A LINEARIZED INVERSE PROBLEM OF ISOTROPIC ELASTICITY IN A SPHERE *

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Abstract. The problem of unique determination of the density function $\rho(r,\theta,\varphi)$ depending on three variables of the spherical coordinate system is considered in the linear approximation with known velocities $c(r,\theta,\varphi)=(\lambda+2\mu)/\rho$ and $a(r,\theta,\varphi)=\mu/\rho$ of longitudinal and transverse waves, which can be determined as solutions of respective inverse kinematic seismic problems. The conditional stability estimate is obtained.

Key words: *isotropic elasticity, direct and inverse problems, existence, uniqueness, conditional stability.*

AMS 1991 Subject Classification: Primary: 35R30. Secondary: 73D50, 35L55, 34L20, 73C15.

1. Introduction

German geophysicists G. Herglotz [1], E. Weichert and K. Zoeppritz [2] have studied the first inverse kinematic seismic problem for the spherically symmetric Earth model in 1905–1907 under the assumption that the wave velocity function depends on the distance to the Earth center only. A. S. Alekseev for the first time have studied the dynamic formulation of one-dimensional inverse problems for systems of differential equations [3, 4]. In 1962 he set up the inverse dynamic seismic problem for the plane model of the Earth under the assumption that the density and elastic Lamé parameters depend on the depth only. Various formulations of one-dimensional and multi-dimensional problems for the plane model of the Earth were studied later on by A. S. Blagoveshchenskii [5], V. G. Romanov [6–8], E. A. Volkova [9], V. G. Yakhno [10, 11]. To solve the dynamic inverse problems for hyperbolic equations the method of linearization is widely employed. Beginning with the paper by M. M. Lavrent'ev and V. G. Romanov [12, 13] the method of linearization was actively used in the problem of determination of hyperbolic equation coefficients by V. G. Yakhno [14], G. Jobert [15], T. V. Bugueva [16] and by other authors. Linearized multi-dimensional inverse problems for isotropic elasticity system and for elastic anisotropic media were described by V. G. Romanov and E.A. Volkova [17,

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18]. Other methods for solving inverse dynamic problems for elastic media were suggested by W. Symes [19, 20], M. Grasselli, S.I. Kabanikhin, A. Lorenzi [21, 22] and others.

In the works [14, Chapter 3, pp. 85–111; 23–25] together with V. G. Yakhno we considered the inverse problem for the system of differential equations of isotropic elasticity in the sphere Ω

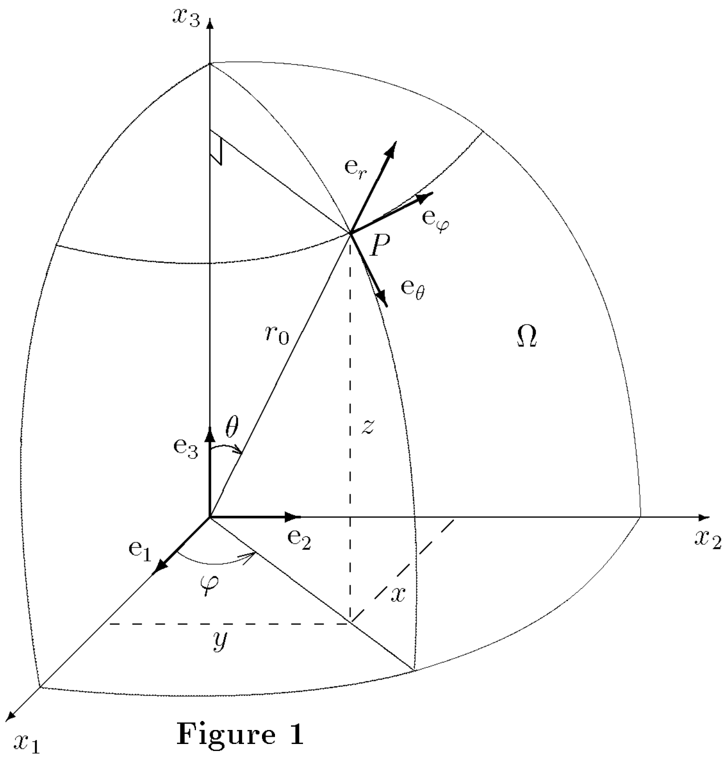
$$\rho\frac{\partial^2\vec{u}}{\partial t^2} = \mu\Delta\vec{u} + (\lambda+\mu)\nabla\mathrm{div}\vec{u} + \nabla\lambda\mathrm{div}\vec{u} + \nabla\mu\cdot(\nabla\vec{u} + \vec{u}\,\nabla). \tag{1.1}$$

We supposed that the Lamé parameters μ, λ and the density function ρ depend only on the radial variable r. A theorem on unique solvability for the one-dimensional inverse dynamic problem for the system of differential equations in a sphere was proved "in small". A conditional stability estimate was obtained for the problem of determination of the Lamé parameters and density function.

Let r_0 be a given positive number; $0 \le \varphi < 2\pi$, $0 \le \theta \le \pi$ and let

$$\mathbf{S}^2 = \Big\{\nu(\theta,\varphi) \;\Big|\; \nu = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)\Big\}$$

be the unit sphere in $\mathbf{R}^3$. Then $\Omega = [0, r_0]\times\mathbf{S}^2$ is a sphere of radius r_0.



$$P = P(x,y,z) \in \partial\Omega,$$

$$x = r_0\sin\theta\cos\varphi,$$

$$y = r_0\sin\theta\sin\varphi,$$

$$z = r_0\cos\theta.$$

Figure 1

The operators entering (1.1) are presented in the monography by A. Ben-Menahem and S. J. Singh [26]

$$\Delta\vec{u} = \mathrm{e}_r\left(\Delta u_r - \frac{2}{r^2}u_r - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta} - \frac{2\mathrm{ctg}\theta}{r^2}u_\theta - \frac{2}{r^2\sin\theta}\frac{u_\varphi}{\partial\varphi}\right) +$$

$$+\mathrm{e}_\theta\left(\Delta u_\theta+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}-\frac{1}{r^2\sin^2\theta}u_\theta-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\varphi}{\partial\varphi}\right)+$$

$$+\mathrm{e}_\varphi\left(\Delta u_\varphi+\frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\varphi}-\frac{1}{r^2\sin^2\theta}u_\varphi+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\varphi}\right),$$

$$\mathrm{div}\vec{u}=\frac{\partial u_r}{\partial r}+\frac{2}{r}u_r+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{\mathrm{ctg}\theta}{r}u_\theta+\frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi},$$

$$\nabla=\mathrm{e}_r\frac{\partial}{\partial r}+\mathrm{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}+\mathrm{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi},$$

$$(\nabla\vec{u}+\vec{u}\,\nabla)=\begin{pmatrix}\varepsilon_{rr}&\varepsilon_{r\theta}&\varepsilon_{r\varphi}\\ \varepsilon_{\theta r}&\varepsilon_{\theta\theta}&\varepsilon_{\theta\varphi}\\ \varepsilon_{\varphi r}&\varepsilon_{\varphi\theta}&\varepsilon_{\varphi\varphi}\end{pmatrix},$$

where

$$\varepsilon_{rr}=2\frac{\partial u_r}{\partial r},\quad \varepsilon_{\theta\theta}=\frac{2}{r}\left(\frac{\partial u_\theta}{\partial r}+u_r\right),\quad \varepsilon_{\varphi\varphi}=\frac{2}{r}\left(\frac{1}{\sin\theta}\frac{\partial u_\varphi}{\partial\varphi}+u_r+u_\theta\mathrm{ctg}\theta\right),$$

$$\varepsilon_{r\theta}=\varepsilon_{\theta r}=\frac{1}{r}\frac{\partial u_r}{\partial\theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r},$$

$$\varepsilon_{r\varphi}=\varepsilon_{\varphi r}=\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\varphi}+\frac{\partial u_\varphi}{\partial r}-\frac{u_\varphi}{r},$$

$$\varepsilon_{\theta\varphi}=\varepsilon_{\varphi\theta}=\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial u_\theta}{\partial\varphi}+\frac{\partial u_\varphi}{\partial\theta}-u_\varphi\mathrm{ctg}\theta\right).$$

Here u_r, u_θ, u_φ are the coefficients of expansion of the vector $\vec{u}$ in terms of the spherical basis $(\mathrm{e}_r,\mathrm{e}_\theta,\mathrm{e}_\varphi)$, which is defined by the formulas

$$\mathrm{e}_r=\mathrm{e}_1\sin\theta\cos\varphi+\mathrm{e}_2\sin\theta\sin\varphi+\mathrm{e}_3\cos\theta,$$

$$\mathrm{e}_\theta=\mathrm{e}_1\cos\theta\cos\varphi+\mathrm{e}_2\cos\theta\sin\varphi-\mathrm{e}_3\sin\theta,$$

$$\mathrm{e}_\varphi=-\mathrm{e}_1\sin\varphi+\mathrm{e}_2\cos\varphi$$

where e_1, e_2, e_3 is a basis in $\mathbf{R}^3$.

Let us consider the initial data for the system of differential equations (1.1)

$$\vec{u}|_{t<0}=0 \tag{1.2}$$

and the boundary condition

$$\sigma_r|_{r=r_0}=\vec{f}(t,\theta,\varphi), \tag{1.3}$$

here σ_r is the stress vector acting on the area element whose normal vector is parallel to the axis e_r

$$\sigma_r=\lambda\mathrm{e}_r\mathrm{div}\vec{u}+\mu\mathrm{e}_r\cdot(\nabla\vec{u}+\vec{u}\nabla), \tag{1.4}$$

where the vector $\mathrm{e}_r \cdot (\nabla\vec{u} + \vec{u}\nabla)$ is determined by the formula

$$\mathrm{e}_r \cdot (\nabla\vec{u} + \vec{u}\nabla) = 2\frac{\partial\vec{u}}{\partial r} + \mathrm{e}_r \times \mathrm{rot}\vec{u},$$

$\vec{f}(t,\theta,\varphi)$ is a source function, $t \in [0,T]$, $\nu(\theta,\varphi) \in \mathbf{S}^2$.

Let us consider the orthonormalized spherical harmonics system (see [26]) $\mathbf{R}_{lm}(\theta,\varphi)$, $\mathbf{S}_{lm}(\theta,\varphi)$, $\mathbf{T}_{lm}(\theta,\varphi)$, $l = 0,1,\ldots,$ $-l \le m \le l$,

$$\mathbf{R}_{lm}(\theta,\varphi) = Y_{lm}(\theta,\varphi)\mathrm{e}_r,$$

$$\mathbf{S}_{lm}(\theta,\varphi) = \frac{1}{\sqrt{l(l+1)}}\left(\mathrm{e}_\theta\frac{\partial}{\partial\theta} + \mathrm{e}_\varphi\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right)Y_{lm},$$

$$\mathbf{T}_{lm}(\theta,\varphi) = \frac{1}{\sqrt{l(l+1)}}\left(\mathrm{e}_\theta\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} - \mathrm{e}_\varphi\frac{\partial}{\partial\theta}\right)Y_{lm}, \tag{1.5}$$

here $\mathbf{S}_{00} = \mathbf{T}_{00} \equiv 0$; $Y_{lm}(\theta,\varphi)$ are the spherical functions of the order l , (see [27]) $l = 0,1,2,\ldots,$ $-l \le m \le l$. These functions satisfy the equality

$$\Delta_{\theta\varphi}Y_{lm} = -l(l+1)Y_{lm}. \tag{1.6}$$

Remark 1.1. Spherical harmonics satisfy the following orthogonality relations [26]

$$\int_0^{2\pi}\int_0^{\pi}\mathbf{R}_{lm}\cdot\mathbf{R}_{l'm'}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = \delta_{ll'}\delta_{mm'},$$

$$\int_0^{2\pi}\int_0^{\pi}\mathbf{S}_{lm}\cdot\mathbf{S}_{l'm'}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = \delta_{ll'}\delta_{mm'},$$

$$\int_0^{2\pi}\int_0^{\pi}\mathbf{T}_{lm}\cdot\mathbf{T}_{l'm'}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = \delta_{ll'}\delta_{mm'},$$

here δ_{ij} is the Kronecker symbol.

The spherical harmonics system $\mathbf{R}_{lm}(\theta,\varphi)$, $\mathbf{S}_{lm}(\theta,\varphi)$, $\mathbf{T}_{lm}(\theta,\varphi)$, $l = 0,1,2,\ldots,$ $-l \le m \le l$, is full, so any vector-function $\vec{f}(\theta,\varphi) = (f_1(\theta,\varphi), f_2(\theta,\varphi), f_3(\theta,\varphi))$ with components from $\mathbf{L}_2(\mathbf{S}^2)$ can be represented in the form of a series convergent in $\mathbf{L}_2(\mathbf{S}^2)$,

$$\vec{f}(\theta,\varphi) = U_{00}\mathbf{R}_{00} + \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\Big[U_{lm}\mathbf{R}_{lm} + V_{lm}\mathbf{S}_{lm} + W_{lm}\mathbf{T}_{lm}\Big]$$

where $U_{lm} = <\vec{f},\mathbf{R}_{lm}>_{\mathbf{L}_2(\mathbf{S}^2)}$, $V_{lm} = <\vec{f},\mathbf{S}_{lm}>_{\mathbf{L}_2(\mathbf{S}^2)}$, $W_{lm} = <\vec{f},\mathbf{T}_{lm}>_{\mathbf{L}_2(\mathbf{S}^2)}$, here $<\cdot,\cdot>_{\mathbf{L}_2(\mathbf{S}^2)}$ is the scalar product in the space $\mathbf{L}_2(\mathbf{S}^2)$.

Let us represent the solution of the system of differential equations (1.1) as the Fourier series expansion in terms of the spherical harmonics system $\mathbf{R}_{lm}(\theta,\varphi)$, $\mathbf{S}_{lm}(\theta,\varphi)$, $\mathbf{T}_{lm}(\theta,\varphi)$, $l = 0,1,\ldots,$ $-l \le m \le l$,

$$\vec{u}(t,r,\theta,\varphi) = U_{00}(r)\mathbf{R}_{00} + \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\Big[U_{lm}(r,t)\mathbf{R}_{lm}(\theta,\varphi) + V_{lm}(r,t)\mathbf{S}_{lm}(\theta,\varphi) +$$

$$+ W_{lm}(r,t)\mathbf{T}_{lm}(\theta,\varphi)\Big]. \tag{1.7}$$

Let us also represent the solution of system (1.1) in the form of expansion in terms of the spherical basis $(\mathrm{e}_r, \mathrm{e}_\theta, \mathrm{e}_\varphi)$

$$\vec{u}(t,r,\theta,\varphi) = u_r(r,\theta,\varphi)\mathrm{e}_r + u_\theta(t,r,\theta,\varphi)\mathrm{e}_\theta + u_\varphi(t,r,\theta,\varphi)\mathrm{e}_\varphi. \tag{1.8}$$

From (1.5), (1.8) we obtain

$$\begin{aligned}\vec{u}(t,r,\theta,\varphi) = {} & \mathrm{e}_r U_{00}(r,t)\mathrm{Y}_{00} + \\ & + \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\Bigg[\mathrm{e}_r U_{lm}(r,t)Y_{lm}(\theta,\varphi) + \\ & + \mathrm{e}_\theta\left(V_{lm}(r,t)\frac{\partial}{\partial\theta} + W_{lm}(r,t)\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right)Y_{lm}(\theta,\varphi) + \\ & + \mathrm{e}_\varphi\left(V_{lm}(r,t)\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} - W_{lm}(r,t)\frac{\partial}{\partial\theta}\right)Y_{lm}(\theta,\varphi)\Bigg].\end{aligned} \tag{1.9}$$

Basing on equations (1.8) and (1.9), we can write

$$\begin{aligned} u_r(r,t) &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} U_{lm}(r,t)Y_{lm}(\theta,\varphi), \\ u_\theta(r,t) &= \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\left(V_{lm}(r,t)\frac{\partial}{\partial\theta} + W_{lm}(r,t)\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right)Y_{lm}(\theta,\varphi), \\ u_\varphi(r,t) &= \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\left(V_{lm}(r,t)\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} - W_{lm}(r,t)\frac{\partial}{\partial\theta}\right)Y_{lm}(\theta,\varphi). \end{aligned} \tag{1.10}$$

Using the representations of the operators $\Delta\vec{u}$, $\nabla\mathrm{div}\vec{u}$, $\mathrm{div}\vec{u}$, $(\nabla\vec{u} + \vec{u}\,\nabla)$ in the spherical coordinates and equations (1.10), we obtain the expansions of these vectors in terms of the spherical harmonics system $\{\mathbf{R}_{lm}, \mathbf{S}_{lm}, \mathbf{T}_{lm} \mid l = 0,1,\ldots;\ -l \le m \le l\}$

$$\begin{aligned}\Delta\vec{u} = {} & \left(\frac{\partial^2 U_{00}}{\partial r^2} + \frac{2}{r}\frac{\partial U_{00}}{\partial r} - \frac{2}{r^2}U_{00}^0\right)\mathbf{R}_{00} + \\ & + \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\Bigg[\mathbf{R}_{lm}\left(\frac{\partial^2 U_{lm}}{\partial r^2} + \frac{2}{r}\frac{\partial U_{lm}}{\partial r} - \frac{2+l(l+1)}{r^2}U_{lm} + \frac{2\sqrt{l(l+1)}}{r^2}V_{lm}\right) + \\ & + \mathbf{S}_{lm}\left(\frac{\partial^2 V_{lm}}{\partial r^2} - \frac{l(l+1)}{r^2}V_{lm} + \frac{2}{r}\frac{\partial}{\partial r}V_{lm} + \frac{2\sqrt{l(l+1)}}{r^2}U_{lm}\right) + \\ & + \mathbf{T}_{lm}\left(\frac{\partial^2 W_{lm}}{\partial r^2} + \frac{2}{r}\frac{\partial W_{lm}}{\partial r} - \frac{l(l+1)}{r^2}W_{lm}\right)\Bigg];\end{aligned} \tag{1.11}$$

$$
\begin{aligned}
\nabla \mathrm{div}\vec{u} = &\left(\frac{\partial^2 U_{00}}{\partial r^2} + \frac{2}{r}\frac{\partial U_{00}}{\partial r} - \frac{2}{r^2}U_{00}\right)\mathbf{R}_{00} + \\
&+ \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\left[\mathbf{R}_{lm}\left(\frac{\partial^2 U_{lm}}{\partial r^2} + \frac{2}{r}\frac{\partial U_{lm}}{\partial r} - \frac{2}{r^2}U_{lm} - \frac{\sqrt{l(l+1)}}{r}\frac{\partial V_{lm}}{\partial r} + \frac{\sqrt{l(l+1)}}{r^2}V_{lm}\right) + \right.\\
&\left. + \sqrt{l(l+1)}\mathbf{S}_{lm}\left(\frac{1}{r}\frac{\partial U_{lm}}{\partial r} + \frac{2}{r^2}U_{lm} - \frac{\sqrt{l(l+1)}}{r^2}V_{lm}\right)\right];
\end{aligned}
\tag{1.12}
$$

$$
\mathrm{div}\vec{u} = \left(\frac{\partial}{\partial r}U_{00} + \frac{2}{r}U_{00}\right)Y_{00} + \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\left(\frac{\partial U_{lm}}{\partial r} + \frac{2}{r}U_{lm} - \frac{\sqrt{l(l+1)}}{r}V_{lm}\right)Y_{lm}; \tag{1.13}
$$

$$
\begin{aligned}
\mathrm{e}_r \cdot (\nabla\vec{u} + \vec{u}\,\nabla) = &\, 2\frac{\partial}{\partial r}U_{00}\mathbf{R}_{00} + \\
&+ \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\left[2\frac{\partial U_{lm}}{\partial r}\mathbf{R}_{lm} + \mathbf{S}_{lm}\left(\frac{\partial V_{lm}}{\partial r} - \frac{V_{lm}}{r} + \frac{\sqrt{l(l+1)}}{r}U_{lm}\right) + \right.\\
&\left. + \mathbf{T}_{lm}\left(\frac{\partial W_{lm}}{\partial r} - \frac{W_{lm}}{r}\right)\right];
\end{aligned}
\tag{1.14}
$$

$$
\begin{aligned}
\mathrm{e}_\theta \cdot (\nabla\vec{u} + \vec{u}\,\nabla) = &\, \mathrm{e}_r \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\left[\frac{1}{\sqrt{l(l+1)}}\left(\frac{\sqrt{l(l+1)}U_{lm} - V_{lm}}{r} + \frac{\partial V_{lm}}{\partial r}\right)\frac{\partial Y_{lm}}{\partial \theta} + \right.\\
&\left. + \frac{1}{\sqrt{l(l+1)}}\left(\frac{\partial W_{lm}}{\partial r} - \frac{W_{lm}}{r}\right)\frac{1}{\sin\theta}\frac{\partial Y_{lm}}{\partial \varphi}\right] + \\
&+ \mathrm{e}_\theta\frac{2}{r}\left\{U_{00}Y_{00} + \right.\\
&+ \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\left[U_{lm}Y_{lm} + \frac{1}{\sqrt{l(l+1)}}\left(V_{lm}\frac{\partial^2}{\partial\theta^2} + \right.\right.\\
&\left.\left.\left. + \frac{1}{\sin\theta}W_{lm}\left(\frac{\partial}{\partial\theta} - \mathrm{ctg}\theta\right)\frac{\partial}{\partial\varphi}\right)Y_{lm}\right]\right\} + \\
&+ \mathrm{e}_\varphi\frac{1}{r}\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\left[\frac{2}{\sqrt{l(l+1)}}V_{lm}\left(\frac{\partial}{\partial\theta} - \mathrm{ctg}\theta\right)\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} - \right.\\
&\left. - \frac{1}{\sqrt{l(l+1)}}W_{lm}\left(2\frac{\partial^2}{\partial\theta^2} + l(l+1)\right)Y_{lm}\right];
\end{aligned}
\tag{1.15}
$$

$$
\begin{aligned}
\mathrm{e}_\varphi\cdot(\nabla\vec{u}+\vec{u}\,\nabla) = \mathrm{e}_r \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\Bigg[&\frac{1}{\sqrt{l(l+1)}}\left(\frac{\sqrt{l(l+1)}U_{lm}-V_{lm}}{r}+\frac{\partial V_{lm}}{\partial r}\right)\frac{1}{\sin\theta}\frac{\partial Y_{lm}}{\partial\varphi} - \\
&-\frac{1}{\sqrt{l(l+1)}}\left(\frac{\partial W_{lm}}{\partial r}-\frac{W_{lm}}{r}\right)\frac{\partial Y_{lm}}{\partial\theta}\Bigg]+ \\
+\mathrm{e}_\theta\frac{1}{r}\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\Bigg[&\frac{2Y_{lm}}{\sqrt{l(l+1)}}\left(\frac{\partial}{\partial\theta}-\mathrm{ctg}\theta\right)\frac{1}{\sin\theta}\frac{\partial Y_{lm}}{\partial\varphi} - \\
&-\frac{1}{\sqrt{l(l+1)}}W_{lm}\left(2\frac{\partial^2}{\partial\theta^2}+l(l+1)\right)Y_{lm}\Bigg]+ \\
+\mathrm{e}_\varphi\frac{2}{r}\Bigg\{&U_{00}Y_{00}+ \\
&+\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\Bigg[U_{lm}Y_{lm}-\frac{1}{\sqrt{l(l+1)}}V_{lm}\left(\frac{\partial^2}{\partial\theta^2}+l(l+1)\right)Y_{lm} - \\
&-\frac{1}{\sqrt{l(l+1)}}W_{lm}\left(\frac{\partial}{\partial\theta}-\mathrm{ctg}\theta\right)\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}Y_{lm}\Bigg]\Bigg\}.
\end{aligned}
\tag{1.16}
$$

2. The inverse problem statement

In this paper it is supposed that the Lamé parameters μ, λ and the density function ρ depend on the spherical coordinates r, θ, φ. Let us rewrite system (1.1) for an isotropic medium in the form

$$
\frac{\partial^2\vec{u}}{\partial t^2}=\frac{\mu}{\rho}\Delta\vec{u}+\frac{\lambda+\mu}{\rho}\nabla\mathrm{div}\vec{u}+\frac{\nabla\lambda}{\rho}\mathrm{div}\vec{u}+\frac{\nabla\mu}{\rho}\cdot(\nabla\vec{u}+\vec{u}\nabla). \tag{2.1}
$$

We will consider this system with the initial data

$$
\vec{u}|_{t<0}=0 \tag{2.2}
$$

and the boundary condition

$$
\sigma_r|_{r=r_0}=\frac{1}{\sqrt{4\pi}}\vec{l}\,\delta(t). \tag{2.3}
$$

Here the stress vector σ_r acting on the area element whose normal vector is parallel to the axis e_r is determined by equation (1.4); the vector $\vec{l}$ has the coordinates $(1,0,0)$ in the spherical basis e_r, e_θ, e_φ; $1/\sqrt{4\pi}$ is the normalization constant.

Let $\vec{h}(t,\theta,\varphi)$ be a known function. As an additional information to solve the inverse problem we consider

$$
\vec{u}\,\Big|_{r=r_0}=\vec{h}(t,\theta,\varphi), \qquad t\in[0,T], \quad \nu(\theta,\varphi)\in\mathbf{S}^2. \tag{2.4}
$$

Consider the functions

$$a^2 = \frac{\mu}{\rho}, \qquad c^2 = \frac{\lambda + 2\mu}{\rho}, \qquad p = \ln \rho. \tag{2.5}$$

From equations (2.5) we have $\lambda/\rho = c^2 - 2a^2$, $\lambda = (c^2 - 2a^2)\rho$, $\mu = a^2\rho$,

$$\frac{\nabla\lambda}{\rho} = \frac{\nabla\lambda}{\lambda}\frac{\lambda}{\rho} = \nabla(c^2 - 2a^2) + (c^2 - 2a^2)\nabla p, \quad \frac{\nabla\mu}{\rho} = \frac{\nabla\mu}{\mu}\frac{\mu}{\rho} = \nabla a^2 + a^2\nabla p. \tag{2.6}$$

Let us rewrite equation (2.1) in terms of the functions a, c and p using equalities (2.5), (2.6)

$$\frac{\partial^2 \vec{u}}{\partial t^2} = a^2\Delta\vec{u} + (c^2 - a^2)\nabla\mathrm{div}\vec{u} + \Big[\nabla(c^2 - 2a^2) + (c^2 - 2a^2)\nabla p\Big]\mathrm{div}\vec{u} +$$

$$+\Big(\nabla a^2 + a^2\nabla p\Big)\cdot(\nabla\vec{u} + \vec{u}\nabla). \tag{2.7}$$

The function σ can be rewritten in terms of the functions a^2 and c^2 as

$$\sigma_r = (c^2 - 2a^2)\rho \mathrm{e}_r \mathrm{div}\vec{u} + a^2\rho\mathrm{e}_r\cdot(\nabla\vec{u} + \vec{u}\nabla). \tag{2.8}$$

Let us suppose that the functions $c^2(r,\theta,\varphi)$, $a^2(r,\theta,\varphi)$, $p(r,\theta,\varphi)$ are represented in the form

$$a^2(r,\theta,\varphi) = a_0^2 + a_1(r,\theta,\varphi),\ c^2(r,\theta,\varphi) = c_0^2 + c_1(r,\theta,\varphi),\ p(r,\theta,\varphi) = p_0 + p_1(r,\theta,\varphi) \tag{2.9}$$

where a_0^2, c_0^2, p_0 are some positive constants and a_1, c_1, p_1 are some functions small as compared with the constants a_0^2, c_0^2, p_0 respectively.

Let us consider the problem (2.1)–(2.3) in the linear approximation. Then we have

$$\rho(r,\theta,\varphi) = \rho_0 + \rho_1(r,\theta,\varphi) \tag{2.10}$$

where

$$\rho_0 = \mathrm{e}^{p_0}, \qquad \rho_1(r,\theta,\varphi) = \rho_0 p_1(r,\theta,\varphi). \tag{2.11}$$

Let us represent the solution $\vec{u}(t,r,\theta,\varphi)$ of the direct problem (2.1)–(2.3) in the form

$$\vec{u}(t,r,\theta,\varphi) = \vec{u}^0(t,r) + \vec{u}^1(t,r,\theta,\varphi), \tag{2.12}$$

here the function $\vec{u}^0(t,r)$ is the solution of the problem

$$\frac{\partial^2\vec{u}^0}{\partial t^2} = a_0^2\Delta\vec{u}^0 + (c_0^2 - a_0^2)\nabla\mathrm{div}\vec{u}^0, \tag{2.13}$$

$$\vec{u}^0\Big|_{t<0} = 0, \tag{2.14}$$

$$\sigma_r^0\Big|_{r=r_0} = \frac{1}{\sqrt{4\pi}}\vec{l}\,\delta(t) \tag{2.15}$$

where

$$\sigma_r^0 = (c_0^2 - 2a_0^2)\rho_0\mathrm{e}_r\mathrm{div}\vec{u}^0 + a_0^2\rho_0\mathrm{e}_r\cdot(\nabla\vec{u}^0 + \vec{u}^0\nabla). \tag{2.16}$$

The function $\vec{u}^1(t,r,\theta,\varphi)$ is the solution of the problem

$$\frac{\partial^2 \vec{u}^1}{\partial t^2} = a_0^2 \Delta \vec{u}^1 + (c_0^2 - a_0^2)\nabla \mathrm{div} \vec{u}^1 + a_1 \Delta \vec{u}^0 + (c_1 - a_1)\nabla \mathrm{div} \vec{u}^0 +$$
$$+\Big[\nabla(c_1 - 2a_1) + (c_0^2 - 2a_0^2)\nabla p_1\Big] \mathrm{div}\vec{u}^0 + \Big[\nabla a_1 + a_0^2 \nabla p_1\Big] \cdot (\nabla \vec{u}^0 + \vec{u}^0 \nabla), \quad (2.17)$$

$$\vec{u}^1\big|_{t<0} = 0, \quad (2.18)$$

$$\sigma_r^1\big|_{r=r_0} = 0. \quad (2.19)$$

Here

$$\sigma_r^1 = (c_0^2 - 2a_0^2)\rho_0 \mathrm{e}_r \mathrm{div}\vec{u}^1 + a_0^2 \rho_0 \mathrm{e}_r \cdot (\nabla \vec{u}^1 + \vec{u}^1 \nabla) +$$
$$+\rho_0 \Big[(c_1 - 2a_1) + (c_0^2 - 2a_0^2) p_1\Big] \mathrm{e}_r \mathrm{div}\vec{u}^0 + \rho_0 \Big(a_1 + a_0^2 p_1\Big) \mathrm{e}_r \cdot (\nabla \vec{u}^0 + \vec{u}^0 \nabla). \quad (2.20)$$

The additional information about the solution of the direct problem (2.1)–(2.3) is represented in the form

$$\vec{h}(t,\theta,\varphi) = \vec{h}_0(t) + \vec{h}_1(t,\theta,\varphi) \quad (2.21)$$

where $\vec{h}_0(t)$ is the trace of the function $\vec{u}^0(t,r)$ when $r = r_0$ and the function $\vec{h}_1(t,\theta,\varphi)$ is defined as

$$\vec{u}^1\big|_{r=r_0} = \vec{h}_1(t,\theta,\varphi). \quad (2.22)$$

Inverse Problem 2.1. Let T, r_0, c_0, a_0, p_0 be given positive constants; $0 < T < 2r_0/c_0$. Determine the function $p_1(r,\theta,\varphi)$ in equations (2.17), (2.19), given the additional information (2.22) about the solution $\vec{u}^1(t,r,\theta,\varphi)$ to the problem (2.17)–(2.19), where $\vec{h}_1(t,\theta,\varphi)$ is a given function for $t \in [0,T]$, $\nu(\theta,\varphi) \in \mathbf{S}^2$; the functions $a_1(r,\theta,\varphi)$ and $c_1(r,\theta,\varphi)$ are known; $\vec{u}^0(t,r)$ is a known function as the solution of the direct problem (2.13)–(2.15).

3. Fourier series representation for the system of Lamé equations. A system of differential relations for the Fourier series coefficients

Since the function $\vec{u}^0(t,r)$ does not depend on the variables θ and φ, therefore, expanding it into the series in terms of the spherical harmonics $\mathbf{R}_{00}$, $\mathbf{R}_{lm}(\theta,\varphi)$, $\mathbf{S}_{lm}(\theta,\varphi)$, $\mathbf{T}_{lm}(\theta,\varphi)$, $l = 1,2,\ldots$, $-l \le m \le l$, we have only one term

$$\vec{u}^0(t,r) = U_{00}^0(t,r)\mathbf{R}_{00} = \frac{1}{\sqrt{4\pi}} U_{00}^0(t,r)\mathrm{e}_r. \quad (3.1)$$

Let us represent the functions $a_1(r,\theta,\varphi)$, $c_1(r,\theta,\varphi)$, $p_1(r,\theta,\varphi)$ as formal series (see [27]) in terms of the normalized spherical functions $Y_{lm}(\theta,\varphi)$ of order l ($l = 0,1,2,\ldots,\ -l \le$

$m \leq l$)

$$a_1(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{1lm}(r)Y_{lm}(\theta,\varphi),$$

$$c_1(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} c_{1lm}(r)Y_{lm}(\theta,\varphi),$$

$$p_1(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} p_{1lm}(r)Y_{lm}(\theta,\varphi) \tag{3.2}$$

where

$$a_{1lm}(r) = <a_1, Y_{lm}>_{\mathbf{L}_2(\mathbf{S}^2)}, \quad c_{1lm}(r) = <c_1, Y_{lm}>_{\mathbf{L}_2(\mathbf{S}^2)}, \quad p_{1lm}(r) = <p_1, Y_{lm}>_{\mathbf{L}_2(\mathbf{S}^2)} \,. \tag{3.3}$$

From (1.7), (1.11)–(1.16), (2.17), (2.20) we obtain the following relations

$$a_1\Delta\vec{u}^0 = \frac{1}{\sqrt{4\pi}}\left(\frac{\partial^2 U^0_{00}}{\partial r^2} + \frac{2}{r}\frac{\partial U^0_{00}}{\partial r} - \frac{2}{r^2}U^0_{00}\right)\sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{1lm}\mathbf{R}_{lm};$$

$$(c_1 - a_1)\nabla\mathrm{div}\vec{u}^0 = \frac{1}{\sqrt{4\pi}}\left(\frac{\partial^2 U^0_{00}}{\partial r^2} + \frac{2}{r}\frac{\partial U^0_{00}}{\partial r} - \frac{2}{r^2}U^0_{00}\right)\sum_{l=0}^{\infty}\sum_{m=-l}^{l} (c_{1lm} - a_{1lm})\mathbf{R}_{lm};$$

$$\nabla(c_1 - 2a_1)\mathrm{div}\vec{u}^0 = \frac{1}{\sqrt{4\pi}}\left(\frac{\partial U^0_{00}}{\partial r} + \frac{2}{r}U^0_{00}\right)\sum_{l=0}^{\infty}\sum_{m=-l}^{l} (c'_{1lm} - 2a'_{1lm})\mathbf{R}_{lm} +$$

$$+\frac{1}{\sqrt{4\pi}}\frac{1}{r}\left(\frac{\partial U^0_{00}}{\partial r} + \frac{2}{r}U^0_{00}\right)\sum_{l=0}^{\infty}\sum_{m=-l}^{l} (c_{1lm} - a_{1lm})\sqrt{l(l+1)}\mathbf{S}_{lm};$$

$$(c_0^2 - 2a_0^2)\nabla p_1\mathrm{div}\vec{u}^0 = \frac{1}{\sqrt{4\pi}}(c_0^2 - 2a_0^2)\left(\frac{U^0_{00}}{\partial r} + \frac{2}{r}U^0_{00}\right)\sum_{l=0}^{\infty}\sum_{m=-l}^{l} p'_{1lm}\mathbf{R}_{lm} +$$

$$+\frac{1}{\sqrt{4\pi}}(c_0^2 - 2a_0^2)\frac{1}{r}\left(\frac{U^0_{00}}{\partial r} + \frac{2}{r}U^0_{00}\right)\sum_{l=0}^{\infty}\sum_{m=-l}^{l} p_{1lm}\sqrt{l(l+1)}\mathbf{S}_{lm};$$

$$\nabla a_1 \cdot (\nabla\vec{u}^0 + \vec{u}^0\nabla) = \frac{1}{\sqrt{4\pi}}2\frac{\partial U^0_{00}}{\partial r}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} a'_{1lm}\mathbf{R}_{lm} +$$

$$+\frac{1}{\sqrt{4\pi}}\frac{2}{r^2}U^0_{00}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{1lm}\sqrt{l(l+1)}\mathbf{S}_{lm};$$

$$a_0^2\nabla p_1 \cdot (\nabla\vec{u}^0 + \vec{u}^0\nabla) = \frac{1}{\sqrt{4\pi}}a_0^2 2\frac{\partial U^0_{00}}{\partial r}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} p'_{1lm}\mathbf{R}_{lm} +$$

$$+\frac{1}{\sqrt{4\pi}}a_0^2\frac{2}{r^2}U_{00}^0\sum_{l=0}^{\infty}\sum_{m=-l}^{l}p_{1lm}\sqrt{l(l+1)}\mathbf{S}_{lm};$$

$$(c_1-2a_1)\rho_0\mathrm{e}_r\mathrm{div}\vec{u}^0=\frac{1}{\sqrt{4\pi}}\rho_0\left(\frac{\partial U_{00}^0}{\partial r}+\frac{2}{r}U_{00}^0\right)\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(c_{1lm}-2a_{1lm})\mathbf{R}_{lm};$$

$$(c_0^2-2a_0^2)\rho_0p_1\mathrm{e}_r\mathrm{div}\vec{u}^0=\frac{1}{\sqrt{4\pi}}\rho_0(c_0^2-2a_0^2)\left(\frac{\partial U_{00}^0}{\partial r}+\frac{2}{r}U_{00}^0\right)\sum_{l=0}^{\infty}\sum_{m=-l}^{l}p_{1lm}\mathbf{R}_{lm};$$

$$\rho_0a_1\mathrm{e}_r\cdot(\nabla\vec{u}^0+\vec{u}^0\nabla)=\frac{1}{\sqrt{4\pi}}\rho_0 2\frac{\partial U_{00}^0}{\partial r}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}a_{1lm}\mathbf{R}_{lm};$$

$$a_0^2\rho_0p_1\mathrm{e}_r\cdot(\nabla\vec{u}^0+\vec{u}^0\nabla)=\frac{1}{\sqrt{4\pi}}\rho_0a_0^2 2\frac{\partial U_{00}^0}{\partial r}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}p_{1lm}\mathbf{R}_{lm}; \tag{3.4}$$

here $c_{1lm}(r)$, $a_{1lm}(r)$, $p_{1lm}(r)$ are the Fourier coefficients for the functions $c_1(r,\theta,\varphi)$, $a_1(r,\theta,\varphi)$, $p_1(r,\theta,\varphi)$ in the spherical functions system $Y_{lm}(\theta,\varphi)$.

Let us rewrite equations (2.13)–(2.15) and (2.17)–(2.19) in terms of the harmonics $U_{00}^0(t,r)$, $U_{lm}^1(t,r)$, $V_{lm}^1(t,r)$, $W_{lm}^1(t,r)$, using equalities (1.11)–(1.16) and (3.1)–(3.4),

$$\frac{\partial^2}{\partial t^2}U_{00}^0=c_0^2\frac{\partial^2}{\partial r^2}U_{00}^0-\widehat{k}_1(r)\frac{\partial}{\partial r}U_{00}^0+\widehat{k}_0(r)U_{00}^0, \tag{3.5}$$

$$U_{00}^0\big|_{t<0}=0, \tag{3.6}$$

$$\left[\frac{\partial}{\partial r}U_{00}^0+\widehat{h}_0U_{00}^0\right]\bigg|_{r=r_0}=\widehat{g}_0\delta(t), \tag{3.7}$$

$$\frac{\partial^2}{\partial t^2}U_{lm}^1=c_0^2\frac{\partial^2}{\partial r^2}U_{lm}^1-\widehat{k}_1(r)\frac{\partial}{\partial r}U_{lm}^1+\widehat{k}_{2l}(r)U_{lm}^1+\widehat{d}_{1l}(r)\frac{\partial}{\partial r}V_{lm}^1+$$

$$+\widehat{d}_{2l}(r)V_{lm}^1+\widehat{k}_{6l}(r)\frac{\partial^2}{\partial r^2}U_{00}^0+\widehat{y}_{1lm}(r)\frac{\partial}{\partial r}U_{00}^0+\widehat{y}_{2lm}(r)U_{00}^0, \tag{3.8}$$

$$U_{lm}^1\big|_{t<0}=0, \tag{3.9}$$

$$\left[\frac{\partial U_{lm}^1}{\partial r}+\widehat{h}_0U_{lm}^1+\widehat{d}_{3l}(r_0)V_{lm}^1+\widehat{y}_{3lm}(r_0)\frac{\partial U_{00}^0}{\partial r}+\widehat{y}_{4lm}(r_0)U_{00}^0\right]\bigg|_{r=r_0}=0, \tag{3.10}$$

$$\frac{\partial^2}{\partial t^2}V_{lm}^1=a_0^2\frac{\partial^2}{\partial r^2}V_{lm}^1-\widehat{d}_4(r)\frac{\partial}{\partial r}V_{lm}^1+\widehat{d}_{5l}(r)V_{lm}^1+\widehat{k}_{3l}(r)\frac{\partial}{\partial r}U_{lm}^1+$$

$$+\widehat{k}_{4l}(r)U_{lm}^1+\widehat{y}_{5lm}\frac{\partial}{\partial r}U_{00}^0+\widehat{y}_{6lm}U_{00}^0, \tag{3.11}$$

$$V_{lm}^1\big|_{t<0}=0, \tag{3.12}$$

$$\left[\frac{\partial}{\partial r} V_{lm}^1 + \widehat{d}_6(r_0) V_{lm}^1 + \widehat{k}_{5l}(r_0) U_{lm}^1 \right]\Big|_{r=r_0} = 0. \tag{3.13}$$

The relations for determination of the coefficients of equations (3.5)–(3.13) are written out below.

Equations (2.17)–(2.19) for the harmonics $W_{lm}^1(t,r)$ are written as

$$\frac{\partial^2}{\partial t^2} W_{lm}^1 = a_0^2 \frac{\partial^2}{\partial r^2} W_{lm}^1 + \frac{2a_0^2}{r} \frac{\partial}{\partial r} W_{lm}^1 - l(l+1) \frac{a_0^2}{r^2} W_{lm}^1, \tag{3.14}$$

$$W_{lm}^1 \big|_{t<0} = 0, \tag{3.15}$$

$$\left[\frac{\partial}{\partial r} W_{lm}^1 - W_{lm}^1 \right]\Big|_{r=r_0} = 0. \tag{3.16}$$

Introduce the following functions for the given vector-function $\vec{h}_1(t,\theta,\varphi)$

$$F_{1lm}(t) = \int_0^{2\pi} \int_0^{\pi} \vec{h}_1(t,\theta,\varphi) \cdot \mathbf{R}_{lm}(\theta,\varphi) \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi, \tag{3.17}$$

$$G_{1lm}(t) = \int_0^{2\pi} \int_0^{\pi} \vec{h}_1(t,\theta,\varphi) \cdot \mathbf{S}_{lm}(\theta,\varphi) \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi, \tag{3.18}$$

$$H_{1lm}(t) = \int_0^{2\pi} \int_0^{\pi} \vec{h}_1(t,\theta,\varphi) \cdot \mathbf{T}_{lm}(\theta,\varphi) \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi, \tag{3.19}$$

here $l = 0, 1, 2, \ldots,\ -l \le m \le l$.

So the vector-function $\vec{h}_1(t,\theta,\varphi)$ is represented in the form of a formal series on spherical harmonics

$$\vec{h}_1(t,\theta,\varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \Big[F_{1lm}(t) \mathbf{R}_{lm}(\theta,\varphi) + G_{1lm}(t) \mathbf{S}_{lm}(\theta,\varphi) + H_{1lm}(t) \mathbf{T}_{lm}(\theta,\varphi) \Big]. \tag{3.20}$$

Here we supposed that $\mathbf{S}_{00}(\theta,\varphi) = \mathbf{T}_{00}(\theta,\varphi) \equiv 0$.

Remark 3.1. It is obvious from (3.14)–(3.16) that for any $l, m,\ l = 0, 1, 2, \ldots,\ -l \le m \le l$, the solution $W_{lm}^1(t,r) \equiv 0$. From here and from (3.19) we have $H_{1lm}(t) \equiv 0$.

So for the vector-function $\vec{h}_1(t,\theta,\varphi)$ the following representation is true

$$\vec{h}_1(t,\theta,\varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \Big[F_{1lm}(t) \mathbf{R}_{lm}(\theta,\varphi) + G_{1lm}(t) \mathbf{S}_{lm}(\theta,\varphi) \Big]. \tag{3.21}$$

Remark 3.2. It is easily seen from (2.22), (3.21) that the additional information for solution of Inverse Problem 2.1 in terms of the Fourier harmonics takes the form

$$U_{lm}^1 \Big|_{r=r_0} = F_{1lm}(t), \qquad V_{lm}^1 \Big|_{r=r_0} = G_{1lm}(t) \tag{3.22}$$

for $l = 0, 1, 2, \ldots,\ -l \le m \le l$.

The coefficients of equations (3.5)–(3.13) are determined by the equalities

$$\widehat{h}_0 = 2\frac{c_0^2 - 2a_0^2}{c_0^2 r_0},$$

$$\widehat{g}_0 = \frac{1}{c_0^2 \rho_0},$$

$$\widehat{k}_0(r) = -2\frac{c_0^2}{r^2},$$

$$\widehat{k}_1(r) = -\frac{2c_0^2}{r},$$

$$\widehat{k}_{2l}(r) = -2\frac{c_0^2}{r^2} - l(l+1)\frac{a_0^2}{r^2},$$

$$\widehat{k}_{3l}(r) = \sqrt{l(l+1)}\,\frac{c_0^2 - a_0^2}{r},$$

$$\widehat{k}_{4l}(r) = \sqrt{l(l+1)}\,\frac{2c_0^2}{r^2},$$

$$\widehat{k}_{5l}(r_0) = \frac{\sqrt{l(l+1)}}{r_0},$$

$$\widehat{k}_{6lm}(r_0) = \frac{1}{\sqrt{4\pi}} c_{1lm}(r),$$

$$\widehat{d}_{1l}(r) = -\frac{\sqrt{l(l+1)}}{r}\,(c_0^2 - a_0^2),$$

$$\widehat{d}_{2l}(r) = \sqrt{l(l+1)}\,\frac{c_0^2 + a_0^2}{r^2},$$

$$\widehat{d}_{3l}(r_0) = -\frac{\sqrt{l(l+1)}}{r_0}\,\frac{c_0^2 - 2a_0^2}{c_0^2},$$

$$\widehat{d}_4(r) = -\frac{2}{r}a_0^2,$$

$$\widehat{d}_{5l}(r) = -l(l+1)\frac{c_0^2}{r^2},$$

$$\widehat{d}_6(r_0) = -\frac{1}{r_0},$$

$$\widehat{y}_{1lm}(r) = \frac{1}{\sqrt{4\pi}}\left[\frac{2c_{1lm}(r)}{r} + c'_{1lm}(r) + c_0^2 p'_{1lm}(r)\right],$$

$$\widehat{y}_{2lm}(r) = \frac{1}{\sqrt{4\pi}}\left[-\frac{2c_{1lm}(r)}{r^2} + \frac{2}{r}c'_{1lm}(r) - \frac{4}{r}a'_{1lm}(r) + \frac{2(c_0^2 - 2a_0^2)}{r}p'_{1lm}(r)\right],$$

$$\widehat{y}_{3lm}(r_0) = \frac{1}{\sqrt{4\pi}} \left[\frac{c_{1lm}(r_0)}{c_0^2} + p_{1lm}(r_0) \right],$$

$$\widehat{y}_{4lm}(r_0) = \frac{1}{\sqrt{4\pi}} \frac{2}{r_0} \left[\frac{c_{1lm}(r_0) - 2a_{1lm}(r_0)}{\rho_0 c_0^2} + \frac{\rho_0(c_0^2 - 2a_0^2)}{c_0^2} p_{1lm}(r_0) \right],$$

$$\widehat{y}_{5lm}(r) = \frac{1}{\sqrt{4\pi}} \frac{\sqrt{l(l+1)}}{r} \Big[(c_{1lm}(r) - 2a_{1lm}(r)) + (c_0^2 - 2a_0^2) p_{1lm}(x) \Big],$$

$$\widehat{y}_{6lm}(r) = \frac{1}{\sqrt{4\pi}} \frac{2\sqrt{l(l+1)}}{r^2} \Big[(c_{1lm}(r) - a_{1lm}(r)) + (c_0^2 - a_0^2) p_{1lm}(r) \Big]. \tag{3.23}$$

Make in equations (3.5)–(3.13) the change of variables

$$x = x(r) = \int_r^{r_0} \frac{\mathrm{d}\xi}{c_0} = \frac{r_0 - r}{c_0}, \qquad r = r(x) = r_0 - \int_0^x c_0 \,\mathrm{d}\xi = r_0 - c_0 x. \tag{3.24}$$

Since c_0 is a known constant, therefore, $x(r)$ and $r(x)$ are known functions.

Define the functions

$$\overline{U}^0_{00}(x,t) = U^0_{00}(r(x),t), \quad \overline{U}^1_{lm}(t,x) = U^1_{lm}(t,r(x)), \quad \overline{V}^1_{lm}(t,x) = V^1_{lm}(t,r(x)) \tag{3.25}$$

and

$$\widehat{p}_{1lm}(x) = p_{1lm}(r(x)), \qquad \widehat{c}_{1lm}(x) = c_{1lm}(r(x)), \qquad \widehat{a}_{1lm}(x) = a_{1lm}(r(x)). \tag{3.26}$$

Equations (3.5)–(3.13) in terms of the variable x are written in the form

$$\frac{\partial^2}{\partial t^2}\overline{U}^0_{00} = \frac{\partial^2}{\partial x^2}\overline{U}^0_{00} - k_1(x)\frac{\partial}{\partial x}\overline{U}^0_{00} + k_0(x)\overline{U}^0_{00}, \tag{3.27}$$

$$\overline{U}^0_{00}\Big|_{t<0} = 0, \tag{3.28}$$

$$\left[\frac{\partial}{\partial x}\overline{U}^0_{00} + h_0\overline{U}^0_{00}\right]\Big|_{x=0} = g_0\delta(t), \tag{3.29}$$

$$\frac{\partial^2}{\partial t^2}\overline{U}^1_{lm} = \frac{\partial^2}{\partial x^2}\overline{U}^1_{lm} - k_1(x)\frac{\partial}{\partial x}\overline{U}^1_{lm} + k_{2l}(x)\overline{U}^1_{lm} + d_{1l}(x)\frac{\partial}{\partial x}\overline{V}^1_{lm} + d_{2l}(x)\overline{V}^1_{lm} +$$

$$+ k_{6lm}(x)\frac{\partial^2}{\partial x^2}\overline{U}^0_{00} + y_{1lm}(x)\frac{\partial}{\partial x}\overline{U}^0_{00} + y_{2lm}(x)\overline{U}^0_{00}, \tag{3.30}$$

$$\overline{U}^1_{lm}\Big|_{t<0} = 0, \tag{3.31}$$

$$\left[\frac{\partial \overline{U}^1_{lm}}{\partial x} + h_0\overline{U}^1_{lm} + d_{3l}(0)\overline{V}^1_{lm} + y_{3lm}(0)\frac{\partial \overline{U}^0_{00}}{\partial x} + y_{4lm}(0)\overline{U}^0_{00}\right]\Big|_{x=0} = 0, \tag{3.32}$$

$$\overline{U}^1_{lm}\Big|_{x=0} = F_{1lm}(t), \tag{3.33}$$

$$\frac{\partial^2}{\partial t^2}\overline{V}^1_{lm} = \frac{a_0^2}{c_0^2}\frac{\partial^2}{\partial x^2}\overline{V}^1_{lm} - d_4(x)\frac{\partial}{\partial x}\overline{V}^1_{lm} + d_{5l}(x)\overline{V}^1_{lm} + k_{3l}(x)\frac{\partial}{\partial x}\overline{U}^1_{lm} +$$

$$+ k_{4l}(x)\overline{U}^1_{lm} + y_{5lm}(x)\frac{\partial}{\partial x}\overline{U}^0_{00} + y_{6lm}\overline{U}^0_{00}, \tag{3.34}$$

$$\overline{V}^1_{lm}\Big|_{t<0} = 0, \tag{3.35}$$

$$\left[\frac{\partial}{\partial x}\overline{V}^1_{lm} + d_6(0)\overline{V}^1_{lm} + k_{5l}(0)\overline{U}^1_{lm}\right]\Big|_{x=0} = 0, \tag{3.36}$$

$$\overline{V}^1_{lm}\Big|_{x=0} = G_{1lm}(t). \tag{3.37}$$

Here

$$h_0 = -c_0\widehat{h}_0 = -\frac{2(c_0^2 - 2a_0^2)}{c_0 r_0},$$

$$g_0 = -c_0\widehat{g}_0 = -\frac{1}{c_0\rho_0},$$

$$k_0(x) = \widehat{k}_0(r(x)) = -2\frac{c_0^2}{r^2(x)},$$

$$k_1(x) = \frac{\widehat{k}_1(r(x))}{c_0} = -2\frac{c_0}{r(x)},$$

$$k_{2l}(x) = \widehat{k}_{2l}(r(x)) = -2\frac{c_0^2}{r^2(x)} - l(l+1)\frac{a_0^2}{r^2(x)},$$

$$k_{3l}(x) = -\frac{\widehat{k}_{3l}(r(x))}{c_0} = -\sqrt{l(l+1)}\,\frac{c_0^2 - a_0^2}{c_0 r(x)},$$

$$k_{4l}(x) = \widehat{k}_{4l}(r(x)) = \sqrt{l(l+1)}\,\frac{2c_0^2}{r^2(x)},$$

$$k_{5l}(0) = -c_0\widehat{k}_{5l}(r_0) = -\sqrt{l(l+1)}\,\frac{c_0}{r_0},$$

$$k_{6lm}(x) = \frac{\widehat{k}_{6lm}(r(x))}{c_0^2} = \frac{1}{\sqrt{4\pi}}\frac{\widehat{c}_{1lm}(x)}{c_0^2},$$

$$d_{1l}(x) = -\frac{\widehat{d}_{1l}(r(x))}{c_0} = \frac{\sqrt{l(l+1)}}{r(x)c_0}(c_0^2 - a_0^2),$$

$$d_{2l}(x) = \widehat{d}_{2l}(r(x)) = \sqrt{l(l+1)}\,\frac{c_0^2 + a_0^2}{r^2(x)},$$

$$d_{3l}(0) = -c_0\widehat{d}_{3l}(r_0) = \sqrt{l(l+1)}\,\frac{c_0^2 - 2a_0^2}{c_0 r_0},$$

$$d_4(x) = -\frac{\widehat{d}_4(r(x))}{c_0} = \frac{2a_0^2}{c_0 r(x)},$$

$$d_{5l}(x) = \widehat{d}_{5l}(r(x)) = -l(l+1)\frac{c_0^2}{r^2(x)},$$

$$d_6(0) = -c_0\widehat{d}_6(r_0) = \frac{c_0}{r_0},$$

$$y_{1lm}(x) = -\frac{\widehat{y}_{1lm}(r(x))}{c_0} = -\frac{1}{\sqrt{4\pi}}\frac{1}{c_0}\left[\frac{2\widehat{c}_{1lm}(x)}{r(x)} - \frac{\widehat{c}'_{1lm}(x)}{c_0} - c_0\widehat{p}'_{1lm}(x)\right],$$

$$y_{2lm}(x) = \widehat{y}_{2lm}(r(x)) = \frac{1}{\sqrt{4\pi}}\left[-\frac{2\widehat{c}_{1lm}(x)}{r^2(x)} - \frac{2\widehat{c}'_{1lm}(x)}{c_0 r(x)} + \frac{4\widehat{a}'_{1lm}(x)}{c_0 r(x)} - \frac{2(c_0^2-2a_0^2)}{c_0 r(x)}\widehat{p}_{1lm}(x)\right],$$

$$y_{3lm}(0) = \widehat{y}_{3lm}(r_0) = \frac{1}{\sqrt{4\pi}}\left[\frac{\widehat{c}_{1lm}(0)}{c_0^2} + \widehat{p}_{1lm}(0)\right],$$

$$y_{4lm}(0) = -c_0\widehat{y}_{4lm}(r_0) = -\frac{1}{\sqrt{4\pi}}\frac{2c_0}{r_0}\left[\frac{\widehat{c}_{1lm}(0) - 2\widehat{a}_{1lm}(0)}{\rho_0 c_0^2} + \frac{\rho_0(c_0^2-2a_0^2)}{c_0^2}\widehat{p}_{1lm}(0)\right],$$

$$y_{5lm}(r) = -\frac{\widehat{y}_{5lm}(r(x))}{c_0} = -\frac{1}{\sqrt{4\pi}}\frac{\sqrt{l(l+1)}}{c_0 r(x)}\Big[\widehat{c}_{1lm}(x) - 2\widehat{a}_{1lm}(x) + (c_0^2-2a_0^2)\widehat{p}_{1lm}(x)\Big],$$

$$y_{6lm}(x) = \widehat{y}_{6lm}(r(x)) = \frac{1}{\sqrt{4\pi}}\frac{2\sqrt{l(l+1)}}{r^2(x)}\Big[\widehat{c}_{1lm}(x) - \widehat{a}_{1lm} + (c_0^2-a_0^2)\widehat{p}_{1lm}(x)\Big]. \tag{3.38}$$

Introduce new functions $u_{00}^0(x,t)$, $u_{lm}^1(x,t)$, $v_{lm}^1(x,t)$

$$u_{00}^0(x,t) = \frac{\overline{U}_{00}^0(x,t)}{S(x)}, \quad u_{lm}^1(x,t) = \frac{\overline{U}_{lm}^1(x,t)}{S(x)}, \quad v_{lm}^1(x,t) = \frac{\overline{V}_{lm}^1(x,t)}{S_1(x)} \tag{3.39}$$

where the functions $S(x)$, $S_1(x)$ are determined from the relations

$$2\frac{S'(x)}{S(x)} = k_1(x), \quad 2\frac{S_1'(x)}{S_1(x)} = \frac{c_0^2}{a_0^2}d_4(x), \qquad S(0) = S_1(0) = 1 \tag{3.40}$$

which yields

$$S(x) = \exp\left(\frac{1}{2}\int_0^x k_1(\xi)\,\mathrm{d}\xi\right), \quad S_1(x) = \exp\left(\frac{c_0^2}{2a_0^2}\int_0^x d_4(\xi)\,\mathrm{d}\xi\right). \tag{3.41}$$

Equalities (3.27)–(3.37) in terms of the functions $u_{00}^0(x,t)$, $u_{lm}^1(x,t)$, $v_{lm}^1(x,t)$ are

$$\frac{\partial^2}{\partial t^2}u_{00}^0 = \frac{\partial^2}{\partial x^2}u_{00}^0 + q_0(x)u_{00}^0, \tag{3.42}$$

$$u_{00}^0\Big|_{t<0} = 0, \tag{3.43}$$

$$\left[\frac{\partial u_{00}^0}{\partial x} + h_1 u_{00}^0\right]\Bigg|_{x=0} = g_0\delta(t), \tag{3.44}$$

$$\frac{\partial^2 u^1_{lm}}{\partial t^2} = \frac{\partial^2 u^1_{lm}}{\partial x^2} + q_{1l}(x) u^1_{lm} + D_{1l}(x) \frac{\partial v^1_{lm}}{\partial x} + D_{2l}(x) v^1_{lm} +$$

$$+ K_{6lm}(x) \frac{\partial^2 u^0_{00}}{\partial x^2} + z_{1lm}(x) \frac{\partial u^0_{00}}{\partial x} + z_{2lm}(x) u^0_{00}, \tag{3.45}$$

$$u^1_{lm} \Big|_{t<0} = 0, \tag{3.46}$$

$$\left[\frac{\partial u^1_{lm}}{\partial x} + h_1 u^1_{lm} + D_3(0) v^1_{lm} + z_3(0) \frac{\partial u^0_{00}}{\partial x} + z_4(0) u^0_{00} \right] \Bigg|_{x=0} = 0, \tag{3.47}$$

$$u^1_{lm} \Big|_{x=0} = F_{1lm}(t), \tag{3.48}$$

$$\frac{\partial^2 v^1_{lm}}{\partial t^2} = \frac{a_0^2}{c_0^2} \frac{\partial^2 v^1_{lm}}{\partial x^2} + q_{2l}(x) v^1_{lm} + K_{3l}(x) \frac{\partial u^1_{lm}}{\partial x} + K_{4l}(x) u^1_{lm} + z_{5lm}(x) \frac{\partial u^0_{00}}{\partial x} + z_{6lm}(x) u^0_{00}, \tag{3.49}$$

$$v^1_{lm} \Big|_{t<0} = 0, \tag{3.50}$$

$$\left[\frac{\partial v^1_{lm}}{\partial x} + h_2 v^1_{lm} + K_5(0) u^1_{lm} \right] \Bigg|_{x=0} = 0, \tag{3.51}$$

$$v^1_{lm} \Big|_{x=0} = G_{1lm}(t). \tag{3.52}$$

Here

$$q_0(x) = \frac{k_1'(x)}{2} - \frac{k_1^2(x)}{4} + k_0(x),$$

$$q_{1l}(x) = \frac{k_1'(x)}{2} - \frac{k_1^2(x)}{4} + k_{2l}(x),$$

$$q_{2l}(x) = \frac{d_4'(x)}{2} - \frac{c_0^2}{a_0^2} \frac{d_4^2(x)}{4} + d_{5l}(x),$$

$$g_0 = -\frac{1}{c_0 \rho_0},$$

$$h_0 = -\frac{2(c_0^2 - 2a_0^2)}{c_0 r_0},$$

$$h_1 = \frac{k_1(0)}{2} + h_0,$$

$$h_2 = \frac{c_0^2}{a_0^2} \frac{d_4(0)}{2} + d_6(0),$$

$$D_{1l}(x) = d_{1l}(x) \frac{S_1(x)}{S(x)},$$

$$D_{2l}(x) = \left[d_{1l}(x) d_4(x) \frac{c_0^2}{2a_0^2} + d_{2l}(x) \right] \frac{S_1(x)}{S(x)},$$

$$
\begin{aligned}
D_{3l}(0) &= d_{3l}(0),\\
K_{3l}(x) &= k_{3l}(x)\frac{S(x)}{S_1(x)},\\
K_{4l}(x) &= \left[\frac{k_1(x)k_{3l}(x)}{2} + k_{4l}(x)\right]\frac{S(x)}{S_1(x)},\\
K_{5l}(0) &= k_{5l}(0),\\
K_{6lm}(x) &= k_{5lm}(x) = \frac{1}{\sqrt{4\pi}}\frac{\widehat{c}_{1lm}(x)}{c_0^2},\\
z_{1lm}(x) &= k_{6lm}(x)k_1(x) + y_{1lm}(x) =\\
&= k_{6lm}(x)k_1(x) - \frac{1}{\sqrt{4\pi}}\left(\frac{2\widehat{c}_{1lm}(x)}{r(x)c_0} - \frac{\widehat{c}'_{1lm}(x)}{c_0^2}\right) + \frac{1}{\sqrt{4\pi}}\widehat{p}'_{1lm}(x),\\
z_{2lm}(x) &= k_{6lm}(x)\left(\frac{k_1'(x)}{2} + \frac{k_1^2(x)}{4}\right) + y_{1lm}(x)\frac{k_{1lm}(x)}{2} + y_{2lm}(x) =\\
&= k_{6lm}(x)\left(\frac{k_1'(x)}{2} + \frac{k_1^2(x)}{4}\right) - \frac{1}{\sqrt{4\pi}}\frac{k_1(x)}{2c_0}\left(\frac{2\widehat{c}_{1lm}(x)}{r(x)} - \frac{\widehat{c}'_{1lm}(x)}{c_0}\right) +\\
&\quad + \frac{1}{\sqrt{4\pi}}\left(-\frac{2\widehat{c}_{1lm}(x)}{r^2(x)} - \frac{2\widehat{c}'_{1lm}(x)}{r(x)c_0} + \frac{4\widehat{a}'_{1lm}(x)}{c_0 r(x)}\right) +\\
&\quad + \frac{1}{\sqrt{4\pi}}\frac{k_1(x)}{2}\widehat{p}'_{1lm}(x) - \frac{1}{\sqrt{4\pi}}\frac{2(c_0^2 - 2a_0^2)}{c_0 r(x)}\widehat{p}_{1lm}(x),\\
z_{3lm}(0) &= y_{3lm}(0) = \frac{1}{\sqrt{4\pi}}\left(\frac{\widehat{c}_{1lm}(0)}{c_0^2} + \widehat{p}_{1lm}(0)\right), \qquad (3.53)\\
z_{4lm}(0) &= y_{3lm}(0)\frac{k_1(0)}{2} + y_{4lm}(0) = \frac{1}{\sqrt{4\pi}}\left(\frac{k_1(0)}{2}\frac{\widehat{c}_{1lm}(0)}{c_0^2} - \frac{2c_0}{r_0}\frac{\widehat{c}_{1lm}(0) - 2\widehat{a}_{1lm}(0)}{\rho_0 c_0^2}\right) +\\
&\quad + \frac{1}{\sqrt{4\pi}}\left(\frac{k_1(0)}{2} - \frac{2\rho_0(c_0^2 - 2a_0^2)}{r_0 c_0}\right)\widehat{p}_{1lm}(0), \qquad (3.54)\\
z_{5lm}(x) &= y_{5lm}(x)\frac{S(x)}{S_1(x)} = -\frac{S(x)}{S_1(x)}\frac{1}{\sqrt{4\pi}}\frac{\sqrt{l(l+1)}}{c_0 r(x)}\Big(\widehat{c}_{1lm}(x) - 2\widehat{a}_{1lm}(x)\Big) -\\
&\quad - \frac{S(x)}{S_1(x)}\frac{1}{\sqrt{4\pi}}\frac{\sqrt{l(l+1)}}{c_0 r(x)}(c_0^2 - 2a_0^2)\widehat{p}_{1lm}(x),\\
z_{6lm}(x) &= \left[y_{5lm}(x)\frac{k_1(x)}{2} + y_{6lm}(x)\right]\frac{S(x)}{S_1(x)} =\\
&= -\frac{S(x)}{S_1(x)}\sqrt{\frac{l(l+1)}{4\pi}}\frac{k_1(x)}{2c_0 r(x)}\Big(\widehat{c}_{1lm}(x) - 2\widehat{a}_{1lm}(x)\Big) +
\end{aligned}
$$

$$+\frac{S(x)}{S_1(x)}\sqrt{\frac{l(l+1)}{4\pi}}\frac{2}{r^2(x)}\Big(\widehat{c}_{1lm}(x)-\widehat{a}_{1lm}(x)\Big)+$$

$$+\frac{S(x)}{S_1(x)}\sqrt{\frac{l(l+1)}{4\pi}}\left[\frac{2}{r^2(x)}(c_0^2-a_0^2)-\frac{k_1(x)}{2c_0r(x)}(c_0^2-2a_0^2)\right]\widehat{p}_{1lm}(x).$$

Define new known functions and constants

$$n_{1lm}(x)=k_{6lm}(x)k_1(x)-\frac{1}{\sqrt{4\pi}}\left(\frac{2\widehat{c}_{1lm}(x)}{r(x)c_0}-\frac{\widehat{c}'_{1lm}(x)}{c_0^2}\right),$$

$$n_2=\frac{1}{\sqrt{4\pi}},$$

$$n_{3lm}(x)=k_{6lm}(x)\left(\frac{k'_1(x)}{2}+\frac{k_1^2(x)}{4}\right)-\frac{1}{\sqrt{4\pi}}\frac{k_1(x)}{2c_0}\left(\frac{2\widehat{c}_{1lm}(x)}{r(x)}-\frac{\widehat{c}'_{1lm}(x)}{c_0}\right)+$$

$$+\frac{1}{\sqrt{4\pi}}\left(-\frac{2\widehat{c}_{1lm}(x)}{r^2(x)}+\frac{2\widehat{c}'_{1lm}(x)}{r(x)c_0}-\frac{4\widehat{a}'_{1lm}(x)}{c_0r(x)}\right),$$

$$n_4(x)=-\frac{1}{\sqrt{4\pi}}\frac{2(c_0^2-2a_0^2)}{c_0r(x)},$$

$$n_5(x)=\frac{1}{\sqrt{4\pi}}\frac{k_1(x)}{2},$$

$$n_{6lm}(0)=\frac{1}{\sqrt{4\pi}}\frac{\widehat{c}_{1lm}(0)}{c_0^2},$$

$$n_{7lm}(0)=\frac{1}{\sqrt{4\pi}}\left(\frac{k_1(0)}{2}\frac{\widehat{c}_{1lm}(0)}{c_0^2}-\frac{2c_0}{r_0}\frac{\widehat{c}_{1lm}(0)-2\widehat{a}_{1lm}(0)}{\rho_0c_0^2}\right),$$

$$n_8(0)=\frac{1}{\sqrt{4\pi}}\left(\frac{k_1(0)}{2}-\frac{2\rho_0(c_0^2-2a_0^2)}{r_0c_0}\right),$$

$$n_{9lm}(x)=-\frac{1}{\sqrt{4\pi}}\frac{S(x)}{S_1(x)}\frac{\sqrt{l(l+1)}}{c_0r(x)}\Big(\widehat{c}_{1lm}(x)-2\widehat{a}_{1lm}(x)\Big),$$

$$n_{10l}(x)=-\frac{1}{\sqrt{4\pi}}\frac{S(x)}{S_1(x)}\frac{\sqrt{l(l+1)}}{c_0r(x)}(c_0^2-2a_0^2),$$

$$n_{11lm}(x)=\frac{\sqrt{l(l+1)}}{\sqrt{4\pi}}\frac{S(x)}{S_1(x)}\left[\frac{2}{r^2(x)}\Big(\widehat{c}_{1lm}(x)-\widehat{a}_{1lm}(x)\Big)-\frac{k_1(x)}{2c_0r(x)}\Big(\widehat{c}_{1lm}(x)-2\widehat{a}_{1lm}(x)\Big)\right],$$

$$n_{12l}(x)=\frac{\sqrt{l(l+1)}}{\sqrt{4\pi}}\frac{S(x)}{S_1(x)}\left[\frac{2}{r^2(x)}(c_0^2-a_0^2)-\frac{k_1(x)}{2c_0r(x)}(c_0^2-2a_0^2)\right], \tag{3.55}$$

then unknown functions and constants $z_{1lm}(x)$, $z_{2lm}(x)$, $z_{3lm}(0)$, $z_{4lm}(0)$, $z_{5lm}(x)$, $z_{6lm}(x)$

can be written as

$$
\begin{aligned}
&z_{1lm}(x)=n_{1lm}(x)+n_2\widehat{p}'_{1lm}(x), && z_{2lm}(x)=n_{3lm}(x)+n_4(x)\widehat{p}_{1lm}(x)+n_5(x)\widehat{p}'_{1lm}(x),\\
&z_{3lm}(0)=n_{6lm}(0)+n_2\widehat{p}_{1lm}(0), && z_{4lm}(0)=n_{7lm}(0)+n_8(0)\widehat{p}_{1lm}(0),\\
&z_{5lm}(x)=n_{9lm}(x)+n_{10l}(x)\widehat{p}_{1lm}(x), && z_{6lm}(x)=n_{11lm}(x)+n_{12l}(x)\widehat{p}_{1lm}(x).
\end{aligned}
\tag{3.56}
$$

Using representations (3.56) one can write equations (3.42)–(3.52) as

$$\frac{\partial^2}{\partial t^2}u^0_{00}=\frac{\partial^2}{\partial x^2}u^0_{00}+q_0(x)u^0_{00}, \tag{3.57}$$

$$u^0_{00}\Big|_{t<0}=0, \tag{3.58}$$

$$\left[\frac{\partial u^0_{00}}{\partial x}+h_1u^0_{00}\right]\Big|_{x=0}=g_0\delta(t), \tag{3.59}$$

$$
\begin{aligned}
\frac{\partial^2 u^1_{lm}}{\partial t^2}=\frac{\partial^2 u^1_{lm}}{\partial x^2}&+q_{1l}(x)u^1_{lm}+D_{1l}(x)\frac{\partial v^1_{lm}}{\partial x}+D_{2l}(x)v^1_{lm}+\\
&+K_{6lm}(x)\frac{\partial^2 u^0_{00}}{\partial x^2}+\Big(n_{1lm}(x)+n_2\widehat{p}'_{1lm}(x)\Big)\frac{\partial u^0_{00}}{\partial x}+\\
&+\Big(n_{3lm}(x)+n_4(x)\widehat{p}_{1lm}(x)+n_5(x)\widehat{p}'_{1lm}(x)\Big)u^0_{00},
\end{aligned}
\tag{3.60}
$$

$$u^1_{lm}\Big|_{t<0}=0, \tag{3.61}$$

$$
\begin{aligned}
\Bigg[\frac{\partial u^1_{lm}}{\partial x}&+h_1u^1_{lm}+D_{3l}(0)v^1_{lm}+\Big(n_{6lm}(0)+n_2\widehat{p}_{1lm}(0)\Big)\frac{\partial u^0_{00}}{\partial x}+\\
&+\Big(n_{7lm}(0)+n_8(0)\widehat{p}_{1lm}(0)\Big)u^0_{00}\Bigg]\Bigg|_{x=0}=0,
\end{aligned}
\tag{3.62}
$$

$$u^1_{lm}\Big|_{x=0}=F_{1lm}(t), \tag{3.63}$$

$$
\begin{aligned}
\frac{\partial^2 v^1_{lm}}{\partial t^2}=\frac{a_0^2}{c_0^2}\frac{\partial^2 v^1_{lm}}{\partial x^2}&+q_{2l}(x)v^1_{lm}+K_{3l}(x)\frac{\partial u^1_{lm}}{\partial x}+K_{4l}(x)u^1_{lm}+\\
&+\Big(n_{9lm}(x)+n_{10l}(x)\widehat{p}_{1lm}(x)\Big)\frac{\partial u^0_{00}}{\partial x}+\Big(n_{11lm}(x)+n_{12l}(x)\widehat{p}_{1lm}(x)\Big)u^0_{00},
\end{aligned}
\tag{3.64}
$$

$$v^1_{lm}\Big|_{t<0}=0, \tag{3.65}$$

$$\left[\frac{\partial v^1_{lm}}{\partial x}+h_2v^1_{lm}+K_{5l}(0)u^1_{lm}\right]\Big|_{x=0}=0, \tag{3.66}$$

$$v^1_{lm}\Big|_{x=0} = G_{1lm}(t). \tag{3.67}$$

Inverse Problem 3.1. Let T, r_0, c_0, a_0, p_0 be given positive constants; $0 < T < 2r_0/c_0$. Find the function $\widehat{p}_{1lm}(x) \in \mathbf{C}^1[0, a_0T/(2c_0)]$ in equations (3.60), (3.62), (3.64), given the additional information (3.63), (3.67) about the solution $(u^1_{lm}(x,t), v^1_{lm}(x,t))$ to the direct problem (3.60)–(3.62), (3.64)–(3.66) where $F_{1lm}(t)$, $G_{1lm}(t)$ are given functions for $t \in [0,T]$; the functions $\widehat{a}_{1lm}(x)$, $\widehat{c}_{1lm}(x)$ are known. The function $u^0_{00}(x,t)$ is known as the solution of the direct problem (3.57)–(3.59).

4. Solving the direct problem (3.57)–(3.59) in the domain $\mathcal{D}^0_T = \{(x,t) \mid 0 < x < t < T - x\}$ by the characteristic method

Remark 4.1. Let r_0, c_0, p_0, T, $0 < T < 2r_0/c_0$, be given positive numbers, then $q_0(x) \in \mathbf{C}^\infty[0, T/2]$ is a known function. It follows from (3.57)–(3.59) that the structure of the solution in the domain $\mathcal{D}^0_T$ is

$$u^0_{00}(x,t) = \alpha_0(x)\theta_0(t-x) + \alpha_1(x)\theta_1(t-x) + \alpha_2(x)\theta_2(t-x) + \dots \tag{4.1}$$

where the coefficients $\alpha_j(x)$ are smooth functions on the interval $[0, r_0/c_0]$.

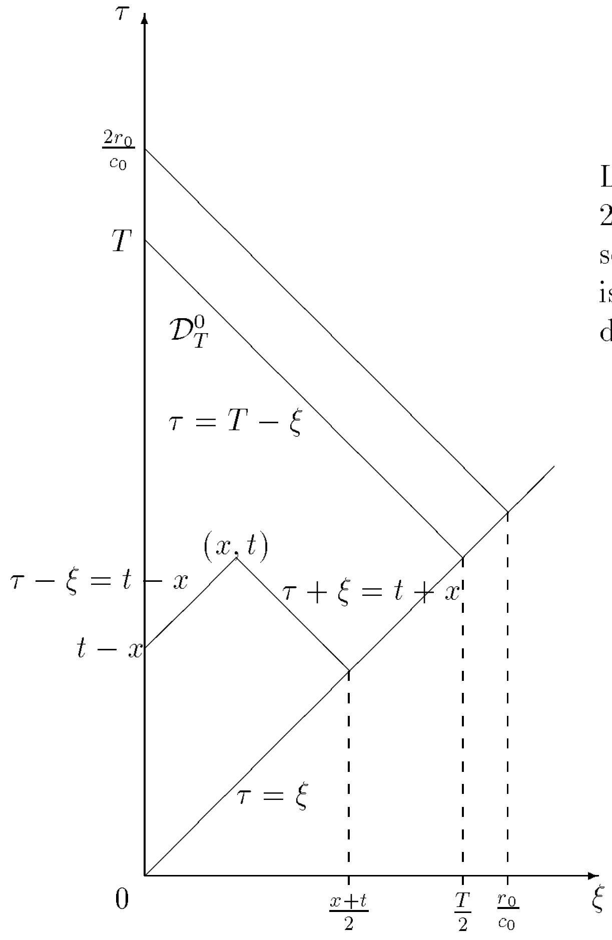


Figure 2

Let T be a given positive number, $0 < T < 2r_0/c_0$. It follows from Remark 4.1 that the solution of the direct problem (3.57)–(3.59) is continuous in the domain $\mathcal{D}^0$ and has a discontinuity at the characteristic $t - x = 0$.

We can determine the coefficients $\alpha_j(x)$ for $j = 0, 1, 2, \dots$ substituting (4.1) into (3.57)–(3.59) and equating the coefficients at the same singularities,

$$-2\alpha_0'(x) = 0, \quad -\alpha_0(0) = g_0; \tag{4.2}$$

$$-2\alpha_1'(x) + \alpha_0''(x) + q_0(x)\alpha_0(x) = 0, \quad -\alpha_1(0) + \alpha_0'(0) + h_1\alpha_0(0) = 0; \tag{4.3}$$

$$-2\alpha_2'(x) + \alpha_1''(x) + q_0(x)\alpha_1(x) = 0, \quad -\alpha_2(0) + \alpha_1'(0) + h_1\alpha_1(0) = 0. \tag{4.4}$$

Solving the corresponding Cauchy problems we obtain

$$\alpha_0(x) = -g_0, \tag{4.5}$$

$$\alpha_1(x) = -g_0 h_1 - \frac{g_0}{2}\int_0^x q_0(\xi)\,\mathrm{d}\xi, \tag{4.6}$$

$$\alpha_2(x) = -\frac{g_0}{2}q_0(0) - g_0 h_1^2 - \frac{g_0}{4}q_0(x) + \frac{1}{2}\int_0^x \alpha_1(\xi)q_0(\xi)\,\mathrm{d}\xi. \tag{4.7}$$

Introduce a new function

$$\widehat{u}_{00}^0(x,t) = \frac{\partial u_{00}^0}{\partial t}. \tag{4.8}$$

Now representation (4.1) for the function $\widehat{u}^0_{00}(x,t)$ can be written as

$$\widehat{u}^0_{00}(x,t) = \alpha_0(x)\delta(t-x) + \alpha_1(x)\theta_0(t-x) + \alpha_2(x)\theta_1(t-x) + \ldots \tag{4.9}$$

We can rewrite this representation in the form

$$\widehat{u}^0_{00}(x,t) = \alpha_0(x)\delta(t-x) + \overline{u}^0_{00}(x,t)\theta_0(t-x) \tag{4.10}$$

where $\overline{u}^0_{00}(x,t)$ is a regular function and, by virtue of (4.9), (4.10), the following equalities are true

$$\frac{\partial \overline{u}^0_{00}}{\partial t}(x, x+0) = \alpha_2(x), \qquad \frac{\partial \overline{u}^0_{00}}{\partial x}(x, x+0) = \alpha_1'(x) - \alpha_2(x). \tag{4.11}$$

From (4.10) it is seen that

$$\widehat{u}^0_{00}(x,t) = \overline{u}^0_{00}(x,t) \quad \text{for } t > x > 0. \tag{4.12}$$

Rewrite the direct problem (3.57)–(3.59) in the domain $\mathcal{D}^0_T$ in terms of the function $\overline{u}^0_{00}$

$$\frac{\partial^2}{\partial t^2}\overline{u}^0_{00} = \frac{\partial^2}{\partial x^2}\overline{u}^0_{00} + q_0(x)\overline{u}^0_{00}, \quad t > x, \tag{4.13}$$

$$\overline{u}^0_{00}\Big|_{t=x} = \alpha_1(x), \quad x \geq 0, \tag{4.14}$$

$$\left[\frac{\partial \overline{u}^0_{00}}{\partial x} + h_1\overline{u}^0_{00}\right]\Bigg|_{x=0} = 0, \quad t > 0. \tag{4.15}$$

Lemma 4.1. *Let r_0, c_0, p_0, T be given positive numbers; $T < 2r_0/c_0$. Then the solution $\overline{u}^0_{00}(x,t)$ of the direct problem (4.13)–(4.15) exists and is unique in the functional class $\mathbf{C}^\infty(\mathcal{D}^0_T)$.*

Proof. Rewrite equations (4.13)–(4.15) in the following form

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\overline{u}^0_{00}(x,t) = q_0(x)\overline{u}^0_{00}(x,t), \quad t > x, \tag{4.16}$$

$$\overline{u}^0_{00}\Big|_{t=x} = \alpha_1(x), \quad x \geq 0, \tag{4.17}$$

$$\frac{\partial \overline{u}^0_{00}}{\partial x}\Big|_{x=0} = -h_1\overline{u}^0_{00}(0,t), \quad t > 0. \tag{4.18}$$

Let us integrate both parts of equation (4.16) along the characteristic of the differential operator $\partial/\partial t - \partial/\partial x$. From (4.11) we can write

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\overline{u}^0_{00}(x,t) = \int\limits_x^{(x+t)/2} q_0(\xi)\overline{u}^0_{00}(\xi, -\xi + t + x)\,\mathrm{d}\xi + \alpha_1'\Big(\frac{x+t}{2}\Big), \quad t \geq x. \tag{4.19}$$

Substituting equality (4.18) into (4.19) we obtain

$$\left(\frac{\partial}{\partial t}\overline{u}^0_{00} - h_1\overline{u}^0_{00}\right)\Bigg|_{x=0} = \int\limits_0^{t/2} q_0(\xi)\overline{u}^0_{00}(\xi, -\xi + t)\,\mathrm{d}\xi + \alpha_1'(t/2) \tag{4.20}$$

which implies

$$\overline{u}^0_{00}(0,t) = \alpha_1(0)\mathrm{e}^{h_1 t} + \int\limits_0^t \mathrm{e}^{h_1(t-\tau)} \left[\int\limits_0^{\tau/2} q_0(\xi)\overline{u}^0_{00}(\xi,-\xi+\tau)\,\mathrm{d}\xi + \alpha_1'(\tau/2) \right] \mathrm{d}\tau \equiv$$

$$\equiv f(t). \tag{4.21}$$

Integrate equality (4.19) along the characteristic of the differential operator $\partial/\partial t + \partial/\partial x$,

$$\overline{u}^0_{00}(x,t) = f(t-x) + \int\limits_{t-x}^{t} \int\limits_{\tau+x-t}^{(2\tau+x-t)/2} q_0(\xi)\overline{u}^0_{00}(\xi,-\xi+2\tau+x-t)\,\mathrm{d}\xi\,\mathrm{d}\tau =$$

$$= \int\limits_0^{t-x} \mathrm{e}^{h_1(t-\tau-x)} \int\limits_0^{\tau/2} q_0(\xi)\overline{u}^0_{00}(\xi,-\xi+\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau +$$

$$+ \int\limits_{t-x}^{t} \int\limits_{\tau+x-t}^{(2\tau+x-t)/2} q_0(\xi)\overline{u}^0_{00}(\xi,-\xi+2\tau+x-t)\,\mathrm{d}\xi\,\mathrm{d}\tau + J(x,t), \tag{4.22}$$

the known function $J(x,t)$ will be defined below. Making in the integrals entering (4.22) the changes of variables

$$\xi_1 \longrightarrow \xi, \qquad -\xi_1 + \tau_1 \longrightarrow \tau$$

and

$$\xi_1 \longrightarrow \xi, \qquad -\xi_1 + 2\tau_1 + x - t \longrightarrow \tau,$$

respectively, and writing them as double integrals we obtain

$$\overline{u}^0_{00}(x,t) = \iint\limits_{\Omega_1(x,t)} \mathrm{e}^{h_1(t-\tau-\xi-x)} q_0(\xi)\overline{u}^0_{00}(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau + \frac{1}{2} \iint\limits_{\Omega_2(x,t)} q_0(\xi)\overline{u}^0_{00}(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau + J(x,t). \tag{4.23}$$

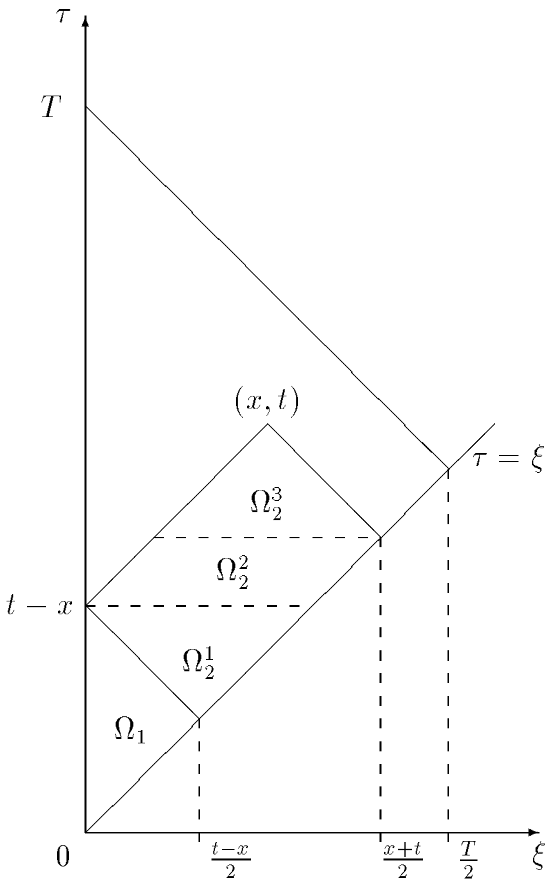


Figure 3

The domains $\Omega_1(x,t)$ and $\Omega_2^1(x,t)$, $\Omega_2^2(x,t)$, $\Omega_2^3(x,t)$ are defined below.

$$
\begin{aligned}
&\Omega_1(x,t)=\left\{(\xi,\tau)\;\Big|\;0<\tau<t-x,\quad 0<\xi<\frac{t-x}{2}-\left|\frac{t-x}{2}-\tau\right|\right\},\\
&\Omega_2(x,t)=\Omega_2^1(x,t)\cup\Omega_2^2(x,t)\cup\Omega_2^3(x,t),\\
&\Omega_2^1(x,t)=\left\{(\xi,\tau)\;\Big|\;\frac{t-x}{2}<\tau<t-x,\quad t-x-\tau<\xi<\tau\right\},\\
&\Omega_2^2(x,t)=\left\{(\xi,\tau)\;\Big|\;t-x<\tau<\frac{t+x}{2},\quad \tau+x-t<\xi<\tau\right\},\\
&\Omega_2^3(x,t)=\left\{(\xi,\tau)\;\Big|\;\frac{t+x}{2}<\tau<t,\quad \tau+x-t<\xi<-\tau+x+t\right\}.
\end{aligned}
\tag{4.24}
$$

The known function $J(x,t)$ is defined as

$$
J(x,t)=\alpha_1(0)\mathrm{e}^{h_1(t-x)}+\int_0^{t-x}\mathrm{e}^{h_1(t-\tau-x)}\alpha_1'(\tau/2)\,\mathrm{d}\tau+\alpha_1\Big(\frac{x+t}{2}\Big)-\alpha_1\Big(\frac{t-x}{2}\Big).
$$

Since $q_0(x) \in \mathbf{C}^\infty[0, T/2]$ and $J(x,t) \in \mathbf{C}^\infty(\mathcal{D}^0_T)$, therefore, the unique solution $\overline{u}^0_{00}(x,t)$ of the Volterra equation (4.23) of the second kind exists in the function class $\mathbf{C}^\infty(\mathcal{D}^0_T)$. ■

Remark 4.2. From (4.8) and (4.10) we can conclude that in the domain $\mathcal{D}^0_T$ the solution $u^0_{00}(x,t)$ of the direct problem (3.57)–(3.59) is in the function class $\mathbf{C}^\infty(\mathcal{D}^0_T)$ and has the form

$$u^0_{00}(x,t) = \alpha_0(x)\theta_0(t-x) + \widetilde{u}^0_{00}(x,t) \tag{4.25}$$

where the function $\widetilde{u}^0_{00}(x,t)$ is defined as

$$\widetilde{u}^0_{00}(x,t) = \int_0^t \overline{u}^0_{00}(x,\tau)\theta_0(\tau - x)\,\mathrm{d}\tau = \theta_0(t-x)\int_x^t \overline{u}^0_{00}(x,\tau)\,\mathrm{d}\tau.$$

5. Investigation of the direct problem (3.60)–(3.62), (3.64)–(3.66) solution structure in the domain $\mathcal{D}^0_T$

Let $\mathcal{D}^0_{*T} = \mathcal{D}^0_{1T} \cup \mathcal{D}^0_{2T}$ where

$$\mathcal{D}^0_{1T} = \left\{(x,t)\,|\,0 < \frac{c_0}{a_0}x < t < T - x\right\}, \quad \mathcal{D}^0_{2T} = \left\{(x,t)\,|\,0 < \frac{a_0}{c_0}t < x < t < T - x\right\}.$$

From here it follows that

$$\mathcal{D}^0_T = \mathcal{D}^0_{*T} \cup \left\{(x,t)\,|\,0 < x < \frac{a_0 T}{a_0 + c_0},\ t = \frac{c_0 x}{a_0}\right\}.$$

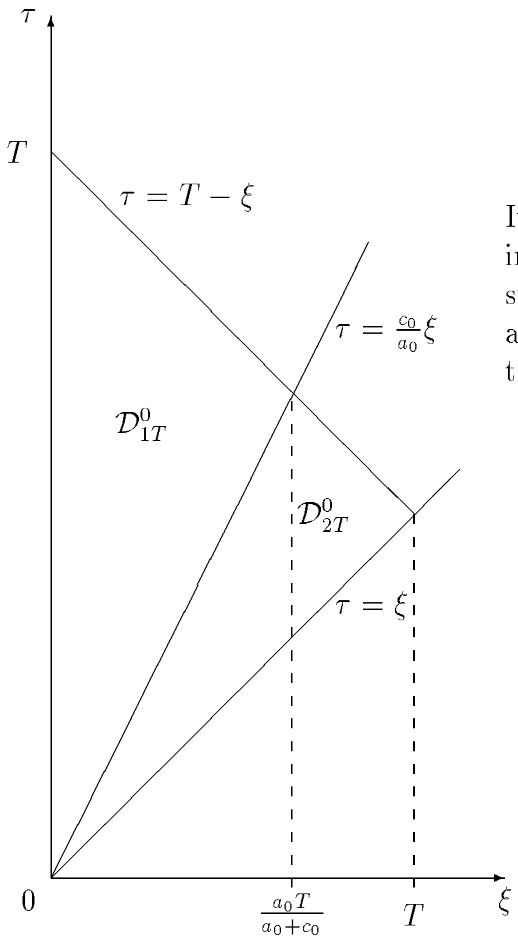


Figure 4

It is obvious from Figure 4 that in investigating the inverse problem (3.60)–(3.62), (3.64)–(3.66) solution structure in the domain $\mathcal{D}^0_T$ it is necessary to take into account the discontinuities of the solution and its derivatives on the characterictics $t - x = 0$ and $t - c_0 x/a_0 = 0$.

Lemma 5.1. *Let r_0, a_0, c_0, p_0 be given positive numbers. Suppose that all coefficients for the functions $u^0_{00}(x,t)$, $u^1_{lm}(x,t)$, $v^1_{lm}(x,t)$ entering equations (3.60)–(3.62), (3.64)–(3.66) are known and sufficiently smooth. Then the structure of the solution $(u^1_{lm}(x,t), v^1_{lm}(x,t))$ of the direct problem (3.60)–(3.62), (3.64)–(3.66) in the domain $\mathcal{D}^0$ is as follows:*

$$u^1_{lm}(t,x) = \beta_{-1lm}(x)\delta(t-x) + \beta_{0lm}(x)\theta_0(t-x) + \beta_{1lm}(x)\theta_1(t-x) + \ldots +$$
$$+\gamma_{2lm}(x)\theta_2\Big(t - \frac{c_0}{a_0}x\Big) + \ldots; \tag{5.1}$$

$$v^1_{lm}(t,x) = \eta_{1lm}(x)\theta_1\Big(t - \frac{c_0}{a_0}x\Big) + \eta_{2lm}(x)\theta_2\Big(t - \frac{c_0}{a_0}x\Big) + \ldots +$$
$$+\text{æ}_{0lm}(x)\theta_0(t-x) + \text{æ}_{1lm}(x)\theta_1(t-x) + \ldots \tag{5.2}$$

where θ_0 is the Heaviside function, $\theta_k(t) = \theta_0(t)t^k/k!$; the coefficients $\beta_{jlm}(x)$, $\gamma_{jlm}(x)$, $\eta_{jlm}(x)$, $\text{æ}_{jlm}(x)$ are smooth functions on the interval $[0, r_0/c_0]$.

Proof. Equalities (3.60)–(3.62), (3.64)–(3.66), (4.1) imply that the structure of the solution of the direct problem (3.60)–(3.62), (3.64)–(3.66) in the domain $\mathcal{D}^0$ can be written as

$$u^1_{lm}(t,x) = \beta_{-1lm}(x)\delta(t-x) + \beta_{0lm}(x)\theta_0(t-x) + \beta_{1lm}(x)\theta_1(t-x) + \ldots +$$

$$+\gamma_{1lm}(x)\theta_1\Big(t-\frac{c_0}{a_0}x\Big)+\ldots; \tag{5.3}$$

$$v^1_{lm}(t,x)=\eta_{0lm}(x)\theta_0\Big(t-\frac{c_0}{a_0}x\Big)+\eta_{1lm}(x)\theta_1\Big(t-\frac{c_0}{a_0}x\Big)+\ldots+$$

$$+æ_{0lm}(x)\theta_0(t-x)+æ_{1lm}(x)\theta_1(t-x)+\ldots \tag{5.4}$$

Substituting the representations (4.1), (5.3)–(5.4) into equalities (3.60)–(3.62), (3.64)–(3.66) and equating the coefficients at the same singularities we can write the equalities for finding the function $\beta_{jlm}(x)$ for $j=-1,0,1$:

$$2\beta'_{-1lm}(x)=K_{6lm}(x)\alpha_0(x), \tag{5.5}$$

$$0=-\beta_{-1lm}(0); \tag{5.6}$$

$$2\beta'_{0lm}(x)=\beta''_{-1lm}(x)+q_{1l}(x)\beta_{-1lm}(x)-D_{1l}(x)æ_{0lm}(x)+$$

$$+K_{6lm}(x)\Big(\alpha_1(x)-2\alpha'_0(x)\Big)-\Big(n_{1lm}(x)+n_2(x)\widehat{p}'_{1lm}(x)\Big)\alpha_0(x), \tag{5.7}$$

$$0=-\Big(\beta_{0lm}(0)-\beta'_{-1lm}(0)\Big)+h_1\beta_{-1lm}(0)-\alpha_0(0)(n_{6lm}(0)+n_2\widehat{p}_{1lm}(0)); \tag{5.8}$$

$$2\beta'_{1lm}(x)=\beta''_{0lm}(x)+q_{1l}(x)\beta_{0lm}(x)-D_{1l}(x)\Big(æ_{1lm}(x)-æ'_{0lm}(x)\Big)+$$

$$+D_{2l}(x)æ_{0lm}(x)+K_{6lm}(x)\Big(\alpha_2(x)-2\alpha'_1(x)+\alpha''_0(x)\Big)-$$

$$-\Big(n_{1lm}(x)+n_2\widehat{p}'_{1lm}(x)\Big)\Big(\alpha_1(x)-\alpha'_0(x)\Big)+$$

$$+\Big(n_{3lm}(x)+n_4(x)\widehat{p}_{1lm}(x)+n_5(x)\widehat{p}'_{1lm}(x)\Big)\alpha_0(x), \tag{5.9}$$

$$0=-\Big(\beta_{1lm}(0)-\beta'_{0lm}(0)\Big)-\frac{c_0}{a_0}\gamma_{1lm}(0)+h_1\beta_{0lm}(0)+$$

$$+D_{3l}(0)\Big(\eta_{0lm}(0)+æ_{0lm}(0)\Big)-(n_{6lm}(0)+n_2\widehat{p}_{1lm}(0))\times$$

$$\times\Big(\alpha_1(0)-\alpha'_0(0)\Big)+(n_{7lm}(0)+n_{8lm}(0)\widehat{p}_{1lm}(0))\alpha_0(0). \tag{5.10}$$

The function $\gamma_{1lm}(x)$ can be determined from the equality

$$\gamma_{1lm}(x)=\frac{c_0^2}{a_0^2}\gamma_{1lm}(x)-\frac{c_0}{a_0}D_{1lm}(x)\eta_{0lm}(x). \tag{5.11}$$

One can find the functions $\eta_{jlm}(x)$, $j=0,1$, from the equalities

$$\frac{2c_0}{a_0}\eta'_{0lm}(x)=0, \tag{5.12}$$

$$0=-\frac{c_0}{a_0}\eta_{0lm}(0)-æ_{0lm}(0)+K_{5lm}\beta_{-1lm}(0); \tag{5.13}$$

$$2\frac{c_0}{a_0}\eta'_{1lm}(x)=\eta''_{0lm}(x)+q_2(x)\eta_{0lm}(x)-\frac{c_0}{a_0}K_{3l}(x)\gamma_{1lm}(x), \tag{5.14}$$

$$0=-\Big(\frac{c_0}{a_0}\eta_{1lm}(0)-\eta'_{0lm}(0)\Big)-\Big(æ_{1lm}(0)-æ'_{0lm}(0)\Big)+$$

$$+h_2\Big(\eta_{0lm}(0)+æ_{0lm}(0)\Big)+K_{5l}(0)\beta_{0lm}(0). \tag{5.15}$$

To find the functions $æ_{jlm}(x)$, $j=0,1$, we can use the following equalities

$$æ_{0lm}(x)=\frac{a_0^2}{c_0^2}æ_{0lm}(x)-K_{3l}(x)\beta_{-1}(x), \tag{5.16}$$

$$æ_{1lm}(x)=\frac{a_0^2}{c_0^2}\Big(æ_{1lm}(x)-2æ'_{0lm}(x)\Big)-K_{3l}(x)\Big(\beta_{0lm}(x)-\beta'_{-1lm}(x)\Big)$$

$$+K_{4l}(x)\beta_{-1lm}(x)-\alpha_0(x)\Big(n_{9l}(x)+n_{10l}(x)\widehat{p}_{1lm}(x)\Big). \tag{5.17}$$

Equalities (4.5) and (5.5), (5.6), (5.16) imply that $\beta_{-1lm}(x)$ and $æ_{0lm}(x)$ are known functions, moreover,

$$æ_{0lm}(x)=\frac{c_0^2}{a_0^2-c_0^2}K_{3l}(x)\beta_{-1lm}(x). \tag{5.18}$$

From (5.18) we have

$$æ_{0lm}(0)=0. \tag{5.19}$$

From equalities (5.12), (5.13), (5.19) we conclude that

$$\eta_{0lm}(x)\equiv 0. \tag{5.20}$$

From (5.11) and (5.20) we obtain

$$\gamma_{1lm}(x)\equiv 0. \tag{5.21}$$

From (5.18) we can write

$$æ'_{0lm}(x)=\frac{c_0^2}{a_0^2-c_0^2}\Big[K'_{3l}(x)\beta_{-1lm}(x)+K_{3l}(x)\beta'_{-1lm}(x)\Big]. \tag{5.22}$$

From (5.22) and (5.5) we derive

$$æ'_{0lm}(0)=\frac{g_0c_0^2}{c_0^2-a_0^2}K_{3l}(0)K_{6lm}(0). \tag{5.23}$$

From (5.17) and the above equalities we conclude that

$$æ_{1lm}(0)=\frac{c_0^2}{c_0^2-a_0^2}\Bigg[\frac{2a_0^2g_0}{a_0^2-c_0^2}K_{3l}(0)K_{6lm}(0)-$$

$$-K_{3l}(0)\Big(\frac{g_0}{2}K_{6lm}(0)+g_0\Big(n_{6lm}(0)+n_2\widehat{p}_{1lm}(0)\Big)\Big)-$$

$$-\frac{g_0}{2}K_{3l}(0)K_{6lm}(0)+g_0\Big(n_{9lm}(0)+n_{10l}(0)\widehat{p}_{1lm}(0)\Big)\Bigg] \tag{5.24}$$

is a known constant. From here and from (5.14), (5.15), (5.19)–(5.21) we conclude that

$$\eta_{1lm}(x) \equiv \eta_{1lm}(0) = æ'_{0lm}(0) - æ_{1lm}(0) \tag{5.25}$$

is a known constant.

Let $u^1_{lm\,x}$, $u^1_{lm\,t}$, $v^1_{lm\,x}$, $v^1_{lm\,t}$ denote the partial derivatives of the functions $u^1_{lm}(x,t)$, $v^1_{lm}(x,t)$ with respect to x and t. The representations (5.3), (5.4) and equalities (5.20), (5.21), (5.25) imply

$$u^1_{lm}\Big|_{t=x+0} = \beta_{0lm}(x), \tag{5.26}$$

$$u^1_{lm\,t}\Big|_{t=x+0} = \beta_{1lm}(x), \tag{5.27}$$

$$u^1_{lm\,x}\Big|_{t=x+0} = \beta'_{0lm}(x) - \beta_{1lm}(x), \tag{5.28}$$

$$\left[u^1_{lm}\right]_{t=c_0x/a_0} \equiv 0, \tag{5.29}$$

$$\left[u^1_{lm\,x}\right]_{t=c_0x/a_0} = -\frac{c_0}{a_0}\gamma_{1lm}(x) \equiv 0, \tag{5.30}$$

$$\left[u^1_{lm\,t}\right]_{t=c_0x/a_0} = \gamma_{1lm}(x) \equiv 0, \tag{5.31}$$

$$v^1_{lm}\Big|_{t=x+0} = æ_{0lm}(x), \tag{5.32}$$

$$v^1_{lm\,t}\Big|_{t=x+0} = æ_{1lm}(x), \tag{5.33}$$

$$v^1_{lm\,x}\Big|_{t=x+0} = æ'_{0lm}(x) - æ_{1lm}(x), \tag{5.34}$$

$$\left[v^1_{lm}\right]_{t=c_0x/a_0} = \eta_{0lm}(x) \equiv 0, \tag{5.35}$$

$$\left[v^1_{lm\,x}\right]_{t=c_0x/a_0} = -\frac{c_0}{a_0}\eta_{1lm}(0), \tag{5.36}$$

$$\left[v^1_{lm\,t}\right]_{t=c_0x/a_0} = \eta_{1lm}(0). \tag{5.37}$$

So, the lemma is proved. ■

Remark 5.1. From (5.29)–(5.31), (5.35)–(5.37) it follows that the functions u^1_{lm}, $u^1_{lm\,x}$, $u^1_{lm\,t}$, v are continuous on the characteristic $t - c_0x/a_0 = 0$, while the functions $v^1_{lm\,x}$, $v^1_{lm\,t}$ have jumps on this characteristic, moreover, the magnitudes of these jumps are known constants independent of the point (x,t).

6. Solving the direct problem (3.60)–(3.62), (3.64)–(3.66) in the domain $\mathcal{D}_T^0$

Lemma 6.1. *Let r_0, a_0, c_0, p_0, T be given positive numbers; $T < 2r_0/c_0$. Let $\widehat{p}_{1lm}(x) \in \mathbf{C}^1[0, T/2]$, and all the coefficients entering equalities (3.60)–(3.62), (3.64)–(3.66) be known sufficiently smooth functions. Then the direct problem (3.60)–(3.62), (3.64)–(3.66) has a unique generalized solution $(u_{lm}^1(x,t),\, v_{lm}^1(x,t))$ in the function class $\mathbf{C}^2(\mathcal{D}_{*T}^0) \cap \mathbf{C}(\overline{\mathcal{D}}_T^0)$.*

For simplicity, here and in the next section we introduce the notations

$$u^1(x,t) = u_{lm}^1(x,t), \quad v^1(x,t) = v_{lm}^1(x,t), \quad u^0(x,t) = u_{00}^0(x,t); \tag{6.1}$$

$$\Phi_{1lm}(x,t) = \Big(n_{3lm}(x) + n_4(x)\widehat{p}_{1lm}(x) + n_5(x)\widehat{p}_{1lm}'(x)\Big)u^0 +$$
$$+\Big(n_{1lm}(x) + n_2\widehat{p}_{1lm}'(x)\Big)u_x^0 + K_{6lm}(x)u_{xx}^0;$$
$$\Phi_{2lm}(x,t) = \Big(n_{9lm}(x) + n_{10l}(x)\widehat{p}_{1lm}(x)\Big)u_x^0 + \Big(n_{11lm}(x) + n_{12l}(x)\widehat{p}_{1lm}(x)\Big)u^0;$$
$$\Phi_{3lm}(t) = \Big(n_{6lm}(0) + n_2\widehat{p}_{1lm}(0)\Big)u_x^0 + \Big(n_{7lm}(0) + n_8\widehat{p}_{1lm}(0)\Big)u^0;$$
$$P_{1lm}(x,t) = q_{1l}(x)u^1 + D_{1l}(x)v_x^1 + D_{2l}(x)v^1 + \Phi_{1lm}(x,t);$$
$$P_{2lm}(x,t) = q_{2l}(x)v^1 + K_{3l}(x)u_x^1 + K_{4l}(x)u^1 + \Phi_{2lm}(x,t). \tag{6.2}$$

Then equation (3.60) can be written as

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)u^1(x,t) = P_{1lm}(x,t), \quad t > x. \tag{6.3}$$

Using the structure (5.1) and equalities (5.26)–(5.31) we can write

$$\begin{gathered} u_x^1\Big|_{t=x+0} = \beta_{0lm}'(x) - \beta_{1lm}(x), \qquad u_t^1\Big|_{t=x+0} = \beta_{1lm}(x), \\ \Big[u^1\Big]_{t=c_0x/a_0} \equiv 0, \qquad \Big[u_x^1\Big]_{t=c_0x/a_0} \equiv 0, \qquad \Big[u_t^1\Big]_{t=c_0x/a_0} \equiv 0. \end{gathered} \tag{6.4}$$

Equation (3.62) can be written in the form

$$\left(u_x^1 + h_1u^1\right)\Big|_{x=0} = -\Big(D_{3l}(0)v^1(0,t) + \Phi_{3lm}(t)\Big), \quad t > 0. \tag{6.5}$$

Equation (3.64) can be written in the form

$$\left(\frac{\partial}{\partial t} - \frac{a_0}{c_0}\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \frac{a_0}{c_0}\frac{\partial}{\partial x}\right)v^1(x,t) = P_{2lm}(x,t), \quad t > x. \tag{6.6}$$

Using the representation (5.2) and equalities (5.32)–(5.37) we can write

$$\begin{gathered} v^1\Big|_{t=x+0} = æ_{0lm}(x), \;\; v_x^1\Big|_{t=x+0} = æ_{0lm}'(x) - æ_{1lm}(x), \;\; v_t^1\Big|_{t=x+0} = æ_{1lm}(x), \\ \Big[v_x^1\Big]_{t=c_0x/a_0} = -\frac{c_0}{a_0}\eta_{1lm}(0), \quad \Big[v_t^1\Big]_{t=c_0x/a_0} = \eta_{1lm}(0). \end{gathered} \tag{6.7}$$

Then equation (3.66) assumes the form

$$\left(v_x^1 + h_2 v^1\right)\Big|_{x=0} = -K_{5l}(0)u^1(0,t). \tag{6.8}$$

We will solve the direct problem (6.3)–(6.5), (6.6)–(6.8) in the domain $\mathcal{D}_T^0$ by the characteristics method. We will derive a system of integral equations to solve this problem. The analytic writing in each of the domains $\mathcal{D}_{1T}^0$ and $\mathcal{D}_{2T}^0$ is different.

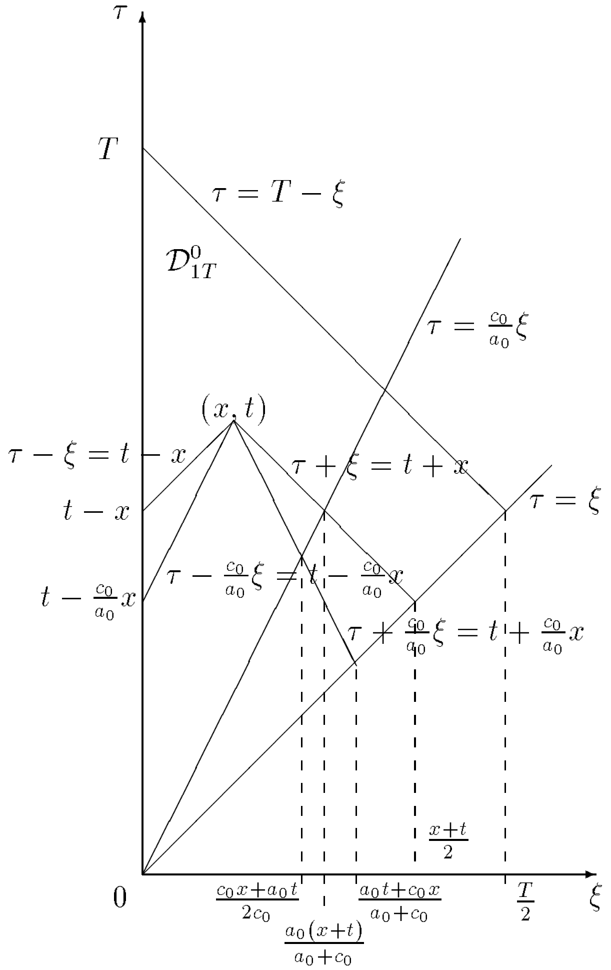


Figure 5

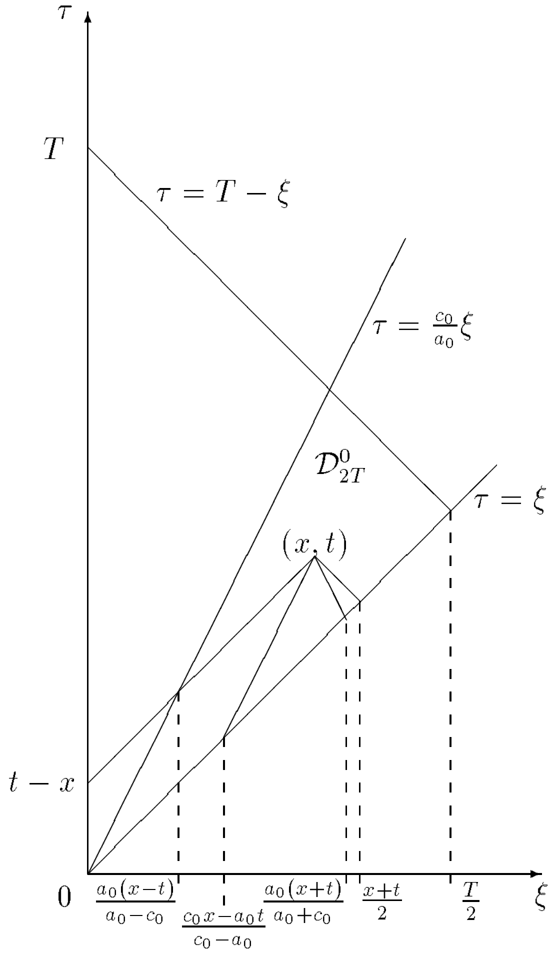


Figure 6

Remark 6.1. Note that in the result of integration along the characteristics of the differential operators $\partial/\partial t - \partial/\partial x$ and $\partial/\partial t + \partial/\partial x$ the solution $u^1(x,t)$ of the problem (6.3)–(6.5) for $(x,t) \in \mathcal{D}_{1T}^0$ and $(x,t) \in \mathcal{D}_{2T}^0$ will be given by the same formulas since in the process of integrating along the characteristics the jumps of the function v_x^1, entering the function $P_{1lm}(x,t)$, do not influence the form of the resulting solution.

Let (x,t) be an arbitrary point in the domain $\mathcal{D}_T^0$. Integrate both parts of equalities (6.3) along the characteristic of the differential operator $\partial/\partial t - \partial/\partial x$ using (6.4). We have the equality

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) u^1(x,t) = \beta'_{0lm}\Big(\frac{x+t}{2}\Big) + \int\limits_x^{(x+t)/2} P_{1lm}(\xi, -\xi+t+x)\,\mathrm{d}\xi \equiv f_1(x,t), \quad (x,t) \in \mathcal{D}_T^0. \tag{6.9}$$

From (6.5) we can write

$$u^1_x(0,t) = -h_1 u^1(0,t) - \Big(D_{3l}(0)v^1(0,t) + \Phi_{3lm}(t)\Big). \tag{6.10}$$

Substituting (6.10) into equality (6.9) we get

$$u^1_t(0,t) - h_1 u^1(0,t) = \Big(D_{3l}(0)v^1(0,t) + \Phi_{3lm}(t)\Big) + \beta'_{0lm}(t/2) +$$
$$+ \int_0^{t/2} P_{1lm}(\xi, -\xi + t)\,\mathrm{d}\xi, \qquad t > 0. \tag{6.11}$$

From here we have

$$u^1(0,t) = \beta_{0lm}(0)\mathrm{e}^{h_1 t} + \int\limits_0^t \mathrm{e}^{h_1(t-\tau)}\Bigg[D_{3l}(0)v^1(0,\tau) + \Phi_{3lm}(\tau) + \beta'_{0lm}(\tau/2) -$$
$$- \int\limits_0^{\tau/2} P_{1lm}(\xi, -\xi + \tau)\,\mathrm{d}\xi\Bigg]\,\mathrm{d}\tau, \qquad (x,t) \in \mathcal{D}^0_T. \tag{6.12}$$

Integrate equality (6.9) along the characteristic of the differential operator $\partial/\partial t + \partial/\partial x$

$$u^1(x,t) = u^1(0, t-x) + \int\limits_{t-x}^{t} f_1(\tau + x - t, \tau)\,\mathrm{d}\tau, \quad (x,t) \in \mathcal{D}^0_T. \tag{6.13}$$

By (6.9), (6.12) and (6.2), equation (6.13) can be written in the form

$$u^1(x,t) = \int\limits_0^{t-x} \exp\Big\{h_1(t-x-\tau)\Big\} D_{3l}(0)v^1(0,\tau)\,\mathrm{d}\tau +$$
$$+ \int\limits_0^{t-x} \exp\Big\{h_1(t-x-\tau)\Big\} \int\limits_0^{\tau/2} \Big(q_{1l}u^1 + D_{1l}v^1_\xi + D_{2l}v^1\Big)(\xi, -\xi+\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau +$$
$$+ \int\limits_0^{t} \int\limits_{\tau+x-t}^{\frac{2\tau+x-t}{2}} \Big[q_{1l}u^1 + D_{1l}v^1_\xi + D_{2l}v^1\Big](\xi, -\xi + 2\tau + x - t)\,\mathrm{d}\xi\,\mathrm{d}\tau +$$
$$+ \widetilde{u}^1(x,t), \quad (x,t) \in \mathcal{D}^0_T, \tag{6.14}$$

where $\widetilde{u}^1(x,t)$ is a known function determined by the formula

$$\widetilde{u}^1(x,t) = \int\limits_0^{t-x} \exp\Big\{h_1(t-x-\tau)\Big\} \times$$

$$\times\left[\Phi_{3lm}(\tau)+\beta'_{0lm}(\tau/2)+\int\limits_0^{\tau/2}\Phi_{1lm}(\xi,-\xi+\tau)\,\mathrm{d}\xi\right]\mathrm{d}\tau+$$

$$+\int\limits_{t-x}^{t}\left[\int\limits_{\tau+x-t}^{\frac{2\tau+x-t}{2}}\Phi_{1lm}(\xi,-\xi+2\tau+x-t)\,\mathrm{d}\xi+\beta'_{0lm}\Big(\frac{2\tau+x-t}{2}\Big)\right]\mathrm{d}\tau+$$

$$+\beta_{0lm}(0)\mathrm{e}^{h_1(t-x)},\quad (x,t)\in\mathcal{D}^0_T. \tag{6.15}$$

Consider the second and third integrals in equality (6.14). Make in them the following changes of variables

$$\xi_1\longrightarrow\xi,\ -\xi_1+\tau_1\longrightarrow\tau\quad\text{and}\quad \xi_1\longrightarrow\xi,\ -\xi_1+2\tau_1+x-t\longrightarrow\tau,$$

respectively, and write them as double integrals

$$\int\limits_0^{t-x}\exp\Big\{h_1(t-x-\tau_1)\Big\}\int\limits_0^{\tau_1/2}\Big(q_{1l}u^1+D_{1l}v^1_\xi+D_{2l}v^1\Big)(\xi,-\xi+\tau_1)\,\mathrm{d}\xi\,\mathrm{d}\tau_1=$$

$$=\iint\limits_{\Omega_1(x,t)}\exp\Big\{h_1(t-\tau-x-\xi)\Big\}\Big(q_{1l}u^1+D_{1l}v^1_\xi+D_{2l}v^1\Big)(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau,$$

$$\int\limits_0^{t}\int\limits_{\tau_1+x-t}^{\frac{2\tau_1+x-t}{2}}\Big[q_{1l}u^1+D_{1l}v^1_\xi+D_{2l}v^1\Big](\xi,-\xi+2\tau_1+x-t)\,\mathrm{d}\xi\,\mathrm{d}\tau_1=$$

$$=\frac{1}{2}\iint\limits_{\Omega_2(x,t)}\Big[q_{1l}u^1+D_{1l}v^1_\xi+D_{2l}v^1\Big](\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau. \tag{6.16}$$

The domains $\Omega_1(x,t)$ and $\Omega_2(x,t)$ are defined by equalities (4.24).

So, (6.15), (6.16) allow us to write equation (6.14) in the form

$$u^1(x,t)=\int\limits_0^{t-x}\exp\Big\{h_1(t-x-\tau)\Big\}D_{3l}(0)v^1(0,\tau)\,\mathrm{d}\tau+$$

$$+\iint\limits_{\Omega_1(x,t)}\exp\Big\{h_1(t-\tau-x-\xi)\Big\}\Big(q_{1l}u^1+D_{1l}v^1_\xi+D_{2l}v^1\Big)(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau+$$

$$+\frac{1}{2}\iint\limits_{\Omega_2(x,t)}\Big[q_{1l}u^1+D_{1l}v^1_\xi+D_{2l}v^1\Big](\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau+\widetilde{u}^1(x,t),\quad (x,t)\in\mathcal{D}^0_T. \tag{6.17}$$

Now consider the system of equalities (6.6)–(6.8). Let $(x,t)\in\mathcal{D}^0_{1T}$ (see Figure 5). Integrate both parts of equality (6.6) along the characteristic of the differential operator

$\partial/\partial t - (a_0/c_0)\partial/\partial x$. From (6.7) we can write

$$\left(\frac{\partial}{\partial t} + \frac{a_0}{c_0}\frac{\partial}{\partial x}\right) v^1(x,t) = \frac{c_0}{a_0} \int\limits_{x}^{\frac{c_0 x + a_0 t}{c_0 + a_0}} P_{2lm}\Big(\xi, t + \frac{c_0}{a_0}(x-\xi)\Big)\, \mathrm{d}\xi +$$

$$+æ'_{0lm}\Big(\frac{a_0 t + c_0 x}{a_0 + c_0}\Big) + \frac{c_0 - a_0}{c_0} æ_{1lm}\Big(\frac{a_0 t + c_0 x}{a_0 + c_0}\Big) \equiv$$

$$\equiv g_1(x,t), \quad (x,t) \in \mathcal{D}^0_{1T}. \tag{6.18}$$

Using (6.8) we arrive at the equality

$$v^1_x(0,t) = -h_2 v^1(0,t) - K_{5l}(0) u^1(0,t). \tag{6.19}$$

Substituting (6.19) into (6.18) we get

$$v^1_t(0,t) - \frac{a_0}{c_0} h_2 v^1(0,t) = \frac{a_0}{c_0} K_{5l}(0) u^1(0,t) + \frac{c_0}{a_0} \int\limits_{x}^{\frac{a_0 t}{c_0 + a_0}} P_{2lm}\Big(\xi, t - \frac{c_0}{a_0}\xi\Big)\, \mathrm{d}\xi +$$

$$+æ'_{0lm}\Big(\frac{a_0 t}{a_0 + c_0}\Big) + \frac{c_0 - a_0}{c_0} æ_{1lm}\Big(\frac{a_0 t}{a_0 + c_0}\Big), \quad t > 0. \tag{6.20}$$

From here we obtain

$$v^1(0,t) = \int\limits_0^t \exp\left\{\frac{a_0 h_2}{c_0}(t-\tau)\right\} \Bigg[\frac{a_0}{c_0} K_{5l}(0) u^1(0,\tau) +$$

$$+ æ'_{0lm}\Big(\frac{a_0 \tau}{a_0 + c_0}\Big) + \frac{c_0 - a_0}{c_0} æ_{1lm}\Big(\frac{a_0 \tau}{a_0 + c_0}\Big)\Bigg]\, \mathrm{d}\tau +$$

$$+\frac{c_0}{a_0} \int\limits_0^t \exp\left\{\frac{a_0 h_2}{c_0}(t-\tau)\right\} \int\limits_0^{\frac{a_0 \tau}{c_0 + a_0}} P_{2lm}\Big(\xi, \tau - \frac{c_0}{a_0}\xi\Big)\, \mathrm{d}\xi\, \mathrm{d}\tau,$$

$$(x,t) \in \mathcal{D}^0_{1T}. \tag{6.21}$$

Now integrate equation (6.18) along the characteristic of the differential operator $\partial/\partial t + (a_0/c_0)\partial/\partial x$.

$$v^1(x,t) = v^1\Big(0, t - \frac{c_0}{a_0}x\Big) + \int\limits_{t - \frac{c_0}{a_0}x}^{t} g_1\Big(x - \frac{a_0}{c_0}(t-\tau), \tau\Big)\, \mathrm{d}\tau, \quad (x,t) \in \mathcal{D}^0_{1T}. \tag{6.22}$$

In view of (6.2) and (6.21), the equation (6.22) can be written in the form

$$v^1(x,t) = \frac{a_0}{c_0} K_5(0) \int\limits_0^{t-\frac{c_0}{a_0}x} \exp\left\{ \frac{a_0 h_2}{c_0}\left(t-\tau-\frac{c_0}{a_0}x\right)\right\} u^1(0,\tau)\,\mathrm{d}\tau +$$

$$+\frac{c_0}{a_0} \int\limits_0^{t-\frac{c_0}{a_0}x} \exp\left\{ \frac{a_0 h_2}{c_0}\left(t-\tau-\frac{c_0}{a_0}x\right)\right\} \times$$

$$\times \int\limits_0^{\frac{a_0\tau}{c_0+a_0}} \Big[q_{2l}v^1 + K_{3l}u^1_\xi + K_{4l}u^1\Big]\Big(\xi, \tau - \frac{c_0}{a_0}\xi\Big)\,\mathrm{d}\xi\,\mathrm{d}\tau +$$

$$+\frac{c_0}{a_0} \int\limits_{t-\frac{c_0}{a_0}x}^{t} \int\limits_{x+\frac{a_0}{c_0}(\tau-t)}^{\frac{c_0x+a_0(2\tau-t)}{c_0+a_0}} \Big[q_{2l}v^1 + K_{3l}u^1_\xi + K_{4l}u^1\Big]\Big(\xi, 2\tau - t + \frac{c_0}{a_0}(x-\xi)\Big)\,\mathrm{d}\xi\,\mathrm{d}\tau +$$

$$+\widetilde{v}^1_1(x,t), \qquad (x,t)\in\mathcal{D}^0_{1T}. \tag{6.23}$$

The known function $\widetilde{v}^1_1(x,t)$ is

$$\widetilde{v}^1_1(x,t) = \frac{c_0}{a_0} \int\limits_0^{t-\frac{c_0}{a_0}x} \exp\left\{ \frac{a_0 h_2}{c_0}\left(t-\tau-\frac{c_0}{a_0}x\right)\right\} \int\limits_0^{\frac{a_0\tau}{c_0+a_0}} \Phi_{2lm}\Big(\xi, \tau - \frac{c_0}{a_0}\xi\Big)\,\mathrm{d}\xi\,\mathrm{d}\tau +$$

$$+\frac{c_0}{a_0} \int\limits_{t-\frac{c_0}{a_0}x}^{t} \int\limits_{x+\frac{a_0}{c_0}(\tau-t)}^{\frac{c_0x+a_0(2\tau-t)}{c_0+a_0}} \Phi_{2lm}\Big(\xi, 2\tau-t+\frac{c_0}{a_0}(x-\xi)\Big)\,\mathrm{d}\xi\,\mathrm{d}\tau +$$

$$+ \int\limits_0^{t-\frac{c_0}{a_0}x} \exp\left\{ \frac{a_0 h_2}{c_0}\left(t-\tau-\frac{c_0}{a_0}x\right)\right\} \times$$

$$\times \left[æ'_{0lm}\Big(\frac{a_0\tau}{a_0+c_0}\Big) + \frac{c_0-a_0}{c_0} æ_{1lm}\Big(\frac{a_0\tau}{a_0+c_0}\Big)\right]\mathrm{d}\tau +$$

$$+ \int\limits_{t-\frac{c_0 x}{a_0}}^{t} \left[æ'_{0lm}\Big(\frac{c_0x + a_0(2\tau-t)}{a_0+c_0}\Big) + \frac{c_0-a_0}{c_0} æ_{1lm}\Big(\frac{c_0x+a_0(2\tau-t)}{a_0+c_0}\Big)\right]\mathrm{d}\tau,$$

$$(x,t)\in\mathcal{D}^0_{1T}. \tag{6.24}$$

Consider the second and third integrals in equality (6.23), make in them the changes of variables

$$\xi_1 \longrightarrow \xi, \quad \tau_1 - \frac{c_0}{a_0}\xi_1 \longrightarrow \tau \qquad \text{and} \qquad \xi_1 \longrightarrow \xi, \quad 2\tau_1 - t + \frac{c_0}{a_0}(x-\xi_1) \longrightarrow \tau,$$

respectively, and rewrite them as double integrals

$$\frac{c_0}{a_0}\int\limits_0^{t-\frac{c_0}{a_0}x} \exp\left\{\frac{a_0 h_2}{c_0}\left(t-\tau-\frac{c_0}{a_0}x\right)\right\} \int\limits_0^{\frac{a_0\tau}{c_0+a_0}} \left[q_{2l}v^1 + K_{3l}u^1_\xi + K_{4l}u^1\right]\left(\xi, \tau - \frac{c_0}{a_0}\xi\right) \mathrm{d}\xi\, \mathrm{d}\tau =$$

$$= \frac{c_0}{a_0}\iint\limits_{\Omega_3(x,t)} \exp\left\{\frac{a_0 h_2}{c_0}\left(t-\tau-\frac{c_0}{a_0}(x+\xi)\right)\right\}\left[q_{2l}v^1 + K_{3l}u^1_\xi + K_{4l}u^1\right](\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau,$$

$$\frac{c_0}{a_0}\int\limits_{t-\frac{c_0}{a_0}x}^{t} \int\limits_{x+\frac{a_0}{c_0}(\tau-t)}^{\frac{c_0x+a_0(2\tau-t)}{c_0+a_0}} \left[q_{2l}v^1 + K_{3l}u^1_\xi + K_{4l}u^1\right]\left(\xi, 2\tau - t + \frac{c_0}{a_0}(x-\xi)\right)\mathrm{d}\xi\,\mathrm{d}\tau =$$

$$= \frac{c_0}{2a_0}\iint\limits_{\Omega_4(x,t)} \left[q_{2l}v^1 + K_{3l}u^1_\xi + K_{4l}u^1\right](\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau. \tag{6.25}$$

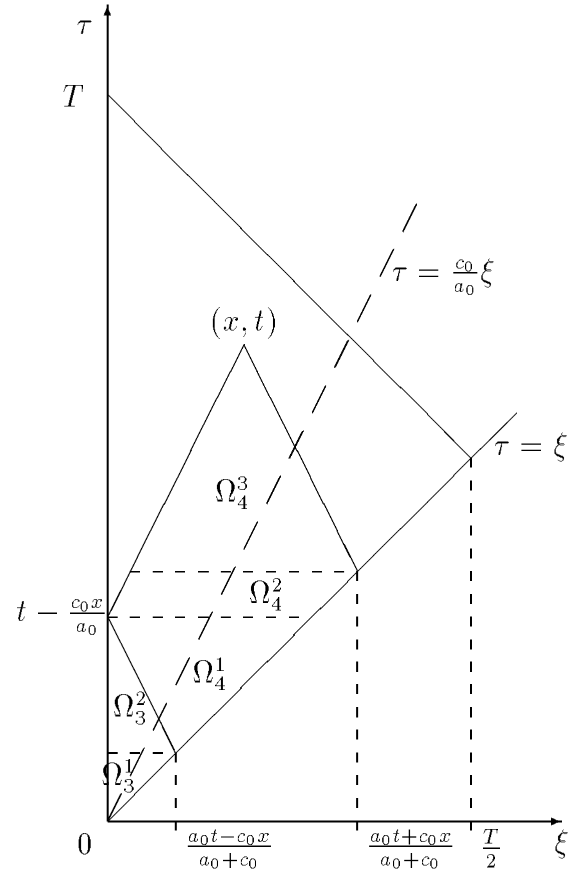


Figure 7

Domains $\Omega_3(x,t)$ and $\Omega_4(x,t)$ are defined below.

$$\Omega_3(x,t) = \Omega_3^1(x,t) \cup \Omega_3^2(x,t), \qquad \Omega_4(x,t) = \Omega_4^1(x,t) \cup \Omega_4^2(x,t) \cup \Omega_4^3(x,t)$$

where (see Figure 7)

$$\Omega_3^1(x,t) = \left\{ (\xi,\tau) \,\Big|\, 0 < \tau < \frac{a_0 t - c_0 x}{a_0 + c_0}, \quad 0 < \xi \ < \tau \right\},$$

$$\Omega_3^2(x,t) = \left\{ (\xi,\tau) \,\Big|\, \frac{a_0 t - c_0 x}{a_0 + c_0} < \tau < t - \frac{c_0 x}{a_0}, \quad 0 < \xi < \frac{a_0}{c_0}(t-\tau) - x \right\},$$

$$\Omega_4^1(x,t) = \left\{ (\xi,\tau) \,\Big|\, \frac{a_0 t - c_0 x}{a_0 + c_0} < \tau < t - \frac{c_0}{a_0}x, \quad \frac{a_0}{c_0}(t-\tau) - x < \xi < \tau \right\},$$

$$\Omega_4^2(x,t) = \left\{ (\xi,\tau) \,\Big|\, t - \frac{c_0}{a_0}x < \tau < \frac{a_0 t + c_0 x}{a_0 + c_0}, \quad x - \frac{a_0}{c_0}(t-\tau) < \xi < \tau \right\},$$

$$\Omega_4^3(x,t) = \left\{ (\xi,\tau) \,\Big|\, \frac{a_0 t + c_0 x}{a_0 + c_0} < \tau < t, \quad x - \frac{a_0}{c_0}(t-\tau) < \xi < x + \frac{a_0}{c_0}(t-\tau) \right\}. \tag{6.26}$$

So in the domain $\mathcal{D}_{1T}^0$ the function $v^1(x,t)$ satisfies the following equation

$$v^1(x,t) = \frac{a_0}{c_0} K_{5l}(0) \int\limits_0^{t - \frac{c_0}{a_0}x} \exp\left\{ \frac{a_0 h_2}{c_0} \left(t - \tau - \frac{c_0}{a_0}x \right) \right\} u^1(0,\tau)\, \mathrm{d}\tau +$$

$$+ \frac{c_0}{a_0} \iint\limits_{\Omega_3(x,t)} \exp\left\{ \frac{a_0 h_2}{c_0} \left(t - \tau - \frac{c_0}{a_0}(x+\xi) \right) \right\} \times$$

$$\times \left[q_{2l} v^1 + K_{3l} u^1_\xi + K_{4l} u^1 \right] (\xi,\tau)\ \mathrm{d}\xi\, \mathrm{d}\tau +$$

$$+ \frac{c_0}{2a_0} \iint\limits_{\Omega_4(x,t)} \left[q_{2l} v^1 + K_{3l} u^1_\xi + K_{4l} u^1 \right] (\xi,\tau)\, \mathrm{d}\xi\, \mathrm{d}\tau + \widetilde{v}_1^1(x,t), \quad (x,t) \in \mathcal{D}_{1T}^0 \tag{6.27}$$

Consider the case when $(x,t) \in \mathcal{D}_{2T}^0$ (see Figure 6) and integrate equation (6.18) along the characteristic of the differential operator $\partial/\partial t - (a_0/c_0)\partial/\partial x$ in the domain $\mathcal{D}_{2T}^0$ using equality (6.7)

$$\left(\frac{\partial}{\partial t} + \frac{a_0}{c_0}\frac{\partial}{\partial t} \right) v^1(x,t) = \frac{a_0}{c_0} æ'_{0lm} \left(\frac{a_0 t + c_0 x}{a_0 + c_0} \right) + \frac{c_0 - a_0}{c_0} æ_{1lm} \left(\frac{a_0 t + c_0 x}{a_0 + c_0} \right) +$$

$$+ \frac{c_0}{a_0} \int\limits_x^{\frac{a_0 t + c_0 x}{a_0 + c_0}} P_{2lm}\left(\xi, t + \frac{c_0}{a_0}(x - \xi) \right) \mathrm{d}\xi \equiv g_2^1(x,t). \tag{6.28}$$

Integrate equation (6.28) along the characteristic of the differential operator $\dfrac{\partial}{\partial t} + \dfrac{a_0}{c_0)}\dfrac{\partial}{\partial x}$

in the domain $\mathcal{D}^0_{2T}$ using equality (6.7)

$$v^1(x,t) = \frac{c_0}{a_0} \int\limits_{\frac{c_0 x - a_0 t}{c_0 - a_0}}^{t} \int\limits_{x + \frac{a_0}{c_0}(\tau - t)}^{\frac{c_0 x + a_0(2\tau - t)}{c_0 + a_0}} \Big[q_{2l} v^1 + K_{3l} u^1_\xi + K_{4l} u^1\Big] \Big(\xi, 2\tau - t + \frac{c_0}{a_0}(x - \xi)\Big)\, \mathrm{d}\xi\, \mathrm{d}\tau +$$

$$+\widetilde{v}^1_2(x,t), \qquad (x,t) \in \mathcal{D}^0_{2T}, \tag{6.29}$$

where $\widetilde{v}^1_2(x,t)$ is a known function,

$$\widetilde{v}^1_2(x,t) = æ_{0lm}\Big(\frac{c_0 x - a_0 t}{c_0 - a_0}\Big) + \int\limits_{\frac{c_0 x - a_0 t}{c_0 - a_0}}^{t} \left[\frac{a_0}{c_0} æ'_{0lm} + \frac{c_0 - a_0}{c_0} æ_{1lm}\right] \Big(\frac{c_0 x + a_0(2\tau - t)}{a_0 + c_0}\Big)\, \mathrm{d}\tau +$$

$$+\frac{c_0}{a_0} \int\limits_{\frac{c_0 x - a_0 t}{c_0 - a_0}}^{t} \int\limits_{x + \frac{a_0}{c_0}(\tau - t)}^{\frac{c_0 x + a_0(2\tau - t)}{c_0 + a_0}} \Phi_{2lm}\Big(\xi, 2\tau - t + \frac{c_0}{a_0}(x - \xi)\Big)\, \mathrm{d}\xi\, \mathrm{d}\tau, \qquad (x,t) \in \mathcal{D}^0_{2T}.$$

Consider the integral in equality (6.29), make in it the change of variables

$$\xi_1 \longrightarrow \xi, \quad 2\tau_1 - t + \frac{c_0}{a_0}(x - \xi_1) \longrightarrow \tau$$

and write it as a double integral

$$\frac{c_0}{a_0} \int\limits_{\frac{c_0 x - a_0 t}{c_0 - a_0}}^{t} \int\limits_{x + \frac{a_0}{c_0}(\tau - t)}^{\frac{c_0 x + a_0(2\tau - t)}{c_0 + a_0}} \Big[q_{2l} v^1 + K_{3l} u^1_\xi + K_{4l} u^1\Big] \Big(\xi, 2\tau - t + \frac{c_0}{a_0}(x - \xi)\Big)\, \mathrm{d}\xi\, \mathrm{d}\tau =$$

$$= \frac{c_0}{2a_0} \iint\limits_{\Omega_5(x,t)} \Big[q_2 v^1 + K_{3l} u^1_\xi + K_{4l} u^1\Big](\xi,\tau)\, \mathrm{d}\xi\, \mathrm{d}\tau \tag{6.30}$$

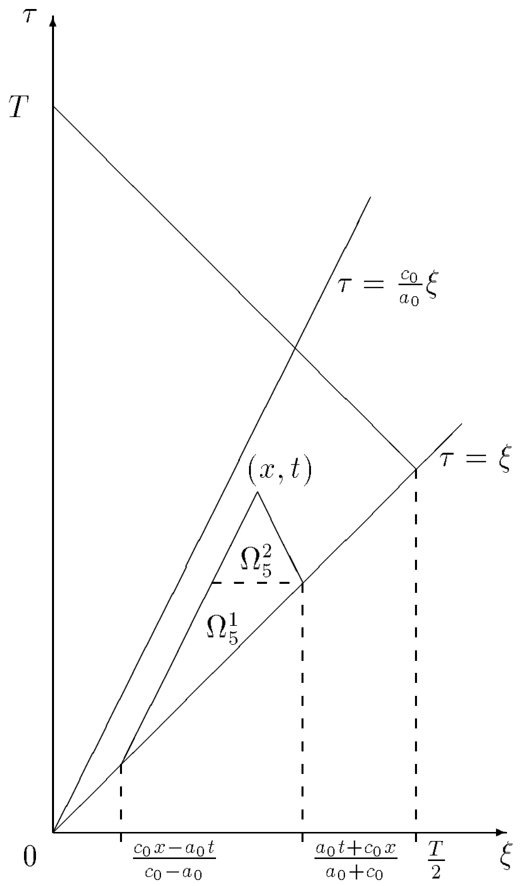


Figure 8

where $\Omega_5(x,t) = \Omega_5^1(x,t) \cup \Omega_5^2(x,t)$. Domains $\Omega_5^1(x,t)$, $\Omega_5^2(x,t)$ is defined below.

$$\Omega_5^1(x,t) = \left\{(\xi,\tau) \;\Big|\; \frac{c_0x - a_0t}{c_0 - a_0} < \tau < \frac{a_0t + c_0x}{a_0 + c_0}, \quad x - \frac{a_0}{c_0}(t-\tau) < \xi < \tau\right\},$$

$$\Omega_5^2(x,t) = \left\{(\xi,\tau) \;\Big|\; \frac{c_0x + a_0t}{c_0 + a_0} < \tau < t, \quad x - \frac{a_0}{c_0}(t-\tau) < \xi < x + \frac{a_0}{c_0}(t-\tau)\right\}.$$

So, by (6.30), in the domain $\mathcal{D}_{2T}^0$ the function $v^1(x,t)$ satisfies the equality

$$v^1(x,t) = \frac{c_0}{2a_0} \iint\limits_{\Omega_5(x,t)} \Big[q_{2l}v^1 + K_{3l}u_\xi^1 + K_{4l}u^1\Big](\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau + \tilde{v}_2^1(x,t), \qquad (x,t) \in \mathcal{D}_{2T}^0. \quad (6.31)$$

Remark 6.2. Note that integrating equalities (6.17), (6.27), (6.31) by parts we get rid of the partial derivatives u_x^1, v_x^1 entering the right-hand parts of these equalities and we arrive at the integral relations for determination of the functions $u^1(x,t)$ and $v^1(x,t)$ in the domain $\mathcal{D}_T^0$.

Consider the vector-functions $\vec{w}(x,t)$, $\vec{w}_0(x,t)$ definded as

$$\vec{w}(x,t) = \Big(u^1(x,t), v^1(x,t)\Big), \qquad \vec{w}_0(x,t) = \Big(\tilde{u}^1(x,t), \tilde{v}^1(x,t)\Big) \qquad (6.32)$$

where $\tilde{v}^1(x,t) = \tilde{v}_1^1(x,t)$ for $(x,t) \in \mathcal{D}_{1T}^0$ and $\tilde{v}^1(x,t) = \tilde{v}_2^1(x,t)$ for $(x,t) \in \mathcal{D}_{2T}^0$.

Consider the operator equation

$$\vec{w}(x,t) = A\vec{w}(x,t) + \vec{w}_0(x,t) \tag{6.33}$$

where the operator A is a Volterra operator of the second kind defined by equalities (6.17), (6.27), (6.31) and Remark 6.2.

This equation defines a unique solution in $\mathcal{D}_T^0$. Differentiating equality (6.33) with respect to x and t we conclude that the partial derivatives $\vec{w}_x$, $\vec{w}_t$, $\vec{w}_{xx}$, $\vec{w}_{xt}$, $\vec{w}_{tt}$ are continuous everywhere except the points of the set $t = c_0x/a_0$. ■

7. Derivation of integral relations for Inverse Problem 3.1

Theorem 7.1. *Let r_0, a_0, c_0, p_0, T be given positive numbers; $0 < T < 2r_0/c_0$. Then for uniqueness of the solution "in small" of Inverse Problem 3.1 in the function class $\widehat{p}(x) \in \mathbf{C}^1[0, Ta_0/(2c_0)]$ it is sufficient that the functions $F_{1lm}(t)$, $G_{1lm}(t)$ satisfy the conditions*

$$F_{1lm}(t) \equiv 0, \quad G_{1lm}(t) \equiv 0 \quad \textit{for} \quad t < 0;$$
$$F_{1lm}(t) \in \mathbf{C}^1[0,T], \qquad G_{1lm}(t) \in \mathbf{C}^1[0,T], \quad G_{1lm}(+0) = 0.$$

Proof. Consider the domain $\mathcal{D}_T^1$,

$$\mathcal{D}_T^1 = \Big\{(x,t) \;\Big|\; 0 < x < Ta_0/(2c_0), \quad x < t < T - c_0x/a_0\Big\}$$

and represent it as the sum $\mathcal{D}_T^1 = \mathcal{D}_{1T}^1 \cup \mathcal{D}_{2T}^1 \cup \{t = c_0x/a_0\}$ where

$$\mathcal{D}_{1T}^1 = \Big\{(x,t) \;\Big|\; 0 < x < Ta_0/(2c_0), \quad c_0x/a_0 < t < T - c_0x/a_0\Big\},$$
$$\mathcal{D}_{2T}^1 = \Big\{(x,t) \;\Big|\; 0 < x < Ta_0/(2c_0), \; x < t < c_0x/a_0\Big\}.$$

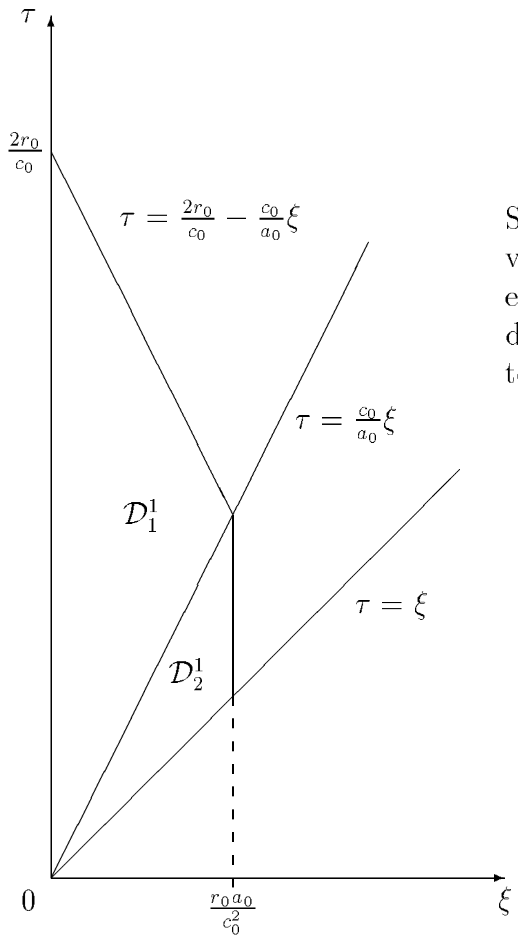


Figure 9

Solving the problem (3.60)–(3.63), (3.64)–(3.67) by virtue of representations (5.1), (5.2) we should consider each of the domains $\mathcal{D}_1^1$ and $\mathcal{D}_2^1$ to take into account discontinuities and jumps of the solution on the characteristics $t - x = 0$ and $t - c_0 x/a_0 = 0$.

Using the definition (6.2) we rewrite equation (3.60) in the form

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right) u^1(x,t) = -P_{1lm}(x,t), \qquad t > x. \tag{7.1}$$

Write the boundary condition (3.62) as

$$u_x^1 \Big|_{x=0} = \Lambda_{1lm}(t), \qquad t > 0, \tag{7.2}$$

where the function $\Lambda_{1lm}(t)$ is defined by the formula

$$\Lambda_{1lm}(t) = -h_1 u^1(0,t) - D_{3l}(0) v^1(0,t) - \Phi_{3lm}(t).$$

The additional information (3.63) can be written as

$$u^1 \Big|_{x=0} = F_{1lm}(t), \qquad t > 0. \tag{7.3}$$

Analogously, we rewrite equation (3.64) as

$$\left(\frac{\partial}{\partial x} + \frac{c_0}{a_0}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} - \frac{c_0}{a_0}\frac{\partial}{\partial t}\right) v^1(x,t) = -\frac{c_0^2}{a_0^2} P_{2lm}(x,t), \qquad t > x. \tag{7.4}$$

The boundary condition (3.66) can be written in the form

$$v_x^1 \Big|_{x=0} = \Lambda_{2lm}(t), \qquad t > 0, \tag{7.5}$$

where the function $\Lambda_{2lm}(t)$ is defined as

$$\Lambda_{2lm}(t) = -h_2 v^1(0,t) - K_{5l}(0)u^1(0,t).$$

The additional information (3.67) can be written in the form

$$v^1 \Big|_{x=0} = G_{1lm}(t), \qquad t > 0. \tag{7.6}$$

Using the information (7.3), (7.6) we can write the functions $\Lambda_{1lm}(t)$, $\Lambda_{2lm}(t)$ in the following form

$$\Lambda_{1lm}(t) = -h_1 F_{1lm}(t) - D_{3l}(0)G_{1lm}(t) - \Phi_{3lm}(t),$$
$$\Lambda_{2lm}(t) = -h_2 G_{1lm}(t) - K_{5l}(0)F_{1lm}(t). \tag{7.7}$$

Remark 7.1. Relation (7.24) follows from the structure (5.1) and equalities (4.5), (5.8). So, from (6.2), (7.7) we can determine the function $\Lambda_{1lm}(t) \in \mathbf{C}^1[0,T]$.

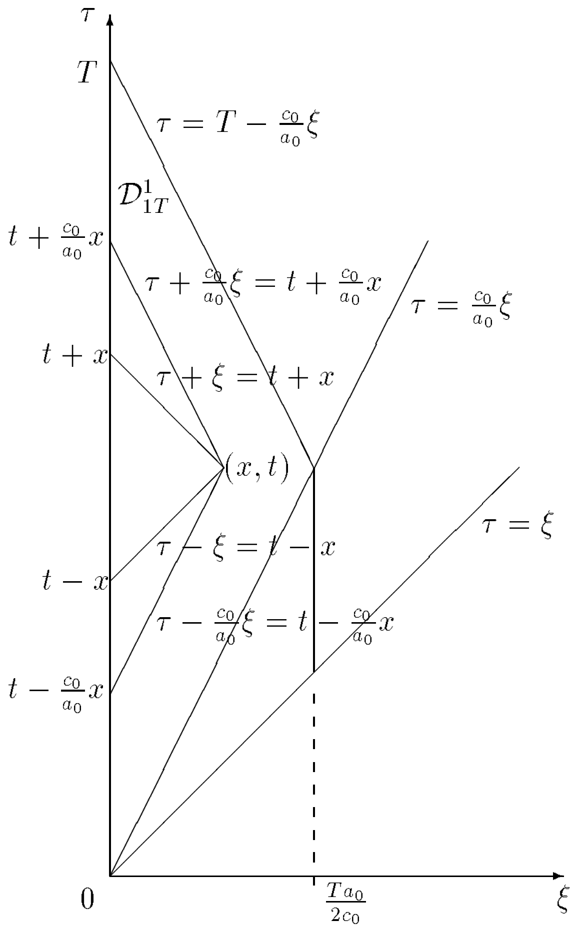


Figure 10

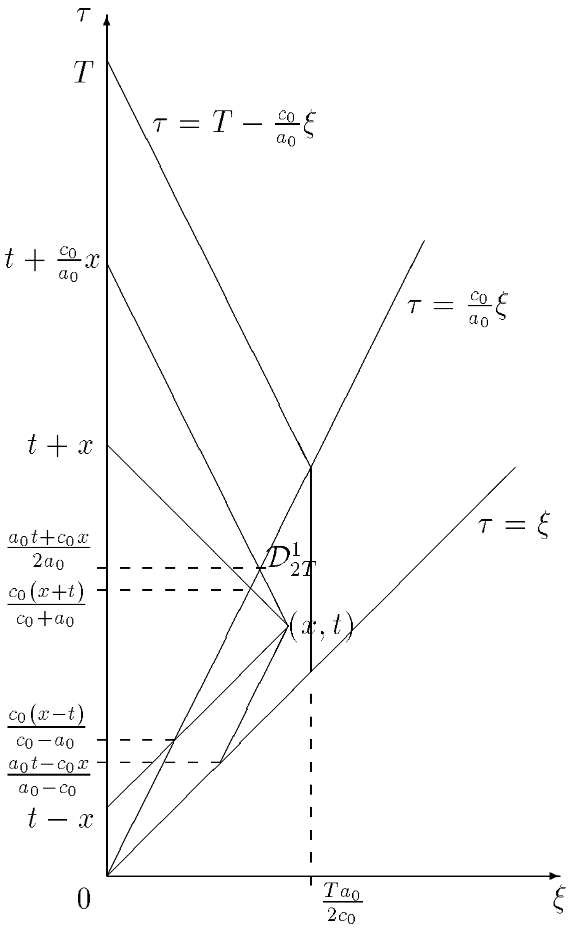


Figure 11

Note that in the result of integrating along the characteristics of the differential operators $\partial/\partial t - \partial/\partial x$ and $\partial/\partial t + \partial/\partial x$ in the domains $\mathcal{D}_{1T}^1$ and $\mathcal{D}_{2T}^1$ the integral equations for the solution of the problem (7.1)–(7.3) have the same form since in the process of integrating along the characteristics the jumps of the function v^1_x, entering the function $P_{1lm}(x,t)$, do not influence the form of the resulting solution. Therefore solving the problem (7.1)–(7.3) by the characteristics method we will consider an arbitrary point $(x,t)\in\mathcal{D}^1_T$.

Let (x,t) be an arbitrary point in the domain $\mathcal{D}^1_T$. Then the solution of the inverse problem (7.1)–(7.3) is given by the d'Alembert formula

$$u^1(x,t)=\frac{1}{2}\Big[F_{1lm}(t-x)+F_{1lm}(t+x)\Big]+\frac{1}{2}\int\limits_{t-x}^{t+x}\Lambda_{1lm}(\xi)\,\mathrm{d}\xi-$$

$$-\frac{1}{2}\int\limits_0^x\int\limits_{t-(x-\xi)}^{t+(x-\xi)}P_{1lm}(\xi,\tau)\,\mathrm{d}\tau\,\mathrm{d}\xi,\qquad (x,t)\in\mathcal{D}^1_T. \tag{7.8}$$

Introducing the notation

$$U^1(x,t)=\frac{1}{2}\Big[F_{1lm}(t-x)+F_{1lm}(t+x)\Big]+\frac{1}{2}\int\limits_{t-x}^{t+x}\Lambda_{1lm}(\xi)\,\mathrm{d}\xi-$$

$$-\frac{1}{2}\int\limits_0^x\int\limits_{t-(x-\xi)}^{t+(x-\xi)}\Big[K_{6lm}u^0_{\xi\xi}+n_{1lm}u^0_\xi+n_{3lm}u^0\Big](\xi,\tau)\,\mathrm{d}\tau\,\mathrm{d}\xi,\qquad (x,t)\in\mathcal{D}^1_T(7.9)$$

this is a known function.
We conclude that in the domain $\mathcal{D}^1_T$ the function $u^1(x,t)$ satisfies the equation

$$u^1(x,t)=-\frac{1}{2}\int\limits_0^x\int\limits_{t-(x-\xi)}^{t+(x-\xi)}\Big[q_{1l}u^1+D_{1l}v^1+D_{2l}v^1_\xi+\widehat{p}'_{1lm}(n_5u^0+n_2u^0_\xi)+$$

$$+n_4\widehat{p}_{1lm}u^0\Big](\xi,\tau)\,\mathrm{d}\tau\,\mathrm{d}\xi+$$

$$+U^1(x,t),\qquad (x,t)\in\mathcal{D}^1_T. \tag{7.10}$$

In the domain $\mathcal{D}^1_{1T}$ (see Figure 10) the solution $v^1(x,t)$ is constructed by the d'Alembert formula

$$v^1(x,t)=\frac{1}{2}\left[G_{1lm}\Big(t-\frac{c_0}{a_0}x\Big)+G_{1lm}\Big(t+\frac{c_0}{a_0}x\Big)\right]+\frac{1}{2}\frac{a_0}{c_0}\int\limits_{t-\frac{c_0}{a_0}x}^{t+\frac{c_0}{a_0}x}\Lambda_{2lm}(\xi)\,\mathrm{d}\xi-$$

$$-\frac{c_0}{2a_0}\int\limits_0^x\int\limits_{t-\frac{c_0}{a_0}(x-\xi)}^{t+\frac{c_0}{a_0}(x-\xi)}P_{2lm}(\xi,\tau)\,\mathrm{d}\tau\,\mathrm{d}\xi,\quad (x,t)\in\mathcal{D}^1_{1T}. \tag{7.11}$$

Now let $(x,t) \in \mathcal{D}^1_{2T}$ (see Figure 11). Integrating both parts of equality (6.6) along the characteristic of the differential operator $\partial/\partial x + (c_0/a_0)\partial/\partial t$ we obtain

$$\Big(\frac{\partial}{\partial x} - \frac{c_0}{a_0}\frac{\partial}{\partial t}\Big)\, v^1(x,t) = -\frac{c_0}{a_0} \int\limits_{\frac{a_0 t - c_0 x}{a_0 - c_0}}^{t} P_{2lm}\Big(x - \frac{a_0}{c_0}(t-\tau), \tau\Big)\, \mathrm{d}\tau, \quad (x,t) \in \mathcal{D}^1_{2T}. \quad (7.12)$$

Integrating equation (7.12) along the characteristic of the differential operator $\partial/\partial x - (c_0/a_0)\partial/\partial t$ we get

$$v^1(x,t) = G_{1lm}\Big(\frac{a_0 t + c_0 x}{a_0}\Big) - \frac{c_0}{2a_0} \int\limits_{0}^{\frac{a_0 t - c_0 x}{a_0 - c_0}} \int\limits_{\xi}^{t + \frac{c_0}{a_0}(x-\xi)} P_{2lm}(\xi,\tau)\, \mathrm{d}\tau\, \mathrm{d}\xi -$$

$$-\frac{c_0}{2a_0} \int\limits_{\frac{a_0 t - c_0 x}{a_0 - c_0}}^{x} \int\limits_{t - \frac{a_0}{c_0}(x-\xi)}^{t + \frac{a_0}{c_0}(x-\xi)} P_{2lm}(\xi,\tau)\, \mathrm{d}\tau\, \mathrm{d}\xi, \qquad (x,t) \in \mathcal{D}^1_{2T}. \tag{7.13}$$

Consider the known functions

$$V^1_1(x,t) = \frac{1}{2}\Big[G_{1lm}\Big(t - \frac{c_0}{a_0}x\Big) + G_{1lm}\Big(t + \frac{c_0}{a_0}x\Big)\Big] + \frac{1}{2}\frac{a_0}{c_0} \int\limits_{t - \frac{c_0}{a_0}x}^{t + \frac{c_0}{a_0}x} \Lambda_{2lm}(\xi)\, \mathrm{d}\xi -$$

$$-\frac{c_0}{2a_0} \int\limits_{0}^{x} \int\limits_{t - \frac{c_0}{a_0}(x-\xi)}^{t + \frac{c_0}{a_0}(x-\xi)} \Big[n_{9lm} u^0_\xi + n_{11lm} u^0\Big](\xi,\tau)\, \mathrm{d}\tau\, \mathrm{d}\xi, \quad (x,t) \in \mathcal{D}^1_{1T};$$

$$V^1_2(x,t) = G_{1lm}\Big(\frac{a_0 t + c_0 x}{a_0}\Big) -$$

$$-\frac{c_0}{2a_0} \int\limits_{0}^{\frac{a_0 t - c_0 x}{a_0 - c_0}} \int\limits_{\xi}^{t + \frac{c_0}{a_0}(x-\xi)} \Big[n_{9lm} u^0_\xi + n_{11lm} u^0\Big](\xi,\tau)\, \mathrm{d}\tau\, \mathrm{d}\xi -$$

$$-\frac{c_0}{2a_0} \int\limits_{\frac{a_0 t - c_0 x}{a_0 - c_0}}^{x} \int\limits_{t - \frac{a_0}{c_0}(x-\xi)}^{t + \frac{a_0}{c_0}(x-\xi)} \Big[n_{9lm} u^0_\xi + n_{11lm} u^0\Big](\xi,\tau)\, \mathrm{d}\tau\, \mathrm{d}\xi, \quad (x,t) \in \mathcal{D}^1_{2T}. \tag{7.14}$$

Now using (6.2), (7.14) we can rewrite equations (7.11), (7.13) as

$$v^1(x,t) = -\frac{c_0}{2a_0}\int\limits_0^x \int\limits_{t-\frac{c_0}{a_0}(x-\xi)}^{t+\frac{c_0}{a_0}(x-\xi)} \Big[q_{2l}v^1 + K_{3l}u^1_\xi + K_{4l}u^1 + \widehat{p}_{1lm}(n_{10l}u^0_\xi + n_{12l}u^0)\Big](\xi,\tau)\,\mathrm{d}\tau\,\mathrm{d}\xi +$$

$$+V_1^1(x,t), \qquad (x,t)\in\mathcal{D}^1_{1T};$$

$$v^1(x,t) = -\frac{c_0}{2a_0}\int\limits_0^{\frac{a_0t-c_0x}{a_0-c_0}} \int\limits_{\xi}^{t+\frac{c_0}{a_0}(x-\xi)} \Big[q_{2l}v^1 + K_{3l}u^1_\xi + K_{4l}u^1 +$$

$$+ \widehat{p}_{1lm}(n_{10l}u^0_\xi + n_{12l}u^0)\Big](\xi,\tau)\,\mathrm{d}\tau\,\mathrm{d}\xi -$$

$$-\frac{c_0}{2a_0}\int\limits_{\frac{a_0t-c_0x}{a_0-c_0}}^{x} \int\limits_{t-\frac{c_0}{a_0}(x-\xi)}^{t+\frac{c_0}{a_0}(x-\xi)} \Big[q_{2l}v^1 + K_{3l}u^1_\xi + K_{4l}u^1 +$$

$$+ \widehat{p}_{1lm}(n_{10l}u^0_\xi + n_{12l}u^0)\Big](\xi,\tau)\,\mathrm{d}\tau\,\mathrm{d}\xi +$$

$$+V_2^1(x,t), \qquad (x,t)\in\mathcal{D}^1_{2T}. \tag{7.15}$$

We should determine u^1_x and v^1_x in order to close the system of equalities (7.10), (7.15). Let $(x,t)\in\mathcal{D}^1_T$, then we can write

$$u^1_x(x,t) = -\frac{1}{2}\int\limits_0^x \Big[q_{1l}u^1 + D_{1l}v^1 + D_{2l}v^1_\xi + \widehat{p}'_{1lm}(n_5u^0 + n_2u^0_\xi) +$$

$$+ n_4\widehat{p}_{1lm}u^0\Big](\xi, t+x-\xi)\,\mathrm{d}\xi -$$

$$-\frac{1}{2}\int\limits_0^x \Big[q_{1l}u^1 + D_{1l}v^1 + D_{2l}v^1_\xi + \widehat{p}'_{1lm}(n_5u^0 + n_2u^0_\xi) +$$

$$+ n_4\widehat{p}_{1lm}u^0\Big](\xi, t-x+\xi)\,\mathrm{d}\xi +$$

$$+U^1_x(x,t), \qquad (x,t)\in\mathcal{D}^1_T. \tag{7.16}$$

Now determine $v^1_x(x,t)$. Let $(x,t)\in\mathcal{D}^1_{1T}$, then

$$v^1_x(x,t)=-\frac{c_0^2}{2a_0^2}\int\limits_0^x\Big[q_{2l}v^1+K_{3l}u^1_\xi+K_{4l}u^1+$$

$$+\widehat{p}_{1lm}(n_{10l}u^0_\xi+n_{12l}u^0)\Big]\Big(\xi,t+\frac{c_0}{a_0}(x-\xi)\Big)\,\mathrm{d}\xi-$$

$$-\frac{c_0^2}{2a_0^2}\int\limits_0^x\Big[q_{2l}v^1+K_{3l}u^1_\xi+K_{4l}u^1+$$

$$\widehat{p}_{1lm}(n_{10l}u^0_\xi+n_{12l}u^0)\Big]\Big(\xi,t-\frac{c_0}{a_0}(x-\xi)\Big)\,\mathrm{d}\xi+$$

$$+\frac{\partial}{\partial x}V^1_1(x,t),\qquad (x,t)\in\mathcal{D}^1_{1T}. \tag{7.17}$$

For $(x,t)\in\mathcal{D}^1_{2T}$ we have

$$v^1_x(x,t)=-\frac{c_0^2}{2a_0^2}\int\limits_0^x\Big[q_{2l}v^1+K_{3l}u^1_\xi+K_{4l}u^1+\widehat{p}_{1lm}(n_{10l}u^0_\xi+n_{12l}u^0)\Big]\Big(\xi,t+\frac{c_0}{a_0}(x-\xi)\Big)\,\mathrm{d}\xi-$$

$$-\frac{c_0^2}{2a_0^2}\int\limits_{\frac{a_0t-c_0x}{a_0-c_0}}^x\Big[q_{2l}v^1+K_{3l}u^1_\xi+K_{4l}u^1+$$

$$+\widehat{p}_{1lm}(n_{10l}u^0_\xi+n_{12l}u^0)\Big]\Big(\xi,t-\frac{c_0}{a_0}(x-\xi)\Big)\,\mathrm{d}\xi+$$

$$+\frac{\partial}{\partial x}V^1_2(x,t),\qquad (x,t)\in\mathcal{D}^1_{2T}. \tag{7.18}$$

From (5.7) we obtain the relation to determine the function $\widehat{p}'_{1lm}(x)$

$$\widehat{p}'_{1lm}(x)=\frac{1}{n_2}\bigg[\frac{1}{g_0}\Big(2\beta'_{0lm}(x)-\beta''_{-1lm}(x)-q_1(x)\beta_{-1lm}(x)+D_1(x)æ_{0lm}(x)-$$

$$-K_{6lm}(x)\alpha_{1lm}(x)\Big)-n_{1lm}(x)\bigg]. \tag{7.19}$$

On the other hand, from (3.24) we can write equation (7.19) in the form

$$\widehat{p}'_{1lm}(x)=\frac{2}{n_2g_0}\frac{\partial u^1(x,x+0)}{\partial x}+\widehat{P}_{1lm}(x) \tag{7.20}$$

where $\widehat{P}_{1lm}(x)$ is a known function,

$$\widehat{P}_{1lm}(x)=\frac{1}{n_2}\left[\frac{1}{g_0}\Big(-\beta''_{-1lm}(x)-q_{1l}(x)\beta_{-1lm}(x)+D_{1l}(x)æ_{0lm}(x)-\right.$$

$$\left.-K_{6lm}(x)\alpha_{1lm}(x)\Big)-n_{1lm}(x)\right]. \tag{7.21}$$

Set $t=x+0$ in equation (7.10) and find the partial derivative $\dfrac{\partial u^1(x,x+0)}{\partial x}$

$$\frac{\partial u^1(x,x+0)}{\partial x}=-\frac{1}{2}\int\limits_x^{2x}\Big[q_{1l}u^1+D_{1l}v^1+D_{2l}v^1_\xi+\widehat{p}'_{1lm}(n_5u^0+n_2u^0_\xi)+n_4\widehat{p}_{1lm}u^0\Big](\xi,\tau)\,\mathrm{d}\tau-$$

$$-\int\limits_0^{x}\Big[q_{1l}u^1+D_{1l}v^1+D_{2l}v^1_\xi+\widehat{p}'_{1lm}(n_5u^0+n_2u^0_\xi)+n_4\widehat{p}_{1lm}u^0\Big](\xi,2x)\,\mathrm{d}\xi+$$

$$+\frac{\partial U^1(x,x+0)}{\partial x}. \tag{7.22}$$

Now we need only one more equation to close the system of equations (7.10), (7.16), (7.22); (7.15), (7.17), (7.18); (7.20)

$$\widehat{p}_{1lm}(x)=\widehat{p}_{1lm}(0)+\int\limits_0^x\widehat{p}'_{1lm}(\xi)\,\mathrm{d}\xi \tag{7.23}$$

where the constant $\widehat{p}_{1lm}(0)$ is determined from equalities (4.5), (5.8) which yield the formula

$$\widehat{p}_{1lm}(0)=\frac{1}{n_2}\left[\frac{1}{g_0}F_{1lm}(+0)+\frac{1}{2}K_{6lm}(0)-n_{6lm}(0)\right]. \tag{7.24}$$

Consider the vector-functions

$$\vec{P}(x)=\Big(\widehat{p}_{1lm}(x),\widehat{p}'_{1lm}(x)\Big),$$

$$\vec{U}(x,t)=\Big(u(x,t),u^1_x(x,t),(u^1(x,x+0))_x\Big),$$

$$\vec{V}(x,t)=\Big(v^1(x,t),v^1_x(x,t)\Big),$$

$$\vec{U}_0(x,t)=\Big(U^1(x,t),U^1_x(x,t),U^1_x(x,x+0)\Big),$$

$$\vec{V}_0(x,t)=\Big(V^1(x,t),V^1_2(x,t),V^1_{1x}(x,t),V^1_{2x}(x,t)\Big),$$

$$\vec{P}_0=\Big(\widehat{p}_{1lm}(0),\widehat{P}_{1lm}(x)\Big). \tag{7.25}$$

Let

$$\vec{W}(x,t)=\Big(\vec{P},\vec{U}^1,\vec{V}^1\Big)(x,t),\qquad \vec{W}_0(x,t)=\Big(\vec{P}_0,\vec{U}^1_0,\vec{V}^1_0\Big)(x,t). \tag{7.26}$$

Consider the operator equation

$$\vec{W}(x,t) = A\vec{W}(x,t) + \vec{W}_0(x,t) \tag{7.27}$$

where the integral operator A is a Volterra type operator of the second order acting with respect to x; the vector-function $\vec{W}_0(x,t)$ is known. Since $F_{1lm}(t),\ G_{1lm}(t) \in \mathbf{C}^1[0,T]$, therefore, for a sufficiently small T there exists a unique solution $\vec{W}(x,t)$ of the second order Volterra equation (7.27).

So, for a sufficiently small T there exist functions $\widehat{p}_{1lm}(x) \in \mathbf{C}^1[0, a_0T/(2c_0)]$, $(u^1(x,t), v^1(x,t)) \in \mathbf{C}^1(\mathcal{D}^1_{*T}) \cap \mathbf{C}(\overline{\mathcal{D}}^1_T)$ which are solutions of the system of integral equations defined by equalities (7.10), (7.16), (7.22); (7.15), (7.17), (7.18); (7.20), (7.23). This implies that for a sufficiently small T the solution of Inverse Problem 3.1 is unique. ■

8. Stability of the solution of Inverse Problem 3.1

Lemma 8.1. *Let r_0, c_0, T, p_0 be given positive numbers, $0 < T < 2r_0/c_0$; $q_0(x) \in \mathbf{C}^\infty[0,T/2]$ be a given function; g_0, h_1 be known constants. Then for the solution of the direct problem (3.57)–(3.59) the following estimate is true*

$$\left\| u^0_{00} \right\|_{\mathbf{C}^2(\mathcal{D}^0_T)} \le \mathbf{B}_1(T+1) \tag{8.1}$$

where $\mathbf{B}_1$ is a positive constant depending on g_0, h_1, $\|q_0\|_{\mathbf{C}[0,T/2]}$.

Proof. Differentiate equality (4.23) with respect to x and t, pass to absolute values, apply the Gronwall inequality to the inequalities obtained, and, finally, use equalities (4.8), (4.12) and Remark 4.2 to derive the required estimate. ■

Let r_0, a_0, c_0, L_*, T be given positive numbers, $0 < T < 2r_0/c_0$; $l = 0,1,2,\ldots$, $-l \le m \le l$. Define the function class

$$\mathbf{Q}'_* = \left\{ \widehat{p}(x,\theta,\varphi),\ \frac{\partial}{\partial x}\widehat{p}(x,\theta,\varphi),\ \Delta_{\theta\varphi}\widehat{p}(x,\theta,\varphi) \in \mathbf{C}\Big([0, a_0T/(2c_0)]; \mathbf{L}_2(\mathbf{S}^2)\Big) \,\Big|\, \|\Delta_{\theta\varphi}\widehat{p}\|_{\mathbf{C}([0,a_0T/(2c_0)];\mathbf{L}_2(\mathbf{S}^2))} \le L_* \right\}.$$

Remark 8.1. By virtue of the representation (3.3), equality (1.6) and the definition of the class $\mathbf{Q}'_*$, we can write for $l = 1,2,\ldots$, $-l \le m \le l$

$$\widehat{p}_{1lm}(x) = \int_0^{2\pi}\int_0^{\pi} \widehat{p}_1(x,\theta,\varphi) Y_{lm}(\theta,\varphi) \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi =$$

$$= -\frac{1}{l(l+1)} \int_0^{2\pi}\int_0^{\pi} \Delta_{\theta\varphi}\widehat{p}_1(x,\theta,\varphi) Y_{lm}(\theta,\varphi) \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi.$$

Square both parts of this equality and apply the Cauchy–Bunyakovskii inequality to the right-hand side of the resulting equality. We obtain

$$|\widehat{p}_{1lm}(x)|^2 \leq \frac{L_*^2}{l^2(l+1)^2}, \qquad l = 1, 2, \dots, \; -l \leq m \leq l. \tag{8.2}$$

If $l = m = 0$, then the following estimate for the function $\widehat{p}_{100}(x)$ is true

$$|\widehat{p}_{100}(x)| \leq L_*. \tag{8.3}$$

The following lemma is true.

Lemma 8.2. *Let r_0, a_0, c_0, p_0, T be given positive numbers, $0 < T < 2r_0/c_0$; and let $\widehat{p}_1(x,\theta,\varphi)$, $\widehat{a}_1(x,\theta,\varphi)$, $\widehat{c}_1(x,\theta,\varphi) \in \mathbf{Q}'_*$ be given functions. Then the solution $(u^1_{lm}(x,t)$, $v^1_{lm}(x,t))$ of the direct problem (3.60)–(3.62), (3.64)–(3.66) for $l = 0, 1, 2, \dots$, $-l \leq m \leq l$ satisfies the following estimates*

$$\left\| u^1_{lm} \right\|_{\mathbf{C}^1(\mathcal{D}^0_T)} \leq \mathbf{B}_2(l+1)\exp\left(\mathbf{B}_3 lT\right), \tag{8.4}$$

$$\left\| v^1_{lm} \right\|_{\mathbf{C}^1(\mathcal{D}^0_{*T})} \leq \mathbf{B}_2(l+1)\exp\left(\mathbf{B}_3 lT\right) \tag{8.5}$$

where $\|\cdot\|_{\mathbf{C}^1(\mathcal{D}^0_{*T})} = \max\{\|\cdot\|_{\mathbf{C}^1(\mathcal{D}^0_{1T})}, \|\cdot\|_{\mathbf{C}^1(\mathcal{D}^0_{2T})}\}$; $\mathbf{B}_2$ is positive constants depending on r_0, a_0, c_0, T, p_0, L_*, $\mathbf{B}_3$ is positive constants depending on r_0, a_0, c_0, p_0, L_*.

Proof. By virtue of (3.38), (3.53)–(3.55) the coefficients entering equations (6.17), (6.27), (6.31), depend on the index l as follows:

$$q_{1l}(x), \; q_{2l}(x) \sim l(l+1),$$

$$D_{1l}(x), \; D_{2l}(x), \; D_{3l}(x) \sim \sqrt{l(l+1)}, \quad K_{3l}(x), \; K_{4l}(x), \; K_{5l}(0) \sim \sqrt{l(l+1)}.$$

The functions $K_{6lm}(x)$, $n_{1lm}(x)$, $n_{3lm}(x)$, $n_{6lm}(0)$, $n_{7lm}(0)$, $n_{9lm}(x)$, $n_{10lm}(x)$, $n_{11lm}(x)$, $n_{12lm}(x)$, $\Phi_{1lm}(x)$, $\Phi_{3lm}(x)$, $\beta_{0lm}(x)$, $æ_{0lm}(x)$ are expressed in terms of the functions $\widehat{a}_{1lm}(x)$, $\widehat{c}_{1lm}(x)$, $\widehat{p}_{1lm}(x)$. Representations of the functions $\Phi_{2lm}(x)$, $æ_{1lm}(x)$ contain the terms

$$\widehat{a}_{1lm}(x)\sqrt{l(l+1)}, \quad \widehat{c}_{1lm}(x)\sqrt{l(l+1)}, \quad \widehat{p}_{1lm}(x)\sqrt{l(l+1)}.$$

Differentiate equations (6.17), (6.27), (6.31) with respect to x and t. Pass to absolute values in the resulting equalities and then apply the Gronwall lemma to the inequalities obtained. Taking into account the type of dependence of the known coefficients on the functions $\widehat{a}_{1lm}(x)$, $\widehat{c}_{1lm}(x)$, $\widehat{p}_{1lm}(x)$ and recalling the definition of the class $\mathbf{Q}'_*$ we prove the lemma. ■

Let

$$\widehat{p}_1(x,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \widehat{p}_{1lm}(x) Y_{lm}(\theta,\varphi), \qquad \widehat{p}_{*1}(x,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \widehat{p}_{*1lm}(x) Y_{lm}(\theta,\varphi).$$

Let $\widehat{p}(x,\theta,\varphi)$, $\widehat{p}_*(x,\theta,\varphi) \in \mathbf{Q}'_*$ be the solutions of Inverse Problem 2.1 corresponding to informations $\vec{h}_1(t,\theta,\varphi)$, $\vec{h}_{*1}(t,\theta,\varphi)$, respectively, for $t \in [0,T]$, $\nu(\theta,\varphi) \in \mathbf{S}^2$.

Theorem 8.1. *Let r_0, a_0, c_0, p_0, T be given positive numbers, $0 < T < 2r_0/c_0$; the functions $\widehat{p}_{1lm}(x)$, $\widehat{p}_{*1lm}(x)$ be the solutions of Inverse Problem 3.1 corresponding to informations $F_{1lm}(t)$, $G_{1lm}(t)$ and $F_{*1lm}(t)$, $G_{*1lm}(t)$, respectively. Then for $l = 0, 1, 2, \ldots$ the following estimates are true*

$$\begin{aligned}
\|\widehat{p}_{1lm} - \widehat{p}_{*1lm}\|_{\mathbf{C}^1[0,\frac{Ta_0}{2c_0}]} &\le \mathbf{B}_4(l+1)\exp(\mathbf{B}_5 l) \times \\
&\qquad \times \Big(\|F_{1lm} - F_{*1lm}\|_{\mathbf{C}^1[0,T]} + \|G_{1lm} - G_{*1lm}\|_{\mathbf{C}^1[0,T]} \Big), \\
\left\| u^1_{lm} - u^1_{*lm} \right\|_{\mathbf{C}^1(\mathcal{D}^1_{*T})} &\le \mathbf{B}_4(l+1)\exp(\mathbf{B}_5 l) \times \\
&\qquad \times \Big(\|F_{1lm} - F_{*1lm}\|_{\mathbf{C}^1[0,T]} + \|G_{1lm} - G_{*1lm}\|_{\mathbf{C}^1[0,T]} \Big), \\
\left\| v^1_{lm} - v^1_{*lm} \right\|_{\mathbf{C}^1(\mathcal{D}^1_{*T})} &\le \mathbf{B}_4(l+1)\exp(\mathbf{B}_5 l) \times \\
&\qquad \times \Big(\|F_{1lm} - F_{*1lm}\|_{\mathbf{C}^1[0,T]} + \|G_{1lm} - G_{*1lm}\|_{\mathbf{C}^1[0,T]} \Big),
\end{aligned} \tag{8.6}$$

where $\|\cdot\|_{\mathbf{C}^1(\mathcal{D}^1_{*T})} = \max\{\|\cdot\|_{\mathbf{C}^1(\mathcal{D}^1_{1T})}, \|\cdot\|_{\mathbf{C}^1(\mathcal{D}^1_{2T})}\}$, $\mathbf{B}_4$, $\mathbf{B}_5$ *depend only on* L_*, r_0, a_0, c_0, p_0, T.

Proof. To prove the estimate (8.6) we pass in the operator equation (7.27) to the differences $\vec{W} - \vec{W}_*$, $\vec{W}_0 - \vec{W}_{0*}$ which are determined in terms of the functions $\widehat{p}_{1lm}$, $\widehat{p}'_{1lm}$, u^1_{lm}, $u^1_{lm\,x}$, v^1_{lm}, $v^1_{lm\,x}$ and $\widehat{p}_{*1lm}$, $\widehat{p}'_{*1lm}$, u^1_{*lm}, $u^1_{*lm\,x}$, v^1_{*lm}, $v^1_{*lm\,x}$ by equations (7.25), (7.26). Going over in equations (7.8), (7.15)–(7.18), (7.21)–(7.23) to the differences $u^1_{lm} - u^1_{*lm}$, $u^1_{lm\,x} - u^1_{*lm\,x}$, $v^1_{lm} - v^1_{*lm}$, $v^1_{lm\,x} - v^1_{*lm\,x}$, $\widehat{p}_{1lm} - \widehat{p}_{*1lm}$, $\widehat{p}'_{1lm} - \widehat{p}'_{*1lm}$ and using the Gronwall lemma and Lemmas 8.1, 8.2 we derive the estimates (8.6). ■

Let $\omega^2 = \|\vec{h}_1 - \vec{h}_{1*}\|^2_{\mathbf{C}^1([0,T],\mathbf{L}_2(\mathbf{S}^2))}$.

Theorem 8.2. *Let T, r_0, a_0, c_0, p_0, L_* be given positive numbers, $0 < T < 2r_0/c_0$; and let $\omega \le \exp(-1/2)$. Let $\widehat{p}_1(x,\theta,\varphi)$, $\widehat{p}_{*1}(x,\theta,\varphi) \in \mathbf{Q}'_*$ be two solutions to Inverse Problem 2.1 corresponding to informations $\vec{h}_1(t,\theta,\varphi)$, $\vec{h}_{*1}(t,\theta,\varphi)$, respectively, for $t \in [0,T]$, $\nu(\theta,\varphi) \in \mathbf{S}^2$. Then the following estimate is true*

$$\|\widehat{p}_1 - \widehat{p}_{*1}\|^2_{\mathbf{C}([0,a_0T/(2c_0)];\mathbf{L}_2(\mathbf{S}^2))} \le \mathbf{B}_6 \left[\ln \omega^{-1}\right]^{-4}$$

where $\mathbf{B}_6$ is some positive value depending on L_*, T, r_0, a_0, c_0, p_0.

Proof. By virtue of (3.2) we can write

$$\begin{aligned}
(\widehat{p} - \widehat{p}_*)(x,\theta,\varphi) &= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (\widehat{p}_{1lm} - \widehat{p}_{*1lm})(x) Y_{lm}(\theta,\varphi) = \\
&= \sum_{l \le N} \sum_{m=-l}^{l} (\widehat{p}_{1lm} - \widehat{p}_{*1lm})(x) Y_{lm}(\theta,\varphi) + \\
&\quad + \sum_{l > N} \sum_{m=-l}^{l} (\widehat{p}_{1lm} - \widehat{p}_{*1lm})(x) Y_{lm}(\theta,\varphi).
\end{aligned}$$

By the completeness and orthogonality of the functions $Y_{lm}(\theta,\varphi)$; $l=0,1,2,\dots$, $-l\le m\le l$, the Parseval equality takes place

$$\|\widehat{p}-\widehat{p}_*\|^2_{\mathbf{L}_2(\mathbf{S}^2)}(x)=\sum_{l\le N}\sum_{m=-l}^{l}(\widehat{p}_{1lm}-\widehat{p}_{*1lm})^2(x)+\sum_{l>N}\sum_{m=-l}^{l}(\widehat{p}_{1lm}-\widehat{p}_{*1lm})^2(x). \tag{8.7}$$

By (8.6), the first term in equation (8.7) can be estimated as

$$\sum_{l\le N}\sum_{m=-l}^{l}\|\widehat{p}_{1lm}-\widehat{p}_{*1lm}\|^2_{\mathbf{C}[0,a_0T/(2c_0)]}\le$$

$$\le \mathbf{B}_7\sum_{l\le N}\sum_{m=-l}^{l}(l^2+1)\exp(2\mathbf{B}_5 l)\Big(\|F_{1lm}-F_{*1lm}\|_{\mathbf{C}^1[0,T]}+\|G_{1lm}-G_{*1lm}\|_{\mathbf{C}^1[0,T]}\Big)^2$$

where $\mathbf{B}_7$ is some positive value.

By the definition of ω^2 we can write

$$\sum_{l\le N}\sum_{m=-l}^{l}\|\widehat{p}_{1lm}-\widehat{p}_{*1lm}\|^2_{\mathbf{C}[0,a_0T/(2c_0)]}\le 2\mathbf{B}_7\sum_{l\le N}\sum_{m=-l}^{l}(l^2+1)\exp(2\mathbf{B}_5 l)\omega^2\le$$

$$\le \mathbf{B}_9 N^4\exp(\mathbf{B}_8 N)\omega^2 \tag{8.8}$$

where $\mathbf{B}_8$, $\mathbf{B}_9$ are some positive values.

By (8.2), the second term in equality (8.7) can be estimated as

$$\sum_{l>N}\sum_{m=-l}^{l}\|\widehat{p}_{1lm}-\widehat{p}_{*1lm}\|^2_{\mathbf{C}[0,a_0T/(2c_0)]}\le\frac{\mathbf{B}_{10}}{N^4} \tag{8.9}$$

where $\mathbf{B}_{10}$ is some value depending on L_*.

Now we return to equality (8.7). By inequalities (8.8) and (8.9), we can write

$$\|\widehat{p}_1-\widehat{p}_{*1}\|^2_{\mathbf{C}([0,a_0T/(2c_0)];\mathbf{L}_2(\mathbf{S}^2))}\le \mathbf{B}_{11}\left(\omega^2N^4\exp\{\mathbf{B}_8N\}+\frac{1}{N^4}\right) \tag{8.10}$$

where $\mathbf{B}_{11}=\max\{\mathbf{B}_9,\mathbf{B}_{10}\}$.

Let us show that there exists a number N depending on ω, $N(\omega)\longrightarrow\infty$ as $\omega\longrightarrow 0$, for which the right-hand side of inequality (8.10) tends to zero. To do this we assume that both terms on the right-hand side of (8.10) are approximately equal, then they should "behave" identically

$$\omega^2N^4(\omega)\exp\{\mathbf{B}_8N(\omega)\}\approx\frac{1}{N^4(\omega)}\qquad\text{or}\qquad \ln\Big(\omega^2N^8(\omega)\Big)+\mathbf{B}_8N(\omega)\approx 0.$$

Let us consider the iteration process and assume on its first step that

$$\ln\omega^2+\mathbf{B}_8N_1(\omega)=0.$$

From this equation we can determine the value $N_1(\omega)$

$$N_1(\omega) = \frac{1}{\mathbf{B}_8} \ln \omega^{-2}, \quad N_1(\omega) \underset{\omega\to 0}{\longrightarrow} \infty.$$

On the second step of the iteration process we consider the relation

$$\ln\Big(\omega^2 N_1^8(\omega)\Big) + \mathbf{B}_8 N_2(\omega) = 0$$

which yields

$$N_2(\omega) = \frac{1}{\mathbf{B}_8} \ln \left(\omega^{-2} \left(\frac{1}{\mathbf{B}_8} \ln \omega^{-2} \right)^{-8} \right), \quad N_2(\omega) \underset{\omega\to 0}{\longrightarrow} \infty.$$

Letting $N = N_2(\omega)$ in (8.10) and simplifying the resulting expression we can write

$$\omega^2 N^4 \exp\{\mathbf{B}_8 N\} + \frac{1}{N^4} = \left[\frac{1}{\mathbf{B}_8} \ln \left(\omega^{-2} \left(\frac{1}{\mathbf{B}_8} \ln \omega^{-2} \right)^{-8} \right) \right]^{-4} \times$$
$$\times \left[\left(1 + 8 \ln \mathbf{B}_8 \frac{1}{\ln \omega^{-2}} + \frac{\ln\Big((\ln \omega^{-2})^{-8}\Big)}{\ln \omega^{-2}} \right)^8 + 1 \right]. \tag{8.11}$$

By the choice of ω we have

$$\frac{1}{\ln \omega^{-2}} < 1, \qquad \frac{\ln(\ln \omega^{-2})^{-8}}{\ln \omega^{-2}} < 1. \tag{8.12}$$

By (8.12), the first factor in (8.11) can be written as

$$\left[\frac{1}{\mathbf{B}_8} \ln \left(\omega^{-2} \left(\frac{1}{\mathbf{B}_8} \ln \omega^{-2} \right)^{-8} \right) \right]^{-4} = \left[\frac{1}{\mathbf{B}_8} \Big(\ln \omega^{-2} + \ln \mathbf{B}_8^8 + \ln(\ln \omega^{-2})^{-8} \Big) \right]^{-4} \leq$$
$$\leq \mathbf{B}_{12} \Big(\ln \omega^{-1} \Big)^{-4}$$

where $\mathbf{B}_{12}$ is some constant.
Let $\mathbf{B}_{13} = \mathbf{B}_{11} \Big[(2 + 8 \ln \mathbf{B}_8)^8 + 1 \Big]$, then we can rewrite (8.10) as

$$\|\widehat{p}_1 - \widehat{p}_{*1}\|^2_{\mathbf{C}([0, a_0 T/(2c_0)]; \mathbf{L}_2(\mathbf{S}^2))} \leq \mathbf{B}_6 \left[\ln \omega^{-1} \right]^{-4}$$

where $\mathbf{B}_6 = \mathbf{B}_{12} \mathbf{B}_{13}$.

Thus, Theorem 8.2 is proved. ■

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