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TOTALLY BALANCED AND EXPONENTIALLY BALANCED GRAY CODES

A. J. van Zanten, I. N. Suparta[*)]

The method of Robinson and Cohn to construct balanced and totally balanced Gray codes is discussed, as well as the extended version of this method by Bhat and Savage. We introduce a slight generalization of their construction which enables us to prove a long standing conjecture of Wagner and West about the existence of Gray codes having a specific spectrum of transition counts, i. e., all transition counts are powers of 2 and the exponents of these powers differ at most 1. Such a Gray code can be considered as generalization of a totally balanced Gray code when the length of the codewords is not a 2-power.

1. Introduction

A Gray code of length n is an ordered list of 2^n n-bit strings (codewords) such that successive codewords differ in exactly one bit position. In this paper, $G(n)$ stands for a binary Gray code of length n. A comprehensive review of Gray codes can be found in [7]. The best known example of a Gray code is the binary reflected Gray code which is defined recursively as

$$G(1) = 0, 1; \qquad G(n) := 0G(n-1), 1G^R(n-1), \tag{1}$$

where $iG(n-1)$ is the list $G(n-1)$ such that each codeword is preceded by the integer $i \in \{0, 1\}$ and $G^R(n-1)$ stands for the list $G(n-1)$ in reversed order. This code is sometimes referred to as the standard Gray code; in this paper we shall denote it by $G_{st}(n)$. Thus, the standard Gray code $G_{st}(2)$ of length 2 is equal to the list 00, 01, 11, 10, and the one of length 3 is the list 000, 001, 011, 010, 110, 111, 101, 100.

If the last codeword of a Gray code also differs in only one bit position from the first codeword, we call the Gray code *cyclic*, and otherwise *non-cyclic*. So, the standard Gray code is a cyclic code.

We shall index the codewords in the list of a Gray code of length n from 0 until $2^n - 1$ and denote the ith codeword by $\mathbf{g}_i$. If $G(n)$ is a cyclic code,

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[*)] On leave from Department of Mathematics Singaraja, Bali Indonesia we shall identify $\mathbf{g}_{2^n}$ and $\mathbf{g}_0$. The bit positions will be labeled from *right* to *left* by $1, 2, \ldots, n$. The integer $s_i \in [n] := \{1, 2, \ldots, n\}$, indicating which bit position changes when going from the codeword $\mathbf{g}_{i-1}$ to $\mathbf{g}_i$, is called the *transition number* of the codeword $\mathbf{g}_{i-1}$. The ordered sequence of all transition numbers of a Gray code is called the transition sequence of the code and denoted by S_n. Thus, the transition sequence S_n of a Gray code of length n is equal to the sequence $s_1, s_2, \ldots, s_{2^n-1}, s_{2^n}$ if $G(n)$ is a cyclic code, and it is equal to $s_1, s_2, \ldots, s_{2^n-1}$ if $G(n)$ is a non-cyclic code. Here s_{2^n} is the transition number of the last codeword when going to the first codeword of $G(n)$. Thus, the transition sequence of the standard Gray code of length 3 is $S_3 = 1, 2, 1, 3, 1, 2, 1, 3$. In general, the transition sequence S_n of the standard Gray code $G_{st}(n)$ can be defined recursively by

$$S'_1 = 1, \quad S'_n = S'_{n-1}, n, S'_{n-1}, \quad S_n = S'_n, n, \tag{2}$$

where S'_n stands for the transition sequence of the non-cyclic standard Gray code. The sequences S'_n and S_n are sometimes referred to as the incomplete and the complete transition sequence of $G_{st}(n)$. The number of times the integer i occurs in the transition sequence of an n-bit Gray code is called the *transition count* of the integer i and will be denoted by $TC_n(i)$. With respect to the list of codewords, $TC_n(i)$ refers to the number of times that bit i changes, from 0 to 1 or from 1 to 0, in the column i. If $G(n)$ is a cyclic code, then it will be clear that $TC_n(i)$ is even for every $i \in [n]$, and moreover that $\sum_{i=1}^{n} TC_n(i) = 2^n$. The standard Gray code $G_{st}(n)$ of length n has the following transition counts

$$TC_n(i) = \begin{cases} 2^{n-i}, & \text{if } 1 \leqslant i \leqslant n-1, \\ 2, & \text{if } i = n. \end{cases} \tag{3}$$

The list of transition counts $(TC_n(1), TC_n(2), \ldots, TC_n(n))$ corresponding to some Gray code $G(n)$ will be called its *transition count spectrum*. In some applications (cf. [3–6], [9]), it appears that the more uniform the distribution of the transition counts over the integers in the set $[n]$, i.e., the smaller the differences between the various numbers $TC_n(i), 1 \leqslant i \leqslant n$, the better the code will be. An n-bit Gray code with transition counts satisfying $|TC_n(i) - TC_n(j)| \leqslant 2$ for every $i \geqslant 1$ and $j \leqslant n$ is called *balanced*, and it is called *totally balanced* if $TC_n(i) = TC_n(j)$ for all i and j. Since $\sum_{i=1}^{n} TC_n(i) = 2^n$, a necessary condition for a Gray code to be totally balanced is that n is equal to a power of 2. So, the standard Gray codes of length 1, 2, and 3 are balanced codes and, moreover, those of length 1 and 2 are totally balanced codes. However for $n \geqslant 4$ the standard Gray code $G_{st}(n)$ is not balanced.

In [6] Robinson and Cohn introduced an interesting method for the construction of balanced Gray codes of length n based on a known $(n-2)$-bit balanced Gray code. Robinson and Cohn claimed, without giving a complete proof, that by applying their technique balanced Gray codes can be produced for any $n \geqslant 1$. Their approach requires a special subsequence of the transition sequence of the $(n-2)$-bit Gray code. In [1] Bhat and Savage present a method to construct such a subsequence, thus completing the proof for the existence and the construction of balanced Gray codes for all values of $n \geqslant 1$. In Section 3 we introduce a different and slightly more general method for the construction of subsequences as mentioned above. This method appears to be simpler than the one in [1]. At the end of Section 3 we prove that if n is a 2-power, a balanced Gray code is always totally balanced. In Section 4 we apply our method to the construction of Gray codes with a special transition count spectrum, the existence of which was conjectured by Wagner and West in [10]. Gray codes having such a transition count spectrum can be considered as generalizations of totally balanced Gray codes in the sense that all transition count numbers are 2-powers with exponents as close as possible. More precisely, these exponents are either equal or differ from each other by 1. For this reason one could call such a Gray code an *exponentially balanced* Gray code.

2. The Robinson-Cohn construction

The construction of Robinson and Cohn [6] for obtaining cyclic Gray codes is an extension of the ultra-composite method introduced by Gilbert in [2]. Their approach is based on the well-known fact that a Gray code of length n corresponds to a Hamiltonian cycle in an n-cube. To obtain such a cycle, they combine Hamiltonian cycles in four copies of the $(n-2)$-cube. The four subcubes are labeled by 00, 01, 11 and 10 (cf. Fig.1) which refer to the two leftmost bits in the codewords of length n. The outlines of the construction are as follows (cf. also [1, Section 2]).

Construction A

1. Let

$$S_{n-2} := s_1, s_2, \ldots, s_{2^{n-2}} \tag{4}$$

be the transition sequence of some Gray code of length $n-2$. Select a subsequence

$$T := t_1, t_2, \ldots, t_{l-1}, t_l \tag{5}$$

of S_{n-2} with l even such that t_1 and t_2, as well as t_{l-1} and t_l are consecutive in S_{n-2}.

2. After having inserted four copies of the transition sequence S_{n-2} in the n-cube, one proceeds by deleting a number of t_i-numbers according to the rules:

   - from the subcube 00, all odd-indexed elements $t_1, t_3, \dots, t_{l-1}$ of T are deleted;
   - from the subcube 01 the elements $t_2, t_3, \dots, t_l$ are deleted;
   - from the subcube 11, all even-indexed elements $t_2, t_4, \dots, t_l$ are deleted;
   - from the subcube 10, only t_1 is deleted.

3. The four subcubes are connected as is illustrated in Fig.1.

We remark here that Fig. 1 is a slightly altered version of Fig. 7 in [6].

The dots in this figure are vertices in the n-cube, and hence they represent codewords of length n, and the elements t_a label those edges of the cube which correspond to a transition from one codeword to the next codeword in the relevant subcube. From this picture it will be evident that the resulting path is a Hamiltonian cycle, since it is closed and all vertices of the n-cube are incident with this path precisely once. The resulting transition sequence S_n of the Gray code is obtained by following the path. This sequence S_n can, in a schematic way, be written as

$$S_n = T'^R, T'', \tag{6}$$

where T' and T'' are modified sequences T. The modifications consist of inserting a certain subsequence if there is a "gap"in T, i.e., if $t_i = s_j, t_{i+1} = s_{j+k}$ with $2 \leqslant i \leqslant l-2, k \geqslant 2$, where s_j and s_{j+k} are elements of the sequence S_{n-2} in (4). The treatment of such a gap is handled as sketched in Fig. 2.

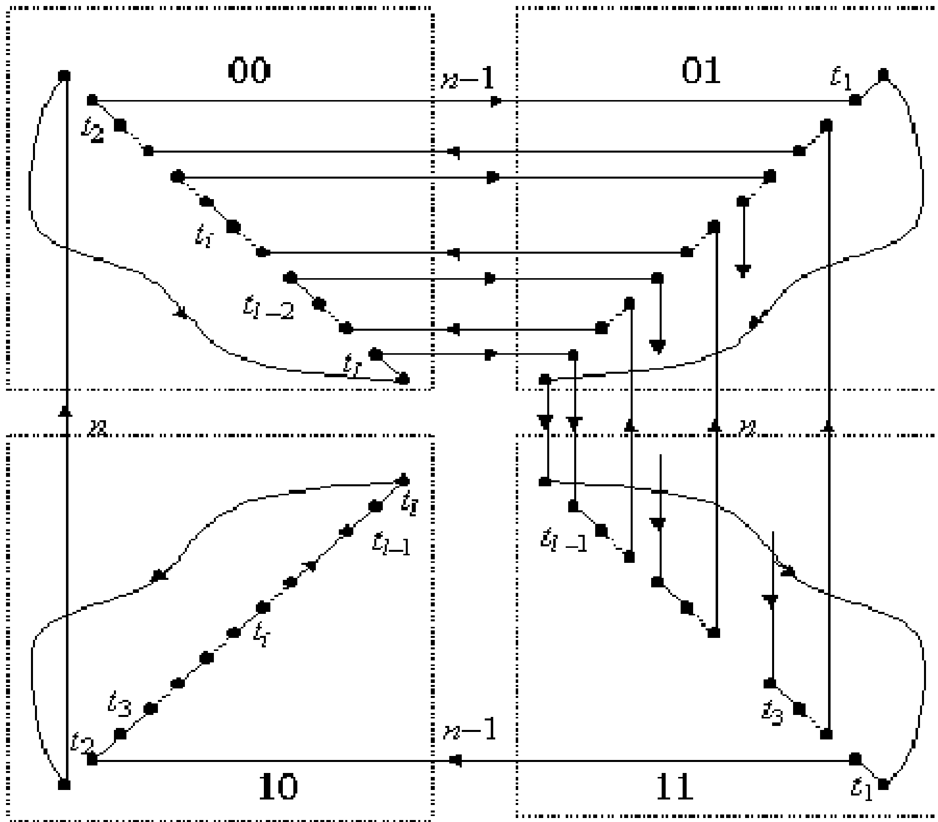

Fig. 1. Gray code construction

Example 1. We take the transition sequence of the standard Gray code $G_{st}(3)$ of length 3: $S_3 = \underline{1}, \underline{2}, \underline{1}, 3, \underline{1}, 2, \underline{1}, \underline{3}$.

Furthermore, we take $l = 6$ and define a subsequence T consisting of the underlined elements of S_3. As one can see the elements t_3, t_4 and t_4, t_5 are not consecutive in S_3. Following the path sketched in Fig. 1 and in Fig. 2, we obtain the following transition sequence of a 5-bit Gray code:

$$\underline{3}, 4, 5, \underline{1}, 2, 5, 2, 4, 2, \underline{1}, 3, 4, 3, 5, 3, \underline{1}, 5, 4, \underline{2}, 4, \underline{1}, 5, \underline{1}, 4, \underline{2}, \underline{1}, 3, \underline{1}, 2, \underline{1}, \underline{3}, 5.$$

This code is a balanced Gray code, since its transition counts are $TC_5(1) = 8$ and $TC_5(2) = TC_5(3) = TC_5(4) = TC_5(5) = 6$. Of course, when starting from some transition sequence S_{n-2}, the resulting Gray code will depend on the selected subsequence $T = t_1, t_2, \ldots, t_l$. Let $TC_n(i)$ denote the transition count of position i in the constructed Gray code of length n, and let $TC_{n-2}(i)$ be its counterpart in the original Gray code of length $n - 2$

for $1 \leqslant i \leqslant n-2$. From the construction rules and from the picture in Fig. 1, it follows that if a position i occurs $b(i)$ times in T, there are $2b(i)$ transitions for that particular position which will be deleted when constructing S_n. Hence, the number of times that i occurs in S_n is equal to

$$TC_n(i) = 4TC_{n-2}(i) - 2b(i) \tag{7}$$

for $1 \leqslant i \leqslant n-2$. Furthermore, it will be obvious, again from the construction rules and from Fig. 1, that

$$TC_n(n-1) = TC_n(n) = l. \tag{8}$$

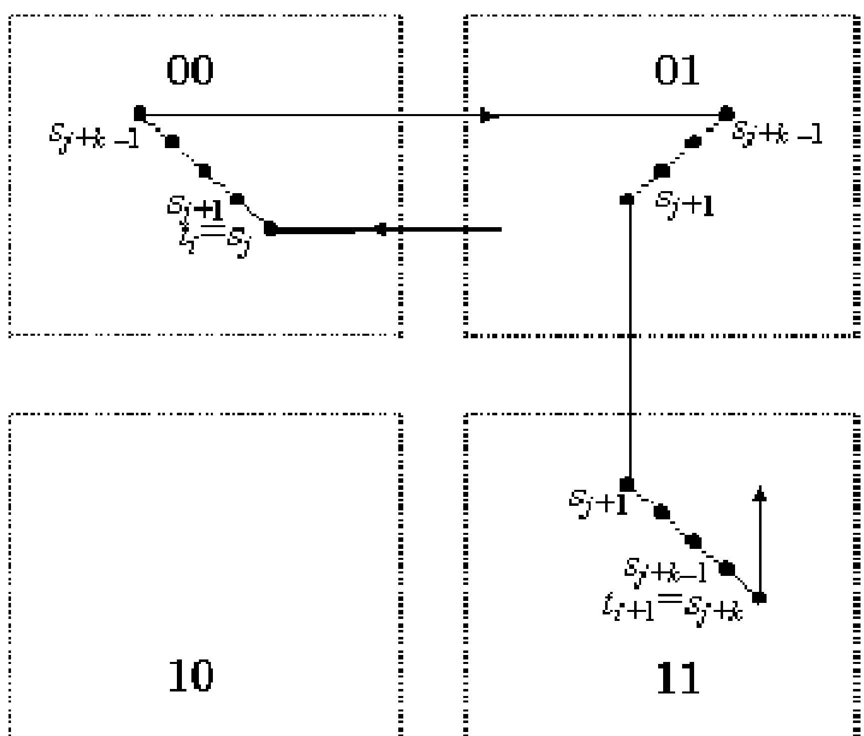

Fig. 2. Gap in detail

Robinson and Cohn in [6] claimed without proof, that T can always be chosen such that if the original Gray code of length $n-2$ is balanced, the produced Gray code of length n is balanced too. Bhat and Savage in [1] proved the existence of such a subsequence T for all $n \geqslant 3$. Actually, they showed in an inductive way that if S_{n-2} is balanced, it always has a subsequence T

with occurrence numbers $b(i)$, $1 \leqslant i \leqslant n-2$, such that the right hand sides of equations (7) and (8) are equal to a_n or to $a_n + 2$, where a_n is defined as the unique even integer satisfying

$$a_n \leqslant \frac{2^n}{n} < a_n + 2. \tag{9}$$

The following theorem holds.

Theorem 1 [Robinson, Cohn, Bhat, Savage]. *For all $n \geqslant 1$, there exists an n-bit balanced Gray code, and if n is a power of 2, there exists an n-bit totally balanced Gray code.*

In [8], along similar lines, a more straightforward proof is given for the existence of a subsequence T with appropriate numbers $b(i)$.

We remark that a totally balanced Gray code of length $n = 2^v$ has transition count spectrum $(2^{n-v}, \ldots, 2^{n-v})$.

In the next section we shall present a completely different proof of a slightly more general theorem. It appears to be possible to weaken the required conditions for the Robinson–Cohn construction, i.e., one can drop the condition that the last two transition numbers t_{l-1} and t_l of T should be consecutive.

3. An extended Robinson-Cohn construction

We consider the transition sequence S_{n-2} of some Gray code $G(n-2)$. If u and v are subsequences of S_{n-2}, the concatenation of u and v will be denoted by uv. For every subsequence u of S_{n-2} we define

$$u(n-1, n) = u, n-1, u^R, n, u \tag{10}$$

and

$$u(n, n-1) = u, n, u^R, n-1, u, \tag{11}$$

where u^R stands for the reversed sequence of u. We emphasize that u may be the empty sequence. Next, we present the construction of a Gray code $G(n)$ based on the transition sequence of a Gray code $G(n-2)$.

Construction B

1. Let l be an even positive integer. Partition S_{n-2} as

$$S_{n-2} = s_{i_1}, u_0, s_{i_2}, u_1, s_{i_3}, u_2, \ldots, s_{i_{l-1}}, u_{l-2}, s_{i_l}, v,$$

   where $i_1 = 1$, $i_2 = 2$, $u_0 = \varnothing$ (the empty sequence) and $u_1, u_2 \ldots, u_{l-2}, v$ are proper subsequences of S_{n-2} which may be empty.

2. Replace $u_0, u_1, u_2, \ldots, u_{l-2}$ by $n-1, u_1(n-1,n), u_2(n,n-1), \ldots,$ $u_{l-2}(n,n-1)$ respectively yielding the sequence U.
3. Let $V = v^R, n, v$ and $W = n-1, S_{n-2}, n$.
4. Interchange $n-1$ and s_1 in W, giving the sequence W'.
5. Define $S_n := U^R, V, W'$.

For the sake of convenience we shall write S instead of S_n in the next.

Theorem 2. *The sequence* $S := U^R, V, W'$ *is the transition sequence of a Gray code* $G(n)$.

Proof.We remark that the occurrences of the integers n and $n-1$ in the sequence S alternate according to the following pattern

$$n-1, n, n, n-1, n-1, n, n, \ldots, n-1, n-1, n, n, n-1, n-1, n, n-1, n.$$

It is obvious that all integers in S occur an even number of times. Below we shall show that any proper subsequence X of S contains at least one integer which occurs an odd number of times. For the sake of convenience we say that X satisfies property P. According to a well-known criterion, the sequence S is the complete transition sequence of a Gray code if and only if property P holds for any proper subsequence of S which consists of a number (> 0) of consecutive elements. A proper subsequence X of S has length less than the length of S, and all its consecutive elements are also consecutive in S. More in particular, if $Z := \{i \in [n] | i$ occurs an odd number of times in $X\}$, then we shall say that X satisfies $P(Z)$. The proof of Theorem will be accomplished by considering a number of distinct cases.

Case 1. $X \subseteq U^R$. If $X = U^R$, then X satisfies $P(\{n-1\})$. This property is the implication of l being an even integer. Whenever $X \neq U^R$ and the integers $n-1$ and n occur an even number of times in X', then, with respect to the occurrence of integers an odd number of times, X is equivalent with a proper subsequence of S_{n-2}. Thus, X satisfies P.

Case 2. $X \subseteq V$. Here it is obvious that X satisfies property P, since if X does not satisfy $P(\{n\})$, then X is a proper subsequence of S_{n-2}.

Case 3. $X \subseteq W'$. Because of rule 4 and the properties of S_{n-2}, it is clear that X satisfies P.

Case 4. $X = X_1, X_2, X_3$ with $X_1 = X \cap U^R$, $X_2 = X \cap V$, $X_3 = X \cap W'$, and $X_1 \neq \varnothing$ or $X_3 \neq \varnothing$. We consider three subcases.

- Let $X_3 = \varnothing$, and hence $X = X_1, X_2$. If $U^R \subseteq X$ or $V \subseteq X'$, then X satisfies at least $P(\{n\})$ or $P(\{n-1\})$. Assume that U^R and V are not subsequences of X'. If X does not contain any of the integers n and $n-1$ an odd number of times, then the sequence X', with respect to the occurrence of integers an odd number of times, is equivalent to a proper subsequence of S_{n-2}. So, X satisfies P.

- Let $X_1 = \varnothing$. If X does not satisfy both $P(\{n\})$ and $P(\{n-1\})$, then X must be some proper subsequence vs_1 of S_{n-2}. Here we use the assumption that S_{n-2} is the transition sequence of a *cyclic* Gray code. So, again X satisfies P.

- Let $X_1 \neq \varnothing$ and $X_3 \neq \varnothing$. It means that $X_2 = V = v^R, n, v$. If $X_3 = s_1$, it will be clear that X satisfies $P(\{n\})$ or $P(\{n-1\})$ or contains a subsequence which is — with respect to the occurrence of integers an odd number of times — equivalent to a proper subsequence of S_{n-2}. Thus X satisfies property P. If X_3 contains the integer $n-1$ but does not contain n, the pattern of the occurrence of the integers $n-1$ and n, as remarked at the beginning of this proof, will imply that X at least satisfies $P(\{n\})$ or $P(\{n-1\})$. If X_3 contains n, or equivalently $X_3 = W'$, then since $X_1 \neq U^R$ (remember that X is a proper subsequence of S), X will satisfy $P(\{n\})$ or $P(\{n-1\})$ or contain a subsequence which is, with respect to the occurrence of integers an odd number of times, equivalent to a proper subsequence of S_{n-2}. Thus X satisfies property P. Theorem 3 is proved.

Remark that the subsequence $s_{i_1}, s_{i_2}, s_{i_3}, \ldots, s_{i_l}$ in Construction B is here considered to be a subsequence of S_{n-2} although not every pair of elements s_j, s_{j+1} need to be consecutive in S_{n-2}. If the subsequence u_{l-2} of S_{n-2} in rule 1 is empty, we have a subsequence T as mentioned in Construction A and our construction is equivalent to Construction A (the Robinson–Cohn construction) with $t_j = s_{i_j}, 1 \leqslant j \leqslant l$. In this sense our construction is a generalization of the Robinson–Cohn construction. From now on, the subsequence $s_{i_1}, s_{i_2}, s_{i_3}, \ldots, s_{i_l}$, is also referred to as subsequence T.

Since the resulting sequence S_n is a complete transition sequence, all transition counts are even positive integers, and at least two transition counts are equal to l, i.e., $TC_n(n-1)$ and $TC_n(n)$. Of course, the subsequence T can be produced if and only if $0 \leqslant b(i) \leqslant TC_{n-2}(i)$ or, applying (7), whenever $2TC_{n-2}(i) \leqslant TC_n(i) \leqslant 4TC_{n-2}(i)$ for all $i, 1 \leqslant i \leqslant n-2$.

Example 2. If we start with a 3-bit Gray code with transition counts (2, 2, 4) and with transition sequence $S_3 = 3, 2, 3, 1, 3, 2, 3, 1,$ then our method can not be used to construct a 5-bit Gray code with transition counts (2,

2, 2, 10, 16), since $10 > 4.2$. Neither can we construct a Gray code with transition counts (2, 4, 6, 8, 12), since all these values are different. However, we can construct a 5-bit Gray code with transition counts (4, 4, 8, 8, 8) by taking the subsequence T equal to S_3 itself. The resulting 5-bit Gray code has transition sequence
$S_5 = 1, 4, 5, 3, 5, 4, 2, 4, 5, 3, 5, 4, 1, 4, 5, 3, 5, 4, 2, 4, 3, 5, 3, 4, 2, 3, 1, 3,\ 2, 3, 1, 5.$

In order to formulate our next theorem, we shall denote the partition of a positive integer N into n positive even integers in non-decreasing order by $E_n(N) = (p_1, p_2, ..., p_n)$, i.e., $p_i \leqslant p_{i+1}, 1 \leqslant i \leqslant n-1$.

Theorem 3. *Let $G(n-2)$ be an $(n-2)$-bit Gray code with transition counts $TC_{n-2}(i)$, $1 \leqslant i \leqslant n-2$, which are ordered in non-decreasing order, and let $E_n(2^n) = (p_1, p_2, \ldots, p_n)$ be a partition of 2^n into n positive integers. Then the extended Robinson-Cohn construction yields an n-bit Gray code with transition count spectrum $(p_1, p_2, \ldots, p_n)$ if and only if*

(i) *$p_k = p_{k+1}$ for some k, $1 \leqslant k \leqslant n-1$;*

(ii) *$2TC_{n-2}(i) \leqslant p_i \leqslant 4TC_{n-2}(i)$ for every i, $1 \leqslant i < k$;*
$2TC_{n-2}(i) \leqslant p_{i+2} \leqslant 4TC_{n-2}(i)$ for every i, $k \leqslant i \leqslant n-2$;

(iii) *there exist at least two bit positions, $j, j' \in [n]\backslash\{k, k+1\}$, such that $p_j < 4TC_{n-2}(i)$ and $p_j' < 4TC_{n-2}(i')$, for some i and i', $1 \leqslant i, i' \leqslant n-2$, and such that i and i' are consecutive integers in the transition sequence S_{n-2} of $G(n-2)$.*

Proof. Let $G(n)$ be the n-bit Gray code constructed by the extended Robinson–Cohn construction with transition counts $p_j, 1 \leqslant j \leqslant n$. According to the construction rules, it is obvious that $G(n)$ has at least two transition counts which are equal to l, the length of sequence T. More precisely, $TC_n(n-1)$ and $TC_n(n)$ are indeed equal to l, and so condition (i) holds for $k = n-1$. Condition (ii) is obvious because of (7). Since s_1, s_2 in the subsequence T are consecutive, by setting $i = s_1$ and $i' = s_2$, we must have at least two bit positions, say j and j' with $j, j' \in [n]\backslash\{k, k+1\}$, satisfying $p_j < 4TC_{n-2}(i)$ and $p_{j'} < 4TC_{n-2}(i')$, and hence condition (iii) holds. So, the only-if-part of Theorem is true. To prove the if-part, we define

$$b(i) = \begin{cases} \dfrac{4TC_{n-2}(i) - p_i}{2}, & 1 \leqslant i < k, \\ \\ \dfrac{4TC_{n-2}(i) - p_{i+2}}{2}, & k \leqslant i \leqslant n-2, \end{cases} \tag{12}$$

where the numbers p_i are the integers from a given partition $E_n(2^n)$. We have that $p_k := \sum\limits_{i=1}^{n-2} b(i) > 1$, since there exist at least two bit positions, j and j'

with $j, j' \in [n] \backslash \{k, k+1\}$, such that $p_j < 4TC_{n-2}(i)$ and $p_{j'} < 4TC_{n-2}(i')$ for some i and i', $1 \leqslant i, i' \leqslant n-2$. Moreover, since i and i' are consecutive, the subsequence T of length p_k can be chosen such that $s_1 = i$ and $s_2 = i'$ are consecutive. Following our construction, the resulting Gray code $G(n)$ will have transition counts $p_j, 1 \leqslant j \leqslant n$. Theorem 3 is proved.

Example 3. Starting with the standard Gray code $G_{st}(3)$ with transition sequence 1, 2, 1, 3, 1, 2, 1, 3 and transition count spectrum (2, 2, 4), we are able to produce a 5-bit Gray code with transition count spectrum (4, 4, 8, 8, 8). As subsequence T we choose the complete transition sequence of $G_{st}(3)$. The resulting Gray code has transition sequence

$$3, 4, 5, 1, 5, 4, 2, 4, 5, 1, 5, 4, 3, 4, 5, 1, 5, 4, 2, 4, 1, 5, 1, 4, 2, 1, 3, 1, 2, 1, 3, 5.$$

We remark that, starting from $G_{st}(3)$, we can construct Gray codes with the following transition count spectra:

(2, 2, 6, 8, 14), (4, 4, 6, 6, 12), (4, 4, 8, 8, 8),

(4, 4, 4, 6, 14), (4, 4, 6, 8, 10), (4, 6, 6, 8, 8),

(4, 4, 4, 8, 12), (4, 6, 6, 6, 10), (6, 6, 6, 6, 8).

Notice that for instance a Gray code with transition count spectrum (2, 2, 8, 8, 12) cannot be produced by Construction B because requirement (iii) of Theorem 3 cannot be satisfied. We remark that the requirement that s_1 and s_2 must be consecutive elements in S_{n-2} (cf. rule 1 Construction B) cannot be dropped, as the following example shall illustrate.

Example 4. We start with $G_{st}(3)$ which has transition sequence $S_3 = \underline{1}, 2, \underline{1}, 3, \underline{1}, 2, \underline{1}, 3$. Take the subsequence T of Construction B consisting of the integers which are underlined. Notice that there are no consecutive integers in T which are consecutive in S_3. Here $u_0 = 2$, and hence if we replace u_0 by $u_0(4) = 2, 4, 2$, and apply Construction B, we find that S_5 is equal to the sequence

$$3, 1, 2, 4, 2, 5, 2, 1, 3, 5, 3, 4, 3, 1, 2, 4, 2, 1, 3, 5, 3, 1, 4, 2, 1, 3, 1, 2, 1, 3, 5,$$

which is not a transition sequence of a Gray code of length 5 because the proper subsequence 3, 4, 3, 1, 2, 4, 2, 1 ($= s_{11}, s_{12}, \ldots, s_{18}$) contains no integer occurring an odd number of times.

At the end of this section we also prove the following result.

Theorem 4. *Let $G(n)$ be a balanced Gray code. Then $G(n)$ is totally balanced if and only if n is a power of 2.*

Proof. If $G(n)$ is a totally balanced Gray code, then for every bit position $i, 1 \leqslant i \leqslant n$, $TC_n(i) = 2^n/n$. Because $TC_n(i)$ is an integer for every i, the number n must be a power of 2. Conversely, let $(p_1, p_2, \ldots, p_n)$ be the transition count spectrum of $G(n)$. Remark that p_i is even for all $i, 1 \leqslant i \leqslant n$, and moreover that $|p_j - p_i| \leqslant 2, 1 \leqslant i, j \leqslant n$. Let i be some fixed index value. Suppose that there are l transition counts p_j such that $p_j - p_i = 2$ with $1 \leqslant l < n = 2^k$. By summation over all j-values, $1 \leqslant j \leqslant n$, we obtain $np_i + 2l = 2^n$, and hence $l = 2^{n-1} - \frac{n}{2}p_i = 2^{n-1} - 2^{k-1}p_i = 2^{k-1}(2^{n-k} - p_i)$. Since $1 \leqslant l < 2^k$, we obtain $1 \leqslant 2^{k-1}(2^{n-k} - p_i) < 2^k$ or $\frac{1}{2^{k-1}} \leqslant 2^{n-k} - p_i < 2$. The number $2^{n-k} - p_i$ must be an integer, and hence $2^{n-k} - p_i = 1$. It implies that $p_i = 2^{n-k} - 1$ is an odd integer. This violates the fact that p_i is even. Hence, we may conclude that $p_i = p_j$ for all i and j. So, $G(n)$ is totally balanced. Theorem 4 is proved.

4. Gray codes with a special transition count spectrum

Let $n = 2^v + u, 0 \leqslant u < 2^v$. The existence of Gray codes with transition counts

$$TC_n(i) = \begin{cases} 2^{n-v-1}, & \text{if } 1 \leqslant i \leqslant 2u, \\ 2^{n-v}, & \text{if } 2u < i \leqslant n \end{cases} \tag{13}$$

was conjectured in [10]. Let Q be a subset of $[n]$ with cardinality $2u$. Then condition (13) is equivalent with

$$TC_n(i) = \begin{cases} 2^{n-v-1}, & \text{if } i \in Q, \\ 2^{n-v}, & \text{if } i \in [n] \backslash Q. \end{cases} \tag{14}$$

A Gray code satisfying (14) will be called an *exponentially balanced* Gray code. By applying the extended Robinson–Cohn construction, we can show that for every $n \geqslant 1$, a Gray code with transition counts as defined in (14) exists.

Example 5. The standard Gray codes of length 1, 2, 3 have transition count spectra satisfying (14). A totally balanced Gray code of length 4 has also a transition count spectrum satisfying (14). Gray codes $G(5)$ and $G(7)$ having the following transition sequences S_5 and S_7 correspond to Gray codes with transition count spectra satisfying (14):

$$S_5 := 3,4,5,1,5,4,2,4,5,1,5,4,3,4,5,1,5,4,2,4,1,5,1,4,2,1,3,1,$$
$$2,1,3,5,$$

$S_7 :=$ 3, 5, 3, 1, 2, 1, 3, 1, 2, 6, 2, 1, 3, 1, 2, 1, 3, 7, 3, 1, 2, 1, 3, 1, 2, 4, 1, 7,
1, 6, 1, 5, 1, 6, 1, 7, 1, 4, 2, 7, 2, 6, 2, 4, 6, 7, 5, 1, 7, 1, 6, 1, 5, 6, 7, 4,
3, 7, 3, 6, 3, 4, 6, 7, 5, 1, 7, 1, 6, 1, 5, 6, 7, 4, 2, 7, 2, 6, 2, 4, 6, 7, 5, 1,
7, 1, 6, 1, 5, 6, 4, 3, 7, 3, 4, 6, 5, 1, 5, 4, 2, 4, 5, 1, 5, 4, 3, 4, 5, 1, 5, 4,
2, 4, 1, 5, 1, 4, 2, 1, 3, 1, 2, 1, 3, 5, 3, 7.

As one can verify, $TC_5 = (2^3, 2^2, 2^2, 2^3, 2^3)$ and $TC_7 = (2^5, 2^4, 2^4, 2^4, 2^4, 2^4, 2^4)$. So, these Gray codes all are examples of exponentially balanced Gray codes.

Theorem 5. *For every $n \geqslant 1$, there exists an exponentially balanced Gray code of length n.*

Proof. We shall distinguish between the cases n is even and n is odd. *Case* I when n is even. We already proved this theorem for the case when n is a 2-power, as can be seen immediately by comparing (14) with the transition count spectrum of a totally balanced Gray code (cf. Theorem 1). The proof for all other cases will be accomplished by (incomplete) induction from one 2-power to the next one. More precisely, starting from a totally balanced Gray code of length 2^v, we shall construct, by applying Construction B, a series of 2^{v-1} Gray codes all of which have a transition count spectrum of type (14).

Let $v \geqslant 1$. If $n = 2^v + u, u < 2^v$, is even, then u is even, and we can write $n = n_0 + 2m - 2$ with $n_0 = 2^v$ and $1 \leqslant m \leqslant 2^{v-1}$. We proceed by induction to m from $m = 1$ until $m = 2^{v-1}$. From Theorem 1, we know that there exists a totally balanced Gray code $G(n_0)$ with transition count spectrum

$$TC_{n_0} = (2^{n_0-v}, 2^{n_0-v}, \ldots, 2^{n_0-v}). \tag{15}$$

Hence, Theorem is true for $m = 1$. Assume that there exists a Gray code $G(n)$ for $n = n_0 + 2m - 2$, where m is some fixed value with $1 \leqslant m \leqslant 2^{v-1} - 1$, and with transition count spectrum

$$TC_n = (\underbrace{2^{n_0+2m-2-v-1}, \ldots, 2^{n_0+2m-2-v-1}}_{4(m-1)}, \underbrace{2^{n_0+2m-2-v}, \ldots, 2^{n_0+2m-2-v}}_{n_0-2m+2}). \tag{16}$$

We shall assume that this code $G(n)$ with transition sequence S_n has been produced by applying Construction B $m-1$ times starting from $G(n_0)$. We shall prove now that we can construct, starting from $G(n)$, a Gray code

$G(n+2)$ with transition count spectrum

$$E_{n+2}(2^{n+2}) = (p_1,\ldots,p_{4m},p_{4m+1},\ldots,p_{n+2}) \\ = (\underbrace{2^{n_0+2m-v-1},\ldots,2^{n_0+2m-v-1}}_{4m},\underbrace{2^{n_0+2m-v},\ldots,2^{n_0+2m-v}}_{n_0-2m}). \quad (17)$$

In order to satisfy condition (i) of Theorem 3 we take $k = n+1$, implying $p_k = p_{k+1} = 2^{n_0+2m-v}$. One can easily verify that for all $i \in [n+2]\backslash\{k, k+1\}$ condition (ii) holds as well. From the above choice for $k+1$ it follows that the length of the subsequence T of S_n must be equal to $2^{n_0+2m-v} = 4.2^{n_0+2m-2-v} = 4TC_n(i), i > 4(m-1)$. So, if T contains precisely four different integers $i_1, i_2, i_3, i_4 \in \{4m-3, 4m-2, \ldots, n\}$, and if all these integers occur in T as often as they do in S_n, i.e., $TC_n(i_j) = 2^{n_0+2m-2-v}$ times, the required length of T is obtained. The only thing we have to prove yet is that condition (iii) of Theorem 3 can be satisfied. To be able to do this, we first show that the number of consecutive pairs $\{n_0+2m-3, n_0+2m-2\}$ in $S_{n_0+2m-2}(= S_n)$ is at least $2^{v-1}-1-(m-1) = 2^{v-1}-m$. One can see this as follows. For $m = 1$, we consider consecutive pairs $\{l, n\}$ in S_n for $n = n_0$. Since each $l < n$ occurs 2^{n-v} times, we have that there is certainly some $l < n$ such that there are at least $\frac{2^{n-v}}{n-1}$ occurrences of consecutive pairs $\{l, n\}$. Without loss of generality we may take $l = n-1$, since the transition count spectrum of $G(n_0)$ is invariant for bit permutations. Since

$$\frac{2^{n-v}}{n-1} = \frac{2^{n-v}}{2^v-1} > \frac{2^{2^v-v}}{2^v} = 2^{2^v-2v} \geqslant 2^{v-1}-1 \text{ for } v \geqslant 1, \quad (18)$$

the inequality holds for $m = 1$. In order to proceed for $m > 1$, we notice that, due to rule 2 of Construction B, any consecutive pair $\{N-1, N\}$ in S_N gives rise to a consecutive pair $\{N+1, N+2\}$ in S_{N+2}, apart from the consecutive pair $\{s_1, s_2\}$. Since we applied Construction B $m-1$ times to obtain (16), it follows that there are at least $2^{v-1}-m > 0$ pairs $\{n_0+2m-3, n_0+2m-2\}$ in S_{n_0+2m-2}. So, we can take a pair $\{i, i'\} = \{n_0+2m-3, n_0+2m-2\}$, providing us with $p_j < 4TC_n(i)$ and $p_{j'} < 4TC_n(i')$, for any pair of indices j and j' taken from $[4m]$. Since the standard Gray code of length 2 has transition count spectrum satisfying (14), we may conclude now that Theorem is proved for the case n is even.

Case II when n is odd. In this case, we shall start with a Gray code of length 2^v+2^v-1 for some $v \geqslant 1$ having transition count spectrum (14). The Gray codes $G_{st}(3)$ and $G(7)$ in Example 5 are of this type. Starting from a Gray code of length $n_0 = 2^v+2^v-1$, $v \geqslant 1$, we now construct a series of 2^v Gray codes of length $n_0+2m, 1 \leqslant m \leqslant 2^v$ with transition count spectrum

(14). Suppose that a Gray code of length $n_0 = 2^v + 2^v - 1$ with transition count spectrum satisfying (14) exits for some fixed $v \geqslant 1$. The transition count spectrum of this Gray code is

$$(\underbrace{2^{n_0-v-1}, \dots, 2^{n_0-v-1}}_{n_0-1}, 2^{n_0-v}). \tag{19}$$

The standard Gray code $G_{st}(3)$ has this type of spectrum with $v = 1$. Based on this Gray code $G(n_0)$, using Construction B, we shall construct a series of 2^v Gray codes of length $n_0 + 2m, 1 \leqslant m \leqslant 2^v$, all of which have a transition count spectrum of type (14). In order to do this, we first shall construct a Gray code with code length $n_1 = n_0 + 2 = 2^{v+1} + 1$ having a transition count spectrum

$$(2^{n_0-v}, 2^{n_0-v}, \underbrace{2^{n_0-v-1}, \dots, 2^{n_0-v-1}}_{n_0}), \tag{20}$$

which clearly is of type (14). From the assumed spectrum of $G(n_0)$ in (19) we know that $TC_{n_0}(n_0) = 2^{n_0-v}$. Since every $l < n_0$ has transition count 2^{n_0-v-1}, it follows, just like in Case I, that there exists an $l < n_0$ such that the number of occurrences of the consecutive pair $\{l, n_0\}$ in S_{n_0} is at least $\dfrac{2^{n_0-v}}{n_0-1}$. Again we may take without loss of generality $l = n_0 - 1$. Now we have that the number of consecutive pairs $\{n_0 - 1, n_0\}$ is equal to

$$\frac{2^{n_0-v}}{n_0-1} = \frac{2^{2^{v+1}-1-v}}{2^{v+1}-2} > \frac{2^{2^{v+1}-v}}{2^{v+2}} = 2^{2^{v+1}-2v-2} \geqslant 2^v, \qquad v \geqslant 2. \tag{21}$$

Consider

$$E_{n_0+2}(2^{n_0+2}) = (p_1, \dots, p_n) = (2^{n_0-v}, 2^{n_0-v}, \underbrace{2^{n_0-v+1}, \dots, 2^{n_0-v+1}}_{n_0}). \tag{22}$$

Take $k = n_1 - 1 = n_0 + 1$. Hence, we have $p_k = p_{k+1} = 2^{n_0-v+1}$ which satisfies condition (i) of Theorem 3. Condition (ii) of that theorem can again be verified easily. For establishing condition (iii) of Theorem 3, we take $\{j, j'\} = \{1, 2\}$. Since $v \geqslant 2$, it is clear that $\{j, j'\} \subseteq [n_0] \backslash \{k, k+1\}$. It follows that for every $i \in [n_0]$ we have $p_j = p_{j'} = 2^{n_0-v} < 4TC_{n_0}(i)$. Since all three conditions of Theorem 3 are satisfied, the existence of a Gray code of length $n_1 = n_0 + 2$ with transition count spectrum (22) is guaranteed. We remark here that the number of occurrences of consecutive pairs of the integers $n_1 - 1$ and n_1 is at least $2^v - 1$, due to rule 2 of Construction B.

Based on this last Gray code, we shall derive a series of 2^v Gray codes all of which have a transition count spectrum of type (14). To this end, we shall apply again (incomplete) induction to m, $1 \leqslant m \leqslant 2^v, v \geqslant 2$, starting from a Gray code of length $n_1 = n_0 + 2 = 2^{v+1} + 1$ until a code of length $n_0 + 2^{v+1} = 2^{v+1} + 2^{v+1} - 1$. Since we constructed a Gray code of length n_1 having transition count spectrum (14), the theorem is true for $m = 1$. We discuss two subcases: $1 \leqslant m \leqslant 2^v - 1$ and $m = 2^v$.

Subcase II.a. $1 \leqslant m \leqslant 2^v - 1, v \geqslant 2$. Assume that Gray codes $G(n)$ of length $n = n_1 + 2m - 2, 1 \leqslant m \leqslant 2^v - 2$, with transition count spectrum

$$TC_n = (\underbrace{2^{n_1+2m-v-4}, \ldots, 2^{n_1+2m-v-4}}_{2(2m-1)}, \underbrace{2^{n_1+2m-v-3}, \ldots, 2^{n_1+2m-v-3}}_{n_1-2m} \qquad (23)$$

have been constructed by applying Construction B $m-1$ times, starting from $G(n_1)$. We shall show now that we can construct, starting from $G(n)$, a Gray code $G(n+2)$ with transition count spectrum

$$\begin{aligned} E_{n+2}(2^{n+2}) &= (p_1, \ldots, p_{4m+2}, p_{4m+3}, \ldots, p_{n+2}) \\ &= (\underbrace{2^{n_1+2m-v-2}, \ldots, 2^{n_1+2m-v-2}}_{2(2m+1)}, \underbrace{2^{n_1+2m-v-1}, \ldots, 2^{n_1+2m-v-1}}_{n_1-2m-2}). \end{aligned} \qquad (24)$$

To satisfy condition (i) of Theorem 3, we take $k = n + 1$, implying $p_k = p_{k+1} = 2^{n_1+2m-v-1}$. One can easily verify that condition (ii) holds for every $i \in [n+2] \backslash \{k, k+1\}$. To prove condition (iii) of Theorem 3, we first show that the number of consecutive pairs $\{n-1, n\}$ in S_n is at least $2^v - m$. We can prove this in the same way as we did for a similar statement in case I. Thus, we can take a pair $\{i, i'\} = \{n-1, n\}$ providing us with $p_j < 4TC_n(i)$ and $p_{j'} < 4TC_n(i')$ for any pair of indices j and j' taken from $[4m+2]$. Notice that for $m = 2^v - 1$ consecutive pairs $\{n_1 + 2m - 1, n_1 + 2m\}$ in S_{n_1+2m} occur at least $2^v - 1 - (2^v - 1 - 1) = 1$ time. Let us define for $m = 2^v - 1, n_2 = n_1 + 2m - 2 = 2^{v+1} + 2^{v+1} - 3$. Then we have that the resulting Gray code has a transition count spectrum

$$\begin{aligned} &(\underbrace{2^{n_1+2m-v-2}, \ldots, 2^{n_1+2m-v-2}}_{2(2m+1)}, \underbrace{2^{n_1+2m-v-1}, \ldots, 2^{n_1+2m-v-1}}_{n_1-2m-2}) = \\ &\qquad (\underbrace{2^{n_2-v-2}, \ldots, 2^{n_2-v-2}}_{n_2-3}, \underbrace{2^{n_2-v-1}, \ldots, 2^{n_2-v-1}}_{3}) \end{aligned} \qquad (25)$$

Subcase II.b. $m = 2^v$. We know that in the transition sequence S_{n_2} of the code with transition count spectrum (25) there occurs at least one pair of consecutive integers $\{n_2 - 1, n_2\}$. Assume that we want to construct a Gray code of length $n_2 + 2 = 2^{v+1} + 2^{v+1} - 1$ with transition count spectrum prescribed by

$$E_{n_2+2}(2^{n_2+2}) = (p_1, \ldots, p_{n_2+2}) = (\underbrace{2^{n_2-v}, \ldots, 2^{n_2-v}}_{n_2+1}, 2^{n_2-v+1}).$$

Again we want to establish conditions (i), (ii), and (iii) of Theorem 3. To satisfy condition (i) we take $k = n_2$ which gives $p_k = p_{k+1} = p_{n_2+1} = 2^{n_2-v}$. For these values of k and $k+1$, condition (ii) holds, as can be verified easily. For validating condition (iii), take $\{j, j'\} \subseteq [n_2+1] \backslash \{k, k+1\}$. By taking $\{i, i'\} = \{n_2 - 1, n_2\}$, we can see that $p_j = p_{j'} = 2^{n_2-v} < 4.2^{n_2-v-1} = 4TC_{n_2}(i) = 4TC_{n_2}(i')$. Since we know that the integers $n_2 - 1$ and n_2 are consecutive in S_{n_2} , we conclude that the three conditions of Theorem 3 are satisfied.

Until here, we showed that a series of $2^v - 1$ Gray codes exists with transition count spectra satisfying (14), starting from a similar type of Gray code of length $n_1 = n_0+2 = 2^{v+1}+1$. Remark that the last Gray code derived in this case (Subcase II.b) has length $n_2+2 = 2^{v+1}+2^{v+1}-1$ and, therefore, it is of the same type as the Gray code of length n_0 we started with in Case II. So, we proved the Theorem for the odd length case for $n \geqslant 2^v + 2^v - 1$, with $v = 2$. Since $G_{st}(1), G_{st}(3)$ and $G(5)$ (cf. Example 5) all have a transition count spectrum of type (14), the Theorem has been proved now for all odd values of n. Theorem 5 is proved.

Remark. It came to our attention (private communication of Prof. A. A. Evdokimov, Novosibirsk State University) that some of the results in this paper and in [1, 6, 10] were already found by Bakos (achieved about 1955), and published, in quite a different context, in A. Ádám, *"Truth Function and the Problem of their Realization by Two-Terminal Graphs Akadémiai Kiadó,* Budapest (1968).

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Адрес авторов:
Department of Mathematics
Faculty of Electrical Engineering,
Mathematics and Computer Sciences
Delft University of Technology
P.O. BOX 5031, 2600 GA Delft,
The Netherlands

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