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S. N. GLAZATOV

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В работе рассматриваются некоторые неклассические краевые задачи для линейных и нелинейных уравнений переменного типа. С этой точки зрения изучаются нелинейные уравнения трансзвуковой газовой динамики и нелинейные параболические уравнения с меняющимся направлением времени. Также рассматривается некоторый класс вариационных неравенств, связанных с упомянутыми уравнениями. Библиогр. 45.

Адрес: Институт математики им. С. Л. Соболева СО РАН, пр. Академика Коптюга, 4. 630090 Новосибирск, Россия.

NONLOCAL BOUNDARY VALUE PROBLEMS FOR LINEAR AND NONLINEAR EQUATIONS OF VARIABLE TYPE

Sergey N. Glazatov

Prospekt Akademika Koptyuga 4, Sobolev Institute of Mathematics of SB RAS, 630090, Novosibirsk - 90, Russia; e - mail: glaz@math.nsc.ru

Abstract. In the article we consider a class of nonclassical boundary value problems for linear and nonlinear equations of variable type. From this viewpoint we treat nonlinear equations of transonic gas dynamics and nonlinear forward - backward parabolic equations. We also deal with a class of variational inequalities relative to the above equations. Bibl. 45.

Key words: linear equation of variable type; nonlocal boundary value problem; transonic gas dynamics; the von Kàrmàn equation; nonlinear forward - backward parabolic equation; variational inequality; local existence theorem.

AMS Subject classification: 35M10, 76H05, 35K60, 35K85, 35L85.

The present article contains a review of results obtained in the theory of nonlocal boundary value problems (BVPs) for linear and nonlinear partial differential equations of variable type. Along with this, here, a number of new theorems concerning this topic is proven. Here BVPS are called nonlocal if one sets certain relations between traces of a solution and of its derivatives at different portions of the boundary of a domain (for instance, a periodicity condition) instead of posing the traces themselves.

We are going to discuss how to pose a correct BVP for an equation of variable type which is nonlinear in the principle part (such as the von Kàrmàn - Fal'kovich equation arising in transonic gas dynamics). It is a nonlocal BVP that occurs to be well - posed in the case when correctness of classical BVPs such as the Cauchy problem or the Dirichlet problem seems to be rather doubtful.

In certain cases nonlocal BVPs select the very solutions of quasilinear equations of variable type at which the equations actually change their type.

The last section of the article contains a small introduction to the theory of variational inequalities for nonlinear operators of variable type. It is shown how one can arrive at a local existence theorem for a solution of an IBVP on the basis of solvability for an appropriate variational inequality.

1. NONLOCAL BOUNDARY - VALUE PROBLEMS FOR LINEAR EQUATIONS OF VARIABLE TYPE

In Dezin's [1, 2] nonlocal boundary conditions were introduced in order to construct solvable expansions for a kind of nonclassical operators such as

$$u_t - u_{xy}, \quad u_{xt} - u_{yz}.$$

Obviously, the "A" Petrovskiĭ condition which guarantees correctness of the Cauchy problem in classes of functions of finite smoothness fails to hold here (see [3, Theorems 4.6 and 5.2]). Further this approach was developed in [4 - 10].

Probably, Frankl's problem for the Chaplygin equation is the first known nonlocal BVP for an equation of variable type. At the present time this problem is completely studied (see [11, Ch.4, sec.2, item 5]). In [12, 13] the Tricomi and the Lavrent'ev - Bitsadze equations were considered in the characteristical domain. At the "elliptic" portion of the boundary the Dirichlet condition was posed, and a nonlocal boundary condition was posed at the "hyperbolic" one.

In [14], the author established incorrectness (in general) of the Dirichlet problem for a variable type equation of the first kind (a Tricomi type equation) in a cylindrical domain in R^{n+1}, (see, also, [15] concerning the Dirichlet problem for the Tricomi equation in the characteristical domain). It was the reason to seek for others, (in particular, nonlocal) BVPs with boundary data posed at entire boundary for equations of variable type.

Well - posed nonlocal BVPs for a wide class of multidimensional linear equations of variable type were proposed in [16 - 23]. Equations were considered in a cylindrical domain in R^{n+1} or in the layer $\Pi = R^n \times (0,T), 0 < T < +\infty$.

Present here an example (see [16]). Consider the variable type equation

$$L_1 u \equiv k(t)u_{tt} - \Delta_x u + a(x,t)u_t + c(x,t)u = f(x,t) \tag{1.1}$$

in the cylinder $Q_T = D \times (0,T), D \subset R^n$ is a bounded domain with smooth boundary γ, $S = \gamma \times (0,T)$. Here and further in Section 1 we assume sufficient smoothness of the coefficients of (1.1) and of all other equations treated.

The function $k(t)$ may change its sign on $[0,T]$. Let $k(T) > 0 > k(0)$. In this case the following BVP can be posed: find in Q_T a solution of (1.1) satisfying the boundary conditions

$$u|_{t=0} = \gamma u|_{t=T}, \quad 0 < |\gamma| < 1, \gamma \in R \tag{1.2}$$

$$u|_S = 0 \tag{1.3}$$

Introduce the function space $H(Q_T)$ with the norm

$$||u||_{H(Q_T)} = (\int_0^T \int_D ([(ku_t)_t]^2 + \sum_{|\alpha|=2} |D_x^\alpha u|^2 + \sum_{i=1}^n u_{xit}^2) dxdt)^{\frac{1}{2}} + ||u||_{W_2^1(Q_T)}.$$

THEOREM 1.1.[16]*Assume that in Q_T the following conditions hold: $2a - k_t \geq \delta > 0, c \geq 0, c_t \leq 0$. Assume, also, that there exists a constant λ such that $0 > \lambda > max\{\frac{2\ln|\gamma|}{T}, -\frac{\delta}{2\max|k|}\}$ and such that for every $x \in D$ the inequality $exp(\lambda T)c(x,T) - \gamma^2 c(x,0) \geq 0$ is valid.*

Then, for every function $f(x,t) \in L^2(Q_T)$ such that $\nabla_x f \in L^2(Q_T)$, $f|_S = 0$ there exists a unique solution of problem (1.1) - (1.3) in the space $H(Q_T)$.

If, in addition to the above conditions, in Q_T the inequality $2a - |k_t| \geq \delta > 0$ holds, and $f_t \in L^2(Q_T)$, then, the solution obtained belongs to $W_2^2(Q_T)$.

In [16], nonlocal BVPs for (1.1) were proposed in cases of various behaviour of $k(t)$ at the points $t = 0$ and $t = T$. Under the conditions analogous to those of Theorem 1.1 correctness of the above problems was established in the spaces $H(Q_T)$ and $W_2^2(Q_T)$. It was found out that posing of nonlocal BVPs (i. e. a number and a kind of boundary conditions posed) essentially depends on the sign of $k(t)$ at the points $t = 0$ and $t = T$. However, the similar phenomenon takes place in the case of local (pointwise) BVPs for (1.1) in Q_T.

In [17], a nonlocal BVP for (1.1) was posed in the domain $\Omega_T = (0, 2\pi)^n \times (0, T)$, provided that $k(T) > 0 > k(0)$. Together with (1.2) periodicity of the solution and of all its normal derivatives with respect to all variables $x_i (i = 1 \dots n)$ was required. Coefficients of (1.1) could depend on t only. Correctness of the above problem in $H(Q_T)$ was proven under conditions close to those of Theorem 1.1 (periodicity of the function $f(x,t)$ with respect to all $x_i, i = 1 \dots n$ was required instead of the condition $f|_S = 0$).

The articles [18, 19] dealt with the equation of general form

$$L_2 u \equiv k(x,t)u_{tt} - \sum_{i,j=1}^{n} a_{ij}(x,t)u_{x_i x_j} + \sum_{i=1}^{n} b_i(x,t)u_{x_i} + b(x,t)u_t + c(x,t)u = f(x,t) \quad (1.4)$$

in the domain Q_T. The form $a_{ij}\xi_i\xi_j$ was assumed to be positive, and the function $k(x,t)$ could change its sign in Q_T. In [18], the case $k(x,0) = k(x,T) \equiv 0$ was treated, and the same boundary conditions (1.2), (1.3) were posed with the same restriction on the parameter γ. In [19], equation (1.4) had hyperbolic - parabolic type and, in addition, $k(x,T) = k(x,0) > 0$ for all $x \in D$. The boundary conditions (1.2) and (1.3) were supplemented by $u_t|_{t=0} = \gamma u_t|_{t=T}$. In the above articles for every function $f(x,t) \in L^2(Q_T)$ the author proved existence and uniqueness of generalized solutions in $W_2^1(Q_T)$ of the problems posed. Coefficients of (1.4) were assumed to satisfy some natural conditions, in particular, the inequality $2a - k_t \geq \delta > 0$ was to hold in Q_T. In [20], the author studied smoothness of a solution of the BVP, treated in [18].

In [21, 22], nonlocal BVPs for (1.4) were studied in the layer $\Pi = R^n \times (0,T)$ with the principle part of the equation independent of the variables x. The form $a_{ij}\xi_i\xi_j$ was assumed to be positive, and the indefinite function $k(t)$ could behave variously at $t = 0$ and $t = T$, and $b_i \equiv 0, i = 1 \dots n$. In the articles existence and uniqueness theorems for smooth solutions in $W_2^2(\Pi)$ of the BVPs posed were proven.

Now, state a new result concerning solvability of a $t-$ periodic BVP for the equation

$$L_3 u \equiv k(x,t)u_{tt} - \Delta_x u + a(x,t)u_t + b(x)u = f(x,t), \quad (1.5)$$

which is considered in the above specified cylinder $Q_T \subset R^{n+1}$.

Assume, that $k(x,T) = k(x,0)$ for all $x \in D$. Denote $\Gamma = \{x \in D : k(x,T) = k(x,0) = 0\}$. Given the boundary conditions

$$u|_{t=0} = u|_{t=T} \quad (1.6)$$

$$u_t|_{\overline{\{t=0\}\setminus\Gamma}} = u_t|_{\overline{\{t=T\}\setminus\Gamma}} \quad (1.7)$$

$$u|_S = 0 \tag{1.8}$$

or

$$\frac{\partial u}{\partial n}|_S = 0. \tag{1.8'}$$

THEOREM 1.2. *Let the inequality $2a - k_t \geq \delta > 0$ hold in Q_T, assume, that $b(x) \geq 0$ (in the case of the condition (1.8)) or $b(x) \geq \delta > 0$ (in the case of the condition (1.8')) for all $x \in D$. Then, a solution of BVP (1.5) - (1.8) ((1.5) - (1.8')) is unique in the space $H(Q_T)$) or in the space $W_2^2(Q_T)$.*

Proof. Integration by parts in the equality $(L_3u, u_t)_0 = 0$ yields the conclusion of the theorem.

THEOREM 1.3. *Let f and $f_t \in L^2(Q_T)$, and $f|_{t=0} = f|_{t=T}$. Let $k_t^j(x,T) = k_t^j(x,0), a_t^j(x,T) = a_t^j(x,0)(j = 0,1)$ for all $x \in D$. Assume also that in Q_T the inequality $2a - |k_t| \geq \delta > 0$ holds, and for all $x \in D$ the inequalities $b(x) \geq 0$ (in the case of (1.8)) or $b(x) \geq \delta > 0$ (in the case of (1.8')) are valid. Then, there exists a solution $u(x,t)$ in the space $W_2^2(Q_T)$ of BVP (1.5 - 1.8) ((1.5 - 1.8')),and, moreover, $u_t|_{t=0} = u_t|_{t=T}$ for all $x \in D$.*

We shall not prove the theorem here, because the proof is based on just the same arguments as those we are going to use to prove the main Theorem 2.1 in Section 2. One is able to reconstruct details easily.

We should also mention the article [24] in connection with the already discussed example of incorrectness of the Dirichlet problem for a variable type equation of the first kind. The author dealt with a semiperiodic BVP for the equation

$$L_4 u \equiv u_{tt} + k(x,t)u_{x_0x_0} + \sum_{i=1}^{n} u_{x_ix_i} + a(x,t)u_{x_0} + c(x,t)u = f(x,t) \tag{1.9}$$

in the parallelepiped $Q_T = (0, 2\pi)^{n+1} \times (0,T)$, with x_0 a distinguished spacial variable. Therein, the independent of x_0 function $k(x,t)$ might change its sign in a prescribed way. Correctness of the above nonlocal BVP for (1.9) was established in a Sobolev space.

2. A PERIODIC BOUNDARY VALUE PROBLEM FOR AN EQUATION OF TRANSONIC GAS DYNAMICS

The equation

$$A_1 u \equiv u_x u_{xx} - u_{yy} = f(x,y) \tag{2.1}$$

was proposed to describe small perturbations in a transonic gas flow in the presence of exterior mass forces. Equation (2.1) is usually called the von Kàrmàn - Fal'kovich equation. It is derived from the system of equations of gas dynamics, for instance, in the monograph [25]. The function $u(x,y)$ has a physical meaning of a velocity potential of a gas flow.

In [26], the equation

$$A_2 u \equiv u_x u_{xx} - u_{yy} + a u_x = f(x,y), \tag{2.2}$$

with a a positive constant, was derived. Equation (2.2) differs from (2.1) by the presence of a dissipative term.

The case of three spacial variables was studied in the monograph [27], where the author considered the equations

$$A_3 u \equiv u_x u_{xx} - \Delta_y u = f(x, y) \tag{2.3}$$

and

$$A_4 u \equiv u_x u_{xx} - \Delta_y u + a u_x = f(x, y), \tag{2.4}$$

where a also is a positive constant, and $\Delta_y u = u_{y_1 y_1} + u_{y_2 y_2}$.

In the present time the question of posing BVPs for (2.1) - (2.4) is, in actuality, open, though, there exist a number of interesting results in this direction.

At first, pay attention to equations (2.1) and (2.3) without dissipation. As it was noticed in [25], these equations become quasilinear hyperbolic when overcoming the sonic velocity, and their solutions can possess weak discontinuities. Therefore, here, one will scarcely succeed to pose a correct BVP with the aim to obtain a proper transonic regime (at least, in a class of smooth functions). The situation can not be improved even if "smallness " of boundary data and of the right - hand side is required. In particular, it is confirmed by a well - known example in [28] (an equation considered therein contained some other nonlinearity, but, obviously, it does not matter).

In [29], a Neumann problem was suggested for (2.3) in a cylindrical domain, and the author obtained quite a nice its solution such that $u_x \leq 0$ in the domain considered.

In [30 - 32], a problem of finding a solution of a hyperbolic variational inequality for (2.1) was posed and solved in a rectangle on the plane (x, y). Constraints had the form $u_x \geq 0, |u_{xy}| \leq 1$ a. e. in the rectangle. In [33], an analogous result was obtained for (2.3).

In the monograph [27, P.18-20], the author proposed a BVP with nonhomogeneous boundary conditions for (2.4) with $f \equiv 0$ in the domain $Q = D \times (0, L)$, where $D \subset R^2_y$ is a bounded domain with a smooth boundary. Therein, a priori a smooth solution $\Phi(x, y)$ of (2.4) such that $\Phi_x|_{x=0} < 0, \Phi_x|_{x=L} > 0$ was assumed to exist. Existence of such a solution was justified by certain physical arguments. A solution of the BVP posed was sought in the form $u = \Phi + z$, where $z(x, y)$ is a quantity with a small norm in some function space.

We should also mention [34], wherein the author dealt with a transonic model rather close to those we are discussing here.

It seems reasonable to suggest such a BVP for (2.4), which would be most "indifferent" to the type of the equation.

In the present article, we suggest an $x-$ periodic BVP for equation (2.4) in $Q = D \times (0, L)$, where $D \subset R^2_y$ is a bounded domain with smooth boundary γ. Its solvability is proven in a class of smooth functions, and, in contrast to [27], we do not assume a priori existence of any solutions of (2.4). The solution obtained realizes a proper transonic regime (i. e. equation (2.4) actually changes its type at the solution).

Pose a problem: find a solution of equation (2.4) in Q satisfying the boundary conditions

$$u|_{x=0} = u|_{x=L} \tag{2.5}$$

$$u_x|_{x=0} = u_x|_{x=L} \tag{2.6}$$

$$\frac{\partial u}{\partial n}|_{\gamma\times(0,L)} = 0, \tag{2.7}$$

where $\vec{n}$ is the normal to γ.

Introduce necessary notations: $(.,.)_0$ is the inner product in $L^2(Q)$; $||\cdot||_s$ is the norm in the Sobolev space $W_2^s(Q)$. Denote $|||f|||^2 = \sum_{j=0}^{3} ||D_x^j f||_0^2 + ||\nabla_y f_x||_0^2$. Define the space $H(Q)$ to be the closure with respect to the norm $||u||^2_{H(Q)} = \sum_{j=0}^{2} ||D_x^j u||_2^2 + ||\nabla_y u_x||_2^2$ of infinitely differentiable in $\bar{Q}$ functions satisfying (2.7) and such that $D_x^j u|_{x=0} = D_x^j u|_{x=L}$ for all $j \geq 0$. Also denote $\tilde{B}_\rho = \{z \in H(Q) : ||z||_{H(Q)} \leq \rho, \int\int_Q z dy dx = 0\}$.

Now state the main result of the article

THEOREM 2.1. *Let* $D_x^j f \in L^2(Q)(j = 0,1,2,3), \nabla_y f_x \in L^2(Q), D_x^j f|_{x=0} = D_x^j f|_{x=L}(j = 0,1,2)$ *and* $\int\int_Q f dy dx = 0$. *Then, if* $|||f||| < \alpha$, *(with* α *a positive number depending on* a *and* Q*), there exists a solution* $u(x,y) \in H(Q)$ *of BVP (2.4) - (2.7) such that* $\int\int_Q u dy dx = 0$.

The solution of (2.4) - (2.7) is unique in the class $\tilde{B}_{\rho_0}$, *with* $\rho_0 > 0$ *a sufficiently small number also depending on* a *and* Q.

Proof. At the first step we are going to prove solvability in $H(Q)$ of a linearized problem

$$L_v u \equiv v_x u_{xx} - \Delta_y u + a u_x = f(x,y), \tag{2.8}$$

where $v \in \tilde{B}_{\rho_0}$ is an arbitrary function, and $\rho_0 > 0$ will be specified further.

We are going to look for a solution $u_v(x,y)$ of the problem as a limit in $H(Q)$ of the function sequence $u_N(x,y) = \sum_{k=1}^{N}\sum_{s=0}^{N} u_{ks} \sin\frac{2\pi kx}{L}\omega_s(y) +$
$+ \sum_{k=0}^{N}\sum_{s=0}^{N} \tilde{u}_{ks} \cos\frac{2\pi kx}{L}\omega_s(y)$, where $\omega_s(y)$ are orthonormal in $L^2(D)$ eigenfunctions of the Neumann problem $-\Delta_y\omega_s = \lambda_s^2\omega_s$, $\frac{\partial\omega_s}{\partial n}|_\gamma = 0, s = 0,1\ldots$, and u_{ks} and $\tilde{u}_{ks}$ are constants which should be determined from a certain algebraic system. We approximate the right - hand side $f(x,y)$ by the functions $f_N(x,y) = \sum_{k=1}^{N}\sum_{s=0}^{N} f_{ks}\sin\frac{2\pi kx}{L}\omega_s(y) + \sum_{k=0}^{N}\sum_{s=0}^{N}\tilde{f}_{ks}\cos\frac{2\pi kx}{L}$ $\times\, \omega_s(y)$, in the norm $|||\cdot|||$.

Denote by E_N the finite dimensional linear span of the functions $\{\sin\frac{2\pi lx}{L}\omega_s(y),$ $\cos\frac{2\pi kx}{L}\omega_s(y)\}_{l=1\ldots N, s=0\ldots N, k=0\ldots N}$, denote, also, by $P_N : L^2(Q) \to E_N$ the orthoprojector in $L^2(Q)$ onto E_N.

It is clear, that $f_N = P_N f$. One also can easily verify that if $\varphi \in E_N$, then $\varphi_x \in E_N$ and $\Delta_y\varphi \in E_N$, and P_N commutates with the operators D_x and Δ_y. Moreover, it is obvious that for every $z \in L^2(Q)$ and for every $\varphi \in E_N$ $\int\int_Q z\varphi dy dx = \int\int_Q (P_N z + (z - P_N z))\varphi dy dx =$ $\int\int_Q P_N z\varphi dy dx$ because $z - P_N z \in E_N^\perp$.

Constants u_{ks} and $\tilde{u}_{ks}$ can be determined from the system of linear algebraic equations

$$L_1 u_N \equiv P_N(v_x u_{Nxx}) - \Delta_y u_N + a u_{Nx} = f_N(x,y) \tag{2.9}$$

The necessary condition for solvability of (2.9) is $\tilde{f}_{00} = 0$, but it is guaranteed by the conditions of the theorem. Here, we have an arbitrariness when determining $\tilde{u}_{00}$, and we put $\tilde{u}_{00} = 0$. Solvability of a system for determination of the rest unknowns is a consequence of uniqueness of its solution, and the uniqueness, in turn, is implied by a priori estimates we are ready to obtain.

Suppose, a solution of system (2.9) exists (here, necessarily $\int\int_Q u_N dydx = 0$ for all N). Let us prove its uniqueness and obtain a priori estimates required.

Consider the equality

$$(L_1 u_N, u_{Nx})_0 = (f_N, u_{Nx}).$$

Integrating, wherever necessary, by parts, making use of the above properties of the system of basic functions chosen, and applying the embedding $W_2^2(Q) \to C(\bar{Q})$ and Young's "ε inequality with $\varepsilon = \frac{a}{2}$, we obtain

$$(\frac{a}{2} - \frac{\rho_0 C_1}{2}) \int\int_Q u_{Nx}^2 dydx \leq C_2(a) \int\int_Q f_N^2 dydx, \tag{2.10}$$

where $C_1(Q)$ is a constant in the inequality $||z||_{C(\bar{Q})} \leq C_1(Q)||z||_{W_2^2(Q)}$.

Next, look at the equality

$$((L_1 u_N)_x, u_{Nxx})_0 = (f_{Nx}, u_{Nxx})_0,$$

taking into account commutation of the operators D_x и P_N. Making use of just the same arguments as those applied to obtain (2.10), we derive

$$(\frac{a}{2} - \frac{\rho_0 C_1}{2}) \int\int_Q u_{Nxx}^2 dydx \leq C_2(a) \int\int_Q f_{Nx}^2 dydx \tag{2.11}$$

Now, consider the equality

$$((L_1 u_N)_{xx}, u_{Nxxx})_0 = (f_{Nxx}, u_{Nxxx})_0.$$

Arguments, similar to those above, imply

$$(\frac{a}{2} - \frac{3}{2}\rho_0 C_1) \int\int_Q u_{Nxxx}^2 dydx \leq C_2(a) \int\int_Q f_{Nxx}^2 dydx +$$

$$+ \int\int_Q |v_{xxx} u_{Nxx} u_{Nxxx}| dydx$$

Estimate the last integral in the right - hand side, making use of Young's inequality and of the embedding theorem

$$\int\int_Q |v_{xxx} u_{Nxx} u_{Nxxx}| dydx \leq \frac{a}{4} \int\int_Q u_{Nxxx}^2 dydx +$$

$$+ C_3(a) \int\int_Q v_{xxx}^2 u_{Nxx}^2 dydx \leq$$

$$\leq \frac{a}{4}\int\int_{Q} u^2_{Nxxx}dydx + C_3(a)\max_{\bar{Q}}|u_{Nxx}|^2\rho_0^2 \leq$$

$$\leq \frac{a}{4}\int\int_{Q} u^2_{Nxxx}dydx + C_4(a,Q)\rho_0^2||u_{Nxx}||_2^2$$

Finally, we arrive at

$$(\frac{a}{4} - \frac{3}{2}\rho_0 C_1)\int\int_{Q} u^2_{Nxxx}dydx \leq C_2(a)\int\int_{Q} f^2_{Nxx}dydx +$$

$$+C_4(a,Q)\rho_0^2||u_N||^2_{H(Q)} \tag{2.12}$$

Consider the next equality

$$((L_1u_N)_{xxx}, u_{Nxxxx})_0 = (f_{Nxxx}, u_{Nxxxx})_0.$$

Transforming it in the way analogous to the above one, we derive

$$(\frac{a}{2} - \frac{5}{2}\rho_0 C_1)\int\int_{Q} u^2_{Nxxxx}dydx \leq C_2(a)\int\int_{Q} f^2_{Nxxx}dydx +$$

$$+3\int\int_{Q}|v_{xxx}u_{Nxxx}u_{Nxxxx}|dydx + \int\int_{Q}|v_{xxxx}u_{Nxx}\times$$

$$\times u_{Nxxxx}|dydx.$$

Estimate the last two integrals in the right - hand side, applying the embedding $W^1_2(Q) \to L^4(Q)$

$$I_1 = 3\int\int_{Q}|v_{xxx}u_{Nxxx}u_{Nxxxx}|dydx \leq \frac{a}{8}\int\int_{Q} u^2_{Nxxxx}dydx$$

$$+C_5(a)\int\int_{Q}|v^2_{xxx}u^2_{Nxxx}|dydx \leq \frac{a}{8}\int\int_{Q} u^2_{Nxxxx}dydx$$

$$+C_5(a)(\int\int_{Q} v^4_{xxx}dydx)^{1/2}(\int\int_{Q} u^4_{Nxxx}dydx)^{1/2} \leq$$

$$\frac{a}{8}\int\int_{Q} u^2_{Nxxxx}dydx + C_6(a,Q)||v_{xxx}||_1^2||u_{Nxxx}||_1^2 \leq$$

$$\frac{a}{8}\int\int_{Q} u^2_{Nxxxx}dydx + C_6(a,Q)\rho_0^2||u_N||^2_{H(Q)},$$

$$I_2 = \int\int_{Q}|v_{xxxx}u_{Nxx}u_{Nxxxx}|dydx \leq \frac{a}{8}\int\int_{Q} u^2_{Nxxxx}dydx$$

$$+C_7(a)\int\int_{Q} v^2_{xxxx}u^2_{Nxx}dydx \leq \frac{a}{8}\int\int_{Q} u^2_{Nxxxx}dydx +$$

$$+C_7(a)\max_{\bar{Q}}|u_{Nxx}|^2\int\int_Q v_{xxxx}^2 dydx \le \frac{a}{8}\times$$

$$\times\int\int_Q u_{Nxxxx}^2 dydx + C_8(a,Q)\rho_0^2||u_N||_{H(Q)}^2$$

Finally, we have

$$(\frac{a}{4}-\frac{5}{2}\rho_0 C_1)\int\int_Q u_{Nxxxx}^2 dydx \le C_2(a)\int\int_Q f_{Nxxx}^2 dydx +$$

$$+C_9(a,Q)\rho_0^2||u_N||_{H(Q)}^2. \tag{2.13}$$

Next, consider the equality

$$-(L_1u_N,\Delta_y u_N+\Delta_y u_x)_0 = -(f_N,\Delta_y u_N+\Delta_y u_{Nx})_0.$$

Making use, wherever necessary, of Cauchy's and Young's inequalities, we arrive at

$$\frac{1}{4}\int\int_Q|\Delta_y u_N|^2 dydx + a\int\int_Q|\nabla_y u_{Nx}|^2 dydx \le C_{10}(\varepsilon)\times$$

$$\times\int\int_Q f_N^2 dydx + C_{11}(\varepsilon)\int\int_Q v_x^2 u_{Nxx}^2 dydx + \varepsilon\int\int_Q|\Delta_y u_{Nx}|^2 dydx.$$

Estimating the second integral in the right - hand side of the last inequality with the help of the embedding theorems, we, finally, obtain

$$\frac{1}{4}\int\int_Q|\Delta_y u_N|^2 dydx + a\int\int_Q|\nabla_y u_{Nx}|^2 dydx \le C_{10}(\varepsilon)\times$$

$$\times\int\int_Q f_N^2 dydx + C_{12}(\varepsilon,Q)\rho_0^2||u_N||_{H(Q)}^2 + \varepsilon\int\int_Q|\Delta_y u_{Nx}|^2 dydx \tag{2.14}$$

Now, consider the equality

$$-((L_1u_N)_x,\Delta_y u_{Nx}+\Delta_y u_{Nxx})_0 = -(f_{Nx},\Delta_y u_{Nx}+\Delta_y u_{Nxx})_0.$$

Applying Cauchy's and Young's inequalities, we derive

$$\frac{1}{4}\int\int_Q|\Delta_y u_{Nx}|^2 dydx + a\int\int_Q|\nabla_y u_{Nxx}|^2 dydx \le C_{13}(\varepsilon)\int\int_Q f_{Nx}^2 dydx$$

$$+C_{14}(\varepsilon)\int\int_Q((v_x u_{Nxx})_x)^2 dydx + \varepsilon\int\int_Q|\Delta_y u_{Nxx}|^2 dydx.$$

Repeating arguments, already employed before, we arrive at the next estimate

$$\frac{1}{4}\int\int_Q|\Delta_y u_{Nx}|^2 dydx + a\int\int_Q|\nabla_y u_{Nxx}|^2 dydx \le C_{13}(\varepsilon)\int\int_Q f_{Nx}^2 dydx$$

$$+C_{15}(\varepsilon, Q)\rho_0^2||u_N||^2_{H(Q)} + \varepsilon \int\limits_Q\int |\Delta_y u_{Nxx}|^2 dydx. \tag{2.15}$$

Consider the equality

$$-((L_1 u_N)_{xx}, \Delta_y u_{Nxx})_0 = -(f_{Nxx}, \Delta_y u_{Nxx})_0.$$

Carrying out simple transformations and applying Young's inequality, we have

$$\frac{1}{4}\int\limits_Q\int |\Delta_y u_{Nxx}|^2 dydx \le C_{16}\int\limits_Q\int f^2_{Nxx} dydx + C_{17}\int\limits_Q\int ((v_x u_{Nxx})_{xx})^2 dydx$$

Estimate the last integral with the help of the embedding theorems just as we had done it earlier

$$I_3 = \int\limits_Q\int ((v_x u_{Nxx})_{xx})^2 dydx \le 4(\int\limits_Q\int v_x^2 u^2_{Nxxxx} dydx +$$

$$+2\int\limits_Q\int v^2_{xx} u^2_{Nxxx} dydx + \int\limits_Q\int v^2_{xxx} u^2_{Nxx} dydx) \le$$

$$\le 4(\max_{\bar{Q}} |v_x|^2 \int\limits_Q\int u^2_{Nxxxx} dydx +$$

$$2\max_{\bar{Q}} |v_{xx}|^2 \int\limits_Q\int u^2_{Nxxx} dydx +$$

$$+(\int\limits_Q\int v^4_{xxx} dydx)^{1/2}(\int\limits_Q\int u^4_{Nxx} dydx)^{1/2}) \le$$

$$\le C_{18}(Q)\rho_0^2||u_N||^2_{H(Q)}$$

Finally, we derive

$$\frac{1}{4}\int\limits_Q\int |\Delta_y u_{Nxx}|^2 dydx \le C_{16}\int\limits_Q\int f^2_{Nxx} dydx +$$

$$+C_{18}(Q)\rho_0^2||u_N||^2_{H(Q)} \tag{2.16}$$

To obtain the next estimate, look at the equality

$$-((L_1 u_N)_{xx}, \Delta_y u_{Nxxx})_0 = -(f_{Nxx}, \Delta_y u_{Nxxx})_0$$

Integration by parts yields

$$a\int\limits_Q\int |\nabla_y u_{Nxxx}|^2 dydx \le |(f_{Nxxx}, \Delta_y u_{Nxx})_0| +$$

$$\int\limits_Q\int (v_x u_{Nxx})_{xx}\Delta_y u_{Nxxx} dydx \le C_{19}(\varepsilon)\int\limits_Q\int f^2_{Nxxx} dydx +$$

$$\varepsilon \int\limits_Q\int |\Delta_y u_{Nxx}|^2 dydx + \int\limits_Q\int v_{xxx} u_{Nxx} \Delta_y u_{Nxxx} dydx +$$

$$2\int\limits_Q\int v_{xx} u_{Nxxx} \Delta_y u_{Nxxx} dydx + \int\limits_Q\int v_x u_{Nxxxx} \Delta_y u_{Nxxx} dydx$$

Estimate each of the last three integrals separately on the basis of the technique already employed earlier

$$I_4 = \int\limits_Q\int v_{xxx} u_{Nxx} \Delta_y u_{Nxxx} dydx = -\int\limits_Q\int \nabla_y v_{xxx} u_{Nxx} \nabla_y u_{Nxxx} dydx -$$

$$-\int\limits_Q\int v_{xxx} \nabla_y u_{Nxx} \nabla_y u_{Nxxx} dydx \leq$$

$$C_{20}(a) \max_{\bar{Q}} |u_{Nxx}|^2 \rho_0^2 + \frac{a}{8}\int\limits_Q\int |\nabla_y u_{Nxxx}|^2 dydx +$$

$$+ C_{21}(a)(\int\limits_Q\int |\nabla_y u_{Nxx}|^4 dydx)^{1/2} \rho_0^2 + \frac{a}{8}\int\limits_Q\int |\nabla_y u_{Nxxx}|^2 dydx \leq$$

$$\leq C_{22}(a,Q)\rho_0^2 ||u_N||^2_{H(Q)} + \frac{a}{4}\int\limits_Q\int |\nabla_y u_{Nxxx}|^2 dydx,$$

$$I_5 = 2\int\limits_Q\int v_{xx} u_{Nxxx} \Delta_y u_{Nxxx} dydx = -2(\int\limits_Q\int \nabla_y v_{xx} u_{Nxxx}$$

$$\nabla_y u_{Nxxx} dydx + \int\limits_Q\int v_{xx} |\nabla_y u_{Nxxx}|^2 dydx) \leq$$

$$\leq C_{23}(a)(\int\limits_Q\int |u_{Nxxx}|^4 dydx)^{1/2} \rho_0^2 + \frac{a}{4}\times$$

$$\times \int\limits_Q\int |\nabla_y u_{Nxxx}|^2 dydx + 2\max_{\bar{Q}} |v_{xx}| \int\limits_Q\int |\nabla_y u_{Nxxx}|^2 dydx$$

$$\leq C_{24}(a,Q)\rho_0^2 ||u_N||^2_{H(Q)} +$$

$$+ \frac{a}{4}\int\limits_Q\int |\nabla_y u_{Nxxx}|^2 dydx + 2\rho_0 C_1(Q)\int\limits_Q\int |\nabla_y u_{Nxxx}|^2 dydx,$$

$$I_6 = \int\limits_Q\int v_x u_{Nxxxx} \Delta_y u_{Nxxx} dydx = \frac{1}{2}\int\limits_Q\int v_{xx} |\nabla_y u_{Nxxx}|^2 dydx -$$

$$\int\limits_Q\int \nabla_y v_x u_{Nxxxx} \nabla_y u_{Nxxx} dydx \leq \frac{\rho_0 C_1}{2}\int\limits_Q\int |\nabla_y u_{Nxxx}|^2 dydx +$$

$$+ C_{25}(a) \max_{\bar{Q}} |\nabla_y v_x|^2 \int\limits_Q\int u^2_{Nxxxx} dydx +$$

$$+\frac{a}{4}\int\int_Q |\nabla_y u_{Nxxx}|^2 dydx \le \frac{\rho_0 C_1}{2}\int\int_Q |\nabla_y u_{Nxxx}|^2 dydx +$$

$$C_{26}(a,Q)\rho_0^2 ||u_N||^2_{H((Q)} + \frac{a}{4}\int\int_Q |\nabla_y u_{Nxxx}|^2 dydx.$$

Summing up all the inequalities obtained for I_4, I_5, and I_6 and recalling the input inequality, we, finally, have

$$(\frac{a}{4} - \frac{5\rho_0 C_1}{2})\int\int_Q |\nabla_y u_{Nxxx}|^2 dydx \le C_{19}(\varepsilon)\int\int_Q f^2_{Nxxx} dydx +$$

$$+\varepsilon\int\int_Q |\Delta_y u_{Nxx}|^2 dydx + C_{27}(a,Q)\rho_0^2 ||u_N||^2_{H(Q)}. \qquad (2.17)$$

Now, notice, that if the boundary γ is sufficiently smooth, we have $\omega_s \in W_2^4(D)$ for all $s = 0, 1\ldots,$ and $\frac{\partial(\Delta_y \omega_s)}{\partial n}|_\gamma = 0$. Taking this circumstance into account, consider the equality

$$((L_1 u_N)_x, \Delta_y^2 u_{Nx})_0 = (f_{Nx}, \Delta_y^2 u_{Nx})_0$$

Integrating by parts, we obtain

$$\int\int_Q |\nabla_y \Delta_y u_{Nx}|^2 dydx = -\int\int_Q \nabla_y f_{Nx} \nabla_y \Delta_y u_{Nx} dydx +$$

$$\int\int_Q \nabla_y (v_{xx} u_{Nxx}) \nabla_y \Delta_y u_{Nx} dydx + \int\int_Q \nabla_y (v_x u_{Nxxx}) \nabla_y \Delta_y u_{Nx} dydx$$

Applying Young's inequality to each of the terms in the right - hand side and making use of the embedding theorems and of Hölder's inequality when estimating appropriate integrals, we, finally, arrive at

$$\frac{1}{2}\int\int_Q |\nabla_y \Delta_y u_{Nx}|^2 dydx \le C_{28}\int\int_Q |\nabla_y f_{Nx}|^2 dydx +$$

$$C_{29}(Q)\rho_0^2 ||u_N||^2_{H(Q)} \qquad (2.18)$$

In order to estimate the third derivatives of u_{Nx} with respect to variables y we, afterwards, shall use the inequality for elliptic operators (see [35, Ch.2, sec.5, item 5.1, Theorem 5.1])

$$||u_{Nx}||^2_{W^3_{2,y}(Q)} \le \tilde{C}(||\Delta_y u_{Nx}||^2_{W^1_{2,y}(Q)} + ||u_{Nx}||^2_{W^2_{2,y}(Q)}). \qquad (*)$$

The right - hand side of the inequality (*) is estimated by virtue of (2.10), (2.14), (2.15) and (2.18).

Now, recall the inequality

$$||z||_0^2 \le C(Q)||\nabla_{x,y} z||_0^2, \qquad (**)$$

which is valid for every $z \in W_2^1(Q)$ such that $\int\int_Q z dydx = 0$ (see [27, P. 9]).

Derive the last estimate, applying (**). Consider the equality

$$(L_1 u_N, u_N)_0 = (f_N, u_N)_0.$$

It easily yields (Young's "ε - inequality taken into account)

$$\int\int_Q |\nabla_y u_N|^2 dydx \leq C_{30}(\varepsilon_1) \int\int_Q f_N^2 dydx + C_{31}(\varepsilon_1, Q) \times$$

$$\times \rho_0^2 ||u_N||^2_{H(Q)} + \varepsilon_1 C(Q) (\int\int_Q |\nabla_y u_N|^2 dydx + \int\int_Q u_{Nx}^2 dydx), \tag{2.19}$$

with $\varepsilon_1 = \min(\frac{C^{-1}}{2}, \frac{C^{-1}a}{4})$.

Choose in (2.14), (2.15) and (2.17) $\varepsilon < \frac{1}{8}$, sum up inequalities (2.10) - (2.19) and choose $\rho_0 > 0$ sufficiently small. Making use of the second basic inequality for elliptic operators (see [36, Ch.2, sec.6]) and of the inequalities (*) and (**), we, finally, obtain

$$||u_N||_{H(Q)} \leq \mu(a, Q) |||f_N|||, \tag{2.20}$$

with μ independent of N.

Due to the equivalence of the norms in finite - dimensional spaces, the above estimate guarantees uniqueness of a solution of the system of equations (2.9), and, consequently, its solvability. Moreover, a passage to the limit as $N \to \infty$ in (2.9) is possible now. Indeed, since $f_N \underset{N\to\infty}{\to} f$ in the norm $|||\cdot|||$, it follows that the sequence $\{u_N\}_{N=1,2...}$ is bounded in $H(Q)$ by virtue of (2.20). It means that there exist a function $u_v(x,y) \in H(Q)$ and a subsequence $\{N_k\}_{k\in N}$ such that $u_{N_k} \underset{k\to\infty}{\to} u_v$ weakly in $H(Q)$ and strongly in $W_2^1(Q)$ due to the compactness theorems.

Consider the equality

$$(L_1 u_{N_k}, \varphi_s)_0 = (f_{N_k}, \varphi_s)_0, \tag{2.21}$$

with φ_s a basic function, and $N_k > s$. Since, as we have already seen, $(P_{N_k}(v_x u_{N_k xx}), \varphi_s)_0 = (v_x u_{N_k xx}, \varphi_s)_0$, it follows that our conclusions make possible the passage to the limit in (2.21) as $k \to \infty$. Due to arbitrariness of s, we, finally, conclude that the function $u_v \in H(Q)$ such that $\int\int_Q u_v dydx = 0$ is a solution of equation (2.8).

Also, we have the obvious inequalities

$$||u_v||_{H(Q)} \leq \underline{\lim}_{k\to\infty} ||u_{N_k}||_{H(Q)} \leq \mu(a, Q)|||f|||,$$

and, therefore, there exists $\alpha(a, Q) > 0$ such that if $|||f||| < \alpha$, then we have

$$||u_v||_{H(Q)} \leq \rho_0, \tag{2.22}$$

and the estimate does not depend on the initially chosen function $v \in \tilde{B}_{\rho_0}$.

Uniqueness of a solution $u_v \in H(Q)$ such that $\int\int_Q udydx = 0$ of BVP (2.8), (2.5) - (2.7) as ρ_0 is sufficiently small can be easily derived from the equality $(L_v z, z_x)_0 = 0$, wherein z is a difference of two such solutions.

Next, we consider the iterative process

$$L(u^{n+1}, u^n) \equiv u_x^n u_{xx}^{n+1} - \Delta_y u^{n+1} + a u_x^{n+1} = f(x,y), \tag{2.23}$$

u^{n+1} satisfies boundary conditions (2.5) - (2.7), and $u^0 \equiv 0$.

We have already proven that for each $n = 0, 1 \dots$ BVP (2.23), (2.5) - (2.7) is uniquely solvable in the function class $\tilde{B}_{\rho_0}$, and the uniform in n estimate

$$||u^n||_{H(Q)} \le \rho_0$$

takes place. It means that there exist a function $u(x,y) \in H(Q)$ and a subsequence $\{u^{n_k}\}_{k\in N}$ such that $u^{n_k} \underset{k\to\infty}{\to} u$ weakly in $H(Q)$ and strongly in $W_2^1(Q)$. Now, in order to pass to the limit in (2.23), we are going to prove that the entire consequence $\{u^n\}_{n\in N}$ converges to the function $u(x,y)$ though, may be, in a weaker topology. It will imply that actually $u^n \underset{n\to\infty}{\to} u$ weakly in $H(Q)$ and strongly in $W_2^1(Q)$.

For this goal we consider the Banach space $H_1(Q)$ consisting of functions with the finite norm $||u||^2_{H_1(Q)} = ||u_x||_0^2 + ||\nabla_y u||_0^2$ and with $\int\int_Q u dydx = 0$ and the mapping $S : H_1(Q) \to H_1(Q)$ which is defined over $\tilde{B}_{\rho_0}$ and takes an arbitrary function $v \in \tilde{B}_{\rho_0}$ into a unique solution $u_v \in \tilde{B}_{\rho_0}$ of equation (2.8). Inequality (2.22) implies that the mapping S takes $\tilde{B}_{\rho_0}$ into itself.

Consider the difference

$$L_{v_1}u_1 - L_{v_2}u_2 = v_{1x}u_{1xx} - v_{2x}u_{2xx} - \Delta_y u + a u_x = 0,$$

with $u_i = Sv_i (i = 1,2)$, and $u = u_1 - u_2$. Denote $v = v_1 - v_2$. Add and subtract here the quantity $v_{1x}u_{2xx}$. Obtain, $L_{v_1}u_1 - L_{v_2}u_2 = v_{1x}u_{xx} + v_x u_{2xx} - \Delta_y u + a u_x = 0$.

Now, look at the equality

$$(L_{v_1}u_1 - L_{v_2}u_2, u_x)_0 = 0.$$

Integration by parts and the embedding theorems yield

$$(a - \frac{\rho_0 C_1}{2}) \int\int_Q u_x^2 dydx \le \int\int_Q |v_x u_{2xx} u_x| dydx \le$$

$$\le \frac{a}{2}\int\int_Q u_x^2 dydx + C_{32}(a)\int\int_Q v_x^2 u_{2xx}^2 dydx \le$$

$$\le \frac{a}{2}\int\int_Q u_x^2 dydx + C_{32}(a) \max_{\bar{Q}} |u_{2xx}|^2 \int\int_Q v_x^2 dydx \le$$

$$\le \frac{a}{2}\int\int_Q u_x^2 dydx + C_{33}(a,Q)\rho_0^2 ||v||^2_{H_1(Q)}$$

Finally, we have the estimate

$$(\frac{a}{2} - \frac{\rho_0 C_1}{2}) \int\int_Q u_x^2 dydx \le C_{33}(a,Q)\rho_0^2 ||v||^2_{H_1(Q)} \tag{2.24}$$

Next, consider the equality

$$(L_{v_1}u_1 - L_{v_2}u_2, u)_0 = 0$$

Applying Young's "ε - inequality" and integrating by parts, wherever necessary, we derive

$$\int\!\!\int_Q |\nabla_y u|^2 dydx \le \varepsilon \int\!\!\int_Q u^2 dydx +$$

$$C_{34}(\varepsilon)\int\!\!\int_Q v_x^2 u_{2xx}^2 dydx + \int\!\!\int_Q |v_{1xx}u_x u| dydx +$$

$$\int\!\!\int_Q |v_{1x}u_x^2| dydx$$

Estimate the last three integrals involving the already employed technique

$$I_7 = C_{34}(\varepsilon)\int\!\!\int_Q v_x^2 u_{2xx}^2 dydx$$

$$\le C_{34}(\varepsilon)\max_{\bar{Q}} |u_{2xx}^2| \int\!\!\int_Q v_x^2 dydx \le C_{35}(\varepsilon, Q)\rho_0^2 ||v||^2_{H_1(Q)},$$

$$I_8 = \int\!\!\int_Q |v_{1xx}u_x u| dydx \le \varepsilon \int\!\!\int_Q u^2 dydx + C_{36}(\varepsilon)\int\!\!\int_Q v_{1xx}^2 u_x^2 dydx \le$$

$$\varepsilon \int\!\!\int_Q u^2 dydx + C_{37}(\varepsilon, Q)\rho_0^2 ||u||^2_{H_1(Q)},$$

$$I_9 = \int\!\!\int_Q |v_{1x}u_x^2| dydx \le C_1\rho_0 ||u||^2_{H_1(Q)}$$

Taking into account (**), we, finally, obtain

$$\int\!\!\int_Q |\nabla_y u|^2 dydx \le 2\varepsilon C(Q)\int\!\!\int_Q |\nabla_{y,x} u|^2 dydx +$$

$$+C_{38}(\varepsilon, Q)\rho_0^2 ||v||^2_{H_1(Q)} + C_{39}(\varepsilon, Q)(\rho_0^2 + \rho_0)||u||^2_{H_1(Q)}. \tag{2.25}$$

Sum up inequalities (2.24) and (2.25). First, choose sufficiently small $\varepsilon > 0$, and then diminish, if necessary, ρ_0 with ε fixed, Thus, finally, we obtain

$$||u||_{H_1(Q)} \le q||v||_{H_1(Q)}, \quad \text{with} \quad 0 < q < 1.$$

Hence, we have proven that the mapping S is a contraction , and, consequently $u^n \underset{n\to\infty}{\to} u$ strongly in $H_1(Q)$. As we have already mentioned, it implies that $u^n \underset{n\to\infty}{\to} u$ weakly in $H(Q)$ and strongly in $W_2^1(Q)$. On the basis of the conclusions obtained we are able to pass to the limit as $n \to \infty$ in (2.23) by a standard procedure. Obviously, the function obtained is a solution of equation (2.4) in $\tilde{B}_{\rho_0}$ (boundary conditions (2.5) - (2.7) are satisfied by virtue of a priori estimates and trace theorems).

Uniqueness for a solution of BVP (2.4), (2.5) - (2.7) in $\tilde{B}_{\rho_0}$ can be proven in exactly the same way as contractivity.

Theorem 2.1 is completely proven.

3. ON SOLVABILITY OF BVPS FOR A NONLINEAR FORWARD - BACKWARD PARABOLIC EQUATION

In this section we shall deal with the equation

$$Bu \equiv uu_t - u_{xx} = f(x,t), \tag{3.1}$$

which is considered in the domain $Q_T = (0,l) \times (0,T)$.

Equation (3.1) is a model nonlinear forward - backward parabolic equation, and, from the other hand, it arises in applications (for instance, in the theory of boundary layer). In [37], the first IBVP was studied in Q_T for a more general equation $a(u)u_t - u_{xx} = f(x,t)$ with $a(s) > 0$ for $s > 0$, and $a(0) = 0$. Existence of a unique generalized solution $u(x,t)$ such that $u \in C(\bar{Q_T}), u_x \in L^\infty(Q_T)$ of the problem posed was proven under the condition $f(x,t) \geq 0$ and provided that the initial and the boundary data (nonhomogeneous in general) were nonnegative. The article cited also contained an example of a blow - up for a classical solution of the first IBVP for an equation like (3.1) with arbitrary smooth initial - boundary data and zero right - hand side.

In [38], the following BVP was proposed with $f \equiv 0$: find in Q_T a solution of (3.1) satisfying the boundary conditions

$$u|_{t=0} = u_0(x) \geq 0 \tag{3.2}$$

$$u|_{t=T} = u_1(x) \leq 0 \tag{3.3}$$

$$u|_{x=0} = 0, \quad u|_{x=l} = h(t) \tag{3.4}$$

Treating the problem, the author proved existence of a generalized solution $u \in L^\infty(0,T;L^2)$, $u_x \in L^2(Q_T)$ meeting the boundary conditions, and a uniqueness theorem was proven provided some extra regularity of a solution.

Also notice, that equation (3.1) along with more general "doubly nonlinear " equations was treated in [33, 39] from the viewpoint of variational inequalities for parabolic operators.

In [40], we established existence and uniqueness in Q_T of smooth global positive solutions of the first and of the second IBVPs for equation (3.1) in the class $V(Q_T) = \{v \in L^2(0,T;W_2^2) \bigcap L^\infty(0,T;W_2^1), v_t \in L^2(Q_T)\}$ under conditions of smallness of the parameter l or of smallness of the L^2 - norm of the right - hand side $f(x,t)$ and of the initial - boundary data.

Let us state here a theorem proven in [40]. Given the initial - boundary data

$$u|_{t=0} = u_0(x) \tag{3.5}$$

$$u|_{x=0} = \varphi_1(t), \quad u|_{x=l} = \varphi_2(t) \tag{3.6}$$

The following theorem is valid.

THEOREM 3.1.[40]*Let* $f(x,t) \in L^2(Q_T), u_0(x) \in H^{4+\alpha}[0,l], \varphi_i \in H^{2+\frac{\alpha}{2}}[0,T](i = 1,2)$ *with* $0 < \alpha < 1$. *Let the second order concordance conditions of the initial data and of the boundary conditions hold. Let* $u_0(x) \geq \delta > 0$ *on* $[0,l]$, *and* $\varphi_i(t) \geq \delta > 0$ *on* $[0,T](i = 1,2)$.

Then, if $||f||_{L^2(Q_T)} < \mu_0, \max(\max\limits_{[0,l]} |u_{0xx}|, \max\limits_{[0,T]} |\varphi_{1t}|, \max\limits_{[0,T]} |\varphi_{2t}|) < \beta_0$ *(positive parameters* μ_0 *and* β_0 *depend on* δ, l, T*), there exists a unique positive solution of IBVP (3.1), (3.5), (3.6) in the space* $V(Q_T)$.

The results contained in [40], from our viewpoint, are quite interesting, but a number of questions remained open. Namely, what can one say about solvability (or a blow - up) of INVP (3.1), (3.5), (3.6) in the case when u_0 or φ_i are nonnegative only and may be zero? Together with this, the solutions obtained are positive, and, consequently, (3.1) is strictly parabolic at these solutions and does not actually change a time direction.

We have a rather different situation when considering in Q_T a $t-$ periodic BVP for a regularized equation

$$\tilde{B}u \equiv uu_t - u_{xx} + au = f(x,t), \tag{3.7}$$

where a is a positive constant.

Given the boundary conditions

$$u|_{t=0} = u|_{t=T} \tag{3.8}$$

$$u|_{x=0} = u|_{x=l} = 0 \tag{3.9}$$

or

$$u_x|_{x=0} = u_x|_{x=l} = 0 \tag{3.9'}$$

Define the space $W(Q_T)$ to be a closer in the norm $||u||^2_{W(Q_T)} = ||u||^2_{W_2^2(Q_T)} + ||u_t||^2_{W_2^2(Q_T)}$ of infinitely differentiable in $\bar{Q}_T$ functions, satisfying (3.9) (or (3.9')) such that $D_t^j u|_{t=0} = D_t^j u|_{t=T}$ for all $j \geq 0$. Denote $|||f||| = \sum\limits_{j=0}^{3} ||D_t^j f||_{L^2(Q_T)}$.

The following theorem takes place.

THEOREM 3.2. *Let* $D_t^j \in L^2(Q_T), j = 0,1,2,3, D_t^j f|_{t=0} = D_t^j f|_{t=T}, j = 0,1,2$. *Then, if* $|||f||| < \alpha$, *(with* $\alpha > 0$ *depending on* l, T, a*), there exists a solution of BVP (3.7) - (3.9) (or of BVP (3.7) - (3.9')) in the space* $W(Q_T)$.

In the set of functions such that $||u||_{W(Q_T)} < \rho_0$, *with* $\rho_o(l,T,a)$ *a sufficiently small number, a solution of (3.7) - (3.9) (or of (3.7) - (3.9')) is unique.*

Proof of the theorem repeats mutatis mutandis that of Theorem 2.1 because the structure of (3.7) is rather close to that of equation (2.4). One can obtain all a priori estimates required considering the following equalities

$$(D_t^j(\tilde{P}_N(vu_{Nt}) - u_{Nxx} + au_N), D_t^j u)_0 = (D_t^j \tilde{P}_N f, D_t^j u)_0, j = 0,1,2,3$$

and

$$-(D_t^k(\tilde{P}_N(vu_{Nt}) - u_{Nxx} + au_N), D_t^k u_{Nxx})_0 = -(D_t^k \tilde{P}_N f, D_t^k u_{Nxx})_0, k = 0,1,$$

with $v(x,t) \in W(Q_T)$ an arbitrary given function with a sufficiently small norm, $\tilde{P}_N$ an appropriate orthoprojector in $L^2(Q_T)$, and u_N an appropriate Galërkin approximation.

We arrive at interesting conclusions by means of the following obvious assertion.

THEOREM 3.3. *If* $\int\int_{Q_T} f dxdt = 0$, *and if there exists a smooth nontrivial solution of BVP (3.7) - (3.9'), then the solution necessarily changes its sign in* Q_T.

Proof. Consideration of the equality $\int\int_{Q_T} \tilde{B}udxdt = \int\int_{Q_T} fdxdt = 0$ validates the theorem.

Hence, Theorems 3.2 and 3.3 imply that a periodic BVP selects the very solution of (3.7) at which the equation actually changes its type and is a properly forward - backward parabolic equation.

4. NOTES ON VARIATIONAL INEQUALITIES

Monographs [41 - 43] and the bibliography therein contain main results obtained in the theory and in numerous applications of variational inequalities. There also exist a number of articles devoted to variational inequalities for quasilinear hyperbolic operators (see the bibliography in [32, 33]).

At the same time, the theory of variational inequalities can serve as a tool to study nonlinear equations of variable type, in particular, equations arising in hydrodynamics and in gas dynamics. Thus, for instance, in [32, 33, 44, 45] the technique of variational inequalities was applied in order to deal with certain models of transonic gas dynamics bringing us to nonlinear equations of variable type. In Section 2 we have already mentioned a result obtained when studying the von Kàrmàn - Fal'kovich equation (2.3). Here let us state the theorem.

Consider (2.3) in the same domain Q as in the Section 2. Given the "initial - boundary " conditions

$$u|_{x=0} = u_0(y) \tag{4.1}$$

$$u_x|_{x=0} = u_1(y) \tag{4.2}$$

$$u|_{\gamma\times(0,L)} = 0 \tag{4.3}$$

Introduce two convex sets of constraints $\mathcal{M}_1 = \{z \in \overset{\circ}{W}{}^1_4(D) : z \geq 0 \text{ in } D\}$ and $\mathcal{M}_2 = \{z \in \overset{\circ}{W}{}^1_4(D) : |\nabla_y z| \leq 1 \text{ a. e. in } D\}$. Denote $\mathcal{M} = \mathcal{M}_1 \bigcap \mathcal{M}_2$.

THEOREM 4.1.[33]*For every functions* $u_0 \in \overset{\circ}{W}{}^1_4(D)$ *and* $u_1 \in \mathcal{M}$ *and for every function* $f(x,y)$ *such that* f *and* $f_x \in L^{3/2}(Q)$ *there exists a function* $u(x,y) \in L^\infty(0,L;\overset{\circ}{W}{}^1_4)$ *such that* $u_x \in L^\infty(0,L;\overset{\circ}{W}{}^1_4)$, $(u_x^{3/2})_x \in L^2(Q)$ *possessing the following properties*

a) $u(x,y)$ *satisfies "initial - boundary conditions" (4.1) - (4.3),*

b) $u_x(x,\cdot) \in \mathcal{M}$ *a. e. on* $[0,L]$,

c) for every function $v \in L^\infty(0,L;\overset{\circ}{W}{}^1_4)$ *such that* $v(x,\cdot) \in m$ *a. e. on* $[0,L]$ *the inequality*

$$\int_0^x < A_3 u - f, v - u_\xi > d\xi \geq 0$$

holds for all $x \in [0, L]$.

Here we denote by $< \cdot, \cdot >$ a duality in appropriate function spaces.

In the next example we deal with a more general equation than one treated in Section 3, namely, with

$$B_1 u \equiv u^\alpha u_t - u_{xx} = f(x,t), \tag{4.4}$$

where $\alpha \geq 1$. We consider equation (4.4) in the same domain Q_T as in Section 3. Given the initial - boundary conditions

$$u|_{t=0} = u_0(x) \tag{4.5}$$

$$u_x|_{x=0} = u_x|_{x=l} = 0 \tag{4.6}$$

Introduce the convex set of constraints $\mathcal{K} = \{z \in W_2^1(0,l) : z \geq 0 \text{ on } [0,l]\}$.

THEOREM 4.2. *Let* f *and* $f_t \in L^2(Q_T)$, *and* $u_0 \in \mathcal{K}$. *Then, there exists a function* $u(x,t) \in L^2(0,T; W_2^2) \bigcap L^\infty(0,T; W_2^1)$ *such that* $(u^{\frac{\alpha}{2}+1})_t \in L^2(Q_T)$; $u(x,t)$ *satisfies the initial - boundary conditions (4.5), (4.6), the inclusion* $u(\cdot,t) \in \mathcal{K}$ *holds a. e. on* $[0,T]$, *and for every function* $v(x,t) \in L^\infty(0,T;\mathcal{K})$ *the inequality* $\int\limits_0^t (B_1 u - f, v - u)_0 d\tau \geq 0$ *is fulfilled for all* $t \in [0,T]$.

One can arrive at the proof of the theorem by a simple modification of the proof of Theorem 1 in [39] or of Theorem 2.2 in [33].

COROLLARY 4.1. *Assume all conditions of Theorem 4.2 hold. Let* $\alpha > 2$, *and for all* $x \in [0,l]$ *the inequality* $u_0(x) \geq \delta > 0$ *is valid. Let* $u(x,t)$ *be a solution of the variational inequality, possessing all properties indicated in Theorem 4.2. Then there exists* $t^* \in (0,T]$ *such that the equation* $B_1 u = f$ *holds a. e. in the rectangle* $Q_{t^*} = (0,l) \times (0,t^*)$.

Proof. Consider the function $v(x,t) = u^{\frac{\alpha}{2}+1}(x,t)$. Then, $v_t \in L^2(Q_T)$, and one can easily derive from the embedding theorems that $v \in L^2(0,T;W_2^2)$, when $\alpha > 2$. Next, the interpolational theorem (see [35, Ch.1, sec.3, item 3.1, Theorem 3.1]) yield that $v \in C(0,T;W_2^1)$, and, due to the embedding theorem, we have $v \in C(\bar{Q}_T)$, and, consequently, $u \in C(\bar{Q}_T)$, too. Due to the conditions on the initial function, $u(x,t)$ remains strictly positive in a rectangle Q_{t^*}, and, thereby, obviously, equation (4.4) holds a. e. in Q_{t^*}.

Similar results can be obtained if to pose the Dirichlet conditions instead of (4.6).

And now the last remark.

In the class of nonnegative functions equation (4.4) with $\alpha > 0$ is equivalent to the diffusion equation $v_t - (v^{\frac{1}{1+\alpha}})_{xx} = f$. Thus, one can come to rather interesting conclusions concerning local solvability of IBVPs for the equation which nowadays is an object of numerous publications.

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