Implicit Function Theorem for systems of polynomial equations with vanishing Jacobian and its application to flexible polyhedra and frameworks *

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Abstract

We study the existence problem for a local implicit function determined by a system of nonlinear algebraic equations in the particular case when the determinant of its Jacobian matrix vanishes at the point under consideration. We present a system of sufficient conditions that implies existence of a local implicit function as well as another system of sufficient conditions that guarantees absence of a local implicit function. The results obtained are applied to proving new and classical results on flexibility and rigidity of polyhedra and frameworks.

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1 Introduction

Let $F : \mathbf{R}^l \times \mathbf{R}^m \to \mathbf{R}^n$ be a differentiable mapping, let $t, t_0 \in \mathbf{R}^l$, let $X, X_0 \in \mathbf{R}^m$, and let $F(t_0, X_0) = 0$. The classical Implicit Function Theorem provides conditions which imply that the equation $F(t, X) = 0$ determines an implicit function $X = X(t)$ in a neighborhood of the point (t_0, X_0). The principal condition is invertibility of the operator $F'_X(t_0, X_0)$.

The Inverse Function Theorem has numerous applications and is generalized in various directions. However, the author is not aware of any version of this theorem which guarantees existence of an implicit function in the case when the operator $F'_X(t_0, X_0)$ is not invertible. In the present paper we will partially fill in this gap.

Our study is motivated by that of flexible polyhedra and frameworks. It turns out that mappings F which appear is that field do not depend on the parameter t. We will focus our attention on this particular case. The following system of nonlinear algebraic equations can be considered as a typical example of a system to which our arguments can be applied:

$$\begin{aligned} F_1(t, x_1, x_2, x_3) &\equiv x_1^2 + x_2^2 - x_3^2 - 1 = 0, \\ F_2(t, x_1, x_2, x_3) &\equiv 3x_1 + x_2 - 3x_3 + 1 = 0, \\ F_3(t, x_1, x_2, x_3) &\equiv x_1 - 3x_2 + x_3 + 3 = 0. \end{aligned} \tag{1}$$

The parameter t is not explicitly involved in this system. The point $X_0 = (5, 5, 7)^{\mathrm{T}}$ satisfies (1). The determinant of the Jacobian matrix of (1) vanishes at $X_0 = (x_1, x_{,2}, x_3)$:

$$\det F'_X(t, X_0) = \begin{vmatrix} 2x_1 & 2x_2 & -2x_3 \\ 3 & 1 & -3 \\ 1 & -3 & 1 \end{vmatrix} = \begin{vmatrix} 10 & 10 & -14 \\ 3 & 1 & -3 \\ 1 & -3 & 1 \end{vmatrix} = 0.$$

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Therefore, the classical Implicit Function Theorem cannot be applied to (1). Nevertheless, from the results presented below it will follow that X_0 is not an isolated solution to (1); on the contrary, it belongs to a continuous family of solutions $X = X(t)$ which can be treated as an implicit function determined by system (1) and initial point X_0.

2 Sufficient conditions for existence of an implicit function

Let $X = (x_1, \dots, x_m) \in \mathbf{R}^m$ and let $F(X) = (F_1(X), \dots, F_n(X))$, where each F_k $(k = 1, \dots, n)$ is a polynomial. Without loss of generality, we may assume that the degree of each F_k is at most 2.

To explain the last statement, we assume, for example, that in the system $F(X) = 0$ under consideration each polynomial F_k $(k = 1, \dots, n-1)$ is of degree at most 2 while $F_n(X) = x_1^2 x_2 - 1$. Introduce a new independent variable x_{m+1} and put $\widetilde{X} = (x_1, \dots, x_m, x_{m+1})$. We also introduce new functions $\widetilde{F}_n(\widetilde{X}) = x_{m+1}x_2 - 1$ and $\widetilde{F}_{n+1} = x_{m+1} - x_1^2$ and put $\widetilde{F}(\widetilde{X}) = (F_1(X), \dots, F_{n-1}(X), \widetilde{F}_n(\widetilde{X}), \widetilde{F}_{n+1}(\widetilde{X}))$. Obviously, the system $F(X) = 0$ is equivalent to the system $\widetilde{F}(\widetilde{X})) = 0$ and each equation of the latter system is of degree at most 2.

Thus, without loss of generality, we may assume that the degree of each F_k is at most 2. In this case, F_k can written as

$$F_k(X) = \sum_{i=1}^{m}\sum_{j=1}^{m} \alpha_{ij}^k x_i x_j + \sum_{i=1}^{m} \beta_i^k x_i + \gamma^k,$$

where α_{ij}^k, β_i^k, and γ^k are some reals satisfying $\alpha_{ij}^k = \alpha_{ji}^k$.

It is well known that, if a system of polynomial equations admits a family of solutions which is continuous with respect to a parameter, then this system also admits a family of solutions which depends analytically on a parameter (possibly different); see, for example, [?] or Lemma 18.3 in [?]. Thus, assuming that the system $F(X) = 0$ admits a continuous family of solutions $X = X(t) \equiv (x_1(t), \dots, x_m(t))$, we may assume without loss of generality that this family depends analytically on t, i.e., it can be expanded in a Maclaurin series:

$$x_i(t) = \sum_{k=0}^{\infty} x_{i,k} t^k, \qquad x_{i,k} \in \mathbf{R}.$$

Substituting this expansion into equation $F_k(X) = 0$, we obtain

$$\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\alpha_{ij}^k \left(\sum_{p=0}^{\infty} x_{i,p}t^p\right)\left(\sum_{q=0}^{\infty} x_{i,q}t^q\right)\right] + \sum_{i=1}^{m} \beta_i^k \left(\sum_{p=0}^{\infty} x_{i,p}t^p\right) + \gamma^k = 0$$

or

$$\sum_{p=0}^{\infty}\left[\sum_{i=1}^{m}\sum_{j=1}^{m} \alpha_{ij}^k \sum_{q=0}^{p} x_{i,q}x_{j,p-q}\right] t^p + \sum_{p=0}^{\infty}\left[\sum_{i=1}^{m} \beta_i^k x_{i,p}t^p\right] + \gamma^k = 0.$$

Interpreting the left-hand side of the latter equation as the Maclaurin expansion of the function that equals zero identically, we conclude that, in this expansion, the coefficient of t^p equals zero for each $p \geq 1$, i.e., the equation

$$\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{q=0}^{p} \alpha_{ij}^k x_{i,q}x_{j,p-q} + \sum_{i=1}^{m} \beta_i^k x_{i,p} + \gamma^k = 0 \tag{2}$$

holds for all $p \geq 1$ and $1 \leq k \leq n$.

For each $p \geq 1$, put $X_p = (x_{1,p}, x_{2,p}, \dots, x_{m,p}) \in \mathbf{R}^m$. Let the bilinear mapping $B : \mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}^n$ be defined by the following rule: if $X = (x_1, \dots, x_m) \in \mathbf{R}^m$ and $Y = (y_1, \dots, y_m) \in \mathbf{R}^m$ then the kth coordinate of the vector $B(X, Y)$ is equal to

$$\sum_{i=1}^{m}\sum_{j=1}^{m} \alpha_{ij}^k x_i y_j.$$

Let the linear mapping $A : \mathbf{R}^m \to \mathbf{R}^n$ be defined by the following rule: if $X = (x_1, \ldots, x_m) \in \mathbf{R}^m$ then the kth coordinate of the vector $A(X)$ equals

$$\sum_{i=1}^{m} \beta_i^k x_i.$$

Using this notation, we can rewrite (2) as

$$\sum_{p=0}^{q} B(X_p, X_{q-p}) + AX_q = 0.$$

It follows that, if the vectors X_0, X_1, $\ldots X_{q-1}$ are given and we seek the vector X_q, then we need to solve the following system of linear equations:

$$B(X_0, X_q) + B(X_q, X_0) + AX_q = -\sum_{p=1}^{q-1} B(X_p, X_{q-p}). \tag{3}$$

Let the linear mapping $C : \mathbf{R}^m \to \mathbf{R}^n$ be given by $CX = B(X_0, X) + B(X, X_0) + AX$. Then we can rewrite (3) in concise form:

$$CX_q = -\sum_{p=1}^{q-1} B(X_p, X_{q-p}). \tag{4}$$

In the preceding consideration, the vectors X_p were generated by the Maclaurin coefficients $x_{i,p}$ of a family of exact solutions to $F(X) = 0$. Now we assume that we have an arbitrary finite set Y_0, Y_1, $\ldots$, Y_q of vectors in $\mathbf{R}^m$. We call the expression

$$Y(t) = \sum_{p=0}^{q} Y_p t^p$$

an *approximate solution of degree* q to the system of polynomial equations $F(X) = 0$ if, for each $p = 1, 2, \ldots, q$, the coefficient of t^p in the Maclaurin expansion of the function $F(Y(t))$ is equal to zero. An equivalent formulation of this condition is as follows: for each $p = 1, 2, \ldots, q$, the equation

$$CY_p = -\sum_{l=1}^{p-1} B(Y_l, Y_{p-l})$$

holds.

Now we are ready to formulate sufficient conditions implying existence of an implicit function which is determined by a system of algebraic polynomial equations.

Theorem 1. *Let*

$$\sum_{p=0}^{q} Y_p t^p \tag{5}$$

be an approximate solution of degree q to a system of algebraic polynomial equations $F(X) = 0$. Suppose that there exists a number k $(0 \le k < q)$ such that, for all $i = 1, 2, \ldots, q$ and $j = k, k+1, \ldots, q$, the equation

$$CY = -B(Y_i, Y_j) - B(Y_j, Y_i)$$

has a solution which lies in the linear span of the vectors Y_k, Y_{k+1}, $\ldots Y_q$. Then $F(X) = 0$ has an analytic family of solutions $X(t) = \sum_{p=0}^{\infty} X_p t^p$ whose initial coefficients coincide with the corresponding coefficients of the approximate solution (5), i.e., $X_p = Y_p$ for each $p = 0, 1, \ldots, q$.

Proof. Denote by L the linear span of the vectors Y_k, Y_{k+1}, ..., Y_q.

Let P be the set of all nonnegative integers p for each of which there exists an approximate solution of degree $q+p$,

$$\sum_{l=0}^{q+p} X_l t^l, \tag{6}$$

to $F(X)=0$ such that (i) $X_l = Y_l$ for all $l=0,1,\ldots,q$ and (ii) $X_l \in L$ for each $l=q+1, q+2, \ldots, q+p$.

In view of the conditions of Theorem 1, $0 \in P$. Hence, $P \neq \emptyset$. Verify that P coincides with the set of all nonnegative integers, $\mathbf{N}$. It suffices to show that if $p \in P$ then $p+1 \in P$.

So, let $p \in P$ and let the approximate solution (6) satisfy (i) and (ii). To prove that $p+1 \in P$, it is sufficient to find a vector $X_{q+p+1} \in L$ satisfying the following system of linear algebraic equations:

$$CX_{q+p+1} = -\sum_{l=1}^{q+p} B(X_l, X_{q+p+1-l}). \tag{7}$$

According to (i) and (ii), each of the vectors $X_{q+1}, \ldots, X_{q+p}$ lies in L and, hence, each of the vectors $X_{q+1}, \ldots, X_{q+p}$ is a linear combination of the vectors Y_k, Y_{k+1}, ..., Y_q. Therefore, the right-hand side of (7) is a linear combination of the vectors $B(Y_i, Y_j) + B(Y_j, Y_i)$ with $1 \le i \le q$ and $k \le j \le q$. Then the conditions of Theorem 1 imply that there exists a solution to (7) which lies in L. Hence, $p+1 \in P$ and $P = \mathbf{N}$.

Thus, we see that $F(X)=0$ has approximate solutions of arbitrarily high degree whose initial coefficients coincide with the corresponding coefficients of the approximate solution (5). It remains to prove that these approximate solutions of arbitrarily high degree can be used to construct an exact solution in the form of a power series whose initial coefficients coincide with the corresponding coefficients of (5).

Our approach is based on the following algebraic theorem by M. Artin (see [?] and [?]): *Given a system of polynomial equations $f(x,y)=0$, where $f=(f_1,\ldots,f_k)$, $x=(x_1,\ldots,x_m)$, $y=(y_1,\ldots,y_n)$, there exists an integer $\beta=\beta(m,n,d,\alpha)$ (depending on m, n, on the total degree d of the polynomials f, and on a nonnegative integer α) such that if $f(x,\overline{y}(x)) = 0 \ (mod\, x^\beta)$, $\beta=\beta(m,n,d,\alpha)$, for some polynomial $\overline{y}(x)$ then the system $f(x,y)=0$ has a solution $y(x)$ that can be represented in the form of a convergent power series whose coefficients coincide with those of the polynomial $\overline{y}(x)$ up to the term x^α.*

To complete the proof of Theorem 1, we apply Artin's theorem to $F(X)=0$ in the following way. Put $\alpha = q$ and let β be an integer whose existence is provided by Artin's theorem. ¿From the above it follows that the approximate solution (5) can be extended to an approximate solution of an arbitrarily high degree, in particular, to an approximate solution of degree β. Now Theorem 1 directly follows from Artin's theorem.

We now discuss several examples of using Theorem 1.

Example 1. Let $F: \mathbf{R}^3 \to \mathbf{R}^3$ be given by (1), namely, let

$$F_1(t,x_1,x_2,x_3) \equiv x_1^2 + x_2^2 - x_3^2 - 1 = 0,$$

$$F_2(t,x_1,x_2,x_3) \equiv 3x_1 + x_2 - 3x_3 + 1 = 0,$$

$$F_3(t,x_1,x_2,x_3) \equiv x_1 - 3x_2 + x_3 + 3 = 0$$

and let $X_0 = (5,5,7)^{\mathrm{T}}$. Direct calculations show that

$$(\alpha^1_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}; \qquad (\beta^1_i) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}; \qquad \gamma^1 = -1;$$

$$(\alpha^2_{ij}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad (\beta^2_i) = \begin{pmatrix} 3 \\ 1 \\ -3 \end{pmatrix}; \qquad \gamma^2 = 1;$$

$$(\alpha^3_{ij}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad (\beta^3_i) = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}; \qquad \gamma^3 = 3;$$

$$B(X,Y) = (x_1y_1 + x_2y_2 - x_3y_3, 0, 0)^{\mathrm{T}};$$

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 1 & -3 \\ 1 & -3 & 1 \end{pmatrix}; \qquad C = \begin{pmatrix} 10 & 10 & -14 \\ 3 & 1 & -3 \\ 1 & -3 & 1 \end{pmatrix}; \qquad \det C = 0.$$

Solving the homogeneous system of linear algebraic equations $CX = 0$, we find that the vector $X_1 = (4,3,5)^{\mathrm{T}}$ constitutes a basis for the space of its solutions. Direct calculations show that $B(X_1, X_1) = (0,0,0)^{\mathrm{T}}$. Hence, we can put $X_q = 0$ for all $q \geq 2$. Thus, Theorem 1 can be applied with $q = 2$ and $k = 1$. It follows that X_0 is not an isolated solution to $F(X) = 0$. In the case under consideration, the corresponding family of solutions may be explicitly written as $X(t) = X_0 + tX_1$. Its geometrical sense becomes obvious if we observe that the equation $F_1(X) = 0$ determines a one-sheet hyperboloid while the pair of linear equations $F_2(X) = F_3(X) = 0$ determines its straight line generator which passes through the point X_0.

Example 2. Let $f : \mathbf{R}^2 \to \mathbf{R}^1$ be given by the formula $f(x_1,x_2) = x_1^3 - x_2^2$ and let $X_0 = (0,0)^{\mathrm{T}}$.

Transform the equation $f(X) = 0$ to a system of equations each of which is of degree 2:

$$\begin{aligned} F_1(x_1,x_2,x_3) &\equiv x_1x_3 - x_2^2 = 0, \\ F_2(x_1,x_2,x_3) &\equiv x_1^2 - x_3 = 0. \end{aligned} \tag{8}$$

Then we have $B(X,Y) = (\frac{1}{2}x_1y_3 - x_2y_2 + \frac{1}{2}x_3y_1, x_1y_1)^{\mathrm{T}}$,

$$C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Instead of solving the corresponding equations (4) step by step, we note that the equation $f(X) = 0$ has an obvious analytic family of solutions, $x_1 = t^2$, $x_2 = t^3$. Whence we immediately obtain $X_0 = X_1 = X_5 = X_6 = \ldots = (0,0,0)^{\mathrm{T}}$, $X_2 = (1,0,0)^{\mathrm{T}}$, $X_3 = (0,1,0)^{\mathrm{T}}$, $X_4 = (0,0,1)^{\mathrm{T}}$. Find the smallest values of q and k for which the hypotheses of Theorem 1 are fulfilled.

Direct calculations show that

$$B(X_1,X_i) + B(X_i,X_1) = (0,0)^{\mathrm{T}} \qquad \text{for all } i \geq 1,$$

$$B(X_2,X_i) + B(X_i,X_2) = (0,0)^{\mathrm{T}} \qquad \text{for } i = 3 \text{ or } i \geq 5,$$

$$B(X_3,X_i) + B(X_i,X_3) = (0,0)^{\mathrm{T}} \qquad \text{for all } i \geq 4,$$

$$B(X_4,X_i) + B(X_i,X_4) = (0,0)^{\mathrm{T}} \qquad \text{for all } i \geq 4,$$

$$B(X_2,X_2) = (0,1)^{\mathrm{T}}, \quad B(X_3,X_3) = (-1,0)^{\mathrm{T}},$$

$$B(X_2,X_4) + B(X_4,X_2) = (1,0)^{\mathrm{T}}.$$

Hence, the hypotheses of Theorem 1 are fulfilled for $q = k = 5$ and are not fulfilled for any smaller values of q and k. Thus, as soon as we obtain the approximate solution $X_0 + tX_1 + t^2X_2 + t^3X_3 + t^4X_4 + t^5X_5$, we can assert that it can be extended to an exact solution to (8). However, we cannot make the same assertion by using only the approximate solution $X_0 + tX_1 + t^2X_2 + t^3X_3 + t^4X_4$.

Now we give an example of a system of algebraic equations possessing an analytic family of solutions that cannot be obtained by Theorem 1 for any values of q and k.

Example 3. Let $F : \mathbf{R}^3 \to \mathbf{R}^2$ be given by the formulas

$$\begin{aligned} F_1(x_1,x_2,x_3) &\equiv x_1^2 + x_2^2 + x_3^2 - 4, \\ F_2(x_1,x_2,x_3) &\equiv (x_1 - 1)^2 + x_2^2 - 1 \end{aligned} \tag{9}$$

and let $X_0 = (2,0,0)^{\mathrm{T}}$. Direct calculations show that

$$(\alpha^1_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \qquad (\beta^1_i) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}; \qquad \gamma^1 = -4;$$

$$(\alpha^2_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad (\beta^2_i) = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}; \qquad \gamma^2 = 0;$$

$$B(X,Y) = (x_1y_1 + x_2y_2 + x_3y_3, x_1y_1 + x_2y_2)^{\mathrm{T}};$$

$$A = \begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}; \qquad C = \begin{pmatrix} 4 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

Note that $\operatorname{rank} C = 1$, $\operatorname{im} C = \{(\xi,\eta) \in \mathbf{R}^2 | \xi = 2\eta\}$, $\ker C = \{(u,v,w) \in \mathbf{R}^3 | u = 0\}$, and $\dim\ker C = 2$.

Obviously, (9) determines the Viviani curve and thus admits the following analytic family of solutions:

$$\begin{aligned} x_1(t) &= 1 + \cos t, \\ x_2(t) &= \sin t, \\ x_3(t) &= 2\sin(t/2), \\ X(t) &= (x_1(t), x_2(t), x_3(t)) = \sum_{p=0}^{\infty} t^p X_p. \end{aligned}$$

It is clear that each expression $\sum_{p=0}^{N} t^p X_p$ is an approximate solution of some degree to (9). Suppose it satisfies the hypotheses of Theorem 1 with some q and k.

¿From the proof of Theorem 1 it follows that, for $p \geq k$, the vector X_p is constructed as a linear combination of solutions to the equations

$$CX = -B(X_i, X_j) - B(X_j, X_i)$$

with $1 \leq i \leq q$ and $k \leq j \leq q$. Hence, each vector $B(X_2, X_p) + B(X_p, X_2)$ must lie in the image of the operator C, i.e., its first coordinate must be twice as large as its second coordinate. However, this condition is not satisfied, since $B(X_2, Y) + B(Y, X_2) = (-y_1, -y_1)^{\mathrm{T}}$ and infinitely many X_p have nonzero first coordinate.

Thus, example 3 shows that the conditions of Theorem 1 are not necessary for existence of an implicit function. This means that Theorem 1 cannot be used for proving that a given solution to a system of algebraic equations is isolated. In the next section we give several additional conditions under which the conditions of Theorem 1 are not only sufficient but also necessary for existence of an analytic family of solutions to a system of algebraic equations.

3 Necessary conditions for existence of an implicit function

A primary necessary condition for existence of a continuous family of solutions is known in the theory of bending of smooth surfaces at least since S. Cohn-Vossen [?]. For systems of algebraic equations it can be formulated as follows (we use the notation introduced in the previous section):

Theorem 2. *If zero is the only solution to the system $CX = 0$ then the system of algebraic equations $F(X) = 0$ has no nonconstant analytic family of solutions representable in the form of a convergent power series with prescribed initial term X_0.*

Proof is carried out by way of contradiction. Suppose that $F(X) = 0$ has a nonconstant analytic family of solutions which is represented in the form of a convergent power series

$$X(t) = \sum_{p=0}^{\infty} t^p X_p$$

and suppose that q is the smallest positive number such that $X_q \neq 0$. According to (4), X_q satisfies the following system of linear algebraic equations:

$$CX_q = -\sum_{p=1}^{q-1} B(X_p, X_{q-p}) = 0.$$

However, due to the hypotheses of Theorem 2, the last system has only zero solution. Hence, $X_q = 0$. This contradiction proves Theorem 2.

In the theory of bending of smooth surfaces some more advanced necessary conditions are also known (see, for example, [?], [?]). An algebraic version is stated in the following theorem:

Theorem 3. *If the system of algebraic equations $F(X) = 0$ and the vector X_0 are such that no approximate solution of the first degree $X_0 + tX_1$, with $X_1 \neq 0$, can be extended to an approximate solution of the second degree, then $F(X) = 0$ has no nonconstant analytic family of solutions representable in the form of a convergent power series with initial term X_0.*

Proof is carried out by way of contradiction. Suppose that $F(X) = 0$ has a nonconstant analytic family of solutions which is represented in the form of a convergent power series

$$X(t) = \sum_{p=0}^{\infty} t^p X_p$$

and suppose that q is the smallest positive number such that $X_q \neq 0$. Then X_q lies in the kernel of C and, according to the hypotheses of Theorem 3, it follows that $B(X_q, X_q)$ does not lie in the kernel of C. In view of (4), X_{2q} satisfies the following system of linear equations:

$$CX_{2q} = -\sum_{p=1}^{2q-1} B(X_p, X_{2q-p}) = -B(X_q, X_q).$$

Since $B(X_q, X_q)$ does not lie in the kernel of C, the last system has no solutions. This contradiction proves Theorem 3.

We now discuss several additional conditions under which the conditions of Theorem 1 are not only sufficient but also necessary for existence of an analytic family of solutions to a system of algebraic equations. The results we aim to obtain will generalize Theorems 2 and 3 in a sense. We start with the case in which there are few linearly independent vectors in the sequence $X_1, X_2, \ldots, X_q$ of coefficients of approximate solutions.

Theorem 4. *Suppose that the system of algebraic equations $F(X) = 0$ has an analytic family of solutions which is represented in the form of a convergent power series*

$$X(t) = \sum_{p=0}^{\infty} t^p X_p$$

and suppose that the vectors X_3 and X_4 belong to the linear span of the vectors X_1 and X_2. Then, for all $1 \leq i, j \leq 2$, the system of linear equations

$$CX = -B(X_i, X_j) - B(X_j, X_i)$$

has a solution that belongs to the linear span of the vectors X_1 and X_2.

Proof. The vectors X_1, X_2, X_3, and X_4 satisfy the following equations:

$$\begin{aligned}
&CX_1 = 0, \\
&CX_2 = -B(X_1, X_1), \\
&CX_3 = -B(X_1, X_2) - B(X_2, X_1), \\
&CX_4 = -B(X_1, X_3) - B(X_2, X_2) - B(X_3, X_1).
\end{aligned}$$

Let L denote the linear span of X_1 and X_2. The second equation implies $B(X_1,X_1) \in CL$. Since $X_3 \in L$, the third equation implies $B(X_1,X_2)+B(X_2,X_1) \in CL$. On the other hand, $X_3 \in L$ implies that $X_3 = c_3^1 X_1 + c_3^2 X_2$, with some reals c_3^1 and c_3^2. Hence, the fourth equation can be written as $CX_4 = -2c_3^1 B(X_1,X_1) - c_3^2[B(X_1,X_2)+B(X_2,X_1)] - B(X_2,X_2)$. Here CX_4 belongs to CL according to the hypotheses of Theorem 4, $B(X_1,X_1)$ and $B(X_1,X_2)+B(X_2,X_1)$ belong to CL in view of the above proof. Therefore, $B(X_2,X_2) \in CL$. This completes the proof of Theorem 4.

Theorem 5. *Suppose that the system of algebraic equations $F(X)=0$ has an analytic family of solutions which is represented in the form of a convergent power series*

$$X(t) = \sum_{p=0}^{\infty} t^p X_p$$

and suppose that the vectors X_4, X_5, X_6, and X_7 belong to the linear span of the vectors X_1, X_2, and X_3. Then, for all $1 \le i,j \le 3$, the system of linear equations

$$CX = -B(X_i,X_j) - B(X_j,X_i)$$

has a solution that belongs to the linear span of the vectors X_1, X_2, and X_3.

Proof. Let $\alpha \in \mathbf{R}$ be an arbitrary real. Change the variable $t = \tau + \alpha\tau^2$ in the solution $X(t) = \sum\limits_{p=0}^{\infty} t^p X_p$: $Y(\tau) \equiv X(\tau+\alpha\tau^2) = \sum\limits_{p=0}^{\infty} \tau^p Y_p$. Clearly, $Y(\tau)$ is an analytic family of solutions to the equation $F(Y)=0$ and, thus, the equality

$$CY_q = -\sum_{p=1}^{q-1} B(Y_p, Y_{q-p}) \tag{10}$$

holds for each $q \ge 1$.

On the other hand, Y_p can be written in terms of X_i by collecting similar terms in the expression

$$\sum_{p=0}^{\infty} \tau^p Y_p = \sum_{p=0}^{\infty} (\tau+\alpha\tau^2)^p Y_p.$$

We thus obtain

$$Y_0 = X_0;$$

$$Y_1 = X_1; \tag{11}$$

$$Y_2 = X_2 + \alpha X_1; \tag{12}$$

$$Y_3 = X_3 + 2\alpha X_2; \tag{13}$$

$$Y_4 = X_4 + 3\alpha X_3 + \alpha^2 X_2; \tag{14}$$

$$Y_5 = X_5 + 4\alpha X_4 + 3\alpha^2 X_3; \tag{15}$$

$$Y_6 = X_6 + 5\alpha X_5 + 6\alpha^2 X_4 + \alpha^3 X_3. \tag{16}$$

According to the hypotheses of Theorem 5, each of the vectors X_4, X_5, and X_6 belongs to the linear span of the vectors X_1, X_2, and X_3. It follows that $X_j = c_j^1 X_1 + c_j^2 X_2 + c_j^3 X_3$ for each $4 \le j \le 6$, with some reals c_j^i $(1 \le i \le 3,\ 4 \le j \le 6)$. Taking these formulas into account, we can write (14)–(16) in the following form:

$$\begin{aligned} Y_4 &= (c_4^3+3\alpha)X_3 + (c_4^2+\alpha^2)X_2 + c_4^1 X_1, \\ Y_5 &= (c_5^3+4\alpha c_4^3+3\alpha^2)X_3 + (c_5^2+4\alpha c_4^2)X_2 + (c_5^1+4\alpha c_4^1)X_1, \\ Y_6 &= (c_6^3+5\alpha c_5^3+6\alpha^2 c_4^3+\alpha^3)X_3 + (c_6^2+5\alpha c_5^2+6\alpha^2 c_4^2)X_2 \\ &\quad + (c_6^1+5\alpha c_5^1+6\alpha^2 c_4^1)X_1. \end{aligned} \tag{17}$$

Let L denote the linear span of X_1, X_2, and X_3.

For the case $q = 2$ equation (10) gives $CY_2 = -B(Y_1, Y_1)$. Taking into account (11) and (12), we conclude from here that $CX_2 + \alpha CX_1 = -B(X_1, X_1)$, and finally that $B(X_1, X_1) \in CL$.

For the case $q = 3$ equation (10) gives $CY_3 = -B(Y_1, Y_2) - B(Y_2, Y_1)$. Taking into account (12) and (13), we conclude from here that $CX_3 + 2\alpha CX_2 = -2\alpha B(X_1, X_1) - [B(X_1, X_2) + B(X_2, X_1)]$, and finally that $B(X_1, X_2) + B(X_2, X_1) \in CL$.

For the case $q = 4$, equation (10) yields $CY_4 = -B(Y_1, Y_3) - B(Y_2, Y_2) - B(Y_3, Y_1)$. Taking (14) and (17) into account, we conclude that $(c_4^3 + 3\alpha)CX_3 + (c_4^2 + \alpha^2)CX_2 + c_4^1 CX_1 = -\alpha^2 B(X_1, X_1) - 3\alpha[B(X_1, X_2) + B(X_2, X_1)] - [B(X_1, X_3) + B(X_3, X_1)] - B(X_2, X_2)$ and, finally,

$$[B(X_1, X_2) + B(X_2, X_1)] + B(X_2, X_2) \in CL. \tag{18}$$

Similarly, from (10), we obtain in the case $q = 5$,

$$[B(X_1, X_3) + B(X_3, X_1)] + [B(X_2, X_3) + B(X_3, X_2)] \in CL, \tag{19}$$

in the case $q = 6$,

$$\begin{aligned} &(c_5^3 + 4\alpha c_4^3 + 3\alpha^2)[B(X_1, X_3) + B(X_3, X_1)] \\ &+(2c_4^2 - 10\alpha^2 - 4\alpha c_4^3)B(X_2, X_2) \\ &+(c_4^3 + 3\alpha)[B(X_2, X_3) + B(X_3, X_2)] \in CL, \end{aligned} \tag{20}$$

and, in the case $q = 7$,

$$\begin{aligned} &(c_4^1 - \alpha c_4^2 + 6\alpha^4 + c_6^3 + 5\alpha c_5^3 + 8\alpha^2 c_4^3)[B(X_1, X_3) + B(X_3, X_1)] \\ &+(c_5^2 + 4\alpha c_4^2 - 2\alpha c_5^3 + 8\alpha^2 c_4^3 - 6\alpha^3)B(X_2, X_2) \\ &+(c_5^3 + 2\alpha c_4^3 + c_4^2 + 4\alpha^2)[B(X_2, X_3) + B(X_3, X_2)] \\ &+(c_4^3 + 3\alpha)B(X_3, X_3) \in CL. \end{aligned} \tag{21}$$

We may treat relations (18)–(21) as a system of algebraic linear equations with respect to the following four vector-valued variables: $B(X_1, X_3) + B(X_3, X_1)$, $B(X_2, X_2)$, $B(X_2, X_3) + B(X_3, X_2)$, and $B(X_3, X_3)$. The right-hand sides of the corresponding equations are some vectors in CL. Denote them by U_1, U_2, U_3, and U_4. It suffices to prove that the determinant of the system is not equal to zero: in this case, each of the four vector-valued variables may be represented as a linear combination of U_1, U_2, U_3, and U_4 and, thus, lies in CL.

Direct calculations show that the determinant of the system corresponding to (18)–(21) equals

$$\begin{aligned} &-6\alpha^4 + 18\alpha^3 + 20\alpha^2 c_4^3 + \alpha[2c_4^2 + (c_4^3)^2 - c_5^3] \\ &+[-c_4^1 + c_5^2 - c_4^2 c_4^3 - (c_4^3)^2 + 2c_4^3 c_5^3 - c_6^3]. \end{aligned}$$

Obviously, this polynomial in α cannot be equal to zero identically for any values of the coefficients c_j^i. Hence, we can find a value of α such that the determinant of the system corresponding to (18)–(21) do not vanish. This completes the proof of Theorem 5.

The above example 3 shows that, if the four vectors X_1, X_2, X_3, and X_4 are linearly independent, it may occur that some of the vectors $B(X_i, X_j) + B(X_j, X_i)$ do not lie in the image of C while the system $F(X) = 0$ determines some nonconstant implicit function. This means that there is no direct generalization of Theorems 4 and 5 to the case in which the four vectors X_1, X_2, X_3, and X_4 are linearly independent.

Obviously, this circumstance hampers proving that a system $F(X) = 0$ admits no analytic family of solutions. There are also some other obstacles in proving this. The first obstacle is of purely technical nature and consists in a considerable increase of calculations: if we find that a first-order approximate solution $X_0 + tX_1$ admits an extension $X_0 + tX_1 + t^2 X_2$ to some second-order approximate solution then, for every $\widetilde{X} \in \ker C$, the expression $X_0 + tX_1 + t^2(X_2 + \widetilde{X})$ also gives us some second-order approximate solution; hence, we need to study the possibility of extending a second-order approximate solution to a third-order approximate solution for some family of second-order solutions rather than for a single solution. Another reason is of more principle character. Suppose that we are able to make our way through the above-described increase of calculations and suppose we find a number N such that no first-order approximate solution $X_0 + tX_1$, $X_1 \in \ker\ C$, $X_1 \neq 0$, can be extended to any approximate solution of order N. Can we conclude that the system $F(X) = 0$ defines no implicit function in a neighborhood of the point X_0? No, we cannot! We have to verify that there is no such an extension either for $X_0 + t \cdot 0 + t^2 X_1$, or for $X_0 + t \cdot 0 + t^2 \cdot 0 + t^3 X_1$, or for any other approximate solution which has several vanishing initial coefficients (here $X_1 \in \ker\ C$, $X_1 \neq 0$). Example 2 shows that a system $F(X) = 0$ may admit an exact solution with several vanishing initial coefficients X_p. On the other hand, we have no estimation for the number of zero coefficients in the Maclaurin expansion of an implicit function defined by a system $F(X) = 0$. Hence, we have to study an infinite set of cases caused by "writing zeros" at initial positions of an approximate solution. Therefore, in general, we have no algorithm which can guarantee absence of an implicit function.

Nevertheless, below we will show that such an algorithm does exist in the case $\dim \ker\ C = 1$. First of all, we make the terminology more accurate. As before, let $F(X) = 0$ be a system of algebraic equations each of which is of degree 1 or 2. Let a bilinear operator B and a linear operator C be constructed by means of the system. Suppose that $\dim \ker\ C = 1$. In the domain of C fix a codimension-1 subspace T such that $T \cap \ker\ C = \{0\}$. A formal power series $X(t) = \sum_{p=0}^{\infty} t^p X_p$ is said to be a T*-standard formal solution* to $F(X) = 0$ if the following conditions are fulfilled:

1) $CX_q = -\sum_{p=1}^{q-1} B(X_p, X_{q-p})$ for each $q \geq 1$;

2) $X_1 \neq 0$;

3) $X_p \in T$ for each $p \geq 2$.

The following theorem plays a key role in our approach:

Theorem 6. *Let the system $F(X) = 0$ admit an exact nonconstant solution which can be represented in the form of a convergent power series $X(t) = \sum_{p=0}^{\infty} t^p X_p$, let $\dim \ker\ C = 1$, and let T be a codimension-1 subspace such that $T \cap \ker\ C = \{0\}$. Then $F(X) = 0$ admits a T-standard formal solution $Y(t) = \sum_{p=0}^{\infty} t^p Y_p$ such that $Y_0 = X_0$.*

To avoid interrupting our presentation, we prove Theorem 6 at the end of this section.

The coefficients Y_p of a T-standard formal solution to $F(X) = 0$ can be found as solutions to the following system of linear algebraic equations: $CY_p = -\sum_{l=1}^{p-1} B(Y_l, Y_{p-l})$. The condition $Y_p \in T$ implies that the solution Y_p is unique (if existent, of course) and thus no increase of calculations occurs. On the other hand, if, for some p, a solution Y_p does not exist then there is no T-standard formal solution to $F(X) = 0$. According to Theorem 6, this implies that $F(X) = 0$ has no exact nonconstant solution in the form of a convergent power series (with arbitrarily many vanishing initial coefficients). Thus, we have a finite algorithm which, in some cases, can guarantee absence of an implicit function determined by $F(X) = 0$ in a neighborhood of the point X_0. We present a test example of executing the algorithm proposed.

Example 4. Let $F : \mathbf{R}^3 \to \mathbf{R}^3$ be given by the formulas

$$\begin{aligned} F_1(x_1, x_2, x_3) &\equiv x_1^2 + x_2^2 + x_3^2 - 4, \\ F_2(x_1, x_2, x_3) &\equiv (x_1 - 3)^2 + x_2^2 - 1, \\ F_3(x_1, x_2, x_3) &\equiv x_2 \end{aligned}$$

and let $X_0 = (2, 0, 0)^{\mathrm{T}}$. It is clear that the equation $F_1 = 0$ defines a sphere in $\mathbf{R}^3$ while $F_2 = 0$ defines a cylinder which has a single common point with the sphere, namely, the point X_0. Thus $F(X) = 0$ defines no implicit function. Demonstrate how Theorem 6 can be used to reach the same conclusion.

Direct calculations show that

$$(\alpha^1_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \qquad (\beta^1_i) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}; \qquad \gamma^1 = -4;$$

$$(\alpha^2_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad (\beta^2_i) = \begin{pmatrix} -6 \\ 0 \\ 0 \end{pmatrix}; \qquad \gamma^2 = -1;$$

$$(\alpha^3_{ij}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad (\beta^3_i) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \qquad \gamma^3 = 0;$$

$$B(X,Y) = (x_1y_1 + x_2y_2 + x_3y_3, x_1y_1 + x_2y_2, 0)^{\mathrm{T}};$$

$$A = \begin{pmatrix} 0 & 0 & 0 \\ -6 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \qquad C = \begin{pmatrix} 4 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Note that $\operatorname{rank} C = 2$, $\operatorname{im} C = \{(\xi, \eta, \zeta) \in \mathbf{R}^3 | \xi = 2\eta\}$, $\ker C = \{(u, v, w) \in \mathbf{R}^3 | u = v = 0\}$, $\dim \ker C = 1$, $X_1 = (0, 0, 1)$, and $B(X_1, X_1) = (1, 0, 0)^{\mathrm{T}} \notin \operatorname{im} C$.

The latter relation implies that the approximate solution $X_0 + tX_1$ cannot be extended to any approximate solution of the second order. Hence, there is no T-standard formal solution to $F(X) = 0$. By Theorem 6, $F(X) = 0$ defines no implicit function in a neighborhood of X_0.

Proof of Theorem 6. Let N be the least positive integer p such that $X_p \neq 0$. Let q be the greatest integer such that the exact nonconstant solution to $F(X) = 0$ (which, according to the claims of Theorem 6, is representable in the form of a convergent power series $X(t) = \sum_{p=0}^{\infty} t^p X_p$) possesses the following properties: (i) $X_p = 0$ for all $0 < p \leq q$, $p \neq 0 \pmod N$ and (ii) $X_p \in T$ for all $0 < p \leq q$, $p \neq N$, $p = 0 \pmod N$.

We will verify that there exists a polynomial change of variables $t = t(\tau)$ such that the new exact nonconstant solution $\widetilde{X}(\tau) \equiv X(t(\tau))$ to $F(X) = 0$, which is representable in the form of a convergent power series $\widetilde{X}(\tau) = \sum_{p=0}^{\infty} \tau^p \widetilde{X}_p$, possesses properties (i) and (ii) for $q+1$ and is such that $\widetilde{X}_0 = X_0$, $\widetilde{X}_N = X_N$.

In other words, we will prove that, by means of polynomial changes of variable t, the coefficients X_p can be transformed one after another into the zero vector if p is not divisible by N and into some vectors lying in T if p is divisible by N in such a manner that X_0 and X_N remain unchanged and, after such a transformation, we again obtain an exact solution to $F(X) = 0$ representable in the form of a convergent power series. Accomplishing infinitely many such polynomial changes of variable t and not controlling the radii of convergence of power series which appear in this process, we obtain a formal power series $Z(\tau) = \sum_{p=0}^{\infty} \tau^p Z_p$ whose coefficients Z_p $(p = 0, 1, \ldots)$ possess the following properties:

(a) $Z_0 = X_0$;
(b) $Z_p = 0$ for all $p \neq 0 \pmod N$;
(c) $Z_N = X_N \neq 0$;
(d) $Z_p \in T$ for all $p > 0$, $p = 0 \pmod N$;
(e) $CZ_q = -\sum_{p=1}^{q-1} B(Z_p, Z_{q-p})$ for every $q \geq 1$.

Finally, executing the change of variable $t = \tau^N$ in the formal power series $Z(\tau)$, we obtain a T-standard formal solution $Y(t)$ whose existence is asserted in Theorem 6.

So, to complete the proof of Theorem 6, it remains to prove that a solution $X(t)$ possessing properties (i) and (ii) for some q, can be transformed into a solution $\widetilde{X}(\tau)$ possessing properties (i) and (ii) for $q+1$ and satisfying $\widetilde{X}_0 = X_0$, $\widetilde{X}_N = X_N$.

Put $q = iN + j$, where $0 \le j \le N-1$. Consider a change of variable $t = \tau + \alpha\tau^{q+1-N}$ (here τ is a new variable and α is a constant whose value will be specified later):

$$\begin{aligned}\widetilde{X}(\tau) &= X(\tau + \alpha\tau^{q+1-N}) \\ &= X_0 + (\tau + \alpha\tau^{q+1-N})^N X_N + (\tau + \alpha\tau^{q+1-N})^{2N} X_{2N} + \cdots \\ &\qquad + (\tau + \alpha\tau^{q+1-N})^{iN} X_{iN} + (\tau + \alpha\tau^{q+1-N})^{iN+j} X_{iN+j} + \cdots \\ &= X_0 + \tau^N X_N + \tau^{2N} X_{2N} + \cdots + \tau^{iN} X_{iN} + \tau^q (X_q + N\alpha X_N) + \cdots .\end{aligned}$$

First, consider the case $0 < j \le N-1$. We know that X_p is a solution to the following system of linear algebraic equations:

$$CX_q = -\sum_{r=1}^{iN+j} B(X_r, X_{iN+j-r}). \tag{22}$$

If $r = 0 \pmod N$ then $iN + j - r = j \pmod N$ and, in particular, $iN + j - r \ne 0 \pmod N$. Hence, for arbitrary $1 \le r \le iN + j - 1$, either $X_r = 0$ or $X_{iN+j-r} = 0$. Therefore, the right-hand side of (22) equals zero and $X_q \in \ker C$. On the other hand, $X_N \in \ker C$ and $\dim \ker C = 1$. Consequently, the vectors X_q and X_N are collinear. Since $X_N \ne 0$, there exists α such that $X_q + \alpha N X_N = 0$. Under such a choice of α, the exact solution $\widetilde{X}(\tau) = X(\tau + \alpha\tau^{q+1-N})$ possesses properties (i) and (ii) for $q+1$ as well as $\widetilde{X}_0 = X_0$ and $\widetilde{X}_N = X_N$.

Now consider the case $j = 0$. In this case, if $r = sN$ $(0 \le s \le i)$ then $iN + j - r = (i-s)N$. Hence, (22) can be rewritten as

$$CX_q = -\sum_{s=1}^{i} B(X_{sN}, X_{(i-s)N}).$$

Generally speaking, the right-hand side of the last expression does not equal zero. So, in general, X_q does not lie in $\ker C$. Nevertheless, using the fact that the linear span of the subspaces T and $\ker C$ coincides with the whole space, we can find α such that $X_q + \alpha N X_N \in T$. Under such a choice of α, the exact solution $\widetilde{X}(\tau) = X(\tau + \alpha\tau^{q+1-N})$ possesses properties (i) and (ii) for $q+1$ as well as $\widetilde{X}_0 = X_0$ and $\widetilde{X}_N = X_N$.

Thus, the possibility has been proven of transferring the solution $X(t)$ into a solution $\widetilde{X}(\tau)$ which possesses properties (i) and (ii) as well as $\widetilde{X_0} = X_0$, $\widetilde{X}_N = X_N$.

This completes the proof of Theorem 6.

4 Applications to studying flexible polyhedra and frameworks

Let K be a simplicial complex whose body is an $(n-1)$-dimensional connected compact topological manifold without boundary. A *polyhedron* in the n-dimensional Euclidean space $\mathbf{R}^n$ is, by definition, a continuous mapping $f : K \to \mathbf{R}^n$ which is linear on each simplex. Sometimes, the image of K under f is also referred to as a polyhedron. By a *polyhedral sphere* in $\mathbf{R}^n$ we mean a polyhedron $f : K \to \mathbf{R}^n$ with the body of K homomorphic to the sphere.

We say that a polyhedron has no self-intersections if the mapping f is (globally) injective. In the present article, we consider polyhedra both with and without self-intersections.

Definition. A polyhedron $P = f(K)$ is *flexible* if there exists a family of polyhedra $P_t = (f_t, K)$, $0 \le t \le 1$, which is analytic with respect to the parameter t and for which the following conditions are satisfied:

1) $P = P_0$;

2) for arbitrary 0-dimensional simplices v_j and v_k of K belonging to a 1-dimensional simplex of K, the equality $|f(v_j) - f(v_k)| = |f_t(v_j) - f_t(v_k)|$ holds for all $0 \le t \le 1$ (henceforth $|y|$ stands for the Euclidean norm of a vector $y = (y_1, y_2, \ldots, y_n) \in \mathbf{R}^n$, i.e., $|y|^2 = y_1^2 + y_2^2 + \ldots + y_n^2$);

3) there exist two 0-dimensional simplices v_j and v_k of K which do not belong to any 1-dimensional simplex of K and for which the expression $|f_t(v_j) - f_t(v_k)|$ is not constant in $t \in [0, 1]$.

A family P_t with the above properties 1)–3) is a *nontrivial flexion* of P. Note that the simplicial complex K remains constant during the flexion.

In other words, a polyhedron is flexible if its spatial shape can be changed analytically with respect to a parameter (see condition 3)) without changing its intrinsic metrics (see condition 2)). Incidentally, the analyticity requirement with respect to the parameter can be substantially weekend. Namely, in [?] it is shown that if there exists a deformation of a polyhedron which is continuous with respect to a parameter and obeys the above conditions 1)–3) then there also exists a deformation of the polyhedron which is analytic with respect to (possibly) another parameter and obeys conditions 1)–3).

During the last 25 years, the following two remarkable results were obtained in the theory of flexible polyhedra: in 1977 R. Connelly gave an example of a flexible polyhedral sphere in $\mathbf{R}^3$ without self-intersections [?], and in 1996 I. Kh. Sabitov published a complete proof of the statement asserting that each flexible polyhedron (even with self-intersections) in $\mathbf{R}^3$ preserves the (oriented) volume bounded by it during the flexion [?]. The last statement was known during several decades as the "Bellows conjecture". Other proofs of this statement may be found in [?], [?], and [?].

We are interested in the problem of "practical" recognition whether a given polyhedron $f : K \to \mathbf{R}^n$ is flexible or not. It is sufficient to describe a flexion $f_t : K \to \mathbf{R}^n$ of the polyhedron via motions of its 0-dimensional simplices v_j $(1 \le j \le N)$: $x_j(t) = f_t(v_j) \in \mathbf{R}^n$. Furthermore, the above conditions 1)–3) can be reformulated as follows: 1') the vectors $x_j(0)$ are given; 2') if 0-dimensional simplices v_i and v_j are joint together in K by a 1-dimensional simplex then the equality $|x_i(t) - x_j(t)|^2 = |x_i(0) - x_j(0)|^2$ holds true for all $0 \le t \le 1$; 3') there exist 0-dimensional simplices v_i and v_j that are not joint together in K by a 1-dimensional simplex and such that the expression $|x_i(t) - x_j(t)|^2$ is not constant in $t \in [0, 1]$.

In other words, the problem of whether a given polyhedron is flexible or not is equivalent to the problem of whether the set of vectors $x_j = x_j(0)$, $j = 1, 2, \ldots, N$, is an isolated solution to the system of algebraic equations

$$|x_i - x_j|^2 = |x_i(0) - x_j(0)|^2 \tag{23}$$

or this system defines an implicit function $x_j = x_j(t)$ in a neighborhood of the point $x_j = x_j(0)$, $j = 1, 2, \ldots, N$. It is worth bearing in mind that we are not interested in motions of $f(K)$ as a rigid body in $\mathbf{R}^n$, i.e., we are only looking for solutions with properties 3) or 3'). This requirement can be easily satisfied in the following manner. Fix an $(n-1)$-dimensional simplex of K. Suppose its 0-dimensional simplices are denoted by $v_1, v_2, \ldots, v_n$. We agree that $f_t(v_1)$ lies at the origin during the course of deformation of the polyhedron (and, thus, always has zero coordinates); $f_t(v_2)$ always lies on the first coordinate axis in $\mathbf{R}^n$ (and, thus, all but the first coordinates of it vanish identically); $f_t(v_3)$ always lies in the 2-dimensional plane spanned by the first and second coordinate axis in $\mathbf{R}^n$ (and, thus, all but the first and second coordinates of it vanish identically); and so on. This construction reduces the number of independent variables in (23) but does not preclude the above reduction of the decision problem of whether the polyhedron is flexible to the problem of whether the given solution to the system of algebraic equations is isolated or the system defines an implicit function in a neighborhood of the solution.

The decision problem of whether a framework is flexible in $\mathbf{R}^n$ can be reformulated in a similar way.

A framework in $\mathbf{R}^n$ is a connected graph whose vertices are points in $\mathbf{R}^n$ and edges are straight line segments joining some of its vertices. It is conventional to call the vertices of a framework *joints* and straight line segments *bars*. The set of 0- and 1-dimensional faces of a polyhedron in $\mathbf{R}^n$ can be considered as a typical example of a framework.

A framework in $\mathbf{R}^n$ is said to be *flexible* if it admits a nontrivial analytic deformation, i.e., the positions of its joints can be changed in $\mathbf{R}^n$ analytically with respect to a parameter in such a way that the length of each bar remains constant while the distance between some two joints (which are not joint together by a bar) is not constant.

Sometimes, it is also useful to study so-called *pinched frameworks*, i. e. such frameworks that the spatial positions of some of their joints are fixed and should not be changed during deformations.

As we have just mentioned above, a suitable framework can be associated with any polyhedron which consists of the set of its 0- and 1-dimensional faces. This framework is called a *1-skeleton* of the polyhedron. Obviously, a polyhedron is flexible if and only if its 1-skeleton is flexible (the reason is that, according to our definition, each face of a polyhedron is a simplex). Hence, the decision problem of whether a given polyhedron is flexible or not is a particular case of the decision problem of whether a given framework is flexible or not. So, we will focus our attention on the latter problem.

The spatial position of a framework is determined as soon as the positions of its joints $x_i(0) \in \mathbf{R}^n$ are given. The decision problem of whether a given framework is flexible or not is, obviously, equivalent to the problem of whether the set of vectors $x_i(0)$ is an isolated solution to (23) or defines an implicit function in a neighborhood of $x_i(0)$. Furthermore, is necessary to exclude trivial deformations when the framework moves as a rigid body. This can be done in the same way as it was previously done for polyhedra. This problem does not appear for pinched frameworks at all, since they often do not permit any trivial deformations.

In Section 2, we introduced the notion of an approximate solution of degree q to a system of polynomial equations. As applied to system (23) associated with a framework, the term "infinitesimal flexion of order q" is conventionally used. More precisely, let a framework in $\mathbf{R}^n$ be determined by the positions $x_i(0)$, $i = 1, 2, \ldots, N$, of its joints. The set of vectors $x_{i,p}$, $i = 1, 2, \ldots, N$, $p = 1, 2, \ldots, q$ is said to be an *infinitesimal flexion of order* q of the framework if

$$\left| \sum_{p=1}^{q} x_{i,p} t^p - \sum_{p=1}^{q} x_{j,p} t^p \right| = 0 \pmod{t^q}$$

for all indices i and j such that the joints $x_i(0)$ and $x_j(0)$ are joint by a bar. An infinitesimal flexion is said to be *trivial* if it is an initial part of the Taylor expansion of the trajectories of $x_i(0)$ under the action of some one-parameter group of isometries of $\mathbf{R}^n$.

A framework is said to be *qth-order infinitesimally flexible* if it admits a nontrivial infinitesimal flexion of order q. Otherwise it is called *qth-order infinitesimally rigid.*

Roughly speaking, the following theorem asserts that if a framework admits a "regular" infinitesimal flexion of sufficiently large order then it is flexible.

Theorem 7. *Let a framework P be qth-order infinitesimally flexible and let $\sum_{p=0}^{q} X_p t^p$ be a nontrivial infinitesimal flexion of order q. Let operators B and C be constructed for system (23) which is associated with P and let there exist a number k $(0 \leq k < q)$ such that, for every $i = 1, 2, \ldots, q$ and every $j = k, k+1, \ldots, q$, the equation*

$$CX = -B(X_i, X_j) - B(X_j, X_i)$$

has a solution lying in the linear span of the vectors $X_k, X_{k+1}, \ldots, X_q$. Then P is flexible.

Proof of Theorem 7 follows immediately from Theorem 1.

All flexible octahedra in $\mathbf{R}^3$ were classified by R. Bricard [?] (see also [?]). In [?], it is shown (in slightly different terms) that the conditions of Theorem 7 hold true with $q = 5$ and $k = 1$ for the 1-skeleton of so-called Bricard's flexible octahedra of the first type.

The idea to use infinitesimal rigidity (of some order) of a framework for proving its rigidity is explored for a long time and is based, first of all, on the following

Theorem 8. *Every 1-order infinitesimally rigid framework in $\mathbf{R}^n$ is rigid.*

Proof follows immediately from Theorem 2 and from the above reduction of the decision problem of whether a framework is flexible to the problem of whether a given solution $x_j = x_j(0)$, $j = 1, \ldots, N$, to system (23) of algebraic equations is isolated or the system defines an implicit function in a neighborhood of the solution.

Theorem 8 is one of the corner stones in problems of existence and uniqueness for convex polyhedra in the way of exposition which is used in the classical book [?]. Among recent papers that use Theorem 8, we indicate the articles [?] and [?] where it is shown that both in $\mathbf{R}^2$ and $\mathbf{R}^3$ there exist rigid triangle-free frameworks with all bars having length 1. Other statements about interrelations between rigidity and infinitesimal rigidity may be found in [?], [?], [?], [?], [?], [?], [?].

The following theorem was proven for the first time (by different methods) in [?]. A similar theorem for smooth surfaces was obtained in [?].

Theorem 9. *Every 2-order infinitesimally rigid framework in $\mathbf{R}^n$ is rigid.*

Proof of Theorem 9 ensues immediately from Theorem 3.

The following theorem generalizes Theorems 8 and 9.

Theorem 10. *Let a framework K in $\mathbf{R}^n$ have a single nontrivial linearly independent first-order infinitesimal flexion and let there exist a number $q \geq 1$ such that K is qth-order infinitesimally rigid. Then K is rigid.*

Proof. Eliminate trivial motions as was described above. The kernel of the operator C which is associated with (23) has dimension 1. Put $T = (\ker C)^{\perp}$. According to Theorem 6, (23) has a T-standard formal solution $Y(t) = \sum_{p=0}^{+\infty} Y_p t^p$ such that $Y_0 = K$ and $K + Y_1 t$ is a nontrivial first-order infinitesimal flexion. Furthermore, $Y(t) = \sum_{p=0}^{q} Y_p t^p$ is a nontrivial qth-order infinitesimal flexion. However, this contradicts the hypothesis of Theorem 10 according to which K is qth-order infinitesimally rigid. This contradiction proves Theorem 10.

For smooth surfaces, a theorem similar to Theorem 10 was obtained (by different methods) in [?]. For some other results about interrelations between higher-order infinitesimal rigidity and rigidity see, for example, [?], [?], and [?] (for frameworks) and [?] and [?] (for smooth surfaces).

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