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ISOMORPHISMS OF SOBOLEV SPACES ON CARNOT GROUPS AND QUASICONFORMAL MAPPINGS

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Abstract: We prove that a measurable mapping of domains on a Carnot group induces by the corresponding change of variables an isomorphism of the Sobolev spaces whose integrability exponent is equal to the Hausdorff dimension of the group if and only if the mapping coincides with a quasiconformal mapping almost everywhere.

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Introduction

The article can be regarded as a natural continuation of [1–5]. In these papers, a few different proofs were obtained of the theorem that a measurable mapping in the Euclidean space $\mathbb{R}^n$ inducing an isomorphism of some spaces of differentiable functions coincides with a quasiconformal mapping almost everywhere.

In this article, we give a solution to the similar problem for the measurable mappings of domains on a Carnot group which induce the isomorphisms of horizontal mappings of domains on a Carnot group which give rise to isomorphisms of horizontal Sobolev classes. The method of the article is a modification of the arguments of [5] basing on the results of [6]. In [6], we introduce the main object of study, the class IL^1_p of mappings on a Carnot group.

Definition 1. Let D and D' be domains on a Carnot group $\mathbb{G}$. A measurable mapping $\varphi : D \to D'$ *belongs to* IL^1_p, $p \in [1,\infty]$, if φ induces the composition operator in Sobolev spaces:

$$\varphi^* : L^1_p(D') \cap C^\infty(D') \to L^1_p(D),\ \varphi^*(f) = f \circ \varphi, \quad f \in L^1_p(D') \cap C^\infty(D'), \tag{1}$$

such that

(1) for every $f \in L^1_p(D') \cap C^\infty(D')$ we have

$$K^{-1}\|f \mid L^1_p(D')\| \le \|\varphi^*(f) \mid L^1_p(D)\| \le K\|f \mid L^1_p(D')\|, \tag{2}$$

where K is a constant independent of the choice of f;

(2) $\varphi^*\big(L^1_p(D') \cap C^\infty(D')\big)$ is everywhere dense in $L^1_p(D)$.

It was proved in [6] that item (2) of this definition is independent of item (1).

In this article, we give a full description of the mappings of class IL^1_ν, where ν is the Hausdorff dimension of $\mathbb{G}$; i.e., we obtain a full description for the measurable mappings of domains of Carnot groups inducing isomorphisms of the Sobolev spaces L^1_ν in the sense of Definition 1. In [6], the case of $p \ne \nu$ was studied, and the general scheme was suggested in [7]. The main result of the article is formulated in the following assertion (see the definitions of the main notions after the theorem):

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†) To S. S. Kutateladze on the occasion of his 70th birthday.

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Theorem 2. *Let D and D' be domains on a Carnot group $\mathbb{G}$ and let ν be the Hausdorff dimension of $\mathbb{G}$. A measurable mapping $\varphi : D \to D'$ belongs to IL^1_ν if and only if φ coincides almost everywhere with a quasiconformal mapping $\Phi : D \setminus \{x_0\} \to \mathbb{G}$ for which the domains $\Phi(D \setminus \{x_0\})$ and D' are $(1,\nu)$-equivalent, where $x_0 \in \overline{\mathbb{G}}$ is some point (here $\overline{\mathbb{G}}$ is the one-point compactification $\mathbb{G}$).*[1)]

Definition 3. A homeomorphism $\Phi : D \to D'$ of class $W^1_{\nu,\mathrm{loc}}$ is called *quasiconformal* if there exists a constant K such that $|D\Phi(x)|^\nu \le K|J(x,\Phi)|$ a.e. in D, where $D\Phi(x)$ is the approximate differential [8] of Φ and $J(x,\Phi) = \det D\Phi(x)$.

Definition 4. Two open sets D_1 and D_2 are called $(1,p)$-*equivalent* if the restriction operators $r_i : L^1_p(D_1 \cup D_2) \to L^1_p(D_i)$, $r_i(f) = f|_{D_i}$, where $f \in L^1_p(D_1 \cup D_2)$, are isomorphisms.

This definition is equivalent to those of [9] and [6].

Definition 5 [6, Definition 2]. Two open sets D_1 and D_2 are called $(1,p)$-*equivalent* if the restriction operators $r_i : L^1_p(D_i) \to L^1_p(D_1 \cap D_2)$ and $r_i(f) = f|_{D_1\cap D_2}$, where $f \in L^1_p(D_i)$, are such that $r_2^{-1} \circ r_1$ and $r_1^{-1} \circ r_2$ are isomorphisms.[2)]

In a Euclidean space, a theorem analogous to Theorem 2 was proved in [1] under the condition that D' is bounded. Families of $(1,p)$-equivalent domains were studied in [9] in a Euclidean space and in [10] on a Carnot group.

The proof of Theorem 2 in this article is based on the method of [5] with substantial amendments inevitable for the environment of the article: in [5], as the domains D and D', we consider the Euclidean space $\mathbb{R}^n$, and take a suitable normed function space as the function space.

Note that the classes IL^1_p for $p \ne \nu$ are completely studied in [6], where a detailed history of the question and an extensive bibliography are given. For comparison with Theorem 2, formulate the main result of [6]:

Theorem 6 [6, Theorem 1]. *Let $p \ge 1$, $p \ne \nu$, while D and D' are domains on a Carnot group $\mathbb{G}$ (here ν is the Hausdorff dimension of $\mathbb{G}$). A measurable mapping $\varphi : D \to D'$ belongs to IL^1_p if and only if φ coincides almost everywhere with some quasi-isometry $\Phi : D \to \Phi(D)$ for which $\Phi(D)$ and D' are $(1,p)$-equivalent.*[3)]

1. Prerequisites

1.1. Sobolev spaces on a Carnot group. A Carnot group $\mathbb{G}$ is a connected simply-connected stratified nilpotent Lie group. This means that the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ splits into some direct sum of vector subspaces $\mathfrak{g} = V_1 \oplus \dots \oplus V_m$ such that $[V_1, V_j] = V_{j+1}$ for $j = 1,\dots,m-1$ and $[V_1, V_m] = \{0\}$. Below we use the notation $n = n_1$. Let $X_1,\dots,X_n$ be vector fields composing a basis for V_1. An absolutely continuous piecewise smooth curve $\gamma : [a,b] \to \mathbb{G}$ whose tangent vector $\dot\gamma(t)$ belongs to V_1 for a.e. $t \in [a,b]$ is called a *horizontal* curve. The *length* of a horizontal curve $\gamma : [a,b] \to \mathbb{G}$ is expressed by the integral $l(\gamma) = \int_a^b |\dot\gamma(t)|\,dt$ (here $|\dot\gamma(t)|$ is the length of the tangent vector; the basis $X_1,\dots,X_n$ is assumed to be orthonormal).

Definition 7. The *Carnot–Carathéodory metric* $d(x,y)$ on a Carnot group $\mathbb{G}$ is the infimum of the lengths of all horizontal curves joining x and y.

Now, consider the family Γ_j of the integral curves of the basis horizontal vector field X_j constituting a smooth foliation of an open set $A \subset \mathbb{G}$. If we denote the flow that corresponds to this field by g_s then

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1) Note that the statement of this theorem in [6] contains the misprint: "for which the Sobolev spaces $L^1_\nu(\Phi(D))$ and $L^1_\nu(D')$ are $(1,\nu)$-equivalent" must be replaced by "for which the domains $\Phi(D)$ and D' are $(1,\nu)$-equivalent."

2) This definition in [6] contains the misprint: "are isomorphisms" must be replaced by "such that $r_2^{-1} \circ r_1$ and $r_1^{-1} \circ r_2$ are isomorphisms."

3) Note that the statement of this theorem in [6] contains the misprint: "for which the Sobolev spaces $L^1_p(\Phi(D))$ and $L^1_p(D')$ are $(1,p)$-equivalent" must be replaced by "for which the domains $\Phi(D)$ and D' are $(1,p)$-equivalent."

the fiber has the form $\gamma(s) = g_s(p)$, where p belongs to a surface S_j transversal to the vector field X_j and the parameter s is taken from an interval $I \subset \mathbb{R}$. For the foliation defined by X_j, the measure $d\gamma$ can be obtained as the interior multiplication $i(X_j)$ of the vector field X_j with bi-invariant volume form dx. If $\mathbb{J}_{g_s}$ is the Jacobian of g_s then

$$g_s^* i(X_j)\,dx = \mathbb{J}_{g_s} i(X_j)\,dx \quad \text{or} \quad g_s^*(\mathbb{J}_{g_{-s}} i(X_j)\,dx) = i(X_j)\,dx.$$

Since g_s takes the tangent vector to the one-parameter family of curves γ_t to the tangent vector to the same family, the form $\mathbb{J}_{g_{-s}} i(V)\,dx$ defines a measure $d\gamma$ on the foliation Γ_j. Since X_j is a left-invariant horizontal vector field, g_s is the right shift by $\exp sX_j$: $\mathbb{G} \ni p \mapsto p \exp sX_j$. Since dx is a bi-invariant form, we have $\mathbb{J}_{g_s} = 1$. Using the left invariance and homogeneity under dilations, we find

$$\int\limits_{\gamma \cap B(x,r) \neq \varnothing} d\gamma = c|B(x,r)|^{\frac{\nu-1}{\nu}}.$$

From this we can deduce Fubini's Theorem that will be applied below.

The Sobolev space $L_p^1(D)$ consists of all locally integrable functions $f : D \to \mathbb{R}$ having weak derivatives $X_i f \in L_p(D)$, $i = 1, \dots, n$. The seminorm in $L_p^1(D)$ is defined as

$$\|f \mid L_p^1(D)\| = \|\nabla_{\mathscr{L}} f \mid L_p(D)\| = \Bigl(\int\limits_D |\nabla_{\mathscr{L}} f(x)|^p\,dx\Bigr)^{\frac{1}{p}},$$

where $\nabla_{\mathscr{L}} f(x) = (X_1 f(x), \dots, X_n f(x))$ is the generalized subgradient of f at $x \in D$ and $|\nabla_{\mathscr{L}} f(x)| = \sqrt{(X_1 f(x))^2 + \dots + (X_n f(x))^2}$. The Sobolev space $W_p^1(D)$ consists of locally integrable functions with the finite norm

$$\|f \mid W_p^1(D)\| = \|f \mid L_p(D)\| + \|\nabla_{\mathscr{L}} f \mid L_p(D)\|.$$

We say that f *belongs to* $W_{p,\mathrm{loc}}^1(D)$ if $f \in W_p^1(V)$ for every bounded subdomain $V \subset D$ such that $\overline{V} \subset D$ (in writing $V \Subset D$).

We say (see [11]) that $\varphi : D \to \mathbb{G}$ *belongs to* $W_{p,\mathrm{loc}}^1(D; \mathbb{G})$ if the conditions are fulfilled:

(A) For every $z \in \mathbb{G}$, the function $[\varphi]_z : D \ni x \mapsto d(\varphi(x), z)$ belongs to $W_{p,\mathrm{loc}}^1(D)$.

(B) The family of subgradients $(\nabla_{\mathscr{L}}[\varphi]_z)_{z \in \mathbb{G}}$ has a *dominant* in $L_{p,\mathrm{loc}}(D)$; i.e., there exists $g \in L_{p,\mathrm{loc}}(D)$ independent of z such that $|\nabla_{\mathscr{L}}[\varphi]_z(x)| \le g(x)$ for almost all $x \in D$.

The specifics of this definition in application to Sobolev spaces on a Carnot group are reflected in [8]. In particular, an *equivalent* description of the Sobolev classes is given: $\varphi : D \to \mathbb{G}$ belongs to $W_{p,\mathrm{loc}}^1(D)$ if and only if φ can be changed on a negligible set so that

(1) $[\varphi]_z : D \ni x \mapsto d(\varphi(x), z)$ belongs to $L_{p,\mathrm{loc}}(D)$ for every $z \in \mathbb{G}$;

(2) $\varphi : D \to \mathbb{G}$ is *absolutely continuous* on almost all integral curves of the horizontal vector fields X_j, $j = 1, \dots, n$ ($\varphi \in ACL(D)$);

(3) the derivative $X_j\varphi(x) = \lim_{t\to 0} \delta_{t^{-1}}(\varphi(x)^{-1}\varphi(\exp tX_j))$ exists a.e. in the open set D, belongs to $V_1(\varphi(x))$, and $|X_j\varphi| \in L_{p,\mathrm{loc}}(D)$ for all j.

Recall that $\varphi : D \to \mathbb{G}$ is called *absolutely continuous* on almost all integral curves of the basis horizontal vector fields X_j, $j = 1, \dots, n$, if, for every domain $U \Subset D$ and the foliation Γ_j defined by X_j ($j = 1, \dots, n$), the mapping φ is absolutely continuous on the intersection $\gamma \cap U$ with respect to the one-dimensional Hausdorff measure for $d\gamma$-almost all curves $\gamma \in \Gamma_j$. For such a mapping, almost everywhere in D, there exist derivatives $X_j\varphi$ ($j = 1, \dots, n$) (various proofs of this fact can be found in [12–14]).

Denote by $D\varphi$ the approximate differential of φ [8] and designate as $D_h\varphi$ the horizontal part of the differential. The Jacobian $\det D\varphi$ of φ will be denoted by $J(x, \varphi)$.

The following change-of-variable formula holds:

Proposition 8 [8, Corollary 5.1; 15]. *Suppose that $\varphi : A \to \mathbb{G}$, where $A \subset \mathbb{G}$ is a measurable set, has approximate partial derivatives on A. Then there exists a negligible set $\Sigma_\varphi \subset A$ such that the change-of-variable formula in the Lebesgue integral for every nonnegative measurable function $f : A \to \mathbb{R}$ has the form*

$$\int\limits_A f(x)|J(x,\varphi)|\,dx = \int\limits_{\mathbb{G}} \Big(\sum_{x\in\varphi^{-1}(y)\cap(A\setminus\Sigma_\varphi)} f(x) \Big)\,dy. \tag{3}$$

1.2. John domains and the Poincaré inequality. In this subsection, we apply the Poincaré inequality in John domains on Carnot groups, proved in [16] (earlier results were established in [17–20]). Moreover, we will need this inequality is some special modification (see Lemma 12 below).

Definition 9 [21]. A proper domain $\Omega \subset \mathbb{G}$ is called a *John domain* $J_{\alpha,\beta}$ (briefly, $\Omega \in J_{\alpha,\beta}$), $0 < \alpha \le \beta$, if there exists $x_0 \in \Omega$ such that each point $x \in \Omega$ can be joined with x_0 by a rectifiable curve γ lying in Ω and satisfying the conditions: if $s \in [0,l]$ is a natural parametrization of γ ($\gamma(0) = x$, $\gamma(l) = x_0$) then

$$l \le \beta \quad \text{and} \quad \operatorname{dist}(\gamma(s), \partial\Omega) \ge \frac{\alpha s}{l} \quad \text{for all } s \in [0,l]. \tag{4}$$

Lemma 10 [6, Lemma 3]. *Suppose that D is an arbitrary domain in $\mathbb{G}$, while some balls B_0 and B_1 lie in this domain. Then there is a John domain $\Omega \in J_{\alpha,\beta}$, $\Omega \subset D$, with some parameters α and β, depending on D, B_0 and B_1 such that Ω includes these balls.*

Remark 11. From the proof of Lemma 3 in [6] we obtain the property: If $\operatorname{dist}(\partial D, B_0) > 0$ and $\operatorname{dist}(\partial D, B_1) > 0$ then, for a sufficiently small parameter $\lambda > 0$, we can construct an additional John domain Ω_λ such that $\Omega \Subset \Omega_\lambda \Subset D$; i.e., the domains Ω and Ω_λ are bounded and $\operatorname{dist}(\partial D, \Omega_\lambda) > 0$, $\operatorname{dist}(\partial\Omega, \Omega_\lambda) > 0$. Indeed, the idea of the proof of the Lemma of [6] consists in constructing a rectifiable curve Γ lying in D and joining the centers of B_0 and B_1. The John domain Ω is constructed as the family of balls centered at Γ each with radius at most $\frac{1}{2}\operatorname{dist}(\Gamma, \partial D)$. The domain Ω_λ can be constructed as the union of balls with the same centers by increasing the radius. As such a radius, we can take every number in the interval $\big(\frac{1}{2}\operatorname{dist}(\Gamma, \partial D), \frac{3}{4}\operatorname{dist}(\Gamma, \partial D)\big)$.

Lemma 12 [6, Lemma 4]. *Suppose that U is a John domain $J_{\alpha\beta}$ and a subset $F \subset U$ has positive measure, $|F| > 0$. Then for all $u(x) \in W^1_p(U)$, $p \le q \le \frac{\nu p}{\nu - p}$, $p < \nu$ ($p \le q < \infty$ for $p = \nu$) such that $u|_F = 0$ we have*

$$\Big(\int\limits_U |u(x)|^q\,dx \Big)^{\frac{1}{q}} \le C \frac{|U|^{\frac{1}{q}}}{|F|^{\frac{1}{q}}} \Big(\frac{\alpha}{\beta} \Big)^{\nu} (\operatorname{diam} U)^{1-\frac{\nu}{p}+\frac{\nu}{q}} \Big(\int\limits_U |\nabla u(x)|^p\,dx \Big)^{\frac{1}{p}}. \tag{5}$$

1.3. Properties of mappings of class IL^1_p. These were established for mappings of class IL^1_p in [6]:

Proposition 13. 1. *As the domain of φ, we can take $T = \bigcup_k T_k$, $|D \setminus T| = 0$, where $\{T_k\}$ is an increasing sequence of bounded sets of positive measure consisting of points of positive density.*

2. *φ is continuous on each T_k.*
3. *φ satisfies Luzin's conditions ($\mathscr{N}$) and ($\mathscr{N}^{-1}$) on T.*
4. *$\varphi : T \to D'$ is injective.*
5. *$\varphi(T)$ is dense in D' and $|D' \setminus \varphi(T)| = 0$.*

Operator (1) extends to $L^1_p(D)$, preserving the properties of the composition operator.

Lemma 14 [6, Lemma 10]. *Suppose that a measurable mapping $\varphi : D \to D'$ belongs to IL^1_p. Then the operator $\varphi* : L^1_p(D') \cap C^\infty(D') \to L^1_p(D)$ extends by continuity to an operator $\widetilde{\varphi*} : L^1_p(D') \to L^1_p(D)$ and possesses the properties:*

(1) *the value of $\widetilde{\varphi*} : L^1_p(D') \to L^1_p(D)$ at $[f] \in L^1_p(D')$ can be found by the formula*

$$\widetilde{\varphi^*}([f]) = \begin{cases} f \circ \varphi & \text{for } p \le \nu, \text{ where } f \text{ is an arbitrary representative of } [f], \\ \tilde{f} \circ \varphi & \text{for } p > \nu, \text{ where } \tilde{f} \text{ is a continuous representative of } [f]; \end{cases}$$

(2) $K^{-1}\|f \mid L^1_p(D')\| \le \|\widetilde{\varphi^*}(f) \mid L^1_p(D)\| \le K\|f \mid L^1_p(D')\|$;

(3) $\widetilde{\varphi^*} : L^1_p(D') \to L^1_p(D)$ *is an isomorphism.*

2. The Space $L^1_{\nu,F}$

Throughout the sequel, we study some mapping $\varphi : D \to D'$ of class IL^1_ν. Each of these mappings possesses the properties of Subsection 1.3.

Fix $k_0 \in \mathbb{N}$ and a closed set of positive measure $F \subset T_{k_0}$ without isolated points. We may assume that $F \subset B_F$, where $B_F \subset D$ is a ball. By Remark 15 in [6], we may also assume that $\varphi : F \to \varphi(F)$ is bi-Lipschitz. Such a choice guarantees the same properties for $\varphi(F)$ as for F: The image $\varphi(F)$ is closed, has no isolated points, and the measure of $\varphi(F)$ is positive.

Consider the set of functions

$$L^1_{\nu,F}(D) = \{u \in L^1_\nu(D) : u(x) = 0 \text{ for a.e. } x \in F\}.$$

Note that $L^1_{\nu,F}(D)$ is a closed subspace in $L^1_\nu(D)$ and a normed space with the norm $\|u \mid L^1_{\nu,F}(D)\| = \|u \mid L^1_\nu(D)\|$. The last is easy to prove on using Lemma 12. Therefore, $L^1_{\nu,F}(D)$ is a Banach space.

By analogy to the above, define another Banach space

$$L^1_{\nu,\varphi(F)}(D') = \{v \in L^1_\nu(D') : v(y) = 0 \text{ for a.e. } y \in \varphi(F)\}.$$

Using Proposition 13 and Lemma 14, we can check that $f \in L^1_{\nu,\varphi(F)}(D')$ if and only $f \circ \varphi \in L^1_{\nu,F}(D)$. Consequently,

$$\varphi^*_F : L^1_{\nu,\varphi(F)}(D') \to L^1_{\nu,F}(D), \quad \varphi^*_F(f) = f \circ \varphi, \quad f \in L^1_{\nu,\varphi(F)}(D')$$

is an isomorphism.

The application of the spaces $L^1_{\nu,F}$ enables us to establish the existence of a quasicontinuous representative for φ.

Put $D_F = D \setminus F$ and $D'_F = D' \setminus \varphi(F)$.

3. Capacity

In this section, we give the main properties of capacity in Sobolev spaces that will help us in studying the further properties of φ.

3.1. Capacity in $L^1_{\nu,F}(D)$ and its properties. Let us give the notion of capacity in $L^1_{\nu,F}(D)$ and properties to be used below. For a detailed exposition in application to other function spaces in Section 3, Subsections 4.1 and 4.2, the reader is referred to [5, § 6; 10, § 6; 22]. For the reader's convenience, we present it completely. In parentheses, we give references to the articles with assertions that are close by contexts to those of this article.

The *capacity* $\operatorname{Cap}(K; L^1_{\nu,F}(D))$ of a compact set $K \subset D_F$ in $L^1_{\nu,F}(D)$ is the quantity

$$\operatorname{Cap}(K; L^1_{\nu,F}(D)) = \inf \|g \mid L^1_{\nu,F}(D)\|^\nu, \tag{6}$$

where the infimum is taken over all continuous functions $g \in L^1_{\nu,F}(D)$ with $g \ge 1$ on K.

REMARK 15. The infimum in (6) does not change if we consider *nonnegative* continuous functions in $L^1_{\nu,F}(D)$ such that $g > 1$ on K.

Given $E \subset D_F$, its *inner capacity* of E is defined as

$$\underline{\mathrm{Cap}}\big(E; L^1_{\nu,F}(D)\big) = \sup\big\{\mathrm{Cap}\big(K; L^1_{\nu,F}(D)\big) : K \subset E,\ K \text{ is compact}\big\},$$

and the *outer capacity* of E, as

$$\overline{\mathrm{Cap}}\big(E; L^1_{\nu,F}(D)\big) = \inf\big\{\underline{\mathrm{Cap}}\big(U; L^1_{\nu,F}(D)\big) : E \subset U,\ U \subset D_F \text{ is open}\big\}.$$

In the following lemma, we formulate the main properties of the capacity:

Lemma 16 [10, Theorem 6.1; 5, Lemma 6.1]. *The capacity in $L^1_{\nu,F}(D)$ possesses the properties:*
1. *If $K \subset D_F$ is compact then, for every $\varepsilon > 0$, there exists an open set $U_\varepsilon \subset D_F$ such that $K \subset U_\varepsilon$ and*

$$\mathrm{Cap}\big(K'; L^1_{\nu,F}(D)\big) \le \mathrm{Cap}\big(K; L^1_{\nu,F}(D)\big) + \varepsilon$$

for every compact set $K' \subset U_\varepsilon$.
2. *If $E \subset E'$ then*

$$\underline{\mathrm{Cap}}\big(E; L^1_{\nu,F}(D)\big) \le \underline{\mathrm{Cap}}\big(E'; L^1_{\nu,F}(D)\big), \quad \overline{\mathrm{Cap}}\big(E; L^1_{\nu,F}(D)\big) \le \overline{\mathrm{Cap}}\big(E'; L^1_{\nu,F}(D)\big).$$

3. *Let $K_1, K_2 \subset D_F$ be compact sets. Then*

$$\mathrm{Cap}\big(K_1 \cup K_2; L^1_{\nu,F}(D)\big) + \mathrm{Cap}\big(K_1 \cap K_2; L^1_{\nu,F}(D)\big) \le \mathrm{Cap}\big(K_1; L^1_{\nu,F}(D)\big) + \mathrm{Cap}\big(K_2; L^1_{\nu,F}(D)\big).$$

4. *Suppose that $E_1, \dots, E_k \subset D_F$, $F_i \subset E_i$, $\overline{\mathrm{Cap}}\big(\bigcup_{i=1}^k F_i; L^1_{\nu,F}(D)\big) < \infty$. Then*

$$\overline{\mathrm{Cap}}\Bigg(\bigcup_{i=1}^k E_i; L^1_{\nu,F}(D)\Bigg) - \overline{\mathrm{Cap}}\Bigg(\bigcup_{i=1}^k F_i; L^1_{\nu,F}(D)\Bigg) \le \sum_{i=1}^k \big(\overline{\mathrm{Cap}}\big(E_i; L^1_{\nu,F}(D)\big) - \overline{\mathrm{Cap}}\big(F_i; L^1_{\nu,F}(D)\big)\big).$$

5. *For every increasing sequence of sets $E_1 \subset E_2 \subset \dots \subset E_k \subset \dots \subset D_F$, we have*

$$\overline{\mathrm{Cap}}\Bigg(\bigcup_{k=1}^\infty E_k; L^1_{\nu,F}(D)\Bigg) = \lim_{k\to\infty} \overline{\mathrm{Cap}}\big(E_k; L^1_{\nu,F}(D)\big).$$

6. *Let $\{E_k\} \subset D_F$, $k \in \mathbb{N}$, be a sequence of sets and let $E = \bigcup_{k=1}^\infty E_k$. Then*

$$\overline{\mathrm{Cap}}\big(E; L^1_{\nu,F}(D)\big) \le \sum_{k=1}^\infty \overline{\mathrm{Cap}}\big(E_k; L^1_{\nu,F}(D)\big).$$

PROOF. 1. By the definition of capacity, there exists a continuous function $u \in L^1_{\nu,F}(D)$ such that $u > 1$ on K (see Remark 15) and $\|u \mid L^1_{\nu,F}(D)\|^\nu \le \mathrm{Cap}\big(K; L^1_{\nu,F}(D)\big) + \varepsilon$. Introduce the set $U_\varepsilon = \{x \in D_F : u(x) > 1\}$. Then U_ε is open, $K \subset U_\varepsilon$, and $u > 1$ on every compact set $K' \subset U_\varepsilon$. Hence,

$$\mathrm{Cap}\big(K'; L^1_{\nu,F}(D)\big) \le \|u \mid L^1_{\nu,F}(D)\|^\nu \le \mathrm{Cap}\big(K; L^1_{\nu,F}(D)\big) + \varepsilon.$$

2. If E and E' are compact sets then

$$\mathrm{Cap}\big(E; L^1_{\nu,F}(D)\big) \le \|g \mid L^1_{\nu,F}(D)\|^\nu$$

for every continuous function $g \in L^1_{\nu,F}(D)$ with $u > 1$ on E' (see Remark 15), and so

$$\operatorname{Cap}\big(E; L^1_{\nu,F}(D)\big) \le \operatorname{Cap}\big(E'; L^1_{\nu,F}(D)\big).$$

For arbitrary sets, we have

$$\begin{aligned}\underline{\operatorname{Cap}}\big(E; L^1_{\nu,F}(D)\big) &= \sup_{K\subset E} \operatorname{Cap}\big(K; L^1_{\nu,F}(D)\big)\\ \le \sup_{K\subset E'} \operatorname{Cap}\big(K; L^1_{\nu,F}(D)\big) &= \underline{\operatorname{Cap}}\big(E'; L^1_{\nu,F}(D)\big),\\ \overline{\operatorname{Cap}}\big(E; L^1_{\nu,F}(D)\big) &= \inf_{E\subset U} \underline{\operatorname{Cap}}\big(U; L^1_{\nu,F}(D)\big)\\ \le \inf_{E'\subset U} \underline{\operatorname{Cap}}\big(U; L^1_{\nu,F}(D)\big) &= \overline{\operatorname{Cap}}\big(E'; L^1_{\nu,F}(D)\big),\end{aligned}$$

where the suprema are taken over compact sets and the infima are taken over open sets.

3. Consider continuous functions $g_1, g_2 \in L^1_{\nu,F}(D)$ such that $g_i \ge 1$ on K_i, $i = 1, 2$. Then the functions $\min(g_1, g_2)$ and $\max(g_1, g_2)$ are continuous, belong to $L^1_{\nu,F}(D)$, $\min(g_1, g_2) \ge 1$ on $K_1 \cap K_2$, $\max(g_1, g_2) \ge 1$ on $K_1 \cup K_2$, and

$$\big\|\min(g_1, g_2) \mid L^1_{\nu,F}(D)\big\|^\nu + \big\|\max(g_1, g_2) \mid L^1_{\nu,F}(D)\big\|^\nu \le \big\|g_1 \mid L^1_{\nu,F}(D)\big\|^\nu + \big\|g_1 \mid L^1_{\nu,F}(D)\big\|^\nu.$$

This yields

$$\begin{aligned}&\operatorname{Cap}\big(K_1 \cup K_2; L^1_{\nu,F}(D)\big) + \operatorname{Cap}\big(K_1 \cap K_2; L^1_{\nu,F}(D)\big)\\ &\le \big\|\min(g_1, g_2) \mid L^1_{\nu,F}(D)\big\|^\nu + \big\|\max(g_1, g_2) \mid L^1_{\nu,F}(D)\big\|^\nu\\ &\le \big\|g_1 \mid L^1_{\nu,F}(D)\big\|^\nu + \big\|g_2 \mid L^1_{\nu,F}(D)\big\|^\nu.\end{aligned}$$

Passing to the infimum over all admissible functions g_1 and g_2 in the last inequality, we get what was required.

4. First, prove the relation for compact sets by induction. If $k = 1$ then the equality is obvious. Suppose that it is fulfilled for j sets; i.e.,

$$\begin{aligned}&\overline{\operatorname{Cap}}\bigg(\bigcup_{i=1}^{j} E_i; L^1_{\nu,F}(D)\bigg) - \overline{\operatorname{Cap}}\bigg(\bigcup_{i=1}^{j} F_i; L^1_{\nu,F}(D)\bigg)\\ &\le \sum_{i=1}^{j}\big(\overline{\operatorname{Cap}}\big(E_i; L^1_{\nu,F}(D)\big) - \overline{\operatorname{Cap}}\big(F_i; L^1_{\nu,F}(D)\big)\big).\end{aligned}$$

Let $F_{j+1} \subset E_{j+1}$. Put $A = \bigcup_{i=1}^{j} E_i$ and $B = \bigcup_{i=1}^{j} F_i$. Applying property 3 for the pairs of compact sets A, E_{j+1} and B, F_{j+1}, we get

$$\begin{aligned}&\operatorname{Cap}\big(A \cup E_{j+1}; L^1_{\nu,F}(D)\big) - \operatorname{Cap}\big(B \cup F_{j+1}; L^1_{\nu,F}(D)\big)\\ &+ \operatorname{Cap}\big(A \cap E_{j+1}; L^1_{\nu,F}(D)\big) - \operatorname{Cap}\big(B \cap F_{j+1}; L^1_{\nu,F}(D)\big)\\ &\le \operatorname{Cap}\big(A; L^1_{\nu,F}(D)\big) + \operatorname{Cap}\big(E_{j+1}; L^1_{\nu,F}(D)\big)\\ &- \operatorname{Cap}\big(B; L^1_{\nu,F}(D)\big) - \operatorname{Cap}\big(F_{j+1}; L^1_{\nu,F}(D)\big).\end{aligned}$$

Since $B \cap F_{j+1} \subset A \cap E_{j+1}$, property 2 yields

$$\begin{aligned}&\operatorname{Cap}\big(A \cup E_{j+1}; L^1_{\nu,F}(D)\big) - \operatorname{Cap}\big(B \cup F_{j+1}; L^1_{\nu,F}(D)\big)\\ &\le \operatorname{Cap}\big(A; L^1_{\nu,F}(D)\big) + \operatorname{Cap}\big(E_{j+1}; L^1_{\nu,F}(D)\big)\\ &- \operatorname{Cap}\big(B; L^1_{\nu,F}(D)\big) - \operatorname{Cap}\big(F_{j+1}; L^1_{\nu,F}(D)\big).\end{aligned}$$

By the induction assumption,

$$\begin{aligned}\operatorname{Cap}\big(A\cup E_{j+1};L^1_{\nu,F}(D)\big)-\operatorname{Cap}\big(B\cup F_{j+1};L^1_{\nu,F}(D)\big)\\ \le\sum_{i=1}^{j}\big(\overline{\operatorname{Cap}}\big(E_i;L^1_{\nu,F}(D)\big)-\overline{\operatorname{Cap}}\big(F_i;L^1_{\nu,F}(D)\big)\big)\\ +\operatorname{Cap}\big(E_{j+1};L^1_{\nu,F}(D)\big)-\operatorname{Cap}\big(F_{j+1};L^1_{\nu,F}(D)\big).\end{aligned}$$

Thus, property 4 is proved for compact sets.

If E_i and F_i are open sets then we use the following fact: If $K\subset\bigcup_{i=1}^k E_i$ and $C_i\subset F_i$ are compact subsets, $\bigcup_{i=1}^k C_i\subset K$, then the compact set $K_j=K\setminus\bigcup_{i=1,\,i\ne j}^k E_i$ is a subset in E_j and includes C_j. Moreover, $K=\bigcup_{i=1}^k K_i$. Then

$$\begin{aligned}\overline{\operatorname{Cap}}\Big(\bigcup_{i=1}^k E_i;L^1_{\nu,F}(D)\Big)-\overline{\operatorname{Cap}}\Big(\bigcup_{i=1}^k F_i;L^1_{\nu,F}(D)\Big)\\ \le\operatorname{Cap}\big(K;L^1_{\nu,F}(D)\big)-\operatorname{Cap}\Big(\bigcup_{i=1}^k C_i;L^1_{\nu,F}(D)\Big)\\ =\operatorname{Cap}\Big(\bigcup_{i=1}^k K_i;L^1_{\nu,F}(D)\Big)-\operatorname{Cap}\Big(\bigcup_{i=1}^k C_i;L^1_{\nu,F}(D)\Big)\\ \le\sum_{i=1}^{k}\big(\operatorname{Cap}(K_i;L^1_{\nu,F}(D))-\operatorname{Cap}\big(C_i;L^1_{\nu,F}(D)\big)\big).\end{aligned}$$

Passing to suprema over K_i and C_i in the last expression gives the claim.

Having property 4 for compact and open sets, we can prove it for arbitrary sets.

5. Put $E=\bigcup_{k=1}^\infty E_k$. Using property 2, we have

$$\overline{\operatorname{Cap}}\big(E;L^1_{\nu,F}(D)\big)\ge\lim_{k\to\infty}\overline{\operatorname{Cap}}\big(E_k;L^1_{\nu,F}(D)\big).$$

Prove the reverse inequality. We may assume that $\overline{\operatorname{Cap}}\big(E_k;L^1_{\nu,F}(D)\big)<\infty$ for each k (if this fails then the reverse inequality is obvious). Fix $\varepsilon>0$ and choose an open set U_k such that $E_k\subset U_k\subset D_F$ and

$$\overline{\operatorname{Cap}}\big(U_k;L^1_{\nu,F}(D)\big)\le\overline{\operatorname{Cap}}\big(E_k;L^1_{\nu,F}(D)\big)+2^{-k}\varepsilon.$$

Since $\overline{\operatorname{Cap}}\big(\bigcup_{k=1}^n E_k;L^1_{\nu,F}(D)\big)=\overline{\operatorname{Cap}}\big(E_n;L^1_{\nu,F}(D)\big)<\infty$ for each n, by property 4,

$$\overline{\operatorname{Cap}}\Big(\bigcup_{k=1}^n U_k;L^1_{\nu,F}(D)\Big)-\overline{\operatorname{Cap}}\Big(\bigcup_{k=1}^n E_k;L^1_{\nu,F}(D)\Big)\le\sum_{k=1}^n 2^{-k}\varepsilon<\varepsilon.$$

If K is a compact set in $\bigcup_{k=1}^\infty U_k$, then $K\subset\bigcup_{k=1}^n U_k$ for some n, whence

$$\begin{aligned}\operatorname{Cap}\big(K;L^1_{\nu,F}(D)\big)\le\overline{\operatorname{Cap}}\Big(\bigcup_{k=1}^n U_k;L^1_{\nu,F}(D)\Big)\\ \le\overline{\operatorname{Cap}}\Big(\bigcup_{k=1}^n E_k;L^1_{\nu,F}(D)\Big)+\varepsilon\le\lim_{k\to\infty}\overline{\operatorname{Cap}}\big(E_k;L^1_{\nu,F}(D)\big)+\varepsilon.\end{aligned}$$

Therefore,

$$\begin{aligned}\overline{\operatorname{Cap}}\big(E;L^1_{\nu,F}(D)\big)\le\overline{\operatorname{Cap}}\Big(\bigcup_{k=1}^n U_k;L^1_{\nu,F}(D)\Big)\\ \le\sup_K\operatorname{Cap}\big(K;L^1_{\nu,F}(D)\big)\le\lim_{k\to\infty}\overline{\operatorname{Cap}}\big(E_k;L^1_{\nu,F}(D)\big)+\varepsilon,\end{aligned}$$

where the supremum is taken over all compact sets $K \subset \bigcup_{k=1}^{\infty} U_k$.

6. By property 4, for each finite set, we have

$$\overline{\mathrm{Cap}}\Bigl(\bigcup_{k=1}^{n} E_k; L^1_{\nu,F}(D)\Bigr) \le \sum_{k=1}^{n} \overline{\mathrm{Cap}}\bigl(E_k; L^1_{\nu,F}(D)\bigr).$$

Since the family $\bigcup_{k=1}^{n} E_k$ constitutes an increasing sequence, the desired assertion follows from property 5. □

A set E is called *measurable* with respect to capacity if

$$\underline{\mathrm{Cap}}\bigl(E; L^1_{\nu,F}(D)\bigr) = \overline{\mathrm{Cap}}\bigl(E; L^1_{\nu,F}(D)\bigr).$$

By Lemma 16, the capacity in $L^1_{\nu,F}(D)$ is a Choquet capacity [23]. This implies [23] that all analytic sets (and, in particular, Borel sets) are measurable.

A property is said to be fulfilled *quasieverywhere* if it is fulfilled everywhere but on a set of capacity zero.

Definition 17. A function $f \in L^1_{\nu,F}(D)$ is called *precise* if there exists a sequence $\{f_s\}$, $s \in \mathbb{N}$, of functions in $L^1_{\nu,F}(D) \cap C(D)$ such that

(1) $\|f - f_s \mid L^1_{\nu,F}(D)\| \to 0 \quad \text{as } s \to \infty$;

(2) for every positive $\varepsilon > 0$, there exists an open set $U_\varepsilon \subset D_F$ such that $\overline{\mathrm{Cap}}(U_\varepsilon) < \varepsilon$ and the sequence f_s converges to f uniformly on $D_F \setminus U_\varepsilon$.

Remark 18. 1. Each element in $L^1_{\nu,F}(D)$ contains a precise function (see [5, Corollary 6.4]).

2. Each sequence of precise functions converging in $L^1_{\nu,F}(D)$ to a precise function f contains a subsequence converging to f quasieverywhere (see [5, Corollary 6.7]).

Lemma 19 [10, Lemma 6.4; 5, Lemma 6.5]. *Let $E \subset D_F$ be an arbitrary set and let $f \in L^1_{\nu,F}(D)$ be a precise function such that $|f(x)| \ge \alpha > 0$ quasieverywhere on E. Then*

$$\overline{\mathrm{Cap}}\bigl(E; L^1_{\nu,F}(D)\bigr) \le \frac{\|f \mid L^1_{\nu,F}(D)\|^\nu}{\alpha^\nu}.$$

Proof. By Remark 15, f can be assumed nonnegative. Consider $g(x) = f(x)/\alpha$. Since g is a precise function, there exist a sequence $\{g_k \in L^1_{\nu,F}(D) \cap C(D)\}$ for which $\|g - g_k \mid L^1_{\nu,F}(D)\| \to 0$ as $k \to \infty$ and an open set U_ε, $\overline{\mathrm{Cap}}\bigl(U_\varepsilon; L^1_{\nu,F}(D)\bigr) \le \varepsilon$ for arbitrary $\varepsilon \in (0,1)$ such that, on the complement $D_F \setminus U_\varepsilon$, the sequence $\{g_k\}$ converges uniformly to g.

Let $E_1 = \{x \in D_F : g(x) \ge 1\}$. Then $E \subset E_1 \cup E_0$, where E_0 is a set of capacity zero. Hence,

$$\overline{\mathrm{Cap}}\bigl(E; L^1_{\nu,F}(D)\bigr) \le \overline{\mathrm{Cap}}\bigl(E_1; L^1_{\nu,F}(D)\bigr) + \overline{\mathrm{Cap}}\bigl(E_0; L^1_{\nu,F}(D)\bigr) = \overline{\mathrm{Cap}}\bigl(E_1; L^1_{\nu,F}(D)\bigr).$$

Starting from some number on, $E_{k,\varepsilon} = \{x \in D_F : g_k(x) > 1 - \varepsilon\}$ include $E_1 \setminus U_\varepsilon$. Therefore, for sufficiently large numbers, $\overline{\mathrm{Cap}}\bigl(E_1 \setminus U_\varepsilon; L^1_{\nu,F}(D)\bigr) \le \overline{\mathrm{Cap}}\bigl(E_{k,\varepsilon}; L^1_{\nu,F}(D)\bigr)$, whence

$$\begin{aligned}\overline{\mathrm{Cap}}\bigl(E; L^1_{\nu,F}(D)\bigr) &\le \overline{\mathrm{Cap}}\bigl(E_1; L^1_{\nu,F}(D)\bigr)\\ &\le \overline{\mathrm{Cap}}\bigl(E_1 \setminus U_\varepsilon; L^1_{\nu,F}(D)\bigr) + \overline{\mathrm{Cap}}\bigl(U_\varepsilon; L^1_{\nu,F}(D)\bigr) \le \overline{\mathrm{Cap}}\bigl(E_{k,\varepsilon}; L^1_{\nu,F}(D)\bigr) + \varepsilon.\end{aligned}$$

Note that $\overline{\mathrm{Cap}}\bigl(E_{k,\varepsilon}; L^1_{\nu,F}(D)\bigr) \le \|g_k \mid L^1_{\nu,F}(D)\|^\nu/(1-\varepsilon)^\nu$ for each k. Taking into account the equality $\lim_{k\to\infty}\|g_k \mid L^1_{\nu,F}(D)\| = \|g \mid L^1_{\nu,F}(D)\|$, we infer $\overline{\mathrm{Cap}}\bigl(E; L^1_{\nu,F}(D)\bigr) \le \frac{\|g|L^1_{\nu,F}(D)\|^\nu}{(1-\varepsilon)^\nu} + \varepsilon$. Since ε is arbitrary,

$$\overline{\mathrm{Cap}}\bigl(E; L^1_{\nu,F}(D)\bigr) \le \|g \mid L^1_{\nu,F}(D)\|^\nu = \frac{\|f \mid L^1_{\nu,F}(D)\|^\nu}{\alpha^\nu}. \quad \square$$

Corollary 20 [5, Corollary 6.6; 10, Corollary 6.2]. *Two precise functions belonging to one element of $L^1_{\nu,F}(D)$ coincide quasieverywhere on D_F.*

Proof. Let f and g be two precise functions belonging to one element in $L^1_{\nu,F}(D)$. In particular,

$$\big\|f - g \mid L^1_{\nu,F}(D)\big\| = 0. \tag{7}$$

Put $\Sigma = \{x \in D_F : f(x) \neq g(x)\}$ and $\Sigma_k = \{x \in D_F : |f(x) - g(x)| > 2^{-k}\}$; then

$$\Sigma = \bigcup_{k=1}^{\infty} \Sigma_k.$$

By Lemma 19 and (7), for each $k \in \mathbb{N}$ we have $\overline{\mathrm{Cap}}\big(\Sigma_k; L^1_{\nu,F}(D)\big) = 0$. By the countable semiadditivity of the capacity (see Lemma 16), $\overline{\mathrm{Cap}}\big(\Sigma; L^1_{\nu,F}(D)\big) = 0$. □

Definition 21. Given an arbitrary set $E \subset D_F$, put $A(E) = \big\{f \in L^1_{\nu,F}(D) :$ a precise representative $\tilde f(x)$ is at least 1 quasieverywhere in $E\big\}$. A function $f \in A(E)$ is called *admissible* for E.

Lemma 22 [10, Lemma 6.5]. *Let $E \subset D_F$ be an arbitrary set. The set $A(E)$ of admissible functions is weakly closed and convex in $L^1_{\nu,F}(D)$.*

Proof. If $A(E) = \varnothing$ then there is nothing to prove. Otherwise, consider $f, g \in A(E)$ and their corresponding precise functions $\tilde f$ and $\tilde g$. We have $\tilde f \geq 1$ and $\tilde g \geq 1$ quasieverywhere on E. For any $t \in (0,1)$, we have $t\tilde f + (1-t)\tilde g \geq 1$ quasieverywhere on E. Thus, $tf + (1-t)g \in A(E)$, and the convexity of $A(E)$ is proved.

Prove the weak closedness of $A(E)$. Let $\{f_n \in A(E)\}_{n\in\mathbb{N}}$ converge weakly to $f \in L^1_{\nu,F}(D)$. Use Mazur's Lemma for weakly convergent sequences (see, for example, [10, Lemma 1.6]): there exists a convex combination $\{g_k\}_{k\in\mathbb{N}} \in A(E)$ of $f_1, \dots, f_k$ converging to f in $L^1_{\nu,F}(D)$; i.e., $\big\|f - g_k \mid L^1_{\nu,F}(D)\big\| \to 0$ as $k \to \infty$.

For each $k \in \mathbb{N}$, for g_k there exists a precise function $\tilde g_k$ such that

$$\overline{\mathrm{Cap}}\big(\{x \in E : \tilde g_k < 1\}; L^1_{\nu,F}(D)\big) = 0.$$

Let $\tilde f$ be a precise function for f. Then $\tilde f - \tilde g_k$ is a precise function too. For every $\varepsilon \in (0,1)$, we have

$$\{x \in E : \tilde f(x) \leq 1 - \varepsilon\} \subset \{x \in D_F : |\tilde f(x) - \tilde g_k(x)| \geq \varepsilon\} \cup \bigcup_{i=1}^{\infty} \{x \in E : \tilde g_i(x) < 1\}.$$

Applying Lemmas 16 and 19, we obtain

$$\overline{\mathrm{Cap}}\big(\{x \in E : \tilde f(x) \leq 1 - \varepsilon\}; L^1_{\nu,F}(D)\big) \leq \frac{\big\|g_k - f \mid L^1_{\nu,F}(D)\big\|^{\nu}}{\varepsilon^{\nu}}.$$

Letting k tend to ∞, we see that

$$\overline{\mathrm{Cap}}\big(\{x \in E : \tilde f(x) \leq 1 - \varepsilon\}; L^1_{\nu,F}(D)\big) = 0.$$

Further,

$$\{x \in E : \tilde f(x) < 1\} = \bigcup_{i=1}^{\infty} \{x \in E : \tilde f(x) \leq 1 - j^{-1}\};$$

i.e., the left-hand side is equal to the union of an increasing sequence of sets of capacity zero. Then, by Lemma 16(5), $\overline{\mathrm{Cap}}\big(\{x \in E : \tilde f(x) < 1\}; L^1_{\nu,F}(D)\big) = 0$. Consequently, $\tilde f \geq 1$ quasieverywhere on E and $f \in A(E)$. Thus, $A(E)$ is weakly closed. □

Lemma 22 yields

Corollary 23 [10, Corollary 6.4]. *If $E \in D_F$ and $A(E) \neq \varnothing$ then there exists a unique element $f_E \in A(E)$ such that*

$$\|f_E \mid L^1_{\nu,F}(D)\| = \inf\{\|f \mid L^1_{\nu,F}(D)\| : f \in A(E)\}.$$

Proof. Put $I = \inf\{\|f \mid L^1_{\nu,F}(D)\| : f \in A(E)\}$. Let $\{f_k\}_{k\in\mathbb{N}} \subset A(E)$ be a sequence such that $I = \lim_{k\to\infty}\|f_k \mid L^1_{\nu,F}(D)\|$. From the sequence $\{f_k\}_{k\in\mathbb{N}}$, we can refine a weakly convergent subsequence f_{k_j}. Denote by f_E its weak limit: $f_E \underset{\mathrm{w}}{=} \lim_{j\to\infty} f_{k_j}$. By Lemma 22, $f_E \in A(E)$. Uniqueness can be deduced in a standard manner from the uniform convexity of the norm on $L^1_{\nu,F}(D)$. □

Corollary 24 [10, Corollary 6.5]. *Suppose that $\{E_m\}_{m\in\mathbb{N}}$ be an increasing sequence of sets, $E = \bigcup_{m=1}^{\infty} E_m$, and $A(E_m) \neq \varnothing$ for all m. Then*

$$A(E) = \bigcap_{m=1}^{\infty} A(E_m), \qquad \lim_{m\to\infty}\|f_{E_m} \mid L^1_{\nu,F}(D)\| = \inf\{\|f \mid L^1_{\nu,F}(D)\| : f \in A(E)\}.$$

Proof. Note that $A(E_m) \supset A(E_{m+1}) \supset \dots \supset A(E)$, and so $\|f_{E_m} \mid L^1_{\nu,F}(D)\| \le \|f_{E_{m+1}} \mid L^1_{\nu,F}(D)\|$.

Show that $A(E) = \bigcap_{m=1}^{\infty} A(E_m)$. Indeed, the inclusion $A(E) \subset \bigcap_{m=1}^{\infty} A(E_m)$ is obvious. Let $f \in L^1_{\nu,F}(D)$ belong to $\bigcap_{m=1}^{\infty} A(E_m)$. Then, for each m, the function f defines a precise function $\tilde{f}_m$ such that $\tilde{f}_m \ge 1$ quasieverywhere on E_m. All these precise functions coincide quasieverywhere on D_F. Define the function

$$\tilde{f}(x) = \begin{cases} \tilde{f}_1(x) & \text{if } x \in E_1, \\ \tilde{f}_m(x) & \text{if } x \in E_m \setminus E_{m-1},\ m \ge 2, \\ \tilde{f}_1(x) & \text{if } x \notin \bigcup\limits_{m=1}^{\infty} E_m. \end{cases}$$

Obviously, $\tilde{f}(x)$ coincides with $\tilde{f}_1(x)$ quasieverywhere on D_F and so it is a precise function. Thus, for the given function f, we have found a precise representative $\tilde{f}$ such that $\tilde{f} \ge 1$ quasieverywhere on each E_m, and so $\tilde{f} \ge 1$ quasieverywhere on $E = \bigcup_{m=1}^{\infty} E_m$. Finally, $f \in A(E)$; therefore, $\bigcap_{m=1}^{\infty} A(E_m) \subset A(E)$.

If $A(E) = \varnothing$ then $\inf\{\|f \mid L^1_{\nu,F}(D)\| : f \in A(E)\} = +\infty$. Simultaneously, $\lim_{m\to\infty}\|f_{E_m} \mid L^1_{\nu,F}(D)\| = +\infty$. Otherwise, from the sequence $\{f_{E_m}\}_{m\in\mathbb{N}}$, we would refine a subsequence converging weakly to some $f_0 \in L^1_{\nu,F}(D)$. Prove that $f_0 \in A(E)$. Indeed, it is not hard to check that, by the weak closedness of $A(E_m)$, we have $f_0 \in A(E_m)$ for all $m \in \mathbb{N}$. We infer that $f_0 \ge 1$ quasieverywhere on E_m for all $m \in \mathbb{N}$. Consequently, $f_0 \ge 1$ quasieverywhere on E; therefore, $f_0 \in A(E)$.

If $A(E) \neq \varnothing$ then $\|f_{E_m} \mid L^1_{\nu,F}(D)\| \le \|f_E \mid L^1_{\nu,F}(D)\|$ and $\lim_{m\to\infty}\|f_{E_m} \mid L^1_{\nu,F}(D)\| \le \|f_E \mid L^1_{\nu,F}(D)\|$. Assume that $\lim_{m\to\infty}\|f_{E_m} \mid L^1_{\nu,F}(D)\| < \|f_E \mid L^1_{\nu,F}(D)\|$. Extracting from $\{f_{E_m}\}_{m\in\mathbb{N}}$ a subsequence converging weakly to $f_0 \in A(E)$, we have

$$\|f_0 \mid L^1_{\nu,F}(D)\| \le \varliminf_{m\to\infty} \|f_{E_m} \mid L^1_{\nu,F}(D)\| < \|f_E \mid L^1_{\nu,F}(D)\|,$$

which contradicts the extremality of f_E. □

Theorem 25 [10, Theorem 6.4; 5, Theorem 6.11]. *For an arbitrary set $E \subset D_F$,*

$$\overline{\mathrm{Cap}}(E; L^1_{\nu,F}(D)) = \inf\{\|f \mid L^1_{\nu,F}(D)\|^{\nu} : f \in A(E)\}.$$

If $A(E) \neq \varnothing$ then there is a function f_E such that

$$\overline{\mathrm{Cap}}(E; L^1_{\nu,F}(D)) = \|f_E \mid L^1_{\nu,F}(D)\|^{\nu}.$$

Proof. If $f \in A(E)$ then, by Lemma 19, we have the inequality $\overline{\mathrm{Cap}}\big(E; L^1_{\nu,F}(D)\big) \le \big\|f \mid L^1_{\nu,F}(D)\big\|^\nu$. Therefore, for an arbitrary set E, we have

$$\overline{\mathrm{Cap}}\big(E; L^1_{\nu,F}(D)\big) \le \inf\big\{\big\|f \mid L^1_{\nu,F}(D)\big\|^\nu : f \in A(E)\big\}.$$

For a compact set K, from the definition of capacity, we obtain

$$\inf\big\{\big\|f \mid L^1_{\nu,F}(D)\big\|^\nu : f \in A(K)\big\} \le \mathrm{Cap}\big(K; L^1_{\nu,F}(D)\big)$$

because the infimum on the right is taken over a less set of functions.

Suppose that E is an open set and $\{K_m\}_{m\in\mathbb{N}}$ is an increasing sequence of compact sets such that $E = \bigcup_{m=1}^\infty K_m$ and $\lim_{m\to\infty} \mathrm{Cap}\big(K_m; L^1_{\nu,F}(D)\big) = \overline{\mathrm{Cap}}\big(E; L^1_{\nu,F}(D)\big)$. Then, from the equality $\mathrm{Cap}\big(K_m; L^1_{\nu,F}(D)\big) = \big\|f_{K_m} \mid L^1_{\nu,F}(D)\big\|^\nu$ and Corollary 24, it follows that

$$\overline{\mathrm{Cap}}\big(E; L^1_{\nu,F}(D)\big) = \lim_{m\to\infty}\big\|f_{K_m} \mid L^1_{\nu,F}(D)\big\|^\nu = \inf\big\{\big\|f \mid L^1_{\nu,F}(D)\big\|^\nu : f \in A(E)\big\}.$$

Let E be an arbitrary set and let U be an open set including E. Then, obviously, $A(U) \supset A(E)$ and

$$\inf\big\{\big\|f \mid L^1_{\nu,F}(D)\big\|^\nu : f \in A(E)\big\} \le \inf\big\{\big\|f \mid L^1_{\nu,F}(D)\big\|^\nu : f \in A(U)\big\} = \overline{\mathrm{Cap}}\big(U; L^1_{\nu,F}(D)\big).$$

Hence, $\inf\big\{\big\|f \mid L^1_{\nu,F}(D)\big\|^\nu : f \in A(E)\big\} \le \overline{\mathrm{Cap}}\big(E; L^1_{\nu,F}(D)\big)$, which, combined with the reverse inequality at the beginning of the proof, constitutes the first assertion of the theorem. The second assertion stems from Corollary 23. □

The function f_E of Theorem 25 is called the *capacity function* for E.

Lemma 26. *If $f \in L^1_{\nu,F}(D)$ is a precise function then*

$$f(x) = \lim_{r\to 0} r^{-\nu} \int\limits_{B(x,r)} f(z)\,dz \tag{8}$$

for quasiall $x \in D_F$.

Proof. Indeed, since the result is local, we may assume by truncating that $f \in S^1_\nu(\mathbb{G})$ (the definition of the space $S^1_\nu(\mathbb{G})$ can be found in Subsection 3.2). For functions in the space of potentials, the lemma was proved in [5, Proposition 6.14]. □

Definition 27. Refer to a function f defined quasieverywhere on D_F as *quasicontinuous* if, for every $\varepsilon > 0$, there is an open set $U_\varepsilon \subset D_F$ such that $\overline{\mathrm{Cap}}\big(U_\varepsilon; L^1_{\nu,F}(D)\big) < \varepsilon$ and the restriction of f to the complement $D_F \setminus U_\varepsilon$ is continuous.

Remark 28. In Proposition 34 below, we will prove that a function of class $L^1_{\nu,F}(D)$ is quasicontinuous if and only if it is a precise function.

Definition 29. Let $E \subset D_F$ be a measurable set. A point $x \in D_F$ will be called a *point of nonzero density* for E if

$$\overline{\lim_{r\to 0}} \frac{|B(x,r) \cap E|}{|B(x,r)|} > 0.$$

The set of all $x \in D_F$ that are points of nonzero density for E will be denoted by $\widetilde{E}$.

Lemma 30 [10, Theorem 6.5; 5, Proposition 6.16]. *Let $E \subset D_F$ be a set of positive measure. If $f \in L^1_{\nu,F}(D)$ is quasicontinuous and $f(x) \geq g(x)$ for a.e. $x \in E$, where $g : E \cup \widetilde{E} \to \mathbb{R}$ is a lower semicontinuous function then $f(x) \geq g(x)$ for quasiall $x \in \widetilde{E}$.*

Proof. Since f is quasicontinuous, for every $\varepsilon > 0$ there exists an open set U_ε such that

$$\overline{\mathrm{Cap}}\big(U_\varepsilon; L^1_{\nu,F}(D)\big) < \varepsilon$$

and f is continuous on $D_F \setminus U_\varepsilon$. Let f_m be the nonnegative capacity function for $U_{\frac{1}{m}}$. Since $\big\|f_m \mid L^1_{\nu,F}(D)\big\| \to 0$ as $m \to \infty$, we may assume on passing to a subsequence that

$$\lim_{m\to\infty} f_m(x) = 0 \quad \text{for quasiall } x \in D_F. \tag{9}$$

By Lemma 26, for each m we have

$$f_m(x) = \lim_{r\to 0} r^{-\nu} \int\limits_{B(x,r)} f_m(z)\, dz \quad \text{for quasiall } x \in D_F. \tag{10}$$

Therefore, (9) and (10) are fulfilled for quasiall $x \in \widetilde{E}$. Let $x \in \widetilde{E}$ be a point at which (9) and (10) hold simultaneously. Since x is a positive capacity point of E, there is a number $\rho_0 > 0$ such that $\frac{|E\cap B(x,\rho)|}{|B(x,\rho)|} > \delta > 0$ for all $\rho \in (0, \rho_0)$. Prove that, for all sufficiently large m,

$$|U_{\frac{1}{m}} \cap E \cap B(x,\rho)| < |E \cap B(x,\rho)|$$

for ρ sufficiently small.

Indeed, since $\lim_{m\to\infty} f_m(x) = 0$, $f_m(x) < \delta$ for sufficiently large m. Next,

$$\overline{\lim_{\rho\to 0}} \frac{|U_{\frac{1}{m}} \cap E \cap B(x,\rho)|}{|E \cap B(x,\rho)|} \leq \overline{\lim_{\rho\to 0}} \frac{|B(x,\rho)|}{|E \cap B(x,\rho)|} \overline{\lim_{\rho\to 0}} \rho^{\nu} \int\limits_{U_{\frac{1}{m}} \cap E \cap B(x,\rho)} f_m(y)\, dy$$

$$< \frac{1}{\delta} \lim_{\rho\to 0} \frac{1}{|B(x,\rho)|} \int\limits_{B(x,\rho)} f_m(y)\, dy = \delta^{-1} f_m(x) < 1.$$

Thus, there are $m(x) \in \mathbb{N}$ and $\rho(x) > 0$ such that, for all $m > m(x)$ and every $\rho \in (0, \rho(x))$, the measure of $V_\rho = (E \cap B(x,\rho)) \setminus U_{\frac{1}{m}}$ is positive. Taking into account the continuity of f_m on $D_F \setminus U_{\frac{1}{m}}$ and the fact that $f(y) \geq g(y)$ for a.e. $y \in E$, we get

$$f(x) = \lim_{\rho\to 0} \frac{1}{|V_\rho|} \int\limits_{V_\rho} f(y)\, dy \geq \lim_{\rho\to 0} \frac{1}{|V_\rho|} \int\limits_{V_\rho} g(y)\, dy \geq g(x). \quad \square$$

Corollary 31 [10, Corollary 26.7; 5, Corollary 6.17]. *Suppose that a measurable set $E \subset D_F$ has positive measure. If two quasicontinuous functions $f_1, f_2 \in L^1_{\nu,F}(D)$ coincide almost everywhere on E then f_1 and f_2 coincide quasieverywhere on $\widetilde{E}$.*

Proof. Indeed, by Lemma 30 we infer that $f_1(x) \leq f_2(x)$ and $f_1(x) \geq f_2(x)$ quasieverywhere on $\widetilde{E}$. Consequently, $f_1(x) = f_2(x)$ quasieverywhere on $\widetilde{E}$. $\square$

Corollary 32 [5, Corollary 6.19]. *For every set $E \subset D_F$, we have*

$$\overline{\mathrm{Cap}}\big(E \cup \widetilde{E}; L^1_{\nu,F}(D)\big) = \overline{\mathrm{Cap}}\big(E; L^1_{\nu,F}(D)\big).$$

PROOF. Note that, by Property 2 of Lemma 16,

$$\overline{\mathrm{Cap}}\big(E \cup \widetilde{E}; L^1_{\nu,F}(D)\big) \geq \overline{\mathrm{Cap}}\big(E; L^1_{\nu,F}(D)\big). \tag{11}$$

If $\overline{\mathrm{Cap}}\big(E; L^1_{\nu,F}(D)\big) = \infty$ then the equality is obvious.

Let $\overline{\mathrm{Cap}}\big(E; L^1_{\nu,F}(D)\big) < \infty$. Then, by Theorem 25 and Remark 18, there exists a precise function f_E such that $\overline{\mathrm{Cap}}\big(E; L^1_{\nu,F}(D)\big) = \big\|f_E \mid L^1_{\nu,F}(D)\big\|^\nu$ and $f(x) \geq 1$ quasieverywhere on E. By Lemma 30, the inequality $f(x) \geq 1$ is valid quasieverywhere on $\widetilde{E}$. Thus, $f_E \in A(E \cup \widetilde{E})$. Consequently,

$$\overline{\mathrm{Cap}}\big(E \cup \widetilde{E}; L^1_{\nu,F}(D)\big) \leq \big\|f_E \mid L^1_{\nu,F}(D)\big\|^\nu = \overline{\mathrm{Cap}}\big(E; L^1_{\nu,F}(D)\big),$$

which, together with (11), guarantees the desired equality. □

Corollary 33 [5, Corollary 6.20]. *Suppose that the conditions of Lemma* 30 *are fulfilled. If $f(x) = g(x)$ a.e. on $E \subset D_F$, where f is a function quasicontinuous on D_F and g is a function continuous on $E \cup \widetilde{E}$ then $f(x) = g(x)$ for quasiall points $x \in \widetilde{E}$.*

PROOF. The assertion is immediate from Lemma 30 since g is in particular lower semicontinuous on $E \cup \widetilde{E}$. □

Proposition 34. *Definitions* 17 and 27 are equivalent: every precise function is quasicontinuous, and, conversely, every quasicontinuous function of class $L^1_{\nu,F}(D)$ is precise.

PROOF. Indeed, if f is a precise function then, by condition 2 in Definition 17 then, by condition 2 in Definition 17 for every $\varepsilon > 0$, there exists an open set U_ε of capacity less than $\varepsilon > 0$ such that the sequence of continuous functions $f_n \in L^1_{\nu,F}(D)$ converges uniformly on the complement $D_F \setminus U_\varepsilon$. Therefore, f is continuous on $D_F \setminus U_\varepsilon$.

Suppose that $f \in L^1_{\nu,F}(D)$ is quasicontinuous. Then, by Remark 18, there exists a precise function $\tilde{f}$ coinciding with f a.e. in D_F. By the above, $\tilde{f}$ is quasicontinuous, and so, by Corollary 31, f and $\tilde{f}$ coincide quasieverywhere. It remains to observe that a function coinciding quasieverywhere with a precise function is itself precise. □

3.2. Capacity in the space of potentials. Let Ω be an open connected set on a Carnot group $\mathbb{G}$. The *capacity* $\mathrm{cap}\big(K; W^1_\nu(\Omega)\big)$ of a compact set $K \subset \Omega$ in $W^1_\nu(\Omega)$ is the quantity

$$\mathrm{cap}\big(K; W^1_\nu(\Omega)\big) = \inf\big\|g \mid W^1_\nu(\Omega)\big\|^\nu,$$

where the infimum is taken over all continuous functions $g \in W^1_\nu(\Omega)$ such that $g \geq 1$ on K. For every set $E \subset \Omega$, the *inner capacity* of E equals

$$\underline{\mathrm{cap}}\big(E; L^1_\nu(\Omega)\big) = \sup\big\{\mathrm{cap}\big(K; W^1_\nu(\Omega)\big) : K \subset E,\ K \text{ is compact}\big\},$$

and the *outer capacity* of E,

$$\overline{\mathrm{cap}}\big(E; W^1_\nu(\Omega)\big) = \inf\big\{\underline{\mathrm{cap}}\big(U; W^1_\nu(\Omega)\big) : E \subset U,\ U \text{ is open}\big\}.$$

The properties of capacity on $W^1_\nu(\Omega)$ (see, for example, [5]) are analogous to the above properties of the capacity in $L^1_{\nu,F}(D)$.

The *space of Bessel potentials* on a Carnot group $\mathbb{G}$ is the space $S^\alpha_p(\mathbb{G})$ of functions of the form

$$g(x) = f * J_\alpha(x) = \int\limits_{\mathbb{G}} J_\alpha(y^{-1}x) f(y)\, dy,$$

where $f \in L_p(\mathbb{G})$, $p \in (1,\infty)$, and J_α is a Bessel kernel [24] on $\mathbb{G}$, $\alpha \in (0,\infty)$. Define the norm on the space of potentials as follows: $\|g \mid S_p^\alpha(\mathbb{G})\| = \|f \mid L_p(\mathbb{G})\|$. If $\alpha = k$ is a natural then $S_p^k(\mathbb{G})$ coincides with the Sobolev space $W_p^k(\mathbb{G})$ [24]. Below we will be interested in the case when $\alpha = 1$ because $S_p^1(\mathbb{G})$ coincides with $W_p^1(\mathbb{G})$.

The *Bessel capacity* of an arbitrary set $E \in \mathbb{G}$ is defined as follows (see [25] for details):

$$\operatorname{cap}\big(E; S_p^1(\mathbb{G})\big) = \inf\bigg\{\int\limits_{\mathbb{G}} f(y)^p\,dy : f * J_1(x) \ge 1 \text{ at } x \in E\bigg\}. \tag{12}$$

It was shown in [25] that the capacity in $S_p^1(\mathbb{G})$ is the outer capacity.

Proposition 35 [25, Corollary 2]. *Let $x \in \mathbb{G}$, $r < 1$. The Bessel capacity of a ball meets the equivalence* $\operatorname{cap}\big(B(x,r); S_\nu^1(\mathbb{G})\big) \sim \big(\log \frac{2}{r}\big)^{1-\nu}$.

REMARK 36. By the equivalence of the norms in $S_\nu^1(\mathbb{G})$ and $W_\nu^1(\mathbb{G})$ [24], the capacities $\operatorname{cap}\big(E; S_\nu^1(\mathbb{G})\big)$ and $\overline{\operatorname{cap}}\big(E; W_\nu^1(\mathbb{G})\big)$ are also comparable; i.e., there exist constants m and M such that

$$m\overline{\operatorname{cap}}\big(E; W_\nu^1(\mathbb{G})\big) \le \operatorname{cap}\big(E; S_\nu^1(\mathbb{G})\big) \le M\,\overline{\operatorname{cap}}\big(E; W_\nu^1(\mathbb{G})\big). \tag{13}$$

Lemma 37. *Let $\Sigma \subset D_F$. The following three properties are equivalent:*

$$\overline{\operatorname{Cap}}\big(\Sigma; L_{\nu,F}^1(D)\big) = 0; \quad \overline{\operatorname{cap}}\big(\Sigma; W_\nu^1(\mathbb{G})\big) = 0; \quad \operatorname{cap}\big(\Sigma; S_\nu^1(\mathbb{G})\big) = 0.$$

PROOF. The equivalence of the last two equalities follows from Remark 36.

Let $\overline{\operatorname{Cap}}\big(\Sigma; L_{\nu,F}^1(D)\big) = 0$. By the countable additivity of capacity, we may assume that Σ is included in a ball $B_\Sigma \subset D$ and F lies in a ball $B_F \subset D$; moreover, $\operatorname{dist}(B_F, B_\Sigma) > 0$, $\operatorname{dist}(\partial D, B_F) > 0$, and $\operatorname{dist}(\partial D, B_\Sigma) > 0$.

Since $\overline{\operatorname{Cap}}\big(\Sigma; L_{\nu,F}^1(D)\big) = 0$, there is an inclusion-ordered sequence of open sets $\{U_k\}$ such that

$$B_S \supset U_1 \supset U_2 \dots \supset \Sigma \quad \text{and} \quad \overline{\operatorname{Cap}}\big(U_k; L_{\nu,F}^1(D)\big) \le \frac{1}{2^k}.$$

By Theorem 25, there exists a sequence of functions $h_k \in L_{\nu,F}^1(D)$ such that $h_k \ge 1$ quasieverywhere on U_k and $\|h_k \mid L_{\nu,F}^1(D)\| \le 1/2^k$. Passing to the truncation $\min(1, h_k)$, we may assume that $h_k = 1$ everywhere on U_k.

Suppose that $\Omega \subset D$ is a John domain including the balls B_Σ and B_F and such that $\operatorname{dist}(\partial\Omega, \partial D) > 0$ (Lemma 10). For sufficiently small $\delta > 0$, choose (see Remark 11) an additional John domain $\Omega_\delta \supset \Omega$ so that $\operatorname{dist}(\partial\Omega, \partial\Omega_\delta) \ge \delta$ and $\operatorname{dist}(\partial D, \partial\Omega_\delta) \ge \delta$. By the Poincaré inequality (Lemma 12), we have $\|h_k \mid L_\nu(\Omega_\delta)\| \le C\|h_k \mid L_\nu^1(\Omega_\delta)\|$. Therefore, passing to a subsequence, we may assume that $h_k \to 0$ a.e. on Ω_δ and $\nabla h_k \to 0$ a.e. on Ω_δ. Define a cut-of-function $\eta \in C_0^\infty(\mathbb{G})$ such that $\eta = 1$ on Ω and $\eta = 0$ on $\mathbb{G} \setminus \Omega_\delta$. Then the products $\eta h_k \in W_\nu^1(\mathbb{G})$ are such that

$$\eta h_k(x) = \begin{cases} h_k(x) & \text{for } x \in \Omega, \\ 0 & \text{for } x \in \mathbb{G} \setminus \Omega_\delta. \end{cases}$$

Further, $|\nabla(\eta h_k)| \le |(\nabla\eta)h_k| + |\eta\nabla h_k|$ and $\|\eta h_k \mid W_\nu^1(\mathbb{G})\| \to 0$ as $k \to \infty$. Thus, $\operatorname{cap}\big(U_k; W_\nu^1(\mathbb{G})\big) \to 0$ as $k \to \infty$, whence

$$\overline{\operatorname{cap}}\big(\Sigma; W_\nu^1(\mathbb{G})\big) = 0 \quad \text{and} \quad \operatorname{cap}\big(\Sigma; S_\nu^1(\mathbb{G})\big) = 0. \tag{14}$$

Suppose that (14) holds. Then there exist an inclusion decreasing sequence of open sets $\{W_k\} \subset B_\Sigma$ containing Σ for which $\operatorname{cap}\big(\Sigma; W_\nu^1(\mathbb{G})\big) \le 1/2^{k+1}$ and a sequence of functions $u_k \in W_\nu^1(\mathbb{G})$ such that $u_k = 1$ quasieverywhere on W_k and $\|u_k \mid W_\nu^1(\mathbb{G})\|^\nu \le 1/2^k$.

Define some cut-of-functions $\eta' \in C_0^\infty(\mathbb{G})$ such that $\eta = 1$ on B_Σ and $\eta' = 0$ on $\mathbb{G} \setminus \lambda B_\Sigma$, where $\lambda > 1$, for which $D \supset \lambda B_\Sigma \supset B_\Sigma$ and $\lambda B_\Sigma \cap B_F = \varnothing$. Then the function $\eta' \cdot u_k = f_k \in L^1_{\nu,\varphi(F)}(D)$ satisfies $f_k = 1$ on W_k and $\|f_k \mid L^1_{\nu,\varphi(F)}(D)\|^\nu \le c/2^k$, where c is a constant independent of k. Then $\operatorname{Cap}(\Sigma; L^1_{\nu,F}(D)) = 0$. $\square$

Remark 38. The method of Lemma 37 is applicable for proving the more general assertion: Let $\{U_k\}_1^\infty \subset D_F$ be a sequence of open sets lying in some ball $B(0,R)$ such that $\operatorname{dist}(U_k, \partial D_F) \ge \eta > 0$ for all $k \in \mathbb{N}$. Then the following three equalities are equivalent:

$$\lim_{k\to\infty} \overline{\operatorname{Cap}}(U_k; L^1_{\nu,F}(D)) = 0; \quad \lim_{k\to\infty} \overline{\operatorname{cap}}(U_k; W^1_\nu(\mathbb{G})) = 0; \quad \lim_{k\to\infty} \operatorname{cap}(U_k; S^1_\nu(\mathbb{G})) = 0.$$

In particular, Proposition 35 gives the estimate of the capacity of a ball $B(x,r) \subset D_F$:

$$\overline{\operatorname{Cap}}(B(x,r); L^1_{\nu,F}(D)) = O\left(\left(\log \frac{2}{r}\right)^{1-\nu}\right) = o(1) \quad \text{as } r \to 0.$$

In the following lemma, we describe some characteristic property of sets of capacity zero.

Lemma 39. *A set $\Sigma \subset D_F$ has outer capacity zero if and only if there exists a lower semicontinuous function $u \in L^1_{\nu,F}(D)$ such that $u = \infty$ on Σ. The norm of u can be chosen arbitrarily small.*

Proof. *Necessity:* Step 1. Consider first a special location of a set Σ: $\Sigma \subset B_\Sigma \Subset D_F$, where B_Σ is a ball. From Lemma 37 we obtain

$$\operatorname{cap}(\Sigma; S^1_\nu(\mathbb{G})) = 0.$$

Then there exists a sequence of nonnegative functions $f_k \in L_\nu(\mathbb{G})$ such that $\|f_k \mid L_\nu(\mathbb{G})\| \le 2^{-k}$ and

$$f_k * J_1(x) \ge 1 \quad \text{at all points } x \in \Sigma.$$

The function

$$f = \sum_{k=1}^\infty f_k \tag{15}$$

is nonnegative, belongs to $L_\nu(\mathbb{G})$, and $f * J_1(x) = \infty$ at all points $x \in \Sigma$. Since the kernel $J_1(z)$ is nonnegative on Σ and continuous everywhere but at one point $z = 0$, the convolution $f * J_1(x)$ is lower semicontinuous (by Fatou's Lemma).

Consider the Lipschitz function $\eta : D \to [0,1]$ such that

$$\eta(x) = \begin{cases} 0 & \text{if } x \in F, \\ 1 & \text{if } x \in B_\Sigma. \end{cases}$$

Since $W^1_\nu(\mathbb{G}) = S^1_\nu(\mathbb{G})$, the restriction of the product $\eta(x) \cdot f * J_1(x)$ on D: $u(x) = \eta(x) \cdot f * J_1(x)|_D$ obviously belongs to $L^1_{\nu,F}(D)$ and satisfies all conditions of the lemma.

Note that the norm of $u(x)$ can be made arbitrarily small since the properties of $f * J_1(x)$ do not depend on the number of summands in (15). Therefore, removing, if need be, finitely many summands from (15) and using its uniform convergence, we can make the norm $\|f * J_1 \mid W^1_\nu(D)\| = \|f * J_1 \mid L_\nu(D)\| + \|f * J_1 \mid L^1_\nu(D)\| \le \|f * J_1 \mid W^1_\nu(\mathbb{G})\|$ arbitrarily small. Since

$$|\nabla(\eta \cdot f * J_1)(x)| \le |\nabla\eta(x)| \cdot |f * J_1(x)| + |\eta(x)| \cdot |\nabla(f * J_1)(x)|,$$

the norm $\|u_B \mid L^1_{\nu,F}(D)\|$ also can be made arbitrarily small.

Step 2. Suppose that a set $\Sigma \subset D_F$ of outer capacity zero has an arbitrary location. Fix some family of balls $\{B_k\}$, $k \in \mathbb{N}$, of finite multiplicity such that $D_F = \bigcup_{k=1}^\infty B_k$ and $B_k \Subset D_F$ for each k (the existence of such coverings is proved, for example, in [26]). Then the intersection $\Sigma \cap B_k$ has outer capacity zero and satisfies the conditions of the first step of the proof. Consequently, there exists a nonnegative lower semicontinuous function $u_k \in L^1_{\nu,F}(D)$ such that $u_k = \infty$ on $\Sigma \cap B_k$ and $\|u_k \mid L^1_{\nu,F}(D)\| \leq \frac{\varepsilon}{M2^k}$, where ε is an arbitrary a priori defined number and M is the multiplicity of the covering $\{B_k\}$.

The function $u(x) = \sum_{k=1}^{\infty} u_k(x)$ is lower semicontinuous, belongs to $L^1_{\nu,F}(D)$, $u = \infty$ on Σ, and $\|u \mid L^1_{\nu,F}(D)\| \leq \varepsilon$.

Sufficiency. Suppose that there exists a lower semicontinuous function $u \in L^1_{\nu,F}(D)$ such that $u = \infty$ on some set $\Sigma \subset D_F$. Take arbitrary $\lambda > 0$. Then $U_\lambda = \{x \in D_F : u(x) > \lambda\}$ is open and contains Σ, and the precise function $\frac{u(x)}{\lambda}$ belongs to the class $A(U_\lambda)$ of admissible functions for the estimation of the capacity

$$\overline{\mathrm{Cap}}\big(\Sigma; L^1_{\nu,F}(D)\big) \leq \overline{\mathrm{Cap}}\big(U_\lambda; L^1_{\nu,F}(D)\big) \leq \frac{\|u \mid L^1_{\nu,F}(D)\|^\nu}{\lambda^\nu}$$

(in the last passage we used Lemma 19). Since λ is an arbitrary number, $\overline{\mathrm{Cap}}\big(\Sigma; L^1_{\nu,F}(D)\big)=0$. □

Theorem 40 [25, Theorem 9]. *Let $K \subset \mathbb{G}$ be a compact set and let $h(\rho)$ be a nondecreasing continuous function for which $h(0) = 0$. Suppose that*

$$\int_0^1 h(\rho)^{\frac{1}{\nu-1}} \frac{d\rho}{\rho} < \infty. \tag{16}$$

Then there exists a constant A such that $H^\infty_h(K) \leq A \operatorname{cap}\big(K; S^1_\nu(\mathbb{G})\big)$. Thus, $H^\infty_h(K) = 0$ if $\operatorname{cap}\big(K; S^1_\nu(\mathbb{G})\big) = 0$. (Here $H^\infty_h(K)$ is the Hausdorff content.)

The following assertion is an analog of Lemma 7.19 in [5].

Lemma 41. *Let $\{\gamma_m\}_{m\in\mathbb{N}}$ be a sequence of continua lying in a closed ball $\overline{B}_\gamma \subset D_F$. The limit $\lim_{m\to\infty} \overline{\mathrm{Cap}}\big(\gamma_m; L^1_{\nu,F}(D)\big)$ is zero if and only if $\lim_{m\to\infty} \operatorname{diam} \gamma_m = 0$.*

Proof. Let $\lim_{m\to\infty} \overline{\mathrm{Cap}}\big(\gamma_m; L^1_{\nu,F}(D)\big) = 0$. We may assume that $\operatorname{dist}(B_F, B_\gamma) > 0$. There exists a sequence of continuous functions $f_m \in L^1_{\nu,F}(D)$ such that $f_m = 1$ on γ_m and $\lim_{m\to\infty}\|f_m \mid L^1_{\nu,F}(D)\| = 0$.

Choose a number $\lambda > 1$ such that $\operatorname{dist}(B_F, \lambda B_\gamma) > 0$ and $\lambda B_\gamma \subset D_F$.

Define some cut-of-functions $\eta \in C^\infty_0(\mathbb{G})$ such that $\eta = 1$ on B_γ and $\eta = 0$ on $\mathbb{G} \setminus \lambda B_\gamma$. Then the functions $\eta \cdot f_m = u_m$ belong to $W^1_\nu(\mathbb{G})$, and, by the Poincaré inequality (Lemma 12), we have $\lim_{m\to\infty}\|f_m \mid W^1_\nu(\mathbb{G})\|=0$. We infer

$$\lim_{m\to\infty} \operatorname{cap}\big(\gamma_m; W^1_\nu(\mathbb{G})\big) = 0 \quad \text{and} \quad \lim_{m\to\infty} \operatorname{cap}\big(\gamma_m; S^1_\nu(\mathbb{G})\big) = 0.$$

Putting $h(\rho) = \rho$ in Theorem 40, we infer $\lim_{m\to\infty} H^\infty_1(\gamma_m) = 0$. It remains to observe that $H^\infty_1(E) = \operatorname{diam}(E)$.

The proof of the converse is obvious. □

3.3. Generalized Teichmüller capacity.

Definition 42. Refer as the *generalized Teichmüller capacity* of a ring $D_{r,R} = \{x \in \mathbb{G} : r < d(0,x) < R\}$ to

$$GT(r,R) = \inf_u \int_{D_{r,R}} |\nabla u|^\nu \, dx,$$

where the infimum is taken over all quasicontinuous functions $u \in W^1_\nu(D_{r,R})$ satisfying the conditions $\min u|_{S(0,t)} \leq 0$ and $\max u|_{S(0,t)} \geq 1$ for a.e. $t \in (r,R)$.

A quasicontinuous function is continuous on almost all spheres (see Proposition 56 below), and the maximum and minimum in Definition 42 are taken over such spheres.

Proposition 43 [25, Proposition 7]. *The generalized Teichmüller capacity $GT(r,R)$ is strictly positive for every $0<r<R<\infty$.*

Corollary 44 [25, Corollary 4]. *The generalized Teichmüller capacity satisfies the lower bound*

$$GT(r,R) \geq \gamma_1 \log \frac{R}{r}. \tag{17}$$

Proposition 45. *Suppose that U is a domain in $\mathbb{G}$ and $\gamma_0, \gamma_1 \subset U$ are two connected sets of positive diameter. If γ_0 and γ_1 have a common limit point in U then there is no quasicontinuous function $v \in L^1_\nu(U)$ with $v|_{\gamma_0} = 0$ and $v|_{\gamma_1} = 1$.*

Proof. Suppose on the contrary that such a function exists. Consider a ring $D_{r,R} \subset U$ centered at a common limit point $x \in U$ so that R does not exceed the maximum of the diameters of γ_0 and γ_1. Then the definition of generalized Teichmüller capacity (see Definition 42) and Corollary 44 yield the inequalities

$$\int\limits_U |\nabla v|^\nu \, dx \geq \int\limits_{D_{r,R}} |\nabla v|^\nu \, dx \geq GT(r,R) \geq \gamma_1 \log \frac{R}{r}. \tag{18}$$

It follows from (18) that $\|v \mid L^1_\nu(U)\|^\nu = \infty$ as $r \to 0$, which contradicts the membership of v in $L^1_\nu(U)$. $\square$

4. Properties of φ

Let us continue the study of mappings $\varphi : D \to D'$ of class IL^1_ν. Every φ possesses the properties of Subsection 1.3.

4.1. Construction of a quasicontinuous representative for a mapping φ. In this subsection, we construct a quasicontinuous mapping ψ that will coincide with φ a.e. on D_F.

Lemma 46 [5, Lemma 7.2]. *Suppose that $E \subset D_F$ is a set of positive measure, a mapping φ is continuous on E, and $f \in L^1_{\nu,\varphi(F)}(D')$ is a lower semicontinuous function. If $g = \varphi^* f$ is a precise function in $L^1_{\nu,F}(D)$ then $g(x) \geq f \circ \varphi(x)$ quasieverywhere on $E \cap \widetilde{E}$.*

Proof. Since φ is continuous on E, the function $f \circ \varphi$ is lower semicontinuous on E. By the properties of φ^* (Lemma 14), $g = f \circ \varphi$ a.e. on D and, in particular, $g(x) \geq f \circ \varphi(x)$ for a.e. $x \in E$. By Lemma 30, $g(x) \geq f \circ \varphi(x)$ for quasiall $x \in E \cap \widetilde{E}$. $\square$

From Lemma 46 we obtain

Corollary 47. *Suppose that $E \subset D_F$ is a subset of positive measure consisting of points of positive density, φ is continuous on E, and $f \in L^1_{\nu,\varphi(F)}(D')$ is a lower semicontinuous function. If $g = \varphi^* f$ is a precise function in $L^1_{\nu,F}(D)$ then $g(x) \geq f \circ \varphi(x)$ quasieverywhere on E.*

Lemma 48 [5, Lemma 7.3]. *Let $E \subset D_F$ be a set of positive measure consisting of points of positive density and let φ be continuous on E. If $\Sigma \subset D'_F$ is a set of outer capacity zero then $\varphi^{-1}(\Sigma) \cap E$ has capacity zero.*

Proof. Let $f \in L^1_{\nu,\varphi(F)}(D')$ be a lower semicontinuous function of Lemma 39. By Corollary 47, the precise function $g = \varphi^* f$ is at least the function $f \circ \varphi$ quasieverywhere on E. In particular, $g(x) = \infty$ for quasiall points $x \in \varphi^{-1}(\Sigma) \cap E$. Lemma 19 yields $\overline{\mathrm{Cap}}\big(\varphi^{-1}(\Sigma) \cap E; L^1_{\nu,F}(D)\big) = 0$. $\square$

Lemma 48 implies

Lemma 49 [5, Lemma 7.4]. *Let $f_k \in L^1_{\nu,\varphi(F)}(D')$ be a sequence of precise functions converging quasieverywhere to $f \in L^1_{\nu,\varphi(F)}(D')$. Then the sequence $f_k \circ \varphi$ converges to $f \circ \varphi \in L^1_{\nu,F}(D)$ a.e. on D and quasieverywhere on $T \cap D_F$, where T is the set of Proposition 13.*

Proof. Recall that $T = \bigcup_k T_k$, $|D \setminus T| = 0$, where $\{T_k\}$ is an increasing sequence of bounded sets of positive measure consisting of points of positive density. The mapping φ is continuous on each T_k.

Let $S \subset D'_F$ be a set of outer capacity zero on which there is no convergence. Then, by Lemma 48, $\varphi^{-1}(S) \cap T_k \cap D_F$ has capacity zero for every k. Consequently, by Lemma 16(5), $\varphi^{-1}(S) \cap T \cap D_F$ has capacity zero. This gives the convergence of the sequence $f_k \circ \varphi$ to $f \circ \varphi \in L^1_{\nu,F}(D)$ quasieverywhere on $T \cap D_F$. The convergence of the sequence $f_k \circ \varphi$ to $f \circ \varphi \in L^1_{\nu,F}(D)$ a.e. on D is obvious: $|D \setminus T| = 0$. $\square$

Lemma 50 [5, Lemma 7.6]. *Suppose that $E \subset D_F$ is a set of positive measure consisting of points of positive density, φ is continuous on E, and $f \in L^1_{\nu,\varphi(F)}(D')$ is a precise function. If $g = \varphi^* f$ is a precise function in $L^1_{\nu,F}(D)$ then $g|_E$ coincides quasieverywhere with $f \circ \varphi|_E$.*

Proof. Let $f_k \in L^1_{\nu,\varphi(F)}(D')$ be a sequence of continuous functions converging to f everywhere outside a set Σ of outer capacity zero. Reckoning with Remark 18, we may assume that the sequence of precise functions $g_k = \varphi^* f_k$ converges quasieverywhere to $\varphi^* f$. By Corollary 33, the functions $g_k = \varphi^* f_k$ coincide with $f_k \circ \varphi|_E$ quasieverywhere on E. Thus, g coincides with $f \circ \varphi$ quasieverywhere on E. $\square$

Corollary 51. *Let T be the set of Proposition 13 and let $f \in L^1_{\nu,\varphi(F)}(D')$ be a precise function. If $g = \varphi^* f$ is a precise function in $L^1_{\nu,F}(D)$ then $g|_{T \cap D_F}$ coincides with the function $f \circ \varphi|_{T \cap D_F}$ quasieverywhere.*

Proof. Recall that $T = \bigcup_k T_k$, $|D \setminus T| = 0$, where $\{T_k\}$ is an increasing sequence of bounded sets of positive measure consisting of points of positive density. The mapping φ is continuous on each T_k.

Put $E = T_k \cap D_F$ in Lemma 50. Then $g = \varphi^* f$, precise in $L^1_{\nu,F}(D)$, coincides with $f \circ \varphi$ quasieverywhere on $T_k \cap D_F$. The corollary follows because k is an arbitrary natural. $\square$

Lemma 52 [5, Lemma 7.7]. *There exist a set $S_\varphi \subset D$ of capacity zero and a quasicontinuous mapping $\psi : D_F \setminus S_\varphi \to \overline{D'_F}$ such that $\psi(x) = \varphi(x)$ a.e. on D_F.*

Proof. By Corollary 31, it suffices, for each open ball $Q \Subset D_F$, construct a quasicontinuous mapping $\bar{\varphi} : Q \to \mathbb{G}$ coinciding with φ a.e. on Q.

Suppose that $f \in L^1_{\nu,F}(D)$ is a continuous function and $f \geq 1$ on Q. There exists a precise function $g \in L^1_{\nu,\varphi(F)}(D')$ with $f = \varphi^* g$. By Corollary 51, $f|_{T \cap Q}$ and $g \circ \varphi|_{T \cap Q}$ coincide quasieverywhere. Let $S_Q \subset T \cap Q$ be a set of capacity zero on which the values of $f|_{T \cap Q}$ and $g \circ \varphi|_{T \cap Q}$ do not coincide. Note that $\varphi : T \cap Q \to \mathbb{G}$ satisfies the conditions of Lemmas 46–50 and their corollaries. Moreover, for all points $y \in \varphi(T \cap Q \setminus S_Q)$, we have $g(y) = g(\varphi(x)) = f(x) \geq 1$. Thus, the capacity of $\varphi(T \cap Q \setminus S_Q)$ is finite. Further, we consider the mapping φ only on $T \cap Q \setminus S_Q$, assuming that φ is undefined on $Q \setminus (T \cap Q \setminus S_Q)$. Furthermore, under the image of $V \subset Q$ we must mean $\varphi(V \cap (T \cap Q \setminus S_Q))$.

Put $P_k = \varphi(Q) \cap B(0,k)$ and $CP_k = \varphi(Q) \setminus P_k$, $k \in \mathbb{N}$. Show that

$$\overline{\mathrm{Cap}}\big(CP_k; L^1_{\nu,\varphi(F)}(D')\big) \to 0 \quad \text{as } k \to \infty. \tag{19}$$

Indeed, fix $k_0 \in \mathbb{N}$ and $0 < r < k_0 - 1$ so that, for $k > k_0$, we have $\varphi(F) \subset B(0,r)$ and $CP_k \subset \mathbb{G} \setminus B(0,k-1)$. This implies immediately that

$$\overline{\mathrm{Cap}}\big(CP_k; L^1_{\nu,\varphi(F)}(D')\big) \leq \mathrm{Cap}\big(\mathbb{G} \setminus B(0,k-1); L^1_{\nu,B(0,r)}(\mathbb{G})\big).$$

The capacity on the right-hand side is the ν-capacity of the ring $D_{r,k-1}$: it is proved in [10, Theorems 6.6 and 6.9] that this capacity is equivalent to $\big(\log \frac{k-1}{r}\big)^{1-\nu}$. As $k \to \infty$, we obtain (19).

Let $g_k \in A(CP_k)$ be a sequence of functions such that

$$\big\| g_k \mid L^1_{\nu,\varphi(F)}(D') \big\|^\nu = \overline{\mathrm{Cap}}\big(CP_k; L^1_{\nu,\varphi(F)}(D')\big).$$

By Corollary 51, the precise function $f_k = \varphi^* g_k$ coincides with $g_k \circ \varphi$ quasieverywhere on $T \cap D_F$. Thus, $f_k \in A(\varphi^{-1}(CP_k))$.

Denote by CF_k the subset of the ball Q consisting of the points of $\varphi^{-1}(CP_k)$ and all points of nonzero density in $\varphi^{-1}(CP_k)$. By Corollary 32,

$$\overline{\mathrm{Cap}}\big(CF_k; L^1_{\nu,F}(D)\big) = \overline{\mathrm{Cap}}\big(\varphi^{-1}(CP_k); L^1_{\nu,F}(D)\big). \tag{20}$$

Next,

$$\overline{\mathrm{Cap}}\big(\varphi^{-1}(CP_k); L^1_{\nu,F}(D)\big) \le \big\|f_k \mid L^1_{\nu,F}(D)\big\|^\nu$$
$$\le K^\nu \big\|g_k \mid L^1_{\nu,\varphi(F)}(D'))\big\|^\nu = K^\nu \overline{\mathrm{Cap}}\big(CP_k; L^1_{\nu,\varphi(F)}(D')\big),$$

where K is the norm of φ^*. From (19) and (20) we deduce

$$\lim_{k\to\infty} \overline{\mathrm{Cap}}\big(CF_k; L^1_{\nu,F}(D)\big) = 0. \tag{21}$$

Put $F_k = Q \setminus CF_k$. Note that $F_k \supset S_Q$ and $F_k \supset (D \setminus T) \cap Q$. If $x \in F_k \cap T \setminus S_Q$ then $\varphi(x) \in P_k$ and, for all $x \in F_k$,

$$\lim_{r\to 0} \frac{|F_k \cap B(x,r)|}{|B(x,r)|} = 1. \tag{22}$$

Indeed, for all sufficiently small r (such that $B(x,r) \subset Q$) we obtain $|B(x,r)| = |F_k \cap B(x,r)| + |CF_k \cap B(x,r)|$ or

$$1 = \frac{|F_k \cap B(x,r)|}{|B(x,r)|} + \frac{|CF_k \cap B(x,r)|}{|B(x,r)|}. \tag{23}$$

By the construction of CF_k, for $x \notin CF_k$ ($x \in F_k$) we have

$$|CF_k \cap B(x,r)|/|B(x,r)| \to 0 \quad \text{as } r \to 0,$$

i.e., (23) implies (22).

Consider the cut-of-functions $\eta_k \in C_0^\infty(\mathbb{G})$ such that $\eta_k(x) = 1$, $x \in B(0,k)$, and $\eta_k(x) = 0$, $x \notin B(0,k+1)$, $k \in \mathbb{N}$. Let $\psi_{i,k}$ be a precise function such that $\psi_{i,k} = \varphi^*(y_i \cdot \eta_k)$, where $y_i(\cdot)$ are coordinate functions (in the coordinates of the first kind [26]). From Corollary 51 we deduce that $\psi_{i,k}(x) = (y_i \cdot \eta_k)(\varphi(x))$ for quasiall points $x \in F_k \cap T$. Therefore,

$$(\varphi^*(y_i\eta_k))(x) = (y_i \cdot \eta_k)(\varphi(x)) = y_i(\varphi(x)) = \varphi_i(x)$$

for quasiall $x \in F_k \cap T$. Thus, a.e. on F_k, the coordinate function φ_i coincides with the precise function $\psi_{i,k}$.

Put

$$\bar\varphi_{i,k}(x) = \begin{cases} \psi_{i,k}(x) & \text{if } x \in F_k \setminus S_Q, \\ \varphi_i(x) & \text{if } x \in Q \setminus F_k. \end{cases}$$

Since φ_i changes on a negligible set, for each $k \in \mathbb{N}$ we have $\varphi_i(x) = \bar\varphi_{i,k}(x)$ a.e. on Q.

Let $k < m$. By the construction of F_k, we have $F_k \subset F_m$; therefore, $\varphi^*(y_i\eta_k)$ and $\varphi^*(y_i\eta_m)$ coincide with φ_i quasieverywhere on $F_k \cap T$.

Since by construction all points of F_k have density 1, the precise functions $\psi_{i,m}$ and $\psi_{i,k}$ coincide quasieverywhere on F_k (Corollary 31). This enables us to define

$$\bar\varphi_{iQ}(x) = \begin{cases} \psi_{i,k}(x) & \text{if } x \in F_k \setminus S_Q, \\ \varphi_i(x), & \text{if } x \in Q \setminus \bigcup\limits_k^\infty F_k \end{cases}$$

correctly quasieverywhere on Q. Since $Q \setminus \bigcup_k^\infty F_k = \bigcap_k^\infty CF_k$, by (21) $\bar\varphi_{iQ}$ is defined quasieverywhere on Q.

Show that the function $\bar\varphi_{iQ}$ is quasicontinuous on Q. Fix $\varepsilon > 0$. Then there exist open sets U_1, U_2, and U_3 such that

(1) there exists a number k such that $CF_k \subset U_1$ and $\overline{\mathrm{Cap}}(U_1) < \varepsilon/3$ (by (21) and Lemma 16),

(2) $\psi_{i,k}$ is continuous on $D_F \setminus U_2$, and $\overline{\mathrm{Cap}}(U_2) < \varepsilon/3$ (since $\psi_{i,k}$ is quasicontinuous),

(3) U_3 contains all points of $Q \setminus U_1$ of capacity zero, where the values of $\bar\varphi_{iQ}$ and $\psi_{i,k}$ do not coincide and $\overline{\mathrm{Cap}}(U_3) < \varepsilon/3$ (since $\bar\varphi_{iQ}$ and $\psi_{i,k}$ coincide quasieverywhere on F_k (outside CF_k)).

The set $U = U_1 \cup U_2 \cup U_3$ has capacity $\overline{\mathrm{Cap}}(U) < \varepsilon$, and $\bar{\varphi}_{iQ}$ is continuous on the complement $Q \setminus U$. By the arbitrariness of $\varepsilon > 0$, the function $\bar{\varphi}_{iQ}$ is quasicontinuous. Thus, we have constructed the quasicontinuous mapping $\bar{\varphi}_Q : Q \setminus S_Q \to \overline{D'_F}$, where $S_Q \subset Q$ is a set of capacity zero.

Covering D_F by countably many open balls of finite multiplicity Q_j and using the above procedure at each of the balls Q_j, construct the quasicontinuous mapping

$$\psi(x) = \overline{\varphi}_{Q_j}(x) \quad \text{if } x \in Q_j.$$

The correctness of the definition of $\psi(x)$ is guaranteed by the property: For two balls Q_j and Q_i with nonempty intersection, we have $\varphi_{Q_j}(x) = \varphi_{Q_i}(x)$ for all $x \in Q_i \cap Q_j$ outside some set $\Sigma_{ij} \subset Q_i \cap Q_j$ of capacity zero (see Corollary 31). Remove from D_F

$$S_\varphi = \bigcup_{i \neq j} \Sigma_{ij} \cup \bigcup_j S_{Q_j} \tag{24}$$

of capacity zero. Then ψ is well defined on $D_F \setminus S_\varphi$. Moreover, $\psi(x) = \varphi(x)$ for a.e. $x \in D_F$. $\square$

Assume that the image of $V \subset D_F$ is $\psi(V \setminus S_\varphi)$.

Remark 53. The constructed mapping ψ possesses the property: $\psi(x) = \varphi(x)$ for all $x \in T \setminus Z$, where Z is a set of capacity zero.

4.2. Construction of φ_0. In this subsection, we construct a mapping φ_0 such that $\varphi_0 = \psi$ quasieverywhere and equivalent estimates on the capacities of the image and preimage hold (see Lemma 55 below).

In the following lemma, we describe the properties of ψ and strengthen Lemmas 46, 48, and 50.

Lemma 54 [5, Lemma 7.8]. 1. *Let $f \in L^1_{\nu,\varphi(F)}(D')$ be a lower semicontinuous function. If $g = \varphi^* f$ is a precise function in $L^1_{\nu,F}(D)$ then $g(x) \geq f \circ \psi(x)$ quasieverywhere on $D_F \cap \psi^{-1}(D'_F)$.*

2. *Let $\Sigma \subset D'_F$. If $\overline{\mathrm{Cap}}\big(\Sigma; L^1_{\nu,\varphi(F)}(D')\big) = 0$ then*

$$\overline{\mathrm{Cap}}\big(\psi^{-1}(\Sigma) \cap D_F; L^1_{\nu,F}(D)\big) = 0.$$

3. *If $f \in L^1_{\nu,\varphi(F)}(D') \cap C(D')$ then the precise function $\varphi^* f$ coincides with $f \circ \psi$ quasieverywhere on $D_F \cap \psi^{-1}(D'_F)$.*

4. *Suppose that $f(y) = \sum_{k=1}^\infty f_k(y)$ quasieverywhere in D'_F, where $f_k \in L^1_{\nu,\varphi(F)}(D') \cap C(D')$ and $\sum_{k=1}^\infty \big\| f_k \mid L^1_{\nu,\varphi(F)}(D') \big\| < \infty$. Then the precise function $\varphi^* f$ coincides with the sum $\sum_{k=1}^\infty (f \circ \psi)(x)$ quasieverywhere on $D_F \cap \psi^{-1}(D'_F)$.*

5. *For every precise function $f \in L^1_{\nu,\varphi(F)}(D')$, the precise function $g = \varphi^* f$ coincides with the composition $f \circ \psi$ quasieverywhere on $D_F \cap \psi^{-1}(D'_F)$.*

6. *Arbitrary sets $A \subset D_F$ and $B \subset D'_F$ with $\psi(A) \subset D'_F$ satisfy the estimates*

$$\overline{\mathrm{Cap}}\big(\psi^{-1}(B); L^1_{\nu,F}(D)\big) \leq K^\nu \overline{\mathrm{Cap}}\big(B; L^1_{\nu,\varphi(F)}(D')\big), \tag{25}$$

$$\overline{\mathrm{Cap}}\big(A \cap \psi^{-1}(D'_F); L^1_{\nu,F}(D)\big) \leq K^\nu \overline{\mathrm{Cap}}\big(\psi(A) \cap D'_F; L^1_{\nu,\varphi(F)}(D')\big), \tag{26}$$

where $K = \max(\|\varphi^*\|, \|\varphi^{*-1}\|)$.

Proof. 1. We have $g(x) \geq f \circ \psi(x)$ for a.e. $x \in D_F \cap \psi^{-1}(D'_F)$ (at least at the points $x \in T$). Given $\varepsilon > 0$, choose an open set $U_\varepsilon \subset D_F$ such that ψ is continuous on $D_F \setminus U_\varepsilon$ and $\mathrm{Cap}\big(U_\varepsilon; L^1_{\nu,F}(D)\big) < \varepsilon$. By Corollary 32, $\overline{\mathrm{Cap}}\big(\widetilde{U}_\varepsilon; L^1_{\nu,F}(D)\big) < \varepsilon$. Note that each point in the complement $D_F \setminus \widetilde{U}_\varepsilon$ has positive density; such are also all points of $(D_F \cap \psi^{-1}(D'_F)) \setminus \widetilde{U}_\varepsilon$ since $|D_F \setminus \psi^{-1}(D'_F)| = 0$.

The composition $f\circ\psi(x)$ is lower semicontinuous at all points of $(D_F\cap\psi^{-1}(D'_F))\setminus U_\varepsilon$. By Lemma 46, $g(x)\ge f\circ\psi(x)$ quasieverywhere on $(D_F\cap\psi^{-1}(D'_F))\setminus\widetilde{U}_\varepsilon$. Since ε is arbitrary, $g(x)\ge f\circ\psi(x)$ quasieverywhere on $D_F\cap\psi^{-1}(D'_F)$.

2. Let $f\in L^1_{\nu,\varphi(F)}(D')$ be the lower semicontinuous function constructed in Lemma 39. By item 1 of the lemma, the precise function $g=\varphi^*f$ is at least $f\circ\varphi$ quasieverywhere on $D_F\cap\psi^{-1}(D'_F)$. In particular, $g(x)=\infty$ for quasiall $x\in\varphi^{-1}(\Sigma)\cap D_F\cap\psi^{-1}(D'_F)=\varphi^{-1}(\Sigma)\cap D_F$. Lemma 19 yields $\operatorname{Cap}\bigl(\varphi^{-1}(\Sigma)\cap D_F;L^1_{\nu,F}(D)\bigr)=0$.

3. If $g=\varphi^*f$ is a precise function in $L^1_{\nu,F}(D)$ then, by item 1 of the lemma, we simultaneously have $g(x)\ge f\circ\psi(x)$ and $-g(x)\ge -f\circ\psi(x)$ quasieverywhere on $D_F\cap\psi^{-1}(D'_F)$. Hence, $g(x)=f\circ\psi(x)$ quasieverywhere on $D_F\cap\psi^{-1}(D'_F)$.

4. Suppose that $f_k\in L^1_{\nu,\varphi(F)}(D')\cap C(D')$, $f(y)=\sum_{k=1}^\infty f_k(y)$ quasieverywhere on D'_F, and $\sum_{k=1}^\infty\|f_k\mid L^1_{\nu,\varphi(F)}(D')\|<\infty$. By item 2, the series

$$f\circ\psi(x)=\sum_{k=1}^\infty f_k\circ\psi(x) \tag{27}$$

converges quasieverywhere on $D_F\cap\psi^{-1}(D'_F)$. Moreover,

$$\sum_{k=1}^\infty\|f_k\circ\psi\mid L^1_{\nu,\varphi(F)}(D')\|\le K\sum_{k=1}^\infty\|f_k\mid L^1_{\nu,\varphi(F)}(D')\|<\infty. \tag{28}$$

In the familiar manner (see, for example, [5]), from (28) we deduce that (27) converges uniformly on D_F beyond some open set of arbitrarily small capacity. Thus, the function $f\circ\psi$ is precise, and so $\varphi^*f(x)=\sum_{k=1}^\infty(f\circ\psi)(x)$ quasieverywhere on $D_F\cap\psi^{-1}(D'_F)$.

5. Let $f_k\in L^1_{\nu,\varphi(F)}(D')$ be a sequence of continuous functions converging to f everywhere beyond a set Σ of outer capacity zero. By items 2 and 4 of the lemma, the sequence of precise functions $g_k=\varphi^*f_k$ converges quasieverywhere to the precise function $g=\varphi^*f$. By item 3, the precise functions $g_k=\varphi^*f_k$ coincide with $f_k\circ\varphi|_{D_F\cap\psi^{-1}(D'_F)}$ quasieverywhere on $D_F\cap\psi^{-1}(D'_F)$. Thus, the precise function $g=\varphi^*f$ coincides with $f\circ\psi$ quasieverywhere on $D_F\cap\psi^{-1}(D'_F)$.

6. Let $f_B\in L^1_{\nu,\varphi(F)}(D')$ be the capacity function for B (see Theorem 25); i.e., $\overline{\operatorname{Cap}}\bigl(B;L^1_{\nu,\varphi(F)}(D')\bigr)=\|f_B\mid L^1_{\nu,\varphi(F)}(D')\|^\nu$. The set $\{y\in B\mid f(y)<1\}$ has capacity zero (by item 2 of the lemma). Therefore, the precise function $g=\varphi^*f_B$ possesses the property: $g(x)\ge 1$ quasieverywhere on $\psi^{-1}(B)$. The relation $\|g\mid L^1_{\nu,F}(D)\|\le K\|f_B\mid L^1_{\nu,\varphi(F)}(D')\|$ yields the chain of inequalities

$$\overline{\operatorname{Cap}}\bigl(\psi^{-1}(B);L^1_{\nu,F}(D)\bigr)\le\|g\mid L^1_{\nu,F}(D)\|^\nu\le K^\nu\|f_B\mid L^1_{\nu,\varphi(F)}(D')\|^\nu=K^\nu\overline{\operatorname{Cap}}\bigl(B;L^1_{\nu,\varphi(F)}(D')\bigr),$$

whence estimate (25) follows.

Let $f_{\psi(A)}\in L^1_{\nu,\varphi(F)}(D')$ be the capacity function of $\psi(A)\cap D'_F$. The set $\{y\in B\mid f_{\psi(A)}(y)<1\}$ has capacity zero (by item 2 of the lemma). Therefore, the property is fulfilled for the precise function $g=\varphi^*f_{\psi(A)}$: $g(x)\ge 1$ quasieverywhere on $A\cap\psi^{-1}(D'_F)$. The relation $\|g\mid L^1_{\nu,F}(D)\|\le K\|f_{\psi(A)}\mid L^1_{\nu,\varphi(F)}(D')\|$ gives

$$\begin{aligned}\overline{\operatorname{Cap}}\bigl(A\cap\psi^{-1}(D'_F);L^1_{\nu,F}(D)\bigr)&\le\|g\mid L^1_{\nu,F}(D)\|^\nu\\ &\le K^\nu\|f_{\psi(A)}\mid L^1_{\nu,\varphi(F)}(D')\|^\nu=K^\nu\overline{\operatorname{Cap}}\bigl(\psi(A)\cap D'_F;L^1_{\nu,\varphi(F)}(D')\bigr),\end{aligned}$$

which implies (26). $\square$

From now on, fix a countable system

$$\mathscr{B}=\{B_j\} \tag{29}$$

of balls in D_F constituting a base of open sets $U \subset D_F$. We will assume that the balls in (29) have the properties:

(1) $B_j \Subset D_F$ for all $j \in \mathbb{N}$;

(2) along with each ball $B_j = B_j(x_j, r_j)$, the system $\mathscr{B}$ contains also a countable family of balls centered at a point x_j and radii of the form $2^{-k} \operatorname{dist}(x_j, D_F)$, $k \in \mathbb{N}$.

Lemma 55 [5, Lemma 7.9]. *There exist a set $S_\psi \subset D_F$ of capacity zero and a mapping $\varphi_0 : D_F \setminus S_\psi \to \overline{D'_F}$ such that $\varphi_0(x)$ coincides with $\psi(x)$ for quasiall $x \in D_F$. The mapping φ_0 satisfies all assertions of Lemma 54 and the estimate*

$$\overline{\operatorname{Cap}}\big(\varphi_0(B_j) \cap D'_F; L^1_{\nu,\varphi(F)}(D')\big) \le K^{-\nu}\overline{\operatorname{Cap}}\big(B_j; L^1_{\nu,F}(D)\big) \tag{30}$$

for each ball B_j in (29).

Proof. Suppose that $B \in \mathscr{B}$ is an arbitrary ball of the countable base of neighborhoods (29) and $g_B \in A(B)$ is the capacity function of B (see Theorem 25). Since φ^* is an isomorphism (see Lemma 14), there exists a precise function $f_B \in L^1_{\nu,\varphi(F)}(D')$ such that $g_B(x) = f_B \circ \psi(x)$ for quasiall $x \in D_F \cap \psi^{-1}(D'_F)$ by Lemma 54(3).

Consider $\Sigma_B = \{x \in B \cap \psi^{-1}(D'_F) : f_B(\psi(x)) < 1\}$. Since $f(\psi(x)) = g(x) \ge 1$ for quasiall $x \in B \cap \psi^{-1}(D'_F)$, we have $\overline{\operatorname{Cap}}\big(\Sigma_B; L^1_{\nu,F}(D)\big) = 0$.

Since $\psi((B \cap \psi^{-1}(D'_F)) \setminus \Sigma_B) = (\psi(B) \cap D'_F) \setminus \psi(\Sigma_B) \subset D'_F$, the function f_B is admissible for $(\psi(B) \cap D'_F) \setminus \psi(E_B) \subset D'_F$; i.e., $f \in A((\psi(B) \cap D'_F) \setminus \psi(\Sigma_B))$. Therefore,

$$\begin{aligned}\overline{\operatorname{Cap}}\big(\psi((B \cap \psi^{-1}(D'_F)) \setminus \Sigma_B); L^1_{\nu,\varphi(F)}(D')\big) &\le \big\|f_B \mid L^1_{\nu,\varphi(F)}(D')\big\|^\nu \\ &\le K^{-\nu}\big\|g_B \mid L^1_{\nu,F}(D)\big\|^\nu = K^{-\nu}\overline{\operatorname{Cap}}\big(B; L^1_{\nu,F}(D)\big).\end{aligned} \tag{31}$$

Make $\varphi_0(x)$ equal to $\psi(x)$ on $D_F \setminus \bigcup_{B_j \in \mathscr{B}} \Sigma_{B_j}$ and undefined on $\bigcup_{B_j \in \mathscr{B}} \Sigma_{B_j}$. By Lemma 16(6), we have

$$\overline{\operatorname{Cap}}\Big(\bigcup_{B_j \in \mathscr{B}} \Sigma_B; L^1_{\nu,F}(D)\Big) = 0;$$

therefore, φ_0 and ψ coincide quasieverywhere in D_F.

Thus, the mapping is defined on $D_F \setminus S_\psi$, where

$$S_\psi = S_\varphi \cup \bigcup_{B_j \in \mathscr{B}} \Sigma_{B_j}, \tag{32}$$

and S_φ is defined by (24). The validity of all items of Lemma 54 for φ_0 is straightforward. □

Now, by the image $\varphi_0(V)$ of an arbitrary set $V \subset D_F$ we mean $\psi\big(V \setminus S_\psi\big)$.

4.3. The topological properties of φ_0. In this subsection, we proceed with study of the properties of a quasicontinuous mapping φ_0. Note that the balls $B(x,r)$ and the spheres $S(x,r)$ are considered in the Carnot–Carathéodory metric.

Proposition 56 [25, Proposition 5]. 1. *The mapping φ_0 is defined and continuous at all points of $S(x,r)$ for a.e. $r \in (0, \operatorname{dist}(x, \partial D_F))$.*

2. *The mapping φ_0 is continuous on almost all integral curves of the horizontal vector fields: for every ball $B(x,r) \subset D_F$ and almost all integral curves $\gamma \subset B(x,r)$ of the horizontal vector field X_i, $i = 1, \dots, n$, the mapping φ_0 is defined and continuous at all points of the integral curve γ.*

Proposition 57. *There exists a negligible set $\Sigma \subset D_F$ such that, in arbitrarily small neighborhoods of the points $x_1, x_2 \in B \setminus \Sigma$, where $B \subset D_F$ is any open ball, there are points that can be joined by a curve $\gamma \subset B$ on which φ_0 is continuous.*

PROOF. STEP 1. By Lemma 1.40 in [26], there exist constants $C > 0$ and $N \in \mathbb{N}$ such that every two points $x_1, x_2 \in \mathbb{G}$ can be joined by a broken line $\sigma = \sigma_1 \cup \sigma_2 \cup \dots \cup \sigma_N$ consisting of segments of integral curves of the horizontal vector fields. Moreover, the length of σ_i is at most $Cd(x_1, x_2)$.

STEP 2. As the negligible set $\Sigma \subset D_F$, take the set of all points in D_F not belonging to the union of all integral curves of the horizontal vector fields X_i, $i = 1, \dots, n$, on each of which φ_0 is continuous (see Proposition 56).

STEP 3. Let $x_1, x_2 \in B \setminus \Sigma$. Consider a continuous curve $\Gamma \subset B$ joining x_1 and x_2. Put

$$R_\Gamma = \frac{\operatorname{dist}(\Gamma, \partial B)}{NC}.$$

Cover the curve Γ by finitely many balls $B_j \subset B$ with equal radii R_Γ and choose points $x_1 = y_1, y_2, \dots, y_{l+1} = x_2$ on this curve so that two neighboring points y_j, y_{j+1} belong to the ball $\{B_j\}$ (some balls maybe repeated). By the choice of a suitable radius of $\{B_j\}$, the points y_j, y_{j+1} can be joined by the curve $\gamma_j \subset B$ constructed at the first step of the proof. The composed curve $\gamma = \gamma_1 \cup \gamma_2 \cup \dots \cup \gamma_l$ may fail to be a desired one since some points of the family $y_2, \dots, y_l$ may belong to Σ.

STEP 4. For composing a desired curve, take a ball $B(y_1, \varepsilon)$ of so small a radius that the tubular neighborhood $\bigcup_t B(\gamma(t), \varepsilon)$ of γ belongs to B. For every ε, there is a curve in this tubular neighborhood composed of integral curves of the vector fields on each of which the mapping φ_0 is continuous. The initial point of such a curve belongs to $B(x_1, \varepsilon)$, and the final point lies in $B(x_2, \varepsilon)$.

Since ε can be taken arbitrarily small, the proposition is proved. $\square$

Let $x \in T \cap D_F$. Introduce the notation

$$\widehat{B}(x, r) = \Big\{ \bigcup_{\rho \in (0,r)} S(x, \rho) \mid \text{the mapping } \varphi_0 : S(x, \rho) \to \mathbb{G} \text{ is continuous} \Big\} \subset D_F. \tag{33}$$

Thus, $\widehat{B}(x, r)$ differs from $B(x, r)$ only by that all spheres $S(x, \rho)$, $\rho \in \sigma_{x,r} \subset (0, r)$, at which φ_0 is discontinuous are removed from the ball $B(x, r)$; moreover, $\sigma_{x,r}$ of such radius has measure zero on $(0, r)$.

Lemma 58. *Given a sequence of positive numbers $\{r_k\}$ converging to 0 as $k \to \infty$. Suppose also that $x \in D_F$ and there exists $u_k \in \widehat{B}(x, r_k) \cap D_F$ for which $\varphi_0(u_k) \to y \in D'_F$ as $k \to \infty$, where y is some point.*

Then $\varphi_0(\widehat{B}(x, r_k))$ contract to $y \subset D'_F$ as $k \to \infty$:

$$\{y\} = \bigcap_{k \in \mathbb{N}} \overline{\varphi_0(\widehat{B}(x, r_k))} \in D'_F. \tag{34}$$

PROOF. Obviously, (34) is equivalent to the following:

$$\sup_{z \in \widehat{B}(x, r_k) \cap D_F} d(\varphi_0(z), y) \to 0 \quad \text{as } k \to \infty. \tag{35}$$

Suppose on the contrary that (35) fails. Then there exist a number $\vartheta > 0$ and a sequence of radii $\varkappa_k \in (0, r_k) \setminus \sigma_{x, r_k}$ for which

$$\operatorname{diam}(\{y\} \cup \varphi_0(S(x, \varkappa_k))) = \sup_{z \in S(x, \varkappa_k) \cap D_F} d(\varphi_0(z), y) \geq \vartheta, \quad k \in \mathbb{N}. \tag{36}$$

Since $u_k \in \widehat{B}(x, r_k)) \cap D_F$, we have $u_k \in S(x, \tau_k)$, where $\tau_k \in (0, r_k) \setminus \sigma_{x, r_k}$ and $\tau_k \to 0$ as $k \to \infty$.

Clearly, for sufficiently large k, each ball $B(x, r_k)$ lies in some ball $B(x_{j_k}, \rho_{j_k})$ of (29) such that $x_{j_k} \in B(x, r_k)$ and $\rho_{j_k} > 2r_k$ (the last inequality guarantees the inclusion $B(x, r_k) \subset B_k$); moreover, $\rho_{j_k} \to 0$ as $k \to \infty$ (i.e., as r_k decreases to zero and ρ_{j_k} decreases to zero too) (see the description of (29) above).

For each $k \in \mathbb{N}$, consider also the continuous curve $\gamma_k \subset B_k$ as in Proposition 57 with endpoints in the ball $B(x, \min(\tau_k, \varkappa_k))$ and the complement $B_k \setminus B(x, r_k)$ at whose points φ_0 is defined and continuous.

Denote the compact set $S(x, \tau_k) \cup S(x, \varkappa_k) \cup \gamma_k$ by K_k. We have $K_k \subset B_k$. The compact set K_k is connected, and $\varphi_0 : K_k \to \overline{D'_F}$ is continuous.

Using the above choice of the compact set K_k and the balls $B_k = B(x_{j_k}, \rho_{j_k})$ and reckoning with (30), we have

$$\overline{\operatorname{Cap}}\big(\varphi_0(K_k) \cap D'_F; L^1_{\nu,\varphi(F)}(D')\big) \le \overline{\operatorname{Cap}}\big(\varphi_0(\widehat{B}(x, r_k)) \cap D'_F; L^1_{\nu,\varphi(F)}(D')\big)$$
$$\le \overline{\operatorname{Cap}}\big(\varphi_0(B_k) \cap D'_F; L^1_{\nu,\varphi(F)}(D')\big) \le K^{-\nu}\overline{\operatorname{Cap}}\big(B_k; L^1_{\nu,F}(D)\big) = O\left(\left(\log \frac{2}{\rho_{j_k}}\right)^{1-\nu}\right) = o(1) \qquad (37)$$

as $k \to \infty$ (see Remark 38). From (37) we deduce $\overline{\operatorname{Cap}}\big(\varphi_0(K_k) \cap D'_F; L^1_{\nu,\varphi(F)}(D')\big) \to 0$ as $k \to \infty$. Now, applying Theorem 40 to $\varphi_0(K_k)$, we get $\operatorname{diam} \varphi_0(K_k) \to 0$ as $k \to \infty$. With the condition $\varphi_0(u_k) \to y \in D'_F$ as $k \to \infty$ taken into account, we have $\operatorname{diam}(\{y\} \cap \varphi_0(K_k)) \to 0$ as $k \to \infty$. This contradicts (36) because $S(x, \varkappa_k) \subset K_k$. $\square$

In the following assertion, we show that the images of the concentric spheres on which φ_0 continuously shrink to a point as the radius tends to zero.

Corollary 59. *Let $x \in T \cap D_F$. Then*

$$\sup_{y \in \varphi_0(\widehat{B}(x,r)) \cap D_F} d(y, \varphi_0(x)) \to 0 \quad \text{as } r \to 0. \qquad (38)$$

Proof. Fix k for which $x \in T_k \cap D_F$. Suppose on the contrary that (38) fails. Then there exist a number $\vartheta > 0$ and a sequence $r_l \to 0$ as $l \to \infty$ such that

$$\sup_{y \in \varphi_0(\widehat{B}(x,r_l)) \cap D_F} d(y, \varphi_0(x)) \ge 2\vartheta$$

for all $l \in \mathbb{N}$. Extract from here a sequence of radii $\varkappa_l \in (0, r_l) \setminus \sigma_{x,r_l}$ for which

$$\sup_{y \in \varphi_0(S(x,\varkappa_l)) \cap D_F} d(y, \varphi_0(x)) \ge \vartheta, \quad l \in \mathbb{N}. \qquad (39)$$

Since x is a point of positive density, for all r_l there is $\tau_l \in (0, r_l) \setminus \sigma_{x,r_l}$ such that $S(x, \tau_l) \cap T_k \ne \varnothing$. Since φ_0 is continuous on $T_k \cap D_F$ (see Proposition 56), for every choice of points $u_l \in S(x, \tau_l) \cap T_k \ne \varnothing$, we have

$$u_l \to x \quad \text{and} \quad \varphi_0(u_l) \to y = \varphi_0(x) \in D'_F \quad \text{as } l \to \infty. \qquad (40)$$

Hence, the sequence u_l satisfies all hypotheses of Lemma 58, which implies (38). $\square$

From Proposition 56 and Corollary 59, we deduce the properties of φ_0:

Corollary 60. *Let $x \in T \cap D_F$. For every sufficiently small $\rho > 0$, there exists a number $\delta_{x,\rho} > 0$ such that*

(1) *for the spheres $S(x, r) \subset D_F$ of radius $r \in (0, \delta_{x,\rho}) \setminus \sigma_{x,r}$, their images $\varphi_0(S(x, r))$ are included in $B(\varphi_0(x), \rho) \subset D'_F$; i.e.,*

$$\varphi_0(\widehat{B}(x, \delta_{x,\rho})) \subset B(\varphi_0(x), \rho) \subset D'_F; \qquad (41)$$

(2) *for almost all integral curves γ of the horizontal vector fields, the images $\varphi_0(\gamma \cap B(x, \delta_{x,\rho}))$ lie in $B(\varphi_0(x), \rho) \subset D'_F$.*

Corollary 61. *Let $x \in T \cap D_F$. The balls meeting (41) possess the property: for every $y \in B(x, \delta_{x,\rho})$, the images $\varphi_0(\widehat{B}(y,\tau))$ contract to the unique point $z \in \overline{B(\varphi_0(x),\rho)} \subset D'_F$ as $\tau \to 0$.*

Proof. Fix an arbitrary sequence $\tau_k \to 0$ as $k \to \infty$. For $k \in \mathbb{N}$, we can find $u_k \in B(x, \delta_{x,\rho}) \cap \widehat{B}(y,\tau)$. We have the properties: $u_k \to x$ as $k \to \infty$ and $\varphi_0(u_k) \in \overline{B(\varphi_0(x),\rho)} \subset D'_F$. Dropping to a subsequence, we may assume that $\varphi_0(u_k) \to y \in D'_F$. Hence, the sequence u_k meets all conditions of Lemma 58, from which we deduce Corollary 61. □

Definition 62. Let $x \in T \cap D_F$. For sufficiently small $\rho > 0$, by Corollary 60, find $\delta_{x,\rho} > 0$ such that

$$\varphi_0(\widehat{B}(x, \delta_{x,\rho})) \subset B(\varphi_0(x), \rho) \subset D'_F.$$

Any point $y \in B(x, \delta_{x,\rho})$ either belongs to the intersection $T \cap D_F$ or does not belong to it. In the former case, we have $\lim_{z \to y, z \in \widehat{B}(y,\delta_1)} \varphi_0(z) = \varphi_0(y)$, where δ_1 is a sufficiently small positive number. In the latter case, the value of φ_0 at y is undefined but, by Corollary 61, the limit

$$\lim_{z \to y, z \in \widehat{B}(y,\delta_2)} \varphi_0(z) \in D'_F$$

is defined which we will take as $\varphi_0(y)$ (here δ_2 is a sufficiently small positive number).

Consequently, at all points of $B(x, \delta_{x,\rho})$, there is defined some mapping that we will denote by the same symbol φ_0. This mapping has the property

$$\varphi_0(y) \in \overline{B(\varphi_0(x),\rho)} \subset D'_F \quad \text{for every } y \in B(x, \delta_{x,\rho}); \tag{42}$$

moreover,

$$\{\varphi_0(y)\} = \bigcap_{r \to 0} \overline{\varphi_0(\widehat{B}(y,r))} \in D'_F.$$

Proposition 63. *The mapping $\varphi_0 : B(x, \delta_{x,\rho}) \to \overline{B(\varphi_0(x),\rho)} \subset D'_F$, where $x \in T \cap D_F$, is continuous.*

Proof. Case 1: Let $y \in B(x, \delta_{x,\rho}) \cap T \cap D_F$. By Definition 62, for sufficiently small $\tau > 0$, there exists a number $\delta_{y,\tau} > 0$ for which we have from (42): $\varphi_0(\widehat{B}(y, \delta_{y,\tau})) \subset B(\varphi_0(y), \tau) \subset \overline{B(\varphi_0(x),\rho)} \subset D'_F$, which proves the continuity of φ_0 at $y \in B(x, \delta_{x,\rho}) \cap T \cap D_F$.

Case 2: Let $y \in B(x, \delta_{x,\rho}) \setminus (T \cap D_F)$. By Definition 62, for sufficiently small $\tau > 0$, there exists a number $\delta_{y,\tau} > 0$ such that

$$\varphi_0(\widehat{B}(y, \delta_{y,\tau})) \subset B(\varphi_0(y), \tau) \subset \overline{B(\varphi_0(y),\tau)} \subset D'_F.$$

As in the previous case, for every $z \in B(y, \delta_{y,\tau})$, we have

$$\varphi_0(z) \in \overline{B(\varphi_0(y),\tau)} \subset D'_F.$$

From this, by analogy, we obtain the continuity of φ_0 at $y \in B(x, \delta_{x,\rho}) \setminus (T \cap D_F)$. □

Proposition 64. *The mappings*

$$\varphi_0 : B(x, \delta_{x,\rho}) \to \overline{B(\varphi_0(x),\rho)}, \quad \varphi_0 : B(y, \delta_{y,\rho}) \to \overline{B(\varphi_0(y),\rho)},$$

where $x, y \in T \cap D_F$, coincide on $B(x, \delta_{x,\rho}) \cap B(y, \delta_{y,\rho})$ if the latter set is nonempty.

Proof. In correspondence with Definition 62, the value of φ_0 at $z \in B(x, \delta_{x,\rho}) \cap B(y, \delta_{y,\rho})$ can be defined starting either from $\varphi_0 : B(x, \delta_{x,\rho}) \to \overline{B(\varphi_0(x),\rho)}$ or $\varphi_0 : B(y, \delta_{y,\rho}) \to \overline{B(\varphi_0(y),\rho)}$. There exists a ball $B(z, r_z) \subset B(x, \delta_{x,\rho}) \cap B(y, \delta_{y,\rho})$ on which both ways of defining φ_0 at a point z coincide. □

Definition 65. For $x \in T \cap D_F$, consider the family of balls $B(x, \delta_{x,\rho}) \subset D_F$ of Definition 62. By Proposition 64, some continuous mapping is defined on the open set

$$U = \bigcup_{x \in T \cap D_F} B(x, \delta_{x,\rho}),$$

which we will denote by $\widetilde{\varphi}_0$. Moreover, $U \subset D_F$ and $|D_F \setminus U| = 0$.

Obviously, $\widetilde{\varphi}_0 : U \to D'_F$ is the extension of $\varphi_0 : T \cap D_F \to D'_F$ to a continuous mapping of the open set U. Since $T \cap D_F$ is dense in U, such an extension is unique.

Proposition 66. *$\widetilde{\varphi}_0 : U \to D'_F$ is a homeomorphism.*

PROOF. By Proposition 13, $\varphi : T \cap D_F \to D'_F$

(1) is injective;

(2) has the image $\varphi(T \cap D_F)$ dense in D'_F, and $|D'_F \setminus \varphi(T \cap D_F)| = 0$;

(3) satisfies Luzin's $\mathscr{N}$- and $\mathscr{N}^{-1}$-conditions.

Consequently, by Lemma 14, the inverse mapping $\varphi^{-1} : T' \to D_F$, where $T' = \varphi_0(T)$, generates the composition operator $\varphi^{*-1} : L^1_p(D) \cap C^\infty(D) \to L^1_p(D')$.

Applying the above results to $\varphi^{-1} : T' \to D_F$, we obtain the continuous mapping $\widetilde{\varphi^{-1}}_0 : V \to D_F$ on an open set $V \subset D'_F$ with values in D_F; moreover, $|D'_F \setminus V| = 0$. We can achieve that $\widetilde{\varphi}_0(U) = V$.

Since $\varphi_0(T \cap D_F)$ is dense in V, by the above we deduce the injectivity and homeomorphy of $\widetilde{\varphi}_0 : U \to V \subset D'_F$. $\square$

4.4. Quasiconformality of the mapping $\widetilde{\varphi}_0 : U \to V$. In this subsection, $U \subset D_F$ is the open set as in Definition 65 and $V = \widetilde{\varphi}_0(U)$. The main result of this subsection is stated as

Proposition 67. *$\widetilde{\varphi}_0 : U \to V$ is a quasiconformal mapping.*

A proof of this assertion can be found in [27]. However, we will give other arguments which have a wider range of application.

The proof of Proposition 67 is essentially reduced to establishing the absolute continuity of $\widetilde{\varphi}_0 : U \to V$ on almost all integral curves of the horizontal vector fields (briefly, $\varphi_0 \in \mathrm{ACL}(U)$) and the pointwise inequality

$$|D(x,\varphi)| \le K|J(x,\varphi)|^{\frac{1}{\nu}} \quad \text{a.e. in } U. \tag{43}$$

Since $\widetilde{\varphi}_0 : U \to V$ is an approximately differentiable homeomorphism, we deduce from (3) that the Jacobian $J(x,\varphi)$ is locally integrable in U. Moreover, by Hölder's inequality, $J(x,\varphi)^{\frac{1}{\nu}}$ is also locally integrable.

Lemma 68 [6, Lemma 19]. *Suppose that $u \in \mathrm{Lip}_l(D') \cap L^1_\nu(D')$ and $\|u \mid L^1_\nu(D')\| \le 1$. Then*

$$|\nabla_{\mathscr{L}}(u \circ \varphi)|(x) \le KJ(x,\varphi)^{\frac{1}{\nu}} \quad \text{a.e. on } D, \tag{44}$$

where K is a constant.

Lemma 69. *Let $D, D' \subset \mathbb{G}$. If $\varphi : D \to D'$ belongs to IL^1_ν then $\widetilde{\varphi}_0 \in W^1_{\nu,\mathrm{loc}}(U)$.*

PROOF. Let us prove that $\widetilde{\varphi}_0 \in \mathrm{ACL}(U)$. Let $\{z_j\}$ be a countable dense set of points in V. Define the countable family of functions $d^r_{z_j} : V \to \mathbb{R}^+$, $d^r_{z_j}(y) = (r - d_{z_j}(y))^+$, where $r \in \mathbb{Q}^+ = \{x \in \mathbb{Q} \mid x > 0\}$, $(d_{z_j}(y) = d(z_j, y))$. Each of these functions satisfies the pointwise equality $\varphi^* d^r_{z_j}(x) = d^r_{z_j} \circ \widetilde{\varphi}_0(x)$, $r \in \mathbb{Q}^+$, $j \in \mathbb{N}$, for all $x \in U$.

Moreover, each of these functions meets the conditions of Lemma 68. Therefore,

$$\left|\nabla_{\mathscr{L}}(d^r_{z_j} \circ \widetilde{\varphi}_0)\right|(x) \le CJ(x,\varphi)^{\frac{1}{\nu}}$$

for a.e. $x \in U$.

Consider the foliation Γ_s of the open set U generated by a horizontal vector field X_s and an integral curve γ from this foliation. Almost all curves γ in Γ_s satisfy the conditions:

(1) $\widetilde{\varphi}_0$ is continuous on γ (Proposition 56);

(2) the pointwise inequality for measurable functions $|\nabla_{\mathscr{L}}(\varphi^* d^r_{z_j})|(t) \le KJ(t,\varphi)^{\frac{1}{\nu}}$, $r \in \mathbb{Q}^+$, $j \in \mathbb{N}$, holds a.e. on γ, and the function $J(t,\varphi)^{\frac{1}{\nu}}$ is integrable over an arbitrary compact part of γ.

(3) for a.e. $x_0 \in \gamma$, there exists a finite limit $\frac{1}{d(x_0,x)} \int_{[x_0,x]} J(t,\varphi)^{\frac{1}{\nu}}\, d\sigma$ as $x \to x_0$ along γ, equal to $J(x_0,\varphi)^{\frac{1}{\nu}}$ (here $[x_0,x] \subset \gamma$ is a segment of the integral curve);

(4) $\varphi^* d^r_{z_j}$ are absolutely continuous functions on γ for all $j \in \mathbb{N}$ and $r \in \mathbb{Q}^+$.

Fix a curve $\gamma \in \Gamma_s$ on which all four properties are fulfilled.

Let $x_0 \in U \cap \gamma$ be a point of positive linear density on γ and a point at which condition 3 is fulfilled. Put $z = \widetilde{\varphi}_0(x_0)$. Fix a subsequence $\{z_{j_l}\}$ of points in $\{z_j\}$ converging to $z = \widetilde{\varphi}_0(x_0)$ (below we denote the elements of this sequence by z_l). Since $\widetilde{\varphi}_0$ is continuous on γ at x_0, we can choose numbers δ, r, and L such that $\widetilde{\varphi}_0(B(x_0,\delta)\cap\gamma) \subset V$ (Corollary 60) and $d^r_{z_l} \circ \widetilde{\varphi}_0(x) \neq 0$ for all $l \geq L$ and all points $x \in B(x_0,\delta)\cap\gamma$.

Integrating the function $KJ(x,\varphi)^{\frac{1}{\nu}}$ (where K is independent of r and z) over the part of γ from x_0 to x, where $x \in B(x_0,\delta)\cap\gamma$, we infer

$$K \int\limits_{[x_0,x]} J(t,\varphi)^{\frac{1}{\nu}}\,dt \geq \int\limits_{[x_0,x]} \big|\nabla_{\mathscr{L}}(\varphi^* d^r_{z_j})\big|(t)\,dt$$
$$\geq \big|d^r_{z_l}\circ\widetilde{\varphi}_0(x_0) - d^r_{z_l}\circ\widetilde{\varphi}_0(x)\big| = |r - d_{z_l}(\widetilde{\varphi}_0(x_0)) - r + d_{z_l}(\widetilde{\varphi}_0(x))|$$
$$= |-d_{z_l}(\widetilde{\varphi}_0(x_0)) + d_{z_l}(\widetilde{\varphi}_0(x))| \to d_z(\widetilde{\varphi}_0(x)) = d(\widetilde{\varphi}_0(x_0),\widetilde{\varphi}_0(x)) \quad \text{as } l\to\infty.$$

Thus,

$$d(\widetilde{\varphi}_0(x_0),\widetilde{\varphi}_0(x)) \leq K \int\limits_{[x_0,x]} J(t,\varphi)^{\frac{1}{\nu}}\,d\sigma \tag{45}$$

for all $x \in B(x_0,\delta)\cap\gamma$. From (45) and the absolute continuity of the integral we deduce that $\widetilde{\varphi}_0$ is absolutely continuous on $B(x_0,\delta)\cap\gamma$.

By the arbitrariness of the choice of the horizontal field X_j, the integral curve $\gamma \in \Gamma_j$,2 and $z_0 \in \gamma$, the mapping φ is absolutely continuous along almost all horizontal curves.

From (45) we have

$$\frac{d(\widetilde{\varphi}_0(x_0),\widetilde{\varphi}_0(x))}{d(x_0,x)} \leq \frac{K}{d(x_0,x)} \int\limits_{[x_0,x]} J(t,\varphi)^{\frac{1}{\nu}}\,d\sigma. \tag{46}$$

Passing to the limit as $x \to x_0$, we obtain

$$|X_s\widetilde{\varphi}_0(x_0)| \leq KJ(x_0,\varphi)^{\frac{1}{\nu}}. \tag{47}$$

Consequently, $|X_s\varphi| \in L_{\nu,\mathrm{loc}}(D_F)$ for all s and $\widetilde{\varphi}_0 \in W^1_{\nu,\mathrm{loc}}(U)$. $\square$

Other properties of the Sobolev classes on a Carnot group, in particular, the change-of-variable formula, can be found in [28].

Proof of Proposition 67. The membership of $\widetilde{\varphi}_0 : U \to V$ in the Sobolev class $W^1_{\nu,\mathrm{loc}}(U)$ was proved in Lemma 69. The pointwise inequality (43) follows from (47).

4.5. The local connectedness of U and V. Put $S = D_F \setminus U$. Let $x \in S$. Then the two cases are possible:

(1) there exists $r_0 > 0$ such that $\overline{\widetilde{\varphi}_0(\widehat{B}(x,r))} \subset D'_F$ for all $r < r_0$,

(2) $\widetilde{\varphi}_0(S(x,r_k)) \cap \partial D'_F \neq \varnothing$ for some sequence $r_k \to 0$.

If case 1 is fulfilled then we can assign some value at a point x to $\widetilde{\varphi}_0$: put

$$\widetilde{\varphi}_0(x) = \bigcap_{r\to 0} \overline{\widetilde{\varphi}_0(\widehat{B}(x,r))} \in D_F.$$

Under such a definition of $\widetilde{\varphi}_0(x)$, we come to the situation similar to that in Definition 62. Therefore, $\widetilde{\varphi}_0(x)$ can be defined not only at x but also at the points of some ball $B(x,\delta_{x,\rho})$ by analogy to Definition 62. Like in Proposition 63, we prove that $\widetilde{\varphi}_0 : B(x,\delta_{x,\rho}) \to D'_F$ is continuous at all points of $B(x,\delta_{x,\rho})$. Consequently, the domains U and V can be enlarged, and S can be narrowed.

Assume henceforth that property 2 is fulfilled for all $x \in S$.

Let $x \in S$. Then there exists a sequence $\{x_k \in U\}$ converging to x and such that $\widetilde{\varphi}_0(x_k) \to \partial D'$ as $k \to \infty$. In this case,

Lemma 70. *$B(x,r)\cap U$ is connected for every ball $B(x,r)\subset D_F$ with center $x\in S$.*

PROOF. Suppose the contrary, and so $B(x,r)\cap U$ consists of several connected components. Then the image $\widetilde{\varphi}_0(B(x,r))$ is divided by the boundary $\partial D'_F$ into the several connected components: $\widetilde{\varphi}_0(B(x,r))=V_1\cup V_2\cup\cdots$ or $D'_F\setminus\widetilde{\varphi}_0(S(x,r))=V_0\cup V_1\cup V_2\cup\cdots$.

In D_F, consider the smooth cut-of-function

$$\eta=\begin{cases}1 & \text{on } B(x,r/2),\\ 0 & \text{outside } B(x,r).\end{cases}$$

We may assume that $\left|\widetilde{\varphi}_0^{-1}(V_1)\cap B(x,r/2)\right|>0$. Construct a function $g:D'_F\to\mathbb{R}$ such that

$$g(y)=\begin{cases}\eta\circ\widetilde{\varphi}_0^{-1}(y) & \text{on } V_1,\\ 0 & \text{on } V_0\cup V_2\cup V_3\cup\cdots.\end{cases}\tag{48}$$

Obviously, g is continuous on V. Show that $g\in L^1_{\nu,\varphi(F)}(D')$. Since the mapping $\varphi_0: B(x,r)\cap U\to\varphi_0(B(x,r))\cap V$ is quasiconformal, g belongs to $L^1_\nu(\varphi_0(B(x,r))\cap V)$. Consequently, g is locally integrable and has weak derivatives integrable to the power ν in $\varphi_0(B(x,r))\cap V$. In particular, g is absolutely continuous on almost all integral curves of the horizontal vector fields, and there exist derivatives $v_j=X_jg$ a.e. in $\varphi_0(B(x,r))\cap V$, $j=1,2,\dots,n$. It remains to prove that v_j is a weak derivative of g in D'_F; i.e.,

$$\int\limits_{D'_F} g(y)\cdot X_j\psi(y)\,dy=-\int\limits_{D'_F} v_j\cdot\psi\,dy\tag{49}$$

for every test function $\psi\in C_0^\infty(D'_F)$. By Fubini's Theorem,

$$\int\limits_{D'_F} g(y)\cdot X_j\psi(y)\,dy=\int\limits_{\Pr_j D'_F} dy_1\dots\widehat{dy_j}\dots dy_N\int\limits_{\Gamma_j(y)\cap D'_F} g(y)\cdot X_j\psi(y)\,dy_j,$$

where $\Pr_j D'_F$ is the projection of D'_F to the hypersurface transversal to the vector field X_j, while $\Gamma_j(y)$ is the integral line of X_j passing through $y\in\Pr_j D'_F$. Since $g=0$ on $V_0\cup V_2\cup V_3\cup\cdots$, we infer

$$\int\limits_{\Gamma_j(y)\cap D'_F} g(y)\cdot X_j\psi(y)\,dy_j=\int\limits_{\Gamma_j(y)\cap V_0\cup V_1\cap V_2\cup\cdots} g(y)\cdot X_j\psi(y)\,dy_j=\int\limits_{\Gamma_j(y)\cap V_1} g(y)\cdot X_j\psi(y)\,dy_j.$$

The intersection $\Gamma_j(y)\cap V_1$ can be represented as a countable union of curves: $\Gamma_j(y)\cap V_1=\bigcup_l\gamma_l$. Then

$$\int\limits_{\Gamma_j(y)\cap V_1} g(y)\cdot X_j\psi(y)\,dy_j=\sum_l\int\limits_{\gamma_l} g(y)\cdot X_j\psi(y)\,dy_j.$$

Applying the formula for integration by parts, we infer

$$\int\limits_{\gamma_l} g(y)\cdot X_j\psi(y)\,dy_j=g(y)\psi(y)\Big|_{\gamma_l(t_0^l)}^{\gamma_l(t_1^l)}-\int\limits_{\gamma_l} X_jg(y)\cdot\psi(y)\,dy_j.$$

Note that $\gamma_l(t_0^l)\in\partial V_1$, and the two cases are possible:

(1) $\gamma_l(t_0^l)\in D'_F$; then $g(\gamma_l(t_0^l))=0$;

(2) $\gamma_l(t_0^l)\in\partial D'_F$; then $\psi(\gamma_l(t_0^l))=0$.

An analogous situation holds for $\gamma_l(t_1^l)$. Thus,

$$g(y)\psi(y)\Big|_{\gamma_l(t_0^l)}^{\gamma_l(t_1^l)}=0,$$

and (49) is proved. Hence, $g\in L^1_{\nu,\varphi(F)}(D')$.

Further, φ^*g belongs to $L^1_{\nu,F}(D)$ and a.e. on $B(x,r/2)$ takes only two values (0 and 1). Consequently, $\nabla\varphi^*g=0$ a.e. on $B(x,r/2)$, and so $\varphi^*g=g\circ\widetilde{\varphi}_0$ is a constant function on $B(x,r/2)$. The so-obtained contradiction leads us to the conclusion that $B(x,r)\cap U$ is connected. □

An analogous property is also fulfilled in the image:

Lemma 71. *For every ball $B(y,r) \subset D'_F$ centered at $y \in \varphi_0(S)$, the intersection $B(y,r) \cap V$ is connected.*

4.6. Extension of $\widetilde{\varphi}_0$ to S and its properties. In this subsection, we will need

Lemma 72. *Let $\gamma_1, \gamma_2 : [0,1) \to V$ be two curves at a positive distance from one another. Then no point in D_F can be a limit point for each of the preimages $\beta_1 = \widetilde{\varphi}_0{}^{-1}(\gamma_1)$ and $\beta_2 = \widetilde{\varphi}_0{}^{-1}(\gamma_2)$.*

Proof. Suppose on the contrary that there is $y \in D'$ that is a limit point for the preimages $\widetilde{\varphi}_0{}^{-1}(\gamma_1)$ and $\widetilde{\varphi}_0{}^{-1}(\gamma_2)$: there exists a sequence $t_k \in [0,1)$, $t_k \to 1$ ($\tau_k \in [0,1)$, $\tau_k \to 1$) as $k \to \infty$ such that $\beta_1(t_k) \to y$ ($\beta_2(\tau_k) \to y$) as $k \to \infty$. Consider a continuous function $g \in L^1_\nu(D')$ such that $g = 0$ on γ_1 and $g = 1$ on γ_2. Then the composition $f = g \circ \widetilde{\varphi}_0 : U \to \mathbb{R}$ is a continuous function equal to 0 on β_1 and to 1 on β_2. Moreover, $\varphi^*(g) \in L^1_\nu(D)$. The existence of such a function contradicts Proposition 45. $\square$

Let us show that the mapping can be extended to a part of S (excluding the points whose images can go to the infinitely remote point). Let $x \in S$. The two cases are possible:

1. For some sequence $\{x_n \in U\}$ converging to x, the sequence of the images $\widetilde{\varphi}_0(x_n)$ converges to some point $z \in \partial D'_F$.

2. For every $\{x_n \in U\}$ converging to x, we have $d(\widetilde{\varphi}_0(x_n), 0) \to \infty$ (this case will be considered separately below).

In what follows, we will prove that, in the first case $\widetilde{\varphi}_0$ extends by continuity to the point $x \in S$.

Proposition 73. *The mapping $\widetilde{\varphi}_0 : U \to V$ extends by continuity to all points $x \in S$ for each of which there exists a sequence $\{x_n \in U\}$ converging to x such that the sequence of images $\widetilde{\varphi}_0(x_n)$ converges to $z \in \partial D'_F$. The extended mapping is injective.*

Proof. Show that the limit z does not depend on the choice of $\{x_n\}$. Let $U \ni x'_n \to x$ be another sequence and $\widetilde{\varphi}_0(x'_n) \to z' \in \partial D'_F$; moreover, $z \neq z'$. By the local connectedness of V, we can construct two curves $\gamma, \gamma' \subset V$ at a positive distance $\operatorname{dist}(\gamma, \gamma') \geq \delta > 0$ and passing through the images $\widetilde{\varphi}_0(x_n)$ and $\widetilde{\varphi}_0(x'_n)$ respectively (starting from some $n > n_0$). Then the preimages $\widetilde{\varphi}_0{}^{-1}(\gamma)$ and $\widetilde{\varphi}_0{}^{-1}(\gamma')$ have a limit point $x \in D_F$. By Lemma 72, we get a contradiction.

Extend the mapping $\widetilde{\varphi}_0$ at the point x: $\widetilde{\varphi}_0(x) = z$. Thus, construct a continuous mapping $\widetilde{\varphi}_0$ to the set S excluding the points that are mapped to the infinitely remote point. Denote the extension by the same symbol.

Check the injectivity of $\widetilde{\varphi}_0$. Suppose that there is a point $z \in Z$ such that $z = \widetilde{\varphi}_0(x_1) = \widetilde{\varphi}_0(x_2)$, where $x_1 \neq x_2$, $x_1, x_2 \in S$. Consider curves γ_1 and γ_2 passing through points x_1 and x_2 respectively and being at a positive distance ($\delta = \operatorname{dist}(\gamma_1, \gamma_2)$). Consider arbitrary sequences $\{x^1_n \in U\}$ and $\{x^2_n \in U\}$ such that $x^i_n \to x_i$ as $n \to \infty$ and $x^i_n \in \gamma_i$. Construct a sequence of curves σ_n joining $\widetilde{\varphi}_0(x^1_n)$ and $\widetilde{\varphi}_0(x^2_n)$ so that $\operatorname{diam} \sigma_n \to 0$. Then $\operatorname{Cap}(\widetilde{\varphi}_0^{-1}(\sigma_n); L^1_\nu(U)) \to 0$, and $\operatorname{diam} \widetilde{\varphi}_0^{-1}(\sigma_n) \to 0$. We arrive at a contradiction since $\operatorname{diam} \widetilde{\varphi}_0^{-1}(\sigma_n) \geq \delta$. $\square$

Thus, from S, there remain only points satisfying the second of the cases described before Proposition 73. In the following assertion, we prove that if S is nonempty then S is a singleton.

Lemma 74. *There can exist at most one point $x_{\rm inv} \in S$ such that, for every sequence $\{x_n\} \subset U$ converging to $x_{\rm inv}$, we have $d(\widetilde{\varphi}_0(x_n)) \to \infty$ as $n \to \infty$ (the case of inversion).*

Proof. Let us first prove that S has capacity zero. Choose two balls $B(0, r_0)$ and $B(0, R_k)$ such that $\varphi(F) \subset B(0, r_0)$, r_0 is fixed, $R_k > r$, and $\lim_{k\to\infty} R_k = \infty$. Note that $S \subset \bigcap_k \overline{\widetilde{\varphi}_0^{-1}(\mathbb{G} \setminus B(0, R_k))}$.

We have

$$\overline{\operatorname{Cap}}\big(S; L^1_{\nu,F}(D)\big) \leq \overline{\operatorname{Cap}}\big(\widetilde{\varphi}_0^{-1}(\mathbb{G} \setminus B(0, R_k)) \cap D_F; L^1_{\nu,F}(D)\big)$$
$$\leq K^\nu \overline{\operatorname{Cap}}\big((\mathbb{G} \setminus B(0, R_k) \cap D'_F; L^1_{\nu,\varphi(F)}(D')\big) \leq K^\nu \operatorname{Cap}\big(\mathbb{G} \setminus B(0, R_k); L^1_{\nu,B(0,r)}(\mathbb{G})\big).$$

From [10, Theorems 6.6 and 6.9] we obtain that $\operatorname{Cap}\big(\mathbb{G} \setminus B(0,R); L^1_{\nu,B(0,r)}(\mathbb{G})\big)$ is equivalent to $\big(\log \frac{R}{r_0}\big)^{1-\nu}$; therefore,

$$\lim_{k\to\infty} \overline{\operatorname{Cap}}\big(\widetilde{\varphi}_0^{-1}(\mathbb{G} \setminus B(0,R_k)) \cap D_F; L^1_{\nu,F}(D)\big) = 0 \quad \text{and} \quad \overline{\operatorname{Cap}}\big(S; L^1_{\nu,F}(D)\big).$$

Thus, S has capacity zero. Let us show that S cannot consist of more than one point.

Suppose on the contrary that there are two distinct points $x_1, x_2 \in S$ with this property. Consider sequences $\{x_n^1\}, \{x_n^2\} \subset U$ such that $\lim_{n\to\infty} x_n^1 = x_1$ and $\lim_{n\to\infty} x_n^2 = x_2$. Choose some spheres $S(x_1,r_1), S(x_2,r_2) \subset U$ on which $\widetilde{\varphi}_0$ is continuous (see Proposition 56) so that $\overline{B}(x_1,r_1) \cap \overline{B}(x_2,r_2) = \varnothing$.

Since $\widetilde{\varphi}_0$ is a continuous and injective mapping, the image $\widetilde{\varphi}_0(S(x_1,r_1))$ partitions $\mathbb{G}$ into two components (one is bounded and the other is unbounded); moreover, $\widetilde{\varphi}_0(B(x_1,r_1) \setminus S)$ belongs to the unbounded component and $\widetilde{\varphi}_0(U \setminus B(x_1,r_1))$ belongs to the bounded component.

On the other hand, $B(x_2,r_2) \setminus S \subset U \setminus B(x_1,r_1)$; therefore, $\widetilde{\varphi}_0(B(x_2,r_2) \setminus S)$ belongs to the bounded component $\mathbb{G} \setminus \widetilde{\varphi}_0(S(x_1,r_1))$, which contradicts the assumption $d\big(\widetilde{\varphi}_0(x_n^2)\big) \to \infty$ as $n \to \infty$. □

In result, we obtain a continuous injective mapping $\widetilde{\varphi}_0 : D_F \setminus \{x_{\mathrm{inv}}\} \to \overline{D'_F}$.

Proposition 75. *$\widetilde{\varphi}_0 : D_F \setminus \{x_{\mathrm{inv}}\} \to \mathbb{G}$ is a homeomorphism.*

For a proof, it suffices to check that $\widetilde{\varphi}_0 : D_F \setminus \{x_{\mathrm{inv}}\} \to \mathbb{G}$ is open. Indeed, the degree $\mu(\widetilde{\varphi}_0, B(x,r), \varphi_0(x))$ is nonzero for every ball $B(x,r) \subset D_F$. Hence, $\widetilde{\varphi}_0(x)$ is an interior point of the image.

We can now prove that $\widetilde{\varphi}_0$ belongs to the Sobolev class $W^1_{\nu,\mathrm{loc}}(D_F \setminus \{x_{\mathrm{inv}}\})$ (an extension of Lemma 69).

Lemma 76. *Let $D, D' \subset \mathbb{G}$. If $\varphi : D \to D'$ belongs to IL^1_ν then $\widetilde{\varphi}_0 \in W^1_{\nu,\mathrm{loc}}(D_F \setminus \{x_{\mathrm{inv}}\})$.*

Proof. This is a direct corollary of Lemma 69 in whose hypothesis we should take $D_F \setminus \{x_{\mathrm{inv}}\}$ as U. □

The above implies

Proposition 77. *$\widetilde{\varphi}_0 : D_F \setminus \{x_{\mathrm{inv}}\} \to \mathbb{G}$ is quasiconformal.*

Proof. The last assertions imply that the homeomorphism $\widetilde{\varphi}_0$ belongs to $W^1_\nu(D_F \setminus \{x_{\mathrm{inv}}\})$ and $|D(x,\widetilde{\varphi}_0)| \le K|J(x,\varphi)|^{\frac{1}{\nu}}$ a.e. in $D_F \setminus \{x_{\mathrm{inv}}\}$ since $|S| = 0$ (note that $J(x,\varphi) = J(x,\widetilde{\varphi}_0)$ a.e.). Consequently, $\widetilde{\varphi}_0 : D_F \setminus \{x_{\mathrm{inv}}\} \to \mathbb{G}$ is quasiconformal. □

Proposition 78. *$\widetilde{\varphi}_0 : D \setminus \{x_{\mathrm{inv}}\} \to \mathbb{G}$ is quasiconformal.*

Proof. Choose another closed set of positive measure $F_1 \subset T_{k_0}$ without isolated points situated at a positive distance from F and such that $x_{\{\mathrm{inv}\}} \notin F_1$. Repeating the above procedure, prove that $\widetilde{\varphi}_0$ is quasiconformal on the open set $D \setminus \{x_{\mathrm{inv}}\}$. □

4.7. Proof of Theorem 2. Let us prove the main result of this article.

Proof. *Sufficiency.* We may assume that $\varphi : D \to D'$ is quasiconformal. By Definition 3 of this article, the quasiconformal mapping φ locally belongs to the Sobolev class ($\varphi \in W^1_{\nu,\mathrm{loc}}$). Moreover, φ is $\mathscr{P}$-differentiable and satisfies Luzin's $\mathscr{N}$- and $\mathscr{N}^{-1}$-conditions [28].

For an arbitrary $f \in L^1_\nu(D') \cap C^\infty(D')$, the composition $f \circ \varphi$ is absolutely continuous on almost all integral curves of the horizontal vector fields because such is f. Moreover, $\nabla_{\mathscr{L}}(f \circ \varphi) = D_h\varphi^T(x)\nabla_{\mathscr{L}} f(\varphi(x))$ [29, p. 263], where $D_h\varphi(x) = \{X_i\varphi_j(x)\}$, $i,j = 1,\dots,n_1$, is the horizontal part of the $\mathscr{P}$-differential. Hence,

$$\int_D |\nabla_{\mathscr{L}}(f \circ \varphi)|^\nu\, dx = \int_D |D_h\varphi^T(x)\nabla_{\mathscr{L}} f(\varphi(x))|^\nu\, dx \le \int_D |D_h\varphi^T(x)|^\nu \cdot |\nabla_{\mathscr{L}} f(\varphi(x))|^\nu\, dx$$

$$= \int_D |\nabla_{\mathscr{L}} f|^\nu(\varphi(x)) \cdot |D_h\varphi(x)|^\nu\, dx \le K \int_D |\nabla_{\mathscr{L}} f|^\nu(\varphi(x)) \cdot |J(x,\varphi)|\, dx = \int_{D'} |\nabla_{\mathscr{L}} f|^\nu(y)\, dy.$$

Here we used the pointwise inequality $|D_h\varphi(x)|^\nu \leq K|J(x,\varphi)|$ for a.e. $x \in D$ and the change-of-variable formula (3).

By Lemma 14, the above inequality holds for all $f \in L^1_\nu(D')$; i.e.,

$$\|\varphi^*(f) \mid L^1_\nu(D)\| \leq K^{\frac{1}{\nu}}\|f \mid L^1_\nu(D')\|. \tag{50}$$

The mapping φ^{-1} is also quasiconformal. Then, for $g \in L^1_\nu(D)$, we have

$$\|\varphi^{-1*}(g) \mid L^1_\nu(D')\| \leq K_1^{\frac{1}{\nu}}\|g \mid L^1_\nu(D)\|, \tag{51}$$

where K_1 is the quasiconformality coefficient of the inverse mapping. Note that, for $f \in L^1_\nu(D') \cap C^\infty(D')$, we have $\varphi^{-1*}(f\circ\varphi) = f$. Consequently, (51) takes the form $K_1^{-\frac{1}{\nu}}\|f \mid L^1_\nu(D')\| \leq \|\varphi^*(f) \mid L^1_\nu(D)\|$. Thus,

$$K_1^{-\frac{1}{\nu}}\|f \mid L^1_\nu(D')\| \leq \|\varphi^*(f) \mid L^1_\nu(D)\| \leq K^{\frac{1}{\nu}}\|f \mid L^1_\nu(D')\|,$$

where the constants K and K_1 depend on the properties of φ.

Show that the image $\varphi^*\big(L^1_\nu(D') \cap C^\infty(D')\big)$ is dense in $L^1_\nu(D)$. Let $g \in L^1_\nu(D)$. There exists $g_n \in L^1_\nu(D) \cap C^\infty(D)$ such that $\|g - g_n \mid L^1_\nu(D)\| \to 0$. On the other hand, by the two-sided estimate, $g_n \circ \varphi^{-1} \in L^1_\nu(D')$. Hence, there is a sequence $f_{nk} \in L^1_\nu(D') \cap C^\infty(D')$ such that

$$\|g_n \circ \varphi^{-1} - f_{nk} \mid L^1_\nu(D')\| \to 0 \quad \text{as } k \to \infty.$$

Then, for some sequence of naturals l_n, we have $\varphi^* f_{nl_n} \in \varphi^*\big(L^1_\nu(D') \cap C^\infty(D')\big)$ and $\|g - \varphi^* f_{nl_n} \mid L^1_\nu(D)\| \to 0$ as $n \to \infty$.

Necessity: The existence of a quasiconformal mapping Φ is proved in Proposition 77: $\Phi = \widetilde{\varphi}_0 : D \setminus \{x_{\mathrm{inv}}\} \to G$. By the above, the composition operator $\Phi^* : L^1_\nu(\Phi(D \setminus \{x_{\mathrm{inv}}\})) \to L^1_\nu(D \setminus \{x_{\mathrm{inv}}\})$ is an isomorphism. Since, obviously, $L^1_\nu(D \setminus \{x_{\mathrm{inv}}\}) = L^1_p(D)$, this gives the isomorphism $\varphi^{*-1} \circ \Phi^* : L^1_\nu(\Phi(D \setminus \{x_{\mathrm{inv}}\})) \to L^1_\nu(D')$ such that $\varphi^{*-1} \circ \Phi^*(f)(x) = f(x)$ for all $x \in \Phi(D \setminus \{x_{\mathrm{inv}}\}) \cap D'$, where $f \in L^1_\nu(\Phi(D \setminus \{x_{\mathrm{inv}}\}))$ is an arbitrary function.

Therefore, by means of the restriction operator, $L^1_\nu(\Phi(D\setminus\{x_{\mathrm{inv}}\}) \cup D')$ is isomorphic both to $L^1_\nu(\Phi(D\setminus \{x_{\mathrm{inv}}\}))$ and to $L^1_\nu(D')$. Thus, $\Phi(D \setminus \{x_{\mathrm{inv}}\})$ and D' are $(1,\nu)$-equivalent domains.

Similarly to what was proved in [9, Theorem 3.1; 10, Proposition 6.10], we can obtain the properties:

(1) $|\Phi(D)\Delta D'| = 0$;

(2) for every ball $B \subset D'$, the set $B \setminus \Phi(D)\Delta D'$ is connected. □

4.8. Corollary: removable sets for quasiconformal mappings. Recall that a closed set $E \subset D$ is called *removable* for quasiconformal mappings if each quasiconformal mapping $\varphi : D \setminus E \to \mathbb{G}$ extends to a quasiconformal mapping of D.

Corollary 79. *Suppose that U and D are $(1,\nu)$-equivalent and $U \subset D$. Then $D \setminus U$ is removable for quasiconformal mappings.*

Proof. Let $\varphi_1 : U \to \mathbb{G}$ be a quasiconformal mapping. For proving the corollary, we must construct a quasiconformal extension of φ_1 to D.

By Theorem 2, the composition operator $\varphi_1^* : L^1_\nu(\varphi_1(U)) \to L^1_\nu(U)$ is an isomorphism. Since the sets U and D are $(1,\nu)$-equivalent, the restriction $r^* : L^1_\nu(D) \to L^1_\nu(U)$ is an isomorphism too.

Consider the measurable mapping $\varphi : D \to \varphi_1(U)$ such that $\varphi(x) = \varphi_1(x)$ for $x \in U$. The composition operator $\varphi^* : L^1_\nu(\varphi_1(U)) \cap C^\infty(\varphi_1(U)) \to L^1_\nu(D)$ defined by the rule $\varphi^* f = f \circ \varphi$ extends to an isomorphism of $L^1_\nu(\varphi_1(U))$ and $L^1_\nu(D)$ because $\varphi^* f = r^{*-1} \circ \varphi_1^* f$ for $f \in L^1_\nu(\varphi_1(U)) \cap C^\infty(\varphi_1(U))$. By Theorem 2, there exists a quasiconformal mapping $\Phi : D \to \mathbb{G}$ coinciding with φ a.e. Moreover, $\Phi(x) = \varphi(x)$ if $x \in U$. Thus, Φ is a desired extension. □

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