An exact method

for the *(r|p)*–centroid problem

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(r|p)–centroid problem

- Input:

  J is the set of users;

  I is the set of potential facilities;

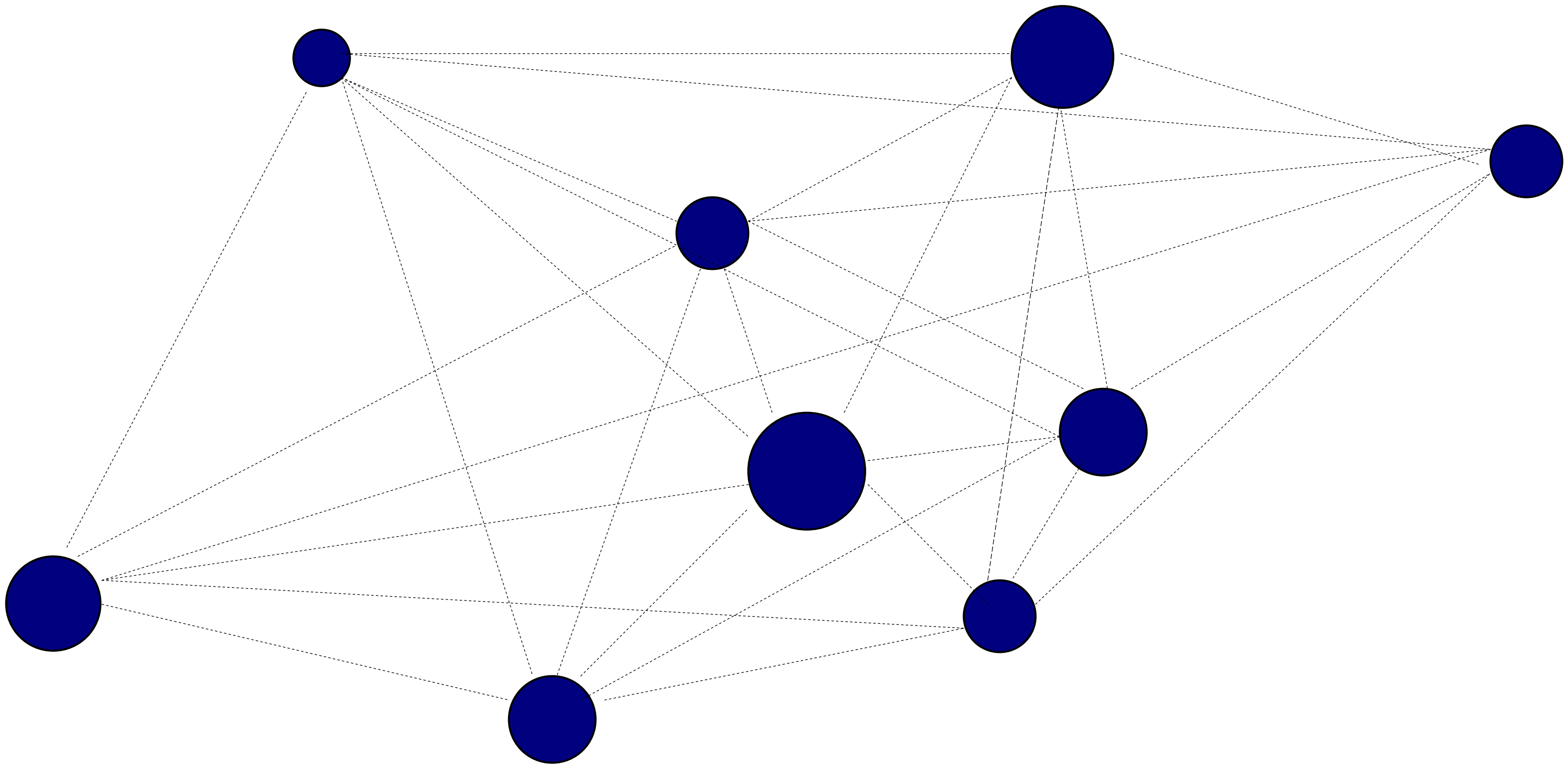
  p is the total number of facilities opened by the Leader;

  r is the total number of facilities opened by the Follower;

  w_j is the profit for servicing of the user j ;

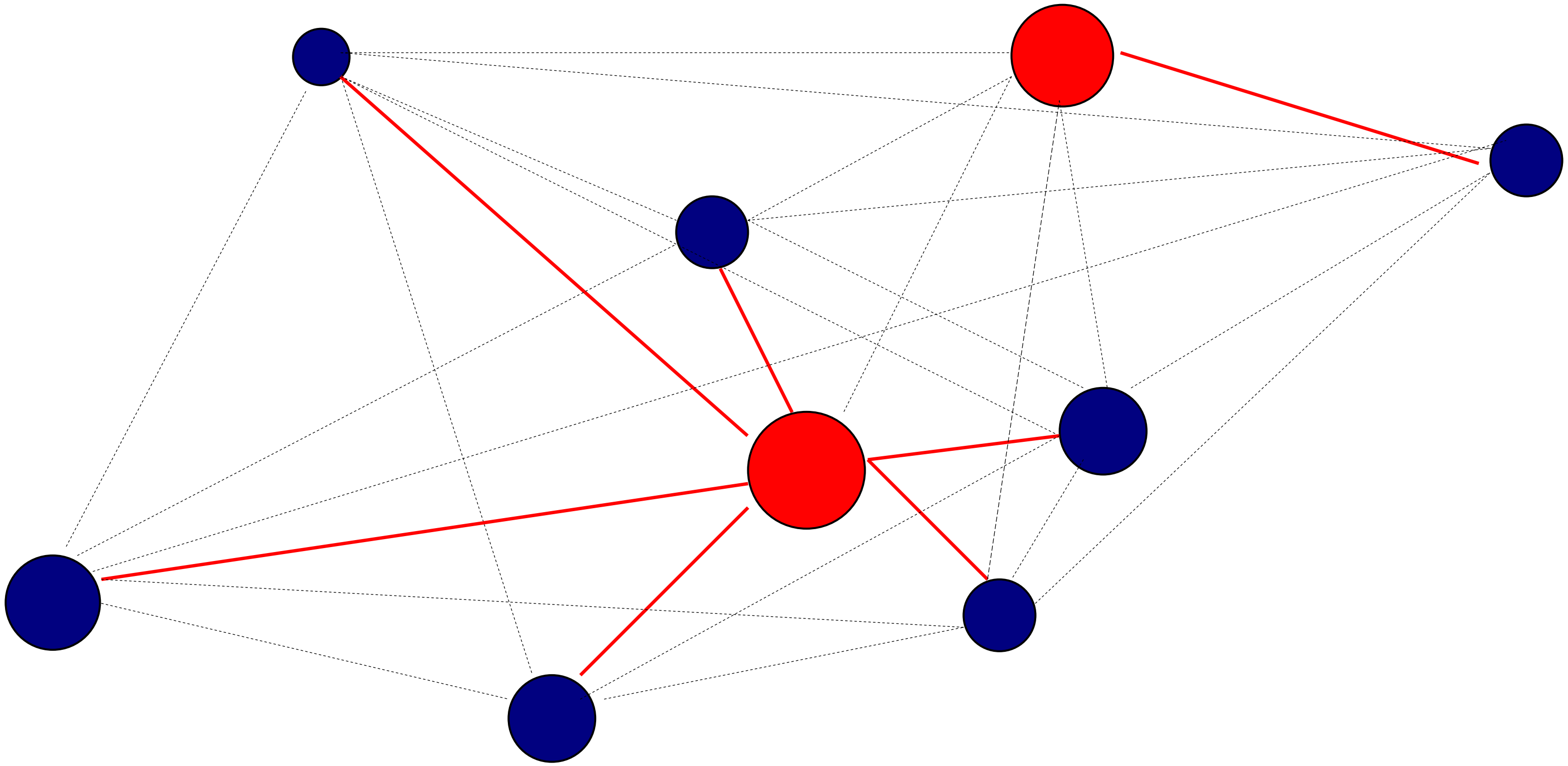
  g_{ij} is the distance between the user j and the facility i;
- Output: p facilities opened by the Leader;
- Goal: maximize the total profit for the Leader.

Example



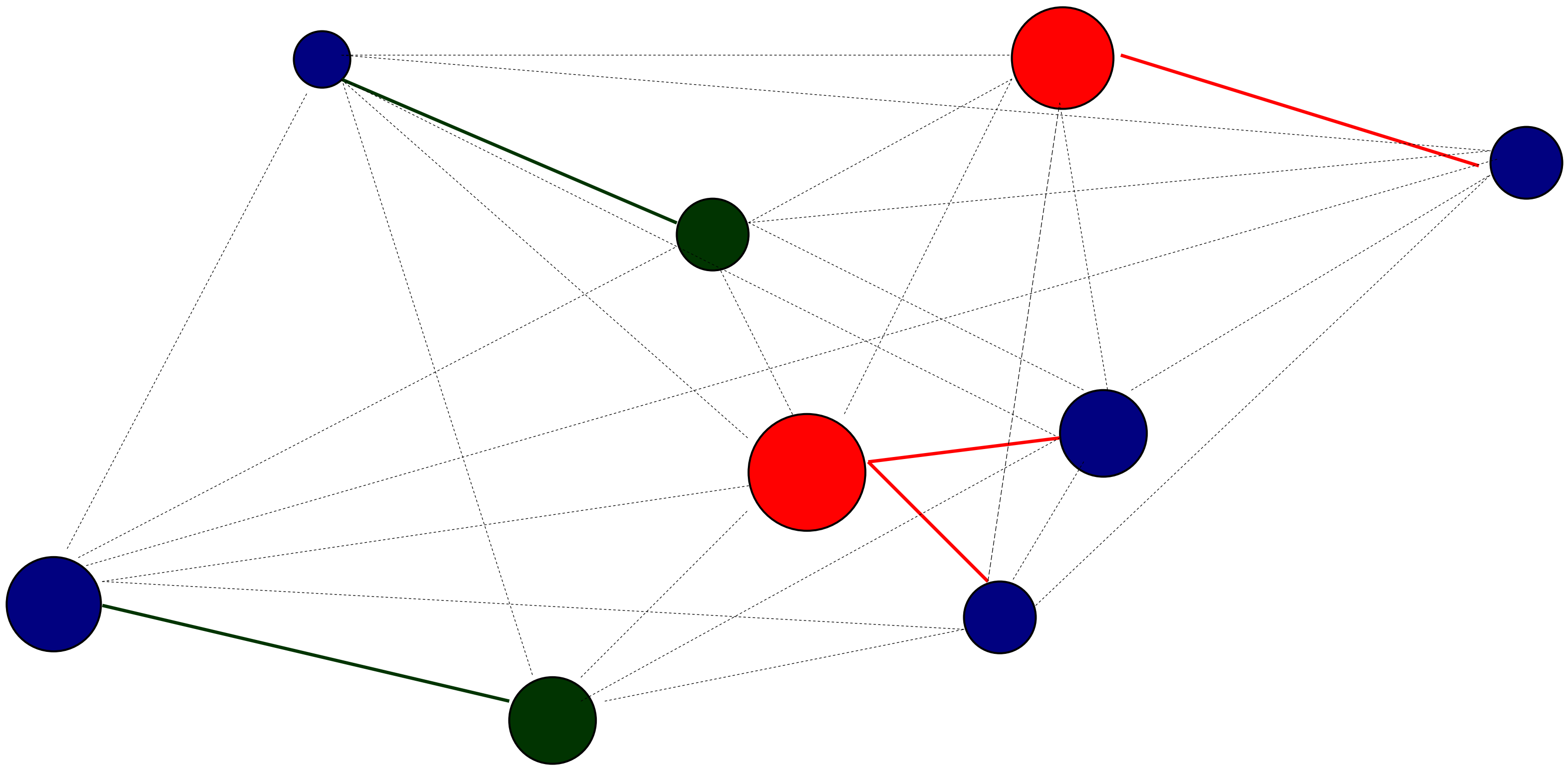
$I = J,\ |I| = 9$

Leader has opened p facilities.
Leader's market share is 100%.



I=J, $|I|$=9, p=2

Follower has opened r facilities.
Leader's market share is 56%.



$I=J,\ |I|=9,\ p=r=2$

Mathematical Formulation

Leader Variables
$$x_i = \begin{cases} 1, & \textit{if the Leader opens facility } i, \\ 0, & \textit{otherwise,} \end{cases}$$

Follower Variables
$$y_i = \begin{cases} 1, & \textit{if the Follower opens facility } i, \\ 0, & \textit{otherwise,} \end{cases}$$

User Variables
$$u_j = \begin{cases} 1, & \textit{if user } j \textit{ is serviced by the Leader,} \\ 0, & \textit{if user } j \textit{ is serviced by the Follower.} \end{cases}$$

For the given solution x_i, $i \in I$ we define the set of facilities

$$I_j(x) = \{ i \in I \mid g_{ij} < \min_{k \in I} (g_{kj} \mid x_k = 1) \}$$

which allows to “capture” the user j by the Follower.

Bilevel 0-1 Model

$$\max_{x} \sum_{j\in J} w_j u_j^*(x,y^*)$$

$$\text{s.t.} \quad \sum_{i\in I} x_i = p, \quad x_i \in \{0,1\},\ i\in I$$

where $u_j^*(x,y^*), y_i^*$ is the optimal solution of the Follower problem:

$$\max_{u_j, y_i} \sum_{j\in J} w_j (1-u_j)$$

$$\text{s.t.} \quad 1-u_j \le \sum_{i\in I_j(x)} y_i,\ j\in J$$

$$\sum_{i\in I} y_i = r$$

$$y_i,\ u_j \in \{0,1\},\ i\in I,\ j\in J$$

Complexity Status

<table>
<tr><td rowspan="4">(r|p)-centroid</td><td colspan="2">NP-hard, S. Hakimi,1990</td></tr>
<tr><td colspan="2">$\sum_2^p$-hard on graph,
H.Noltemeier,J.Spoerhase, H.Wirth,2007</td></tr>
<tr><td>NP-hard on spider</td><td rowspan="4">J.Spoerhase,
H.-C.Wirth,2008</td></tr>
<tr><td>$O(pn^4)$ on path</td></tr>
<tr><td rowspan="2">(1|p)-centroid</td><td>$O(n^2(\log n)^2 \log W)$ on tree</td></tr>
<tr><td>NP-hard on pathwidth bounded graph</td></tr>
<tr><td>(1|1)-centroid</td><td colspan="2">polynomial solvable on graph and on a network
P. Hansen, M. Labbé, 1988</td></tr>
</table>

Computational Methods

- ✓ *Tabu search algorithm*, $|I|=|J|=70, p, r \le 3$

S. Benati, G. Laporte, 1994

- ✓ *An alternating heuristic on the plane*, $|J| \le 100, p, r \le 25$

J. Bhadury, H. A. Eiselt, J. H. Jaramillo, 2001

- ✓ *Hybrid memetic algorithm,* $|I|=|J|=100, p=r \le 10$

E. Alekseeva, N. Kochetova, Y. Kochetov, A. Plyasunov, 2009

- ✓ *The partial enumeration algorithm*, $|I| \le 50, |J| \le 100,\ p, r \le 5$

C.M.C. Rodríguez, J.A.Moreno Pérez, 2008

- ✓ *Three MIP models*, $|I|=|J| \le 25,\ r=1,\ p \ge 1$ (arbitrary)

F. Plastria, L.Vanhaverbeke, 2008

Main Results

New reformulation as Integer Linear Program

An exact algorithm

Computational experiments on the large scale instances

Notations

Let F be the set of all feasible solutions of the Follower.

For $y \in F$ define $I_j(y) = \{i \in I \mid g_{ij} < \min_{k \in I}(g_{kj} \mid y_k = 1)\}, \ j \in J$

the set of the Leader's facilities which allows the Leader to keep client *j* if the Follower will use the solution y.

Introduce new variables:

$$u_j^y = \begin{cases} 1, \textit{if client } j \textit{ is served by the Leader when the Follower} \\ \textit{uses solution } y \\ 0, \textit{if user } j \textit{ is serviced by the Follower when the Follower} \\ \textit{uses solution } y \end{cases}$$

Integer Linear Program

$$\max_{UB,x,u} UB$$

s.t. $$\sum_{i \in I} x_i = p$$

$$\sum_{j \in J} w_j u_j^y \geq UB, \; y \in F$$

$$u_j^y \leq \sum_{i \in I_j(y)} x_i, \; j \in J, \; y \in F$$

$$u_j^y \in \{0,1\}, \; j \in J, \; y \in F$$

$$x_i \in \{0,1\}, \; i \in I$$

Column Generation Method for *(r|p)*-centroid problem

1. Choose an initial family F
2. Find $UB(F)$ and $x(F)$
3. Solve the Follower problem and calculate $LB(F)$
4. If $UB(F) = LB(F)$ then stop
5. Add $y(F)$ in the family F go to the step 2.

Computational Experiments

http://math.nsc.ru/AP/benchmarks/english.html

The sets $I = J$, $|I|=|J|$

The element g_{ij} is an Euclidean distance between points $i \in I$ and $j \in J$, the points are randomly generated following the uniform distribution on a 7000*7000 square

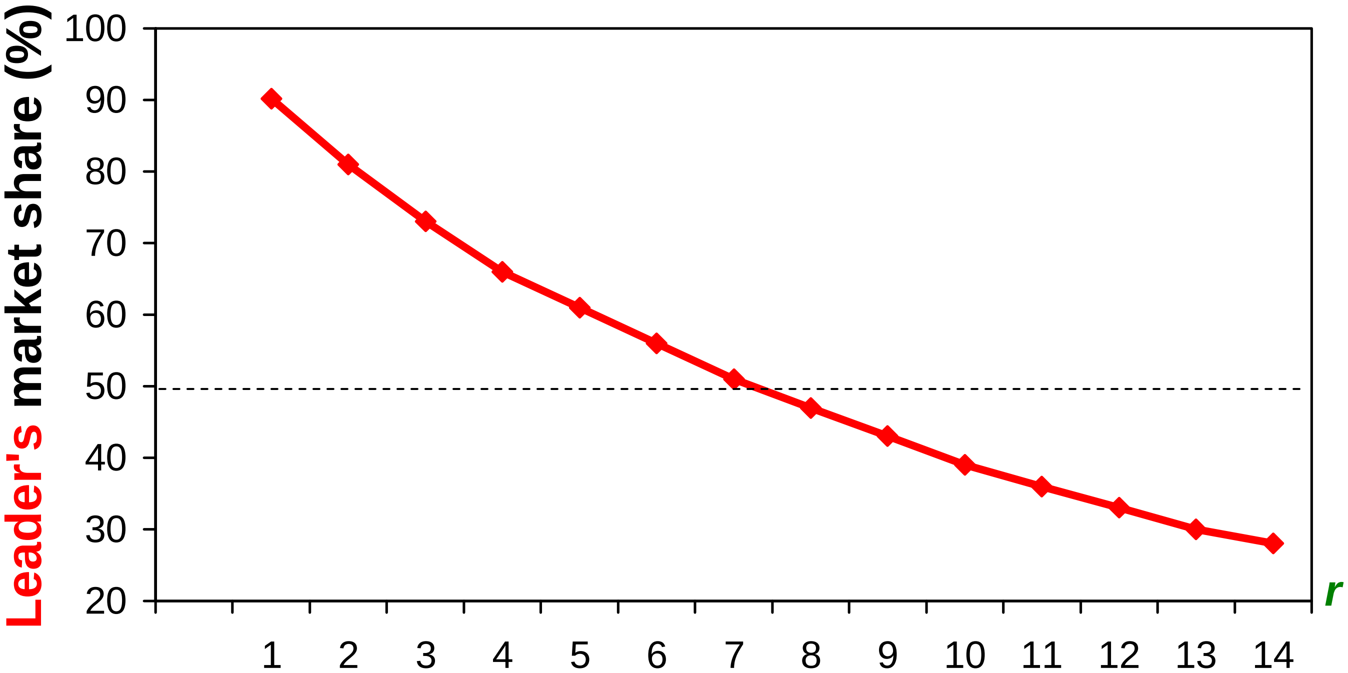
The profit w_j equals one for all $j \in J$ or w_j, $j \in J$ is randomly generated following the uniform distribution on a $(0,200)$ interval

PC Pentium Intel Core 2, 1.87 GHz, RAM 2Gb, Windows XP Professional operating system, GAMS

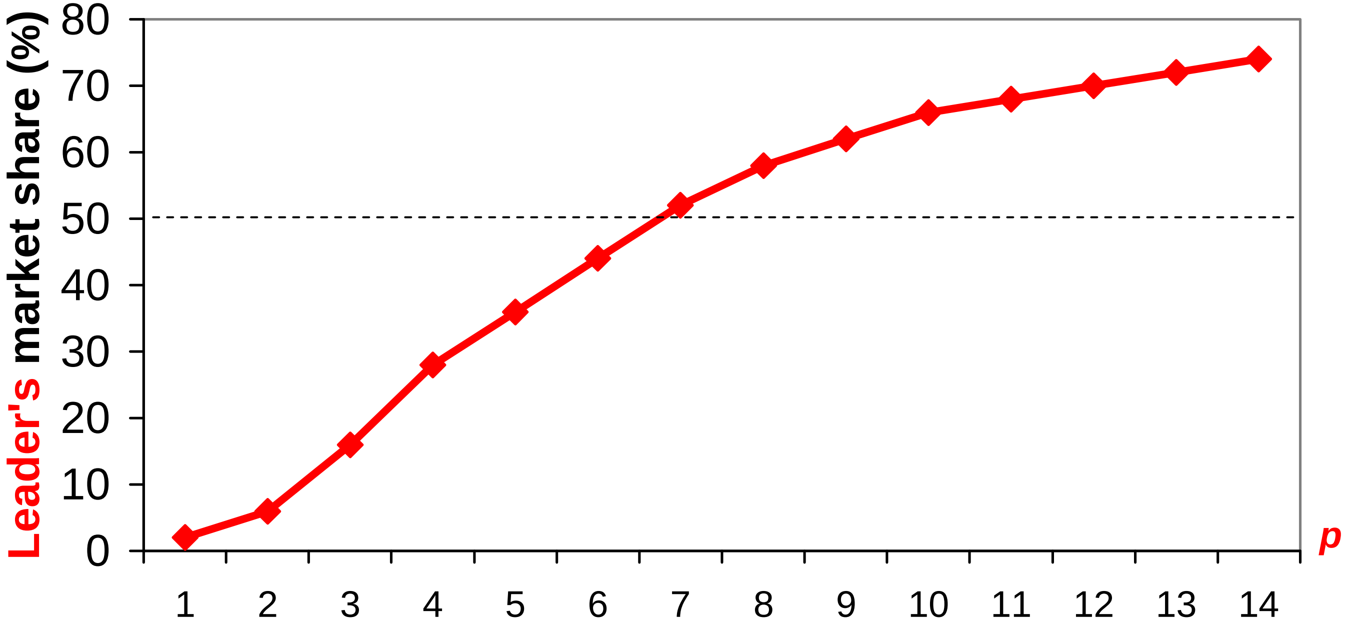
Optimal solutions, |*I*|=100, *p*=*r*=5

$w_j = 1,\ j \in J$		
opt	|*F*|	time (min)
47	123	120
48	69	60
45	231	3600
47	111	150
47	106	120
47	102	90
47	115	180
48	67	42
47	108	160
47	124	165

$w_j \in (0{,}200),\ j \in J$		
opt	|*F*|	time (min)
4139 (47%)	98	65
4822 (45%)	127	37
4215 (45%)	262	5460
4678 (47%)	128	900
4594 (44%)	190	720
4483 (47%)	121	660
5153 (46%)	167	2550
4404 (46%)	190	720
4700 (45%)	247	2520
4923 (48%)	83	30

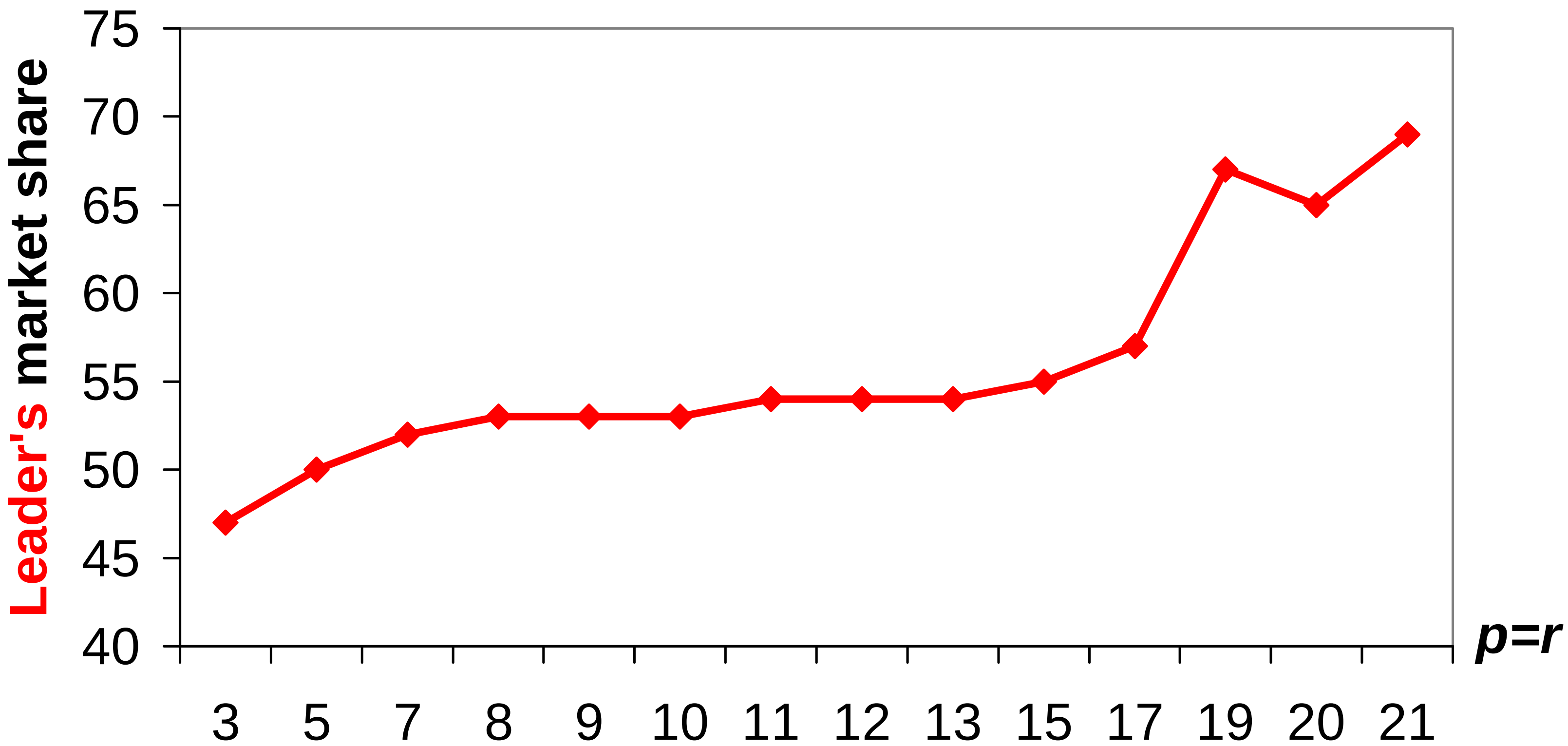

Leader's market share (%)
100
90
80
70
60
50
40
30
20
1
2
3
4
5
6
7
8
9
10
11
12
13
14
r

$|I|=50, p=7$

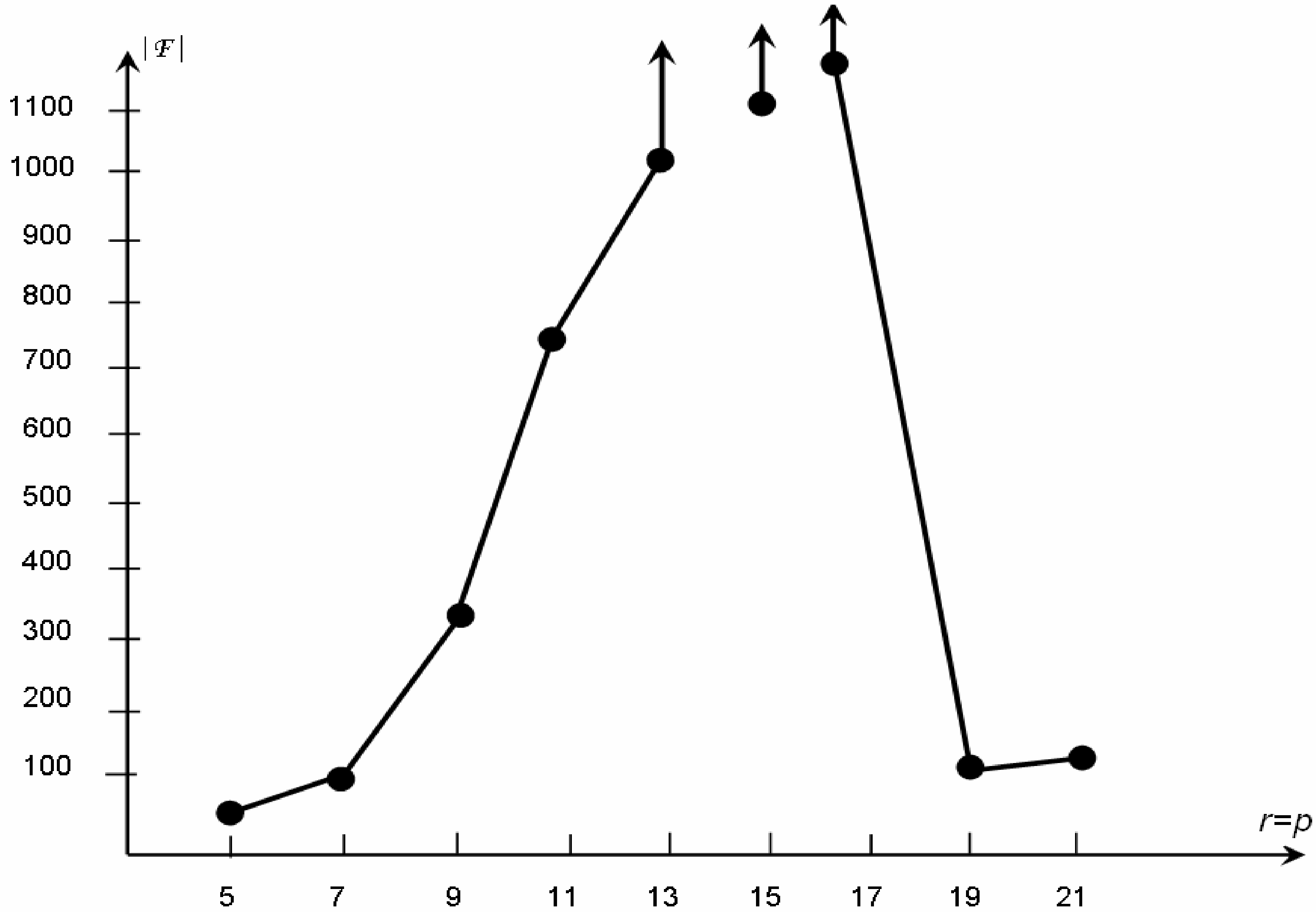

Leader's market share (%)
80
70
60
50
40
30
20
10
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
p

$|I|=50, r=7$

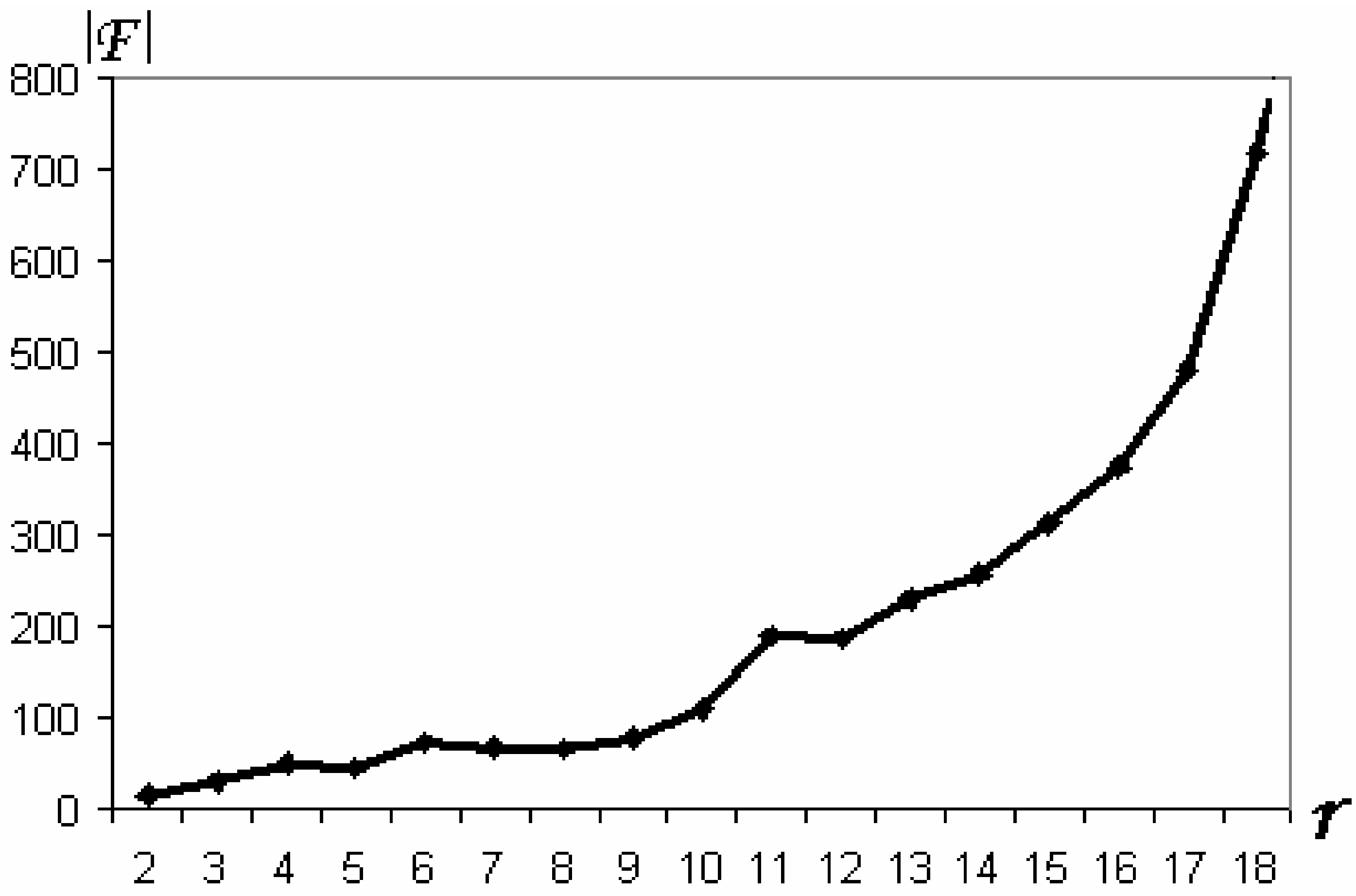
Leader's Market Share, $|I|=50$, $w_j \in (0, 200)$



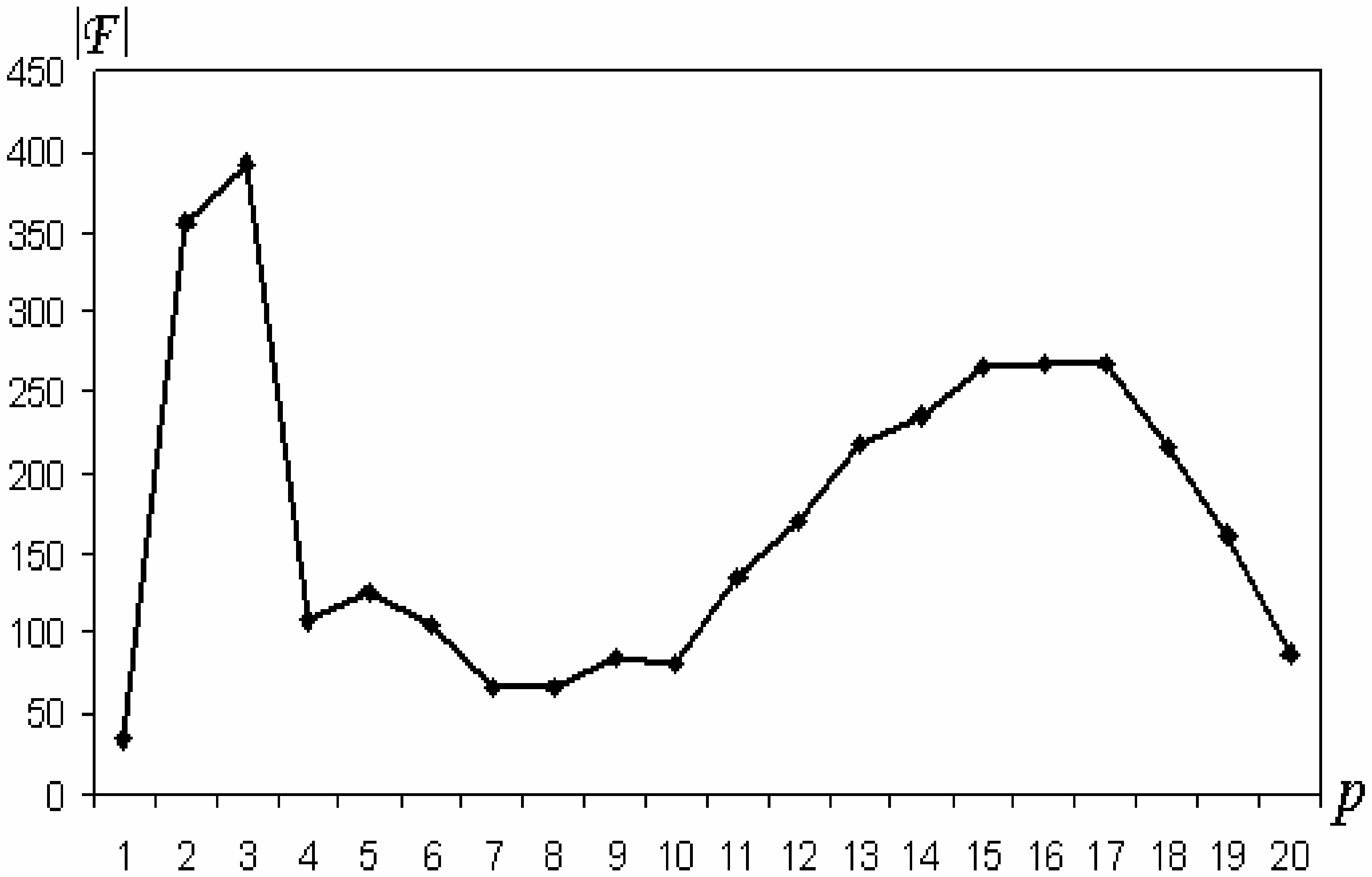
The Number of Iterations, $|F(p)|$, $|I|=50$, $p=r$



The Number of Iterations, |*F(p)*|, |*I*|=50, *p*=7



The Number of Iterations, $|F(p)|$, $|I|=50$, $r=7$



Conclusion

- ✓ $\sum_2^p$-hard problem has been studied
- ✓ A new MIP reformulation with the exp number of constraints has been suggested
- ✓ A new exact method has been proposed
- ✓ The optimal solutions for the instances with $|I|=|J|=100$ and $p=r=5$ have been found