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PERIODIC SOLUTIONS OF A WEAKLY NONLINEAR AUTONOMS WAVE EQUATION

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1. **Introduction and notation.** Let α and β be positive numbers fixed throughout the paper. Let ν be a small positive number. Under the assumption that the parameters ε and δ assume certain values close to 0 we will investigate the existence of a function u satisfying

$$(1.1)\qquad u_{tt}(t,x) - u_{xx}(t,x) = \varepsilon\left\{(\alpha - \beta\int_0^{\pi} u^2(t,\xi)\,d\xi)u(t,x) + \delta F(u)(t,x)\right\},\quad (t,x)\in(-\infty,+\infty)\times(0,\pi),$$

$$(1.2)\qquad u(t,0) = u(t,\pi) = 0\ ,\quad t\in(-\infty,+\infty)\ ,$$

$$(1.3)\qquad u(t+2(\pi-\varepsilon\nu),x) = u(t,x),\quad (t,x)\in(-\infty,+\infty)\times(0,\pi)\ .$$

Here the operator F is given by

$$(1.4)\qquad F(u)(t,x) = f(x,u(t,x)),\quad (t,x)\in(-\infty,+\infty)\times(0,\pi)\ ,$$

and the function $f\in C^3([0,\pi]\times R)$ is supposed to satisfy

$$f(x,-u) = -f(x,u),\quad (x,u)\in[0,\pi]\times R\ .$$

In Section 2, by virtue of a special form of the nonlinearity in (1.1) for $\delta = 0$, the equations (1.1) - (1.3) are split into a sequence of equations which we will call the system of associated equations. Generally, the system of associated equations is formed by an infinite system of nonlinear equations which are obtained

by integrating an infinite system of ordinary differential equations of second order with the periodicity conditions. It is shown that every solution of the associated system yields a solution of (1.1) - (1.3).

In Section 3 the parameter δ is assumed to range in a small neighbourhood of 0 in contrast to Section 4 where δ is taken to be identically 0. There is found a function u^o, generated by a finite number of harmonics, to which the solutions of (1.1) - (1.3) are supposed to converge as $\varepsilon \to 0$ and $\delta \to 0$. In this case the system of associated equations is a genuine infinite dimensional system which, in a convenient space of sequences of functions, we will solve by applying the implicit function theorem (Lemma 5.2) in a form suitable to the problem in question. The parameter ε is hypothesized to run through a discrete subset of R, which has 0 as a point of accumulation, since for these values of ε we are able to prove that the inverse operator to the Frechet derivative of the associated system exists, is continuous and its norm is bounded independently of ε. The result (Theorem 3.1) can be roughly described as follows:

Let $\nu = \pi/s$, s sufficiently large integer. Then for every $\varepsilon = 1/q$, q integer, $|q|$ sufficiently large and every δ close to 0 there exists a solution to (1.1) - (1.3).

In Section 4, the equations (1.1) - (1.3) are investigated for $\delta = 0$. In this case the system of associated equations is actually a finite dimensional one providing the solutions are generated by the same finite system of functions, depending on x, as the limit function u^o. It turns out that we need not restrict to a discrete subset of a neighbourhood of 0 any more and roughly

speaking we will prove:

Let ν be any sufficiently small positive number and $\delta = 0$. Then for every ε close to 0 there is a solution to (1.1) - (1.3).

In Section 5, for the reader's convenience we give a lemma which, applied to the system of associate equations, gives a solution to the treated problem.

The investigated problem has many points in common with that solved in [2], where R.B. Melrose and M. Pemberton have proved the existence of a solution to

$$u_{tt} - u_{xx} + \varepsilon u = \delta f(u), \quad (t,x) \in (-\infty,+\infty) \times (0,\pi),$$
$$u(t,0) = u(t,\pi) = 0, \quad t \in (-\infty,+\infty),$$

which is t-periodic with period $2\pi/(k^2+\varepsilon)^{1/2}$ provided that k is a positive integer, ε runs through a set, depending on k, which has 0 as a point of accumulation and δ is in a neighbourhood of 0.

The existence of t-periodic solutions to wave equations has been investigated by P. Rabinowitz in a series of papers. In the last one, [3], for any rational number λ and any smooth function $f = f(x,u)$ which is monotone in u, $f(x,0) = 0$ and satisfies certain growth condition at $u = 0$ and $u = \pm\infty$, it has been proved that there is a classical nontrivial solution to

$$u_{tt} - u_{xx} = f(x,u), \quad (t,x) \in (-\infty,+\infty) \times (0,\pi),$$
$$u(t,0) = u(t,\pi) = 0, \quad t \in (-\infty,+\infty),$$
$$u(t,x) = u(t+2\pi\lambda,x), \quad (t,x) \in (-\infty,+\infty) \times (0,\pi).$$

Notation. We denote by R and N the sets of real numbers and positive integers respectively. $H^k(0,\pi)$ denotes the Sobolev space of all functions on $(0,\pi)$ whose generalized

derivatives up to order k are square integrable. Given $k \in N$, we denote by $H_0^k(0,\pi)$ the subspace of $H^k(0,\pi)$ of all functions vanishing at 0 and π. Further, we set

$$U_\infty = \bigcap_{j=0}^{2} C^j(R;\ H_0^{2-j}(0,\pi)),$$

$$U = \Big\{u;\ u \in \bigcap_{j=0}^{2} C^j([-2\pi,2\pi];\ H_0^{2-j}(0,\pi)),\ u(t) = \\ = -u(-t) \ \text{ for } |t| \leqq 2\pi\Big\}.$$

To simplify the formulation of the results we denote U_ω the space of all functions from U_∞ which are ω-periodic in t. If we put

$$\|u\|_\omega = \Big(\sup\Big(\sum_{j=0}^{2} \|D_t^j u\|^2_{H^{2-j}(0,\pi)};\ |t| \leqq \omega/2\Big)\Big)^{1/2},$$

then the space U_ω equipped with the norm $\|\cdot\|_\omega$ is a Banach space as well as U is a Banach space if endowed by $\|\cdot\|_U = \|\cdot\|_{2\pi}$.

By $\mathcal{X}_1$ we denote the space of all sequences $\bar{u} = \{u_k\}_{k\in N}$, $u_k \in C^2([-2\pi,2\pi])$, $u_k(t) = -u_k(-t)$ for $|t| \leqq 2\pi$, such that $t \to \sum_{k\in N}(k^4u_k^2(t) + k^2\dot{u}_k^2(t) + \ddot{u}_k^2(t))$ is a finite and continuous function on $[-2\pi,2\pi]$. $\mathcal{X}_1$ is a Banach space if endowed with the norm

$$\|\bar{u}\|_{\mathcal{X}_1} = \Big(\frac{\pi}{2}\sup\Big(\sum_{k\in N}(k^4u_k^2(t) + k^2\dot{u}_k^2(t) + \ddot{u}_k^2(t));\ |t| \leqq 2\pi\Big)\Big)^{1/2}.$$

It is well-known that the mapping $\chi:\ \mathcal{X}_1 \to U$ given by the formula

$$(\chi\bar{u})(t,x) = \sum_{k\in N} u_k(t)\sin kx$$

is an isometric isomorphism between the spaces $\mathcal{X}_1$ and U and

therefore the elements $\bar{u} \in \mathcal{X}_1$ and $u = \chi\,\bar{u} \in U$ will be identified without any particular reference.

We conclude this section by introducing two spaces of sequences of real numbers. We denote by $\mathcal{X}_2$ the space of all sequences $a = \{a_k\}_{k \in N}$, $a_k \in R$ satisfying $\|a\|^2_{\mathcal{X}_2} = \sum_{k \in N} k^4 a_k^2 < +\infty$. Given $\varepsilon \neq 0$, we set

$$(1.5) \qquad \alpha_k^{\varepsilon} = \max(1,\ k|\sin k\varepsilon\nu| / |\varepsilon|)$$

and we define the space $\mathcal{Y}^{\varepsilon}$ to be the set of all sequences $a = \{a_k\}_{k \in N}$, $a_k \in R$ such that

$$(1.6) \qquad \|a\|^2_{\mathcal{Y}^{\varepsilon}} = \sum_{k \in N} k^4 (\alpha_k^{\varepsilon})^{-2} a_k^2 < +\infty .$$

Both the spaces $\mathcal{X}_2$, $\mathcal{Y}^{\varepsilon}$ are Banach spaces, $\mathcal{X}_2 \subset \mathcal{Y}^{\varepsilon}$ and

$$(1.7) \qquad \|a\|_{\mathcal{Y}^{\varepsilon}} \leq \|a\|_{\mathcal{X}_2} .$$

For brevity in the forthcoming sections we denote

$$X = \mathcal{X}_1 \times \mathcal{X}_2 \quad \text{and} \quad Y^{\varepsilon} = \mathcal{X}_1 \times \mathcal{Y}^{\varepsilon}.$$

Eventually, being X_1 and X_2 two Banach spaces, we denote by $L(X_1,X_2)$ the space of all linear bounded mappings of X_1 into X_2 equipped with the norm $\|A\|_{L(X_1,X_2)} = \sup(\|Ax\|_{X_2};\ \|x\|_{X_1} \leq 1)$, $A \in L(X_1,X_2)$ and by $B(x_1, \rho, X_1)$ the open ball in X_1 with the centre x_1 and the radius ρ.

2. System of associated equations. In this section we present a system of equations whose solution yields immediately a solution to (1.1) - (1.3). We denote, $u \in \mathcal{X}_1$,

$$(2.1) \qquad T(u)(t) = \int_0^{\pi} u^2(t,\xi)\,d\xi = \frac{\pi}{2} \sum_{k \in N} u_k^2(t) ,$$

and

$$(2.2) \qquad F_k(u)(t) = \frac{2}{\pi} \int_0^{\pi} F(u)(t,x) \sin kx \, dx .$$

Let $0 < |\varepsilon| \leq 1$ and $0 < \nu < \pi$ be fixed. Let $\delta \in R$. For $u \in \mathcal{X}_1$, $a \in \mathcal{X}_2$ we put, $k \in N$

$$(2.3) \qquad {}^{\varepsilon}G_{1,k}(u,a,\delta)(t) = - u_k(t) + a_k \sin kt +$$

$$+ \frac{\varepsilon}{k} \int_0^t (\alpha - \beta T(u)(\tau))u_k(\tau) \sin k(t-\tau)d\tau +$$

$$+ \frac{\varepsilon\delta}{k} \int_0^t F_k(u)(\tau) \sin k(t-\tau)d\tau , \quad t \in [-2\pi, 2\pi] ,$$

$$(2.4) \qquad {}^{\varepsilon}G_{2,k}(u,a,\delta) = \frac{k \sin k\varepsilon\nu}{\varepsilon} a_k +$$

$$+ \int_0^{\pi-\varepsilon\nu} (\alpha - \beta T(u)(\tau))u_k(\tau) \sin k(\tau + \varepsilon\nu)d\tau +$$

$$+ \delta \int_0^{\pi-\varepsilon\nu} F_k(u)(\tau) \sin k(\tau + \varepsilon\nu)d\tau .$$

Next, denoting

$$(2.5) \qquad {}^{\varepsilon}G_1(u,a,\delta) = \{ {}^{\varepsilon}G_{1,k}(u,a,\delta)\}_{k \in N} ,$$

$$(2.6) \qquad {}^{\varepsilon}G_2(u,a,\delta) = \{ {}^{\varepsilon}G_{2,k}(u,a,\delta)\}_{k \in N}$$

and

$$(2.7) \qquad {}^{\varepsilon}G(u,a,\delta) = ({}^{\varepsilon}G_1(u,a,\delta), {}^{\varepsilon}G_2(u,a,\delta)) ,$$

we verify easily that ${}^{\varepsilon}G$ maps $X \times R$ into Y^{ε} .

Now let us suppose we have $(\tilde{u},\tilde{a}) \in \mathcal{X}_1 \times \mathcal{X}_2 = X$ such that ${}^{\varepsilon}G(\tilde{u},\tilde{a},\delta) = 0$. Show that this $\tilde{u}$ gives a solution to (1.1) - (1.3). Using (2.4) in (2.3), we have

(2.8) $\quad \tilde{u}_k(\pi - \varepsilon\nu) = 0$ for all $k \in N$.

For $|t| < 2\pi$ the functions $\tilde{u}_k$, $k \in N$ satisfy

$$\tilde{u}_k''(t) + k^2 \tilde{u}_k(t) = \varepsilon(\alpha - \beta T(u)(t))\tilde{u}_k(t) + \varepsilon\delta F_k(\tilde{u})(t)$$

which multiplied by $\sin kx$ and summed up for $k \in N$ give

(2.9) $\quad \tilde{u}_{tt} - \tilde{u}_{xx} = \varepsilon(\alpha - \beta T(u)(t))\tilde{u}(t) + \varepsilon\delta F(\tilde{u}))$ for $|t| < 2\pi$.

The function $\tilde{u}$, being an element of $\mathcal{X}_1$, is odd and therefore

$$\tilde{u}'(\pi - \varepsilon\nu) = \tilde{u}'(-\pi + \varepsilon\nu) .$$

By (2.8),

$$\tilde{u}(\pi - \varepsilon\nu) = 0 = \tilde{u}(-\pi + \varepsilon\nu)$$

and (2.9) gives

$$\tilde{u}''(\pi - \varepsilon\nu) = \tilde{u}''(-\pi + \varepsilon\nu) .$$

Hence, there is a function $u \in U_\infty$, $2(\pi - \varepsilon\nu)$-periodic in t , which is equal to $\tilde{u}$ for $|t| \leq \pi - \varepsilon\nu$ and which satisfies (1.1) by virtue of (2.9). Since $u \in U_\infty$, (1.2) is obviously satisfied. Thus we have shown that the problem (1.1) - (1.3) will be solved as soon as we have found $(u,a) \in \mathcal{X}_1 \times \mathcal{X}_2$ satisfying

(2.10) $\quad {}^{\varepsilon}G(u,a,\delta) = 0$,

which is actually a system of equations called the system of associated equations. We now state some fundamental properties of the mapping ${}^{\varepsilon}G$. We have already claimed that ${}^{\varepsilon}G$ maps $X \times R$ into Y^{ε} . It is easy to prove even more:

(2.11) $\quad {}^{\varepsilon}G_1$ is a continuous mapping of $X \times R$ into $\mathcal{X}_1$;

(2.12) For every $\varepsilon \neq 0$, ${}^{\varepsilon}G_2$ is a continuous mapping of $X \times R$ into Y^{ε} .

These two assertions can be verified by applying the method we will demonstrate in proving

(2.13) $\tilde{G}(u) \equiv \left\{ \int_0^t F_k(u)(\tau) \sin k(t-\tau)d\tau \right\}_{k \in N}$ is a continuous mapping of $\mathcal{X}_1$ into $\mathcal{X}_1$.

Really,

$$\frac{2}{\pi} \|\tilde{G}(u^1) - \tilde{G}(u^2)\|^2_{\mathcal{X}_1} =$$

$$= \sup_{|t| \leq 2\pi} \sum_{k \in N} \Big\{ k^4 \Big(\int_0^t (F_k(u^1)(\tau) - F_k(u^2)(\tau)) \sin k(t-\tau)d\tau \Big)^2 +$$

$$+ k^4 \Big(\int_0^t (F_k(u^1)(\tau) - F_k(u^2)(\tau)) \cos k(t-\tau)d\tau \Big)^2 +$$

$$+ \Big(-k^2 \int_0^t (F_k(u^1)(\tau) - F_k(u^2)(\tau)) \sin k(t-\tau)d\tau +$$

$$+ k(F_k(u^1)(t) - F_k(u^2)(t)) \Big)^2 \Big\} \leqq$$

$$\leqq 4\pi \sup_{|t| \leq 2\pi} \Big(\sum_{k \in N} k^4 \int_0^t |F_k(u^1)(\tau) - F_k(u^2)(\tau)|^2 d\tau +$$

$$2 \sup_{|t| \leq 2\pi} \Big(\sum_{k \in N} k^2 |F_k(u^1)(t) - F_k(u^2)(t)|^2 \Big) \leqq$$

$$\leqq (16\pi + 4/\pi) \|F(u^1) - F(u^2)\|^2_{\mathcal{X}_1} .$$

As $F : \mathcal{X}_1 \rightarrow \mathcal{X}_1$ is continuous by the imposed assumptions, (2.13) is proved and it is easy, though rather involved, to verify (2.11). (2.12) can be proved similarly since the operator ${}^{\varepsilon}\hat{G}$ given by

$$^{\varepsilon}\hat{G}(a) = \left\{ \frac{k \sin k\varepsilon\nu}{\varepsilon} a_k \right\}_{k \in N}$$

is an element of $L(\mathcal{X}_2, \mathcal{Y}^{\varepsilon})$.

Remark 2.1. We note that just to ensure $^{\varepsilon}\hat{G} \in L(\mathcal{X}_2, \mathcal{Y}^{\varepsilon})$ has been the reason for defining the norm in $\mathcal{Y}^{\varepsilon}$ by (1.6) since the operator $^{\varepsilon}G_2 - {}^{\varepsilon}\hat{G}$ is easily verified to be a continuous mapping of $X \times R$ even into $\mathcal{X}_2$ and thus, by (1.7), into $\mathcal{Y}^{\varepsilon}$.

Let $\varepsilon \neq 0$ and $\delta \in R$. Then there exists the Frechet derivative $^{\varepsilon}G'(u,a,\delta) \in L(X,Y^{\varepsilon})$ of the mapping $^{\varepsilon}G$ with respect to (u,a) and satisfies the following two conditions which can be verified by applying of the procedure shown in proving (2.13).

(2.14) Given $\varepsilon \neq 0$ and $(u,a) \in X$, then $\delta \to {}^{\varepsilon}G'(u,a,\delta)$ is a continuous mapping of R into $L(X,Y^{\varepsilon})$.

(2.15) Given $\varkappa > 0$ and $(u^o,a^o) \in X$, then there exist $\varepsilon_o > 0$, $\delta_o > 0$ and $r_o > 0$ such that

$$\sup\left\{ \|{}^{\varepsilon}G'(u,a,\delta) - {}^{\varepsilon}G'(u^o,a^o,\delta)\|_{L(X,Y^{\varepsilon})} ;\ 0 < |\varepsilon| \leq \varepsilon_o ,\ |\delta| \leq \delta_o ,\ \|(u,a) - (u^o,a^o)\|_X \leq r_o \right\} \leq \varkappa .$$

We conclude this section by writing explicitly the expressions for the Frechet derivative $^{\varepsilon}G'(u,a,\delta)$ which we will make full use of in the next section. By (2.5) - (2.7) we can write

$$^{\varepsilon}G'(u,a,\delta) = ({}^{\varepsilon}G_1'(u,a,\delta),\ {}^{\varepsilon}G_2'(u,a,\delta))$$

where the single components of $^{\varepsilon}G_i'$ are given by, $k \in N$

$$({}^{\varepsilon}G_1'(u,a,\delta)(v,c))_k(t) = - v_k(t) + c_k \sin kt +$$

$$+ \frac{\varepsilon}{k} \int_0^t (\alpha - \beta T(u)(\tau)) v_k(\tau) \sin k(t-\tau) d\tau +$$

$$+ (-1)\frac{\pi\beta\varepsilon}{k}\int_0^t\Big(\sum_{l=1}^{\infty}u_l(\tau)v_l(\tau)\Big)u_k(\tau)\sin k(t-\tau)d\tau +$$

$$+ \frac{\varepsilon\delta}{k}\int_0^t (F'(u)v)_k(\tau)\sin k(t-\tau)d\tau \ , \quad t\in[-2\pi,2\pi] \ ,$$

$$({}^{\varepsilon}G_2'(u,a,\delta)(v,c))_k = \frac{k\sin k\varepsilon\nu}{\varepsilon}c_k +$$

$$+ \int_0^{\pi-\varepsilon\nu}(\alpha-\beta T(u)(\tau))v_k(\tau)\sin k(\tau+\varepsilon\nu)d\tau +$$

$$+ (-1)\pi\beta\int_0^{\pi-\varepsilon\nu}\Big(\sum_{l=1}^{\infty}u_l(\tau)v_l(\tau)\Big)u_k(\tau)\sin k(\tau+\varepsilon\nu)d\tau +$$

$$+ \delta\int_0^{\pi-\varepsilon\nu}(F'(u)v)_k(\tau)\sin k(\tau+\varepsilon\nu)d\tau \ ,$$

where T is given by (2.1), $F'(u)$ is the Frechet derivative of F and (cf. (2.2))

$$(F'(u)v)_k(t) = \frac{2}{\pi}\int_0^{\pi}(F'(u)v)(t,x)\sin kx\, dx \ .$$

Remark 2.2. In (2.15) the difference of the derivatives can be investigated even in the norm of $L(X,X)$ since the term ${}^{\varepsilon}\hat{G}$ which is a linear operator occurs in both the derivatives and therefore disappears in their difference.

3. Existence of periodic solutions in the case $\delta \neq 0$. In this section we will prove the theorem which is the main result of the paper.

Theorem 3.1. <u>Let</u> $\alpha > 0$ <u>and</u> $\beta > 0$. <u>Let</u> $n\in N$. <u>Then there</u>

exists $s \in N$, $q_o \in N$, $\delta_o > 0$ and a function $u^o \in U_{2\pi}$ of the form

$$u^o(t,x) = \sum_{k=1}^{n} a_k^o \sin kt \sin kx \ , \quad \prod_{k=1}^{n} a_k^o \neq 0$$

such that for every ε, $|\varepsilon| = 1/q$, $q \geq q_o$, $q \in N$ and every δ, $|\delta| \leq \delta_o$ there is a function $u = {}^{\delta}u^{\varepsilon} \in U_{2(\pi - \varepsilon\nu)}$, $\nu = \pi/s$ satisfying (1.1) - (1.3) and $\lim_{\varepsilon \to 0} \|{}^{o}u^{\varepsilon} - u^o\|_{2(\pi - \varepsilon\nu)} =$ $= 0$. Moreover, for fixed ε, $\delta \rightarrow {}^{\delta}u^{\varepsilon}$ is a continuous mapping of $[-\delta_o, \delta_o]$ into $U_{2(\pi - \varepsilon\nu)}$.

Proof. We begin by investigating the existence of $u^o \in \mathcal{X}_1$ and $a^o \in \mathcal{X}_2$ which satisfy

(3.1) $$\lim_{\varepsilon \to 0} \|{}^{\varepsilon}G(u^o, a^o, 0)\|_{Y^{\varepsilon}} = 0 .$$

Provided that $u_k^o = 0$, $a_k^o = 0$ for $k > n$, this is equivalent to solving the system, $k = 1,2,\dots,n$

(3.2) $$- u_k(t) + a_k \sin kt = 0 \ , \quad t \in [-2\pi, 2\pi] \ ,$$

(3.3) $$k^2 \nu a_k + \int_0^{\pi} (\alpha - \frac{2\beta}{\pi} \sum_{l=1}^{n} u_l^2(\tau)) u_k(\tau) \sin k\tau \, d\tau = 0 .$$

Supposing $\prod_{k=1}^{n} a_k \neq 0$ and using (3.2) in (3.3), we obtain

(3.4) $$\sum_{l=1}^{n} (2 + \delta_k^l) a_l^2 = \frac{8\alpha}{\pi\beta} + \frac{16k^2}{\pi^2\beta} \nu \ , \quad k = 1,2,\dots,n .$$

For $\nu = 0$ this system has the solution

for $n \to \infty$, indicates the difficulties connected with this general case.

4. Existence of periodic solutions in the case $\delta = 0$. In this case we shall obtain a result which is much more satisfactory than that given by Theorem 3.1 since we need not restrict ourselves to a discrete set of values of ε. The reason is that the form of the associated equations for $\delta = 0$ allows to reduce the problem to a finite dimensional one. Let $\mathcal{X}_1^n = \{u \in \mathcal{X}_1;\ u_k = 0 \text{ for } k > n\}$ and let $\mathcal{X}_2^n$ and $\mathcal{Y}_n^\varepsilon$ be defined by (3.13) and (3.14). The operator ${}^\varepsilon G(.,.,0)$ can be considered as a mapping of $\mathcal{X}_1^n \times \mathcal{X}_2^n$ into $\mathcal{X}_1^n \times \mathcal{Y}_n^\varepsilon$ and thus a solution $(u,a) \in \mathcal{X}_1^n \times \mathcal{X}_2^n$ to (2.10) with $\delta = 0$ will be obtained by applying the implicit function theorem. As in the preceding section we can verify that all the assumptions of this theorem are satisfied. In particular, it follows from (3.15) that the inverse operator to ${}^\varepsilon G'(u^o,a^o,0)$, the Frechet derivative of the operator ${}^\varepsilon G(u,a,0)$ with respect to (u,a), exists for all $\varepsilon \neq 0$ which are sufficiently close to 0. We sum up this outlined result in a theorem.

Theorem 4.1. Let $\alpha > 0$ and $\beta > 0$. Let $n \in N$ and let $\nu > 0$ be sufficiently small. Then there exist $\varepsilon_o > 0$ and a function $u^o \in U_{2\pi}$ of the form

$$u^o(t,x) = \sum_{k=1}^{n} a_k^o \sin kt \sin kx\ , \quad \prod_{k=1}^{n} a_k^o \neq 0\ ,$$

such that for every ε, $0 < |\varepsilon| \leq \varepsilon_o$ there is a function $u = u^\varepsilon \in U_{2(\pi - \varepsilon\nu)}$ satisfying (1.1) - (1.3) with $\delta = 0$ and such that

$$(3.5)\qquad a_1^2 = a_2^2 = \dots = a_n^2 = 8\alpha/\pi\beta\,(2n+1) > 0$$

which satisfies

$$(3.6)\qquad \frac{\pi}{2}\Big(\alpha - \beta\,\frac{\pi}{4}\sum_{l=1}^{n} a_l^2\Big) = \pi\alpha/2(2n+1)\ .$$

Since $\det \|2+\delta_k^l\|_{k,l=1}^n = 2n+1$, we can choose $s \in N$ such that for

$$(3.7)\qquad \nu = \pi/s$$

(this special form of ν is used in proving (3.16)) there exist $a_1^o, a_2^o, \dots, a_n^o$ satisfying (3.4) and in virtue of (3.5), (3.6) satisfying also

$$\prod_{l=1}^{n} a_l^o \neq 0$$

and

$$(3.8)\qquad \pi\alpha/4(2n+1) \leqq \frac{\pi}{2}\Big(\alpha - \beta\,\frac{\pi}{4}\sum_{l=1}^{n}(a_l^o)^2\Big) \leqq \pi\alpha/(2n+1)\ .$$

Setting

$$u_k^o(t) = a_k^o \sin kt\ ,$$

and $u_k^o = 0$, $a_k^o = 0$ for $k > n$, we have $(u^o, a^o) \in \mathfrak{X}_1 \times \mathfrak{X}_2$ for which (3.1) holds.

The form of ${}^{\varepsilon}G$ implies that given $\varkappa > 0$, there exist $\varepsilon_1 > 0$ and $\delta_1 > 0$ such that

$$(3.9)\qquad \sup\big\{\|{}^{\varepsilon}G(u^o, a^o, \delta)\|_{Y^{\varepsilon}}\ ;\ 0 < |\varepsilon| \leqq \varepsilon_1\ ,\quad |\delta| \leqq \delta_1\big\} < \varkappa\ .$$

Hence by (2.11), (2.12), (2.14), (2.15), (3.1) and (3.9) the theorem will be proved by applying Lemma 5.2 as soon as we have verified the following two assertions:

(3.10) There is a family of operators ${}^{\varepsilon}K \in L(X,Y^{\varepsilon})$ such that given $\varkappa > 0$, there are $\varepsilon_2 > 0$ and $\delta_2 > 0$ for which

$$\sup\{\|{}^{\varepsilon}G'(u^{o},a^{o},\delta)-{}^{\varepsilon}K\|_{L(X,Y^{\varepsilon})};\ 0 < |\varepsilon| \leqq \varepsilon_2,\ |\delta| \leqq \delta_2\} < \varkappa .$$

(3.11) There are $m > 0$ and $q_o > 0$ such that for every ε, $|\varepsilon| = 1/q$, $q \geqq q_o$, $q \in N$ we have

$$\|({}^{\varepsilon}K)^{-1}\|_{L(X,Y^{\varepsilon})} \leqq m .$$

We define the operator $K \in L(\mathcal{X}_1 \times \mathcal{X}_2,\ \mathcal{X}_1 \times \mathcal{Y}^{\varepsilon})$ by setting

$${}^{\varepsilon}K = (K^1, {}^{\varepsilon}K^2)$$

where $K^1 \in L(\mathcal{X}_1 \times \mathcal{X}_2, \mathcal{X}_1)$ and ${}^{\varepsilon}K^2 \in L(\mathcal{X}_1 \times \mathcal{X}_2, \mathcal{Y}^{\varepsilon})$ have the components

$$K_k^1(v,c)(t) = -v_k(t) + c_k \sin kt, \quad t \in [-2\pi, 2\pi],$$

$${}^{\varepsilon}K_k^2(v,c) = \frac{k \sin k\varepsilon\nu}{\varepsilon} c_k + z_k^{\varepsilon} v + \hat{z}_k^{\varepsilon} v$$

with

$$z_k^{\varepsilon} v = \int_0^{\pi} (\alpha - \frac{\pi\beta}{2} \sum_{l=1}^{n} (u_l^o(\tau))^2) v_k(\tau) \sin k(\tau + \varepsilon\nu) d\tau ,$$

$$\hat{z}_k^{\varepsilon} v = (-1)\pi\beta \int_0^{\pi} (\sum_{l=1}^{n} u_l^o(\tau) v_l(\tau)) u_k^o(\tau) \sin k(\tau + \varepsilon\nu) d\tau .$$

The assertion (3.10) can now be verified easily using the standard technique applied in proving (2.13). To prove (3.11) let us introduce an operator ${}^{\varepsilon}\mathcal{K} \in L(\mathcal{X}_2, \mathcal{Y}^{\varepsilon})$ by setting its k-th component

$${}^{\varepsilon}\mathcal{K}_k(c) = {}^{\varepsilon}K_k^2(\{c_l \sin lt\}_{l \in N}, c) =$$

$$= \left(\frac{k \sin k\varepsilon\nu}{\varepsilon} + \frac{\pi}{2}\left(\alpha - \frac{\pi\beta}{8}\sum_{l=1}^{n}(a_l^o)^2(2+\delta_k^l)\right)\cos k\varepsilon\nu\right)c_k -$$

$$- \frac{\pi^2\beta}{8} a_k^o \cos k\varepsilon\nu \sum_{l=1}^{n} a_l^o(2+\delta_k^l)c_l \, .$$

As the norm $\|\cdot\|_{L(\mathcal{X}_1,\mathcal{X}_2)}$ of the mappings z^ε and $\hat{z}^\varepsilon$ can be estimated by a constant independent of ε, it follows that (3.11) is an immediate consequence of the following assertion:

(3.12) There are $m_1 > 0$ and $q_o > 0$ such that for every ε, $|\varepsilon| = 1/q$, $q \geqq q_o$, $q \in N$ we have

$$\|({}^\varepsilon\mathcal{K})^{-1}\|_{L(\mathcal{Y}^\varepsilon,\mathcal{X}_2)} \leqq m_1 \, .$$

Let finite dimensional subspaces of $\mathcal{X}_2$ and $\mathcal{Y}^\varepsilon$ be defined by

(3.13) $\mathcal{X}_2^n = \{c \in \mathcal{X}_2;\ c_k = 0 \text{ for } k > n\}$,

(3.14) $\mathcal{Y}_n^\varepsilon = \{d \in \mathcal{Y}^\varepsilon;\ d_k = 0 \text{ for } k > n\}$.

Then $\mathcal{X}_2 = \mathcal{X}_2^n \oplus (\mathcal{X}_2^n)^\perp$, $\mathcal{Y}^\varepsilon = \mathcal{Y}_n^\varepsilon \oplus (\mathcal{Y}_n^\varepsilon)^\perp$ and the spaces $\mathcal{X}_2^n$ and $\mathcal{Y}_n^\varepsilon$ are both isomorphic with R^n. As

$$\lim_{\varepsilon \to 0} {}^\varepsilon\mathcal{K}_k(c) = -\frac{\pi^2\beta}{8} a_k^o \sum_{l=1}^{n} a_l^o(2+\delta_k^l)c_l$$

uniformly for $1 \leqq k \leqq n$, $k \in N$, $\det \|2+\delta_k^l\|_{k,l=1}^n = 2n+1$ and $\prod_k^n a_k^o \neq 0$ we can find $\varepsilon_3 > 0$ and $m_2 > 0$ such that for every $d \in \mathcal{Y}_n^\varepsilon$ there is $c \in \mathcal{X}_2^n$ satisfying ${}^\varepsilon\mathcal{K}(c) = d$ and

(3.15) $\|c\|_{\mathcal{X}_2^n} \leqq m_2 \|d\|_{\mathcal{Y}_n^\varepsilon}$.

For $k > n$, we have

$$\varepsilon \mathcal{K}_k(c) = \left(\frac{k \sin k\varepsilon\nu}{\varepsilon} + \sigma_o \cos k\varepsilon\nu \right) c_k ,$$

where $\sigma_o = \frac{\pi}{2}\left(\alpha - \frac{\pi\beta}{4} \sum_{l=1}^{n} (a_l^o)^2\right)$. Thus by (3.8) σ_o satisfies

$$\pi\alpha/4(2n+1) \leqq \sigma_o \leqq \pi\alpha/(2n+1) .$$

Denoting $\gamma(\varepsilon,k) = \frac{k \sin k\varepsilon\nu}{\varepsilon} + \sigma_o \cos k\varepsilon\nu$, we will prove that there are $m_3 > 0$ and $q_1 > 0$ such that

(3.16) $|\gamma(\varepsilon,k)| \geqq m_3$ for $|\varepsilon| = 1/q$, $q \geqq q_1$, $q \in N$ and $k \in N$.

As ν is given by (3.7) we put $q_1 = [5\pi\alpha/2(2n+1)] + 1$ and we will distinguish three cases.

(i) Let $\sin k\varepsilon\nu = 0$. Then $|\gamma(\varepsilon,k)| = \sigma_o$.

(ii) Let $\sin k\varepsilon\nu \neq 0$ and $|k\varepsilon\nu| \leqq \pi/4$. Then

$$|\gamma(\varepsilon,k)| = \frac{k \sin k\varepsilon\nu}{\varepsilon} + \sigma_o \cos k\varepsilon\nu \geqq \sigma_o \cos \pi/4 .$$

(iii) Let $\sin k\varepsilon\nu \neq 0$ and $|k\varepsilon\nu| > \pi/4$. Since $k\varepsilon\nu = k\frac{\pi}{qs}$, we have $k > \frac{qs}{4} \geqq \frac{q_1 s}{4}$ and therefore

$$|\gamma(\varepsilon,k)| \geqq \frac{k}{|\varepsilon|}|\sin k\varepsilon\nu| - \sigma_o \geqq \frac{k}{|\varepsilon|}\sin\frac{\pi}{qs} - \sigma_o \geqq$$

$$\geqq \frac{2k}{s} - \sigma_o \geqq \frac{q_1}{2} - \sigma_o \geqq \pi\alpha/4(2n+1) > 0 ,$$

which proves (3.16). The estimates (3.16) and (3.15) imply (3.12). This completes the proof.

Remark 3.1. We are not able deal with the existence of periodic solutions bifurcating from a point u^o for which the set $\{k \in N;\ u_k^o \neq 0\}$ is infinite. The relation (3.5), being limited

$$\lim_{\varepsilon \to 0} \|u^{\varepsilon} - u^{0}\|_{2(\pi - \varepsilon\nu)} = 0 .$$

Remark 4.1. For $n = 1$ and for any value of ε and ν we are able to give periodic solutions explicitely by means of elliptic functions.

5. Auxiliary assertions. This section contains two auxiliary assertions. The second one is the implicit function theorem given in a form which is very suitable for proving theorems on the existence of periodic solutions to partial differential equations. This fact has been shown by J.P. Fink and W.S. Hall in [1] .

The first assertion is used in the proof of the second and can be verified immediately.

Lemma 5.1. Let X and Y be Banach spaces. Let $T \in L(X,Y)$ with $T^{-1} \in L(Y,X)$. Let $\Delta \in L(X,Y)$, $\|\Delta\|_{L(X,Y)} \leq$ $\leq (2\|T^{-1}\|_{L(Y,X)})^{-1}$. Then

(i) $(T+\Delta)^{-1} \in L(Y,X)$,

(ii) $\|(T+\Delta)^{-1}\|_{L(Y,X)} \leq 2\|T^{-1}\|_{L(Y,X)}$,

(iii) $\|(T+\Delta)^{-1} - T^{-1}\|_{L(Y,X)} \leq 2\|T^{-1}\|^{2}_{L(Y,X)} \cdot \|\Delta\|_{L(X,Y)}$.

Lemma 5.2. Let $r_0 > \hat{r}$, δ_0 and m be positive numbers. Let X be a Banach space, $x^0 \in X$. Let $E_0 \subset R$ be such that 0 is an accumulation point of E_0 . For every $\varepsilon \in E_0$, let Y^{ε} be a Banach space. Let ${}^{\varepsilon}G : B(x^0,r_0,X) \times [-\delta_0, \delta_0] \to Y^{\varepsilon}$ and ${}^{\varepsilon}K \in$ $\in L(X,Y^{\varepsilon})$ be mappings satisfying:

(i) For every $\varepsilon \in E_0$, ${}^{\varepsilon}G : B(x^0,r_0,X) \times [-\delta_0, \delta_0] \to Y^{\varepsilon}$ is continuous.

(ii) For every $\varepsilon \in E_0$ and $\delta \in [-\delta_0, \delta_0]$, there exists ${}^{\varepsilon}G'_x : B(x^0, r_0, X) \times [-\delta_0, \delta_0] \to L(X, Y^{\varepsilon})$ and for every $\varepsilon \in E_0$, the mapping $\delta \to {}^{\varepsilon}G'_x(x^0, \delta)$ is continuous.

(iii) $\sup\{\| {}^{\varepsilon}G'_x(x, \delta) - {}^{\varepsilon}G'_x(x^0, \delta)\|_{L(X,Y^{\varepsilon})};\ \varepsilon \in E_0,\ \delta \in$ $\in [-\delta_0, \delta_0],\ \|x - x^0\|_X < r_0\} < 1/4m$.

(iv) $\sup\{\| {}^{\varepsilon}G'_x(x^0, \delta) - {}^{\varepsilon}K\|_{L(X,Y^{\varepsilon})};\ \varepsilon \in E_0,\ \delta \in$ $\in [-\delta_0, \delta_0]\} < 1/2m$.

(v) $\sup\{\| {}^{\varepsilon}G(x^0, \delta)\|_{Y^{\varepsilon}},\ \varepsilon \in E_0,\ \delta \in [-\delta_0, \delta_0]\} \leqq 1/4m$.

(vi) $\| {}^{\varepsilon}G(x^0, 0)\|_{Y^{\varepsilon}} \to 0$ as $\varepsilon \to 0$, $\varepsilon \in E_0$.

(vii) For every $\varepsilon \in E_0$ there is $({}^{\varepsilon}K)^{-1} \in L(Y^{\varepsilon}, X)$ such that $\|({}^{\varepsilon}K)^{-1}\|_{L(Y^{\varepsilon}, X)} \leqq m$.

Then for every $\varepsilon \in E_0$ and $\delta \in [-\delta_0, \delta_0]$ there exists a unique ${}^{\delta}x^{\varepsilon} \in B(x^0, r_0, X)$ satisfying ${}^{\varepsilon}G({}^{\delta}x^{\varepsilon}, \delta) = 0$. Moreover, ${}^{0}x^{\varepsilon} \to x^0$ as $\varepsilon \to 0$, $\varepsilon \in E_0$ and for every $\varepsilon \in E_0$ the function $\delta \to {}^{\delta}x^{\varepsilon}$ is continuous.

Proof. The parameters ε and δ are elements of E_0 and $[-\delta_0, \delta_0]$ respectively. By (iv), (vii) and Lemma 5.1 we have

$$(5.1) \quad \|({}^{\varepsilon}G'_x(x^0, \delta))^{-1}\|_{L(Y^{\varepsilon}, X)} \leqq 2m .$$

We put ${}^{\delta}x_0^{\varepsilon} = x^0$ and ${}^{\delta}x_{n+1}^{\varepsilon} = {}^{\delta}x_n^{\varepsilon} - ({}^{\varepsilon}G'_x(x_0, \delta))^{-1}\, {}^{\varepsilon}G({}^{\delta}x_n^{\varepsilon}, \delta)$ if ${}^{\delta}x_k^{\varepsilon} \in B(x^0, r_0, X)$ for $k = 1, 2, \dots, n$. By (5.1) we have, $k = 1, 2, \dots, n$,

$$(5.2) \qquad \|{}^{\delta}x^{\varepsilon}_{k+1} - {}^{\delta}x^{\varepsilon}_{k}\|_X \leq 2m \|{}^{\varepsilon}G({}^{\delta}x^{\varepsilon}_{k}, \delta)\|_{Y^{\varepsilon}} .$$

The relation

$${}^{\varepsilon}G({}^{\delta}x^{\varepsilon}_{k-1}, \delta) + {}^{\varepsilon}G'_x(x^o, \delta)({}^{\delta}x^{\varepsilon}_{k} - {}^{\delta}x^{\varepsilon}_{k-1}) = 0 ,$$

the estimate (8.6.2) of [4] and (iii) yield

$$(5.3) \qquad \|{}^{\varepsilon}G({}^{\delta}x^{\varepsilon}_{k}, \delta)\|_{Y^{\varepsilon}} =$$

$$= \|{}^{\varepsilon}G({}^{\delta}x^{\varepsilon}_{k}, \delta) - {}^{\varepsilon}G({}^{\delta}x^{\varepsilon}_{k-1}, \delta) - {}^{\varepsilon}G'_x(x^o, \delta)({}^{\delta}x^{\varepsilon}_{k} - {}^{\delta}x^{\varepsilon}_{k-1})\|_{Y^{\varepsilon}} \leq \sup\{\|{}^{\varepsilon}G'_x(x, \delta) - {}^{\varepsilon}G'_x(x^o, \delta)\|_{L(X,Y^{\varepsilon})};$$

$$\varepsilon \in E_o, \quad \delta \in [-\delta_o, \delta_o], \quad x \in B(x^o, r_o, X)\} \cdot \|{}^{\delta}x^{\varepsilon}_{k} - {}^{\delta}x^{\varepsilon}_{k-1}\|_X \leq \|{}^{\delta}x^{\varepsilon}_{k} - {}^{\delta}x^{\varepsilon}_{k-1}\|_X /4m .$$

Combining (5.2) and (5.3) we have

$$(5.4) \qquad \|{}^{\delta}x^{\varepsilon}_{k+1} - {}^{\delta}x^{\varepsilon}_{k}\|_X \leq \tfrac{1}{2} \|{}^{\delta}x^{\varepsilon}_{k} - {}^{\delta}x^{\varepsilon}_{k-1}\|_X$$

for $k = 1,2,\dots,n$. By (5.4) and (5.2) we obtain

$$(5.5) \qquad \|{}^{\delta}x^{\varepsilon}_{k+1} - x^o\|_X \leq (1 + \tfrac{1}{2} + \tfrac{1}{4} + \dots) \|{}^{\delta}x^{\varepsilon}_{1} - x^o\|_X \leq$$

$$\leq 4m \|{}^{\varepsilon}G(x^o, \delta)\|_{Y^{\varepsilon}} .$$

By (v) this gives $\|{}^{\delta}x^{\varepsilon}_{k+1} - x^o\|_X \leq \hat{r}$. Thus ${}^{\delta}x^{\varepsilon}_{n} \in B(x^o, r_o, X)$ for all $n \in N$. By (5.4) we can put ${}^{\delta}x^{\varepsilon} = \lim_{n \to +\infty} {}^{\delta}x^{\varepsilon}_{n}$. Then ${}^{\varepsilon}G({}^{\delta}x^{\varepsilon}, \delta) = 0$ is a consequence of (5.3) and (i). Since ${}^{\delta}x^{\varepsilon}_{n}$ is continuous in δ for fixed $\varepsilon \in E_o$, by (i) and (ii), the continuity of ${}^{\delta}x^{\varepsilon}$ follows from (5.4) which ensures uniform convergence of ${}^{\delta}x^{\varepsilon}_{k}$ on $[-\delta_o, \delta_o]$. By (5.5) and (vi) we have

${}^{o}x^{\varepsilon} \longrightarrow x^{o}$ as $\varepsilon \to 0$, $\varepsilon \in E_o$. Uniqueness of ${}^{\delta}x^{\varepsilon}$ can be proved easily. This completes the proof.

References

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