psi_k$.

The similar result for $\Delta = (-1,1)$ and some generalization are also given.

The work is supported in part by RFBR (projects no. 06-01-00341 and no. 07-01-00054).