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URYSOHN'S CONDITION AND CAUCHY SEQUENCES

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A b s t r a c t. In Abelian groups, endowed with convergence, three types of Cauchy sequences are considered in connection with Urysohn's condition. An application to the intrinsic topology in l_q is shown.

1. Let $\mathfrak{X}$ be a set endowed with a convergence such that

F. If $x_n \to x$, then also $x_{p_n} \to x$ for each increasing p_n.

S. If $x_n = x$ for $n=1,2,\dots$, then $x_n \to x$.

H. If $x_n \to x$ and $x_n \to y$, then $x=y$.

By the Urysohn condition we mean

U. If from each subsequence y_n of x_n one can select a subsequence $z_n \to x$, then $x_n \to x$.

If $\mathfrak{X}$ does not satisfy U, then we can complete the set of convergent sequences, by considering as convergent to x each sequence x_n whose each subsequence contains a subsequence $z_n \to x$. The set $\mathfrak{X}$ with so generalized convergence will be denoted by $\mathfrak{X}^*$. The spaces $\mathfrak{X}$ and $\mathfrak{X}^*$ have the same elements, but $\mathfrak{X}^*$ may happen to have more convergent sequences.

2. Assume now that $\mathfrak{X}$ is an Abelian group satisfying:

L. If $x_n \to x$ and $y_n \to y$, then $x_n - y_n \to x - y$.

We shall consider three types of Cauchy sequences x_n:

P. If $p_n \uparrow$, then $x_{p_n} - x_{p_{n+1}} \to 0$.

Q. If $p_n \uparrow$ and $q_n \uparrow$, then $x_{p_n} - x_{q_n} \to 0$.

R. If $p_n \to \infty$ and $q_n \to \infty$, then $x_{p_n} - x_{q_n} \to 0$.

We say that $\mathfrak{X}$ is P-complete, if each P-Cauchy sequence converges to some element in $\mathfrak{X}$. Similarly, we define Q-complete-

ness and R-completeness. (In case of normed groups all the three completenesses are equivalent.)

If x_n is R-Cauchy, then it is Q-Cauchy. If x_n is Q-Cauchy, then it is P-Cauchy. Hence: If $\mathcal{X}$ is P-complete, then it is Q-complete. If $\mathcal{X}$ is Q-complete, then it is R-complete.

(i) *If $\mathcal{X}^*$ is Q-complete, then $\mathcal{X}$ is Q-complete.*

P r o o f. Assume that x_n is Q-Cauchy in $\mathcal{X}$. Then it is Q-Cauchy in $\mathcal{X}^*$. Since $\mathcal{X}^*$ is complete, x_n converges in $\mathcal{X}^*$ to some element x. This implies that there exists a subsequence x_{q_n} converging in $\mathcal{X}$ to x. Now $x_n - x_{q_n} \to 0$, because x_n is Q-Cauchy in $\mathcal{X}$. Consequently, $x_n \to x$ in $\mathcal{X}$.

(ii) *If $\mathcal{X}^*$ is R-complete, then $\mathcal{X}$ is R-complete.*

Proof the same (one only has to replace Q by R).

(iii) *If $\mathcal{X}^* = \mathcal{X}$, then each Q-Cauchy sequence is R-Cauchy, and conversely.*

P r o o f. If $p_n \to \infty$ and $q_n \to \infty$, then from each subsequence y_n of $x_{p_n} - x_{q_n}$ we can select $z_n = x_{p'_n} - x_{q'_n}$ such that $p'_n \uparrow$ and $q'_n \uparrow$. Since x_n is Q-Cauchy, we have $z_n \to 0$, i.e., $x_{p'_n} - x_{q'_n} \to 0$. Hence $x_{p_n} - x_{q_n} \to 0$, because $\mathcal{X}^* = \mathcal{X}$. Hence x_n is R-Cauchy. The concerse inclusion is obvious.

(iv) *If $\mathcal{X}^* = \mathcal{X}$, then the properties P-completeness, Q-completeness and R-completeness are equivalent.*

P r o o f. The equivalence of P-completeness and Q-completeness has been proved in [2]. The equivalence of Q-completeness and R-completeness follows from (iii).

3. Let $q \geqslant 1$ and let l_q be the space of all sequences $x=(\xi_1, \xi_2, \ldots)$ of real numbers such that $|\xi_1|^q + |\xi_2|^q + \ldots < \infty$. By the norm in l_q we mean as usually $\|x\| = (|\xi_1|^q + |\xi_2|^q + \ldots)^{1/q}$. This norm induces in l_q a convergence such that l_q is Q-complete (also P-complete and R-complete). Besides, we consider the spase $\hat{l}_q$ which has the same elements as l_q, but the convergence is defined intrinsicly as follows. We say that a sequence of vectors $x_n \in \hat{l}_q$ converges to x, if it converges coordinatewise to x and is bounded by some vector from $\hat{l}_q$.

It is easy to see that $\hat{l}_q$ satisfies F,S,H,L. However, it does not satisfy U. In fact, the sequence $x_n = n^{-1/q} e_n$ does not converge to 0, but from each its subsequence one can select a subsequence which does.

(v) $\hat{l}_q^* = l_q$.

P r o o f. If $\|x_n - x\| \to 0$, then from each subsequence x_{p_n} one can select a subsequence x_{q_n} such that $\|x_{q_n} - x\| \leqslant \varepsilon_n$, $\varepsilon_1 + \varepsilon_2 + \ldots < \infty$. Then

$$|x_{q_n}| \leqslant |x| + |x_{q_1} - x| + |x_{q_2} - x| + \ldots \in l_q.$$

Since x_{q_n} converges coordinatewise to x, we have $x_{q_n} \to x$ in $\hat{l}_q^*$. Hence $l_q^* \supset l_q$. We know that $l_q^* = l_q$, because l_q is Banach. Since $\hat{l}_q \subset l_q$, it follows that $\hat{l}_q^* \subset l_q^*$. Together with the preceding inclusion this implies (v).

(vi) <u>The spase $\hat{l}_q$ is Q-complete</u>.

P r o o f. The space l_q is Q-complete, because each Banach space so is. Thus $\hat{l}_q^*$ is Q-complete, by (v). Hence $\hat{l}_q$ is Q-complete, by (i).

4. The space $\hat{l}_q$ may be considered as a particular interpretation of a σ-complete vector lattice (see e.g. [1]). We write $\text{O-lim}\, x_n = x$, iff $|x_n - x| \leqslant w_n$, where w_n is a sequence of positive vectors tending monotonically to 0. We have generally

(vii) $\underset{n\to\infty}{\text{O-lim}}\, x_n$ <u>exists, iff</u> $\underset{n,m\to\infty}{\text{O-lim}}(x_n - x_m) = 0$.

In our notation, this theorem can be reformulated as

(vii bis) <u>The space $\hat{l}_q$ is R-complete</u>.

We see that our assertion (vi) is sharper than that deduced from the general theory of vector lattices.

The question remains open, whether $\hat{l}_q$ is P-complete.

R e f e r e n c e s

1. Kôsacu Yosida, Functional Analysis, Springer 1966, p.364-368.
2. Piotr Mikusiński, Cauchy sequences and Abelian groups, to appear in Bull. Pol. Ac. Sci.