

## COMPOSITION OPERATORS IN WEIGHTED SOBOLEV SPACES ON THE CARNOT GROUP

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UDC 517.518:517.54

**Abstract:** We study the properties of the mappings inducing bounded change-of-variables operators in weighted Sobolev spaces on the Carnot group. We obtain an analytical description of these mappings in terms of integrability of the weighted distortion function. In some cases we prove that the mapping inducing a bounded operator is piecewise absolutely continuous on almost all horizontal lines.

**DOI:** 10.1134/S0037446615060087

**Keywords:** composition operator, weighted Sobolev space, Carnot group

### Introduction

Under study are the properties of a measurable mapping  $\varphi : D \rightarrow D'$  inducing the bounded operator in weighted Sobolev spaces on the Carnot group by change of variables:

$$\varphi^* : L_p^1(D', \nu) \cap C_0^\infty(D') \rightarrow L_q^1(D, u), \quad \varphi^*(f) = f \circ \varphi, \quad f \in L_p^1(D', \nu) \cap C_0^\infty(D').$$

Depending on the relation between the summability exponents  $q$  and  $p$ , we manage to obtain either a complete or a partial solution to the problem. In the case  $q = p$  we find necessary conditions for a mapping to induce a bounded composition operator (Theorem 3). Theorem 4 gives sufficient conditions for the boundedness of the composition operator in the general situation  $1 \leq q \leq p < \infty$ . When the summability exponents and the Hausdorff dimension coincide,  $q = p = \nu$ , we solve the problem completely (Theorem 5).

In the Euclidean case the mappings, inducing a bounded composition operator in weighted Sobolev spaces, are fully described in [1]. These are the mappings with finite distortion whose weighted distortion function is summable to some power. A similar description on Carnot groups is obtained in [2] (the unweighted case) under the condition that  $\varphi$  is of class ACL. The weighted case is studied in [3] on assuming that  $\varphi$  is a homeomorphism. In this article we dispense with the a priori regularity properties of mappings and in some cases deduce the ACL-property from the boundedness of the composition operator.

Most methods we apply were developed in [1, 2, 4–7], where some equivalent descriptions were obtained of the mappings inducing bounded operators of the Sobolev spaces both on Euclidean spaces and Carnot groups. In this article we apply a new method to prove the piecewise absolute continuity of the measurable mappings inducing bounded operators in Sobolev spaces.

For a more detailed discussion of the history of this question and extensive bibliography on the study of composition operators in Sobolev spaces with the first generalized derivatives, see [1–6, 8–10].

The article is organized as follows: In Section 1 we recall the main concepts and justify some auxiliary statements. In Section 2 we define a bounded composition operator and characterize its norm in terms of a finitely additive set function. In Section 3 we study the continuity and differentiability properties of a mapping  $\varphi$  inducing a bounded composition operator. In Subsection 3.1 we prove the continuity on almost all lines and the approximate differentiability of  $\varphi$  almost everywhere. In Subsection 3.2 we introduce the concept of piecewise absolute continuity on almost all lines, proposed by Vodopyanov, which

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The author was partially supported by the Government of the Russian Federation (Grant 14.B25.31.0029) and the Russian Foundation for Basic Research (Grant 14–01–00768).

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Novosibirsk. Translated from *Sibirskii Matematicheskii Zhurnal*, Vol. 56, No. 6, pp. 1304–1325, November–December, 2015; DOI: 10.17377/smzh.2015.56.608. Original article submitted July 2, 2015.

is a weakening of the ACL-property which suits our problem. Subsequently this property ensures that the composition  $f \circ \varphi$  belongs to  $\text{ACL}(D)$ . Section 4 contains the main results of the article, Theorems 3–5.

## 1. Preliminaries

### 1.1. Carnot groups.

DEFINITION 1. A *Carnot group*  $\mathbb{G}$  is a connected simply-connected nilpotent Lie group whose Lie algebra  $\mathcal{G}$  is graded:  $\mathcal{G} = V_1 \oplus \cdots \oplus V_m$  with  $\dim V_1 = n_1 \geq 2$  and  $[V_1, V_k] = V_{k+1}$  for  $1 \leq k \leq m-1$  and  $[V_1, V_m] = 0$ . Put  $N = \sum_{i=1}^m \dim V_i$ .

Identify  $g \in \mathbb{G}$  with  $x \in \mathbb{R}^N$  by the exponential mapping  $\exp(\sum x_{ij} X_{ij})$  and call the numbers  $x_{ij}$  the *coordinates of the first kind* of  $g \in \mathbb{G}$ , for  $1 \leq i \leq m$  and  $1 \leq j \leq n_i = \dim V_i$ . Therefore, there exists a global coordinate system on  $\mathbb{G}$  which enables us to identify the points of  $\mathbb{G}$  with those in  $\mathbb{R}^N$ .

In the coordinates of the first kind the mapping

$$\delta_t : (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_m) \mapsto (t\bar{x}_1, t^2\bar{x}_2, \dots, t^m\bar{x}_m), \quad t > 0,$$

determines a one-parameter family of dilations. Here  $\bar{x}_i \in \mathbb{R}^{n_i}$ .

For  $i = 1, \dots, n_1$  the left-invariant vector fields  $X_i = X_{i,1}$ , constituting the standard basis of the subbundle  $V_1$ , are called *horizontal*.

Fix the homogeneous norm on  $\mathbb{G}$  (see [11] for instance):

$$\rho(x) = \left( \sum_{i=1}^m |\bar{x}_i|^{2m!/i} \right)^{1/2m!},$$

where  $|\bar{x}_i|$  is the Euclidean norm on  $V_i$ . This homogeneous norm determines the homogeneous quasimetric  $\rho(x, y) = \rho(x^{-1}y)$  for  $x, y \in \mathbb{G}$ .

An absolutely continuous piecewise smooth line  $\gamma : [a, b] \rightarrow \mathbb{G}$  whose tangent vector lies in  $V_1$  is called a *horizontal line*.

DEFINITION 2. The *Carnot–Carathéodory metric*  $d(x, y)$  on  $\mathbb{G}$  is the greatest lower bound of the length of all horizontal lines connecting  $x$  and  $y$ .

We can show that  $d(x, y)$  and  $\rho(x, y)$  are equivalent (see the arguments in Lemma 1.4 and Proposition 1.5 of [12]); i.e.,

$$c\rho(x, y) \leq d(x, y) \leq C\rho(x, y).$$

The Hausdorff dimension of  $\mathbb{G}$  equals  $\nu = n_1 + 2n_2 + 3n_3 + \cdots + mn_m$ , where  $n_i = \dim V_i$ .

**1.2. Weighted Sobolev spaces.** Consider an open subset  $D$  of a Carnot group  $\mathbb{G}$ . For  $i = 1, \dots, n_1$ , a locally summable function  $g_i : D \rightarrow \mathbb{R}$  is called the *generalized derivative* of a function  $f$  along the vector field  $X_i$  whenever

$$\int_D g_i \psi \, dx = - \int_D f X_i \psi \, dx$$

for an arbitrary compactly supported function  $\psi \in C_0^\infty(D)$ . Henceforth we write  $g_i = X_i f$ .

Refer as a *weight* to a measurable function  $w : \mathbb{G} \rightarrow \mathbb{R}$  taking positive values almost everywhere. Given a measurable set  $A$ , define the weighted measure

$$w(A) = \int_A w(x) \, dx.$$

Here  $dx$  is the Lebesgue measure on  $\mathbb{R}^N$  normalized so that the measure of radius 1 balls (with respect to the quasimetric  $\rho$ ) equals 1. The weighted space  $L_p(D, w)$  consists of the functions summable to the power  $p \in [1, \infty)$  with weight  $w$ :

$$\|f\|_{L_p(D, w)} = \left( \int_D |f(x)|^p w(x) \, dx \right)^{1/p} < \infty.$$

The weighted Sobolev space  $L_p^1(D, w)$  consists of the locally summable functions with finite seminorm

$$\|f\|_{L_p^1(D, w)} = \|\nabla_{\mathcal{L}} f\|_{L_p(D, w)},$$

where  $\nabla_{\mathcal{L}} f(x) = (X_1 f(x), \dots, X_{n_1} f(x))$  is the generalized subgradient of  $f$  at  $x \in D$ .

DEFINITION 3. A weight function  $w$  is of the *Muckenhoupt class*  $A_p$  with  $p \in (1, \infty)$  whenever

$$\sup_{B \subset \mathbb{G}} \left( \frac{1}{|B|} \int_B w \, dx \right) \left( \frac{1}{|B|} \int_B w^{\frac{1}{1-p}} \, dx \right)^{p-1} = c_{w,p} < \infty,$$

where we take the least upper bound over all balls  $B$  in  $\mathbb{G}$ .

We may assume that the weight functions  $u(x)$  and  $v(y)$  belong to appropriate Muckenhoupt classes, but in the main results we indicate more precise conditions on these functions.

For more details on weighted Sobolev spaces; see [13, 14] for instance.

**1.3. The measure of a family of lines.** Consider a family  $\Gamma_j$  of integral lines of a basis horizontal vector field  $X_j$  forming a smooth foliation of an open set  $A \subset \mathbb{G}$ . Denote the corresponding flow by  $g_s$ . The leaves are of the form  $\gamma(s) = g_s(p)$ , where  $p$  lies on a surface  $S_j$  transversal to  $X_j$ , while the parameter  $s$  runs over an interval  $I \subset \mathbb{R}$ . For the foliation determined by  $X_j$ , we can obtain the measure  $d\gamma$  as the contraction  $i(X_j)$  of  $X_j$  with the bi-invariant volume form  $dx$ . Denote by  $\mathbb{J}_{g_s}$  the Jacobian of the flow  $g_s$ . Then

$$g_s^* i(X_j) dx = \mathbb{J}_{g_s} i(X_j) dx \quad \text{or} \quad g_s^* (\mathbb{J}_{g_s} i(X_j) dx) = i(X_j) dx.$$

Since  $g_s$  carries every tangent vector to the one-parameter family of lines  $\gamma_t$  into a tangent vector to the same family, the form  $\mathbb{J}_{g_s} i(X_j) dx$  determines a measure  $d\gamma$  on  $\Gamma_j$ . Since  $X_j$  is a left-invariant horizontal vector field, the flow  $g_s$  is the right translation on  $\exp sX_j$ :  $\mathbb{G} \ni p \mapsto p \exp sX_j$ . Since  $dx$  is a bi-invariant form, we have  $\mathbb{J}_{g_s} = 1$ . Using left-invariance and homogeneity under dilations, we find that

$$\int_{\gamma \cap B(x,r) \neq \emptyset} d\gamma = c |B(x,r)|^{\frac{\nu-1}{\nu}}.$$

Hence, we can deduce Fubini's theorem to be applied below. For more details; see [11].

#### 1.4. Set functions.

DEFINITION 4. A mapping  $\Phi$  defined on the open subsets of an open set  $D \subset \mathbb{G}$  and taking nonnegative values is called a *finitely  $\lambda$ -quasiadditive set function*, with  $1 \leq \lambda < \infty$ , whenever

- (1) for every  $x \in D$  there exists  $\delta$  with  $0 < \delta < \text{dist}(x, \partial D)$  such that  $0 \leq \Phi(B(x, \delta)) < \infty$ ;
- (2)  $\sum_{i=1}^k \Phi(U_i) \leq \lambda \Phi(U)$  for every tuple of disjoint open sets  $U_1, \dots, U_k \subset U \subset D$  for  $i = 1, \dots, k$ .

In the case  $\lambda = 1$  we use the term a *quasiadditive function*.

If instead of the second condition in the definition of set function we impose the equality

$$\sum_{i=1}^k \Phi(U_i) = \Phi\left(\bigcup_{i=1}^k U_i\right)$$

for every finite tuple of disjoint open sets  $U_i \subset D$  then we call this function *finitely additive*. If this equality extends to countable collections then the function is called *countably additive*.

A quasiadditive function  $\Phi$  is called *monotone* if  $\Phi(U_1) \leq \Phi(U_2)$  for all open sets  $U_1 \subset U_2 \subset D$ .

Define the upper and lower derivatives of a quasiadditive function on a collection of open subsets as

$$\overline{\Phi}'(x) = \limsup_{h \rightarrow 0} \sup_{\delta \leq h} \frac{\Phi(B_\delta)}{|B_\delta|} \quad \text{and} \quad \underline{\Phi}'(x) = \liminf_{h \rightarrow 0} \inf_{\delta \leq h} \frac{\Phi(B_\delta)}{|B_\delta|}. \quad (1)$$

Here we take the supremum and infimum over all open balls  $B_\delta$  of radius  $\delta \leq h$  containing  $x$ . If at some point  $x$  the upper and lower derivatives coincide,  $\overline{\Phi}'(x) = \underline{\Phi}'(x)$ , then we call their common value the *derivative*  $\Phi'(x)$  of  $\Phi$ .

**Proposition 1** [2]. *Take a finitely  $\lambda$ -quasiadditive set function  $\Phi$  on the open subsets of a domain  $D \subset \mathbb{G}$ . The following hold:*

(a) *For every open set  $U \subset D$*

$$\int_U \overline{\Phi}'(x) dx \leq \lambda \Phi(U).$$

(b) *At almost every point  $x \in D$  the upper derivative exists, is finite, and*

$$\overline{\Phi}'(x) \leq \lambda \underline{\Phi}'(x).$$

*If  $\lambda = 1$  then at almost every point  $x \in D$  the derivative*

$$\Phi'(x) = \lim_{\delta \rightarrow 0} \frac{\Phi(B_\delta)}{|B_\delta|}$$

*exists and is finite.*

Below we need the following theorem by Lebesgue.

**Theorem 1** [2, Corollary 9]. *Consider a domain  $D$  in a Carnot group  $\mathbb{G}$  and take a function  $f$  in  $L_{1,\text{loc}}(D)$ . Then*

$$\lim_{\delta \rightarrow 0} \frac{1}{|B_\delta|} \int_{B_\delta} |f(y) - f(x)| dy = 0$$

*for almost all  $x \in D$ .*

Let us present two more covering lemmas we will use in establishing the main results.

**Lemma 1** [2, Lemma 9]. *Given an open set  $U \subset \mathbb{R}^N$  with  $U \neq \mathbb{R}^N$ , there exists a countable family  $\mathcal{B} = \{B_j\}$  of Euclidean balls such that*

(1)  $\bigcup_j B_j = U$ ;

(2) if  $B_j = B(x_j, r_j) \in \mathcal{B}$  then  $\text{dist}(x_j, \partial U) = 12r_j$ ;

(3)  $\mathcal{B} = \{B_j\}$  and  $2\mathcal{B} = \{2B_j\}$  constitute a point-finite cover of  $U$ ;

(4) if  $B_i(x_i, r_i) \cap B_j(x_j, r_j) = \emptyset$  then  $\frac{5}{7}r_i \leq r_j \leq \frac{7}{5}r_i$ ;

(5) *it is possible to split the family  $\{2B_j\}$  into finite families depending only on the dimension  $N$  so that the balls in each family are disjoint.*

**Lemma 2** [2, Lemma 10]. *Take a monotone countably additive function  $\Phi$  on the open subsets of an open set  $D \subset \mathbb{R}^N$ . Then for every open set  $U \subset D$  with  $U \neq \mathbb{R}^N$  there exists a sequence of Euclidean balls  $\{B_j\}$  such that*

(1)  $\{B_j\}$  and  $\{2B_j\}$  constitute a point-finite cover of  $U$ ;

(2)  $\sum_{j=1}^{\infty} \Phi(2B_j) \leq \zeta_N \Phi(U)$ , where  $\zeta_N$  is a constant depending only on the dimension  $N$ .

**1.5. The volume derivative.** Consider a measurable mapping  $\varphi : D \rightarrow D'$ . The *Banach indicatrix*  $N(y, \varphi, A) = \#\{x \in A \mid \varphi(x) = y\}$  is the number of preimages of  $y$  in  $A$ . If the number of preimages is infinite then put  $N(y, \varphi, A) = \infty$ . In the case  $A = D$  we write  $N(y, \varphi)$  instead of  $N(y, \varphi, D)$ .

The function  $N(y, \varphi)$  is measurable (see [15] for instance). Hence, we can express  $D'$  as the disjoint union of measurable sets:

$$D' = \bigcup_{i=0}^{\infty} N^{-1}(i, \varphi).$$

Put  $A_i = \varphi^{-1}(N^{-1}(i, \varphi))$ . Then  $D = \bigcup_{i=0}^{\infty} A_i$ .

Given a bounded set  $Q \subset D'$ , define the family of set functions

$$U \mapsto \Psi_i(U; Q) = |\varphi(U) \cap N^{-1}(i, \varphi) \cap Q|,$$

where  $i \in \mathbb{N}$ , while  $U \subset D$  is an open set. It is obvious that each  $\Psi_i$  is a monotone finitely  $i$ -quasiadditive set function. Consequently,  $\bar{\Psi}'_i(x) < \infty$  almost everywhere in  $D$ . By Proposition 1, the inequality

$$\bar{\Psi}'_i(x; Q) \leq i \underline{\Psi}'_i(x; Q) \tag{2}$$

holds for almost all  $x \in D$ .

Denote by  $\Sigma_{i,Q}$  the negligible set on which the derivative  $\bar{\Psi}'_i(\cdot; Q)$  is either undefined or equals  $\infty$ .

**Lemma 3.** *If  $E \subset D \setminus \Sigma_{i,Q}$  is a negligible set then  $|\varphi(E) \cap N^{-1}(i, \varphi) \cap Q| = 0$ .*

PROOF. Put  $M_k = \{x \in D \setminus \Sigma_{i,Q} : \bar{\Psi}'_i(x; Q) < k\}$ . Then  $D \setminus \Sigma_{i,Q} = \bigcup_k M_k$ . Take a negligible set  $E_k$  in  $M_k$ . We may assume that  $E_k$  is bounded. For every  $\varepsilon > 0$  there exists an open set  $U_\varepsilon \supset E_k$  such that  $|U_\varepsilon| < \varepsilon$ . The definition of  $M_k$  and (1) imply that for each  $x \in E_k$  there is  $r_x > 0$  such that  $B(x, r) \subset U_\varepsilon$  and  $|\varphi(B(x, r)) \cap N^{-1}(i, \varphi) \cap Q| < k|B(x, r)|$  for every  $0 < r < r_x$ . By Vitali's lemma, we can refine from the family

$$\mathcal{B} = \{B(x, r) : x \in E_k, B(x, r) \subset U_\varepsilon, 0 < r < r_x\}$$

of balls covering the set  $E_k$  a disjoint countable family of balls  $\{B_j\}$  such that  $E_k \subset \bigcup_j cB_j$ , where  $c$  is a constant depending only on  $\nu$ . Moreover,  $cB(x, r) \subset U_\varepsilon$  and  $|\varphi(cB(x, r)) \cap N^{-1}(i, \varphi) \cap Q| < k|cB(x, r)|$ . Then

$$|\varphi(E_k) \cap N^{-1}(i, \varphi) \cap Q| \leq \sum_{j=1}^{\infty} |\varphi(cB_j) \cap N^{-1}(i, \varphi) \cap Q| \leq k \sum_{j=1}^{\infty} |cB_j| \leq c^\nu k |U_\varepsilon| < c^\nu k \varepsilon, \tag{3}$$

and so  $|\varphi(E_k) \cap N^{-1}(i, \varphi) \cap Q| = 0$  because  $\varepsilon > 0$  is arbitrary.

For every negligible set  $E \subset D \setminus \Sigma_i$  we also have  $|\varphi(E) \cap N^{-1}(i, \varphi) \cap Q| = 0$  since  $E = \bigcup_k E \cap M_k$ .  $\square$

We can restate the lemma as follows.

**Corollary 1.** *If  $E \subset D \setminus \Sigma_{i,Q}$  is a negligible set then*

$$\inf\{\Psi_i(U; Q) : U \text{ is open and } E \subset U\} = 0.$$

REMARK 1. Since  $Q$  is arbitrary,  $|\varphi(E)| = 0$  for every negligible set  $E \subset D \setminus \bigcup_{k=0}^{\infty} \bigcup_{i=0}^{\infty} \Sigma_{i,Q_k}$ , where  $Q_k$  is an increasing sequence with  $\bigcup_{k=0}^{\infty} Q_k \supset D'$ .

Verify that  $\Psi_i$  is absolutely continuous.

**Proposition 2.** *Every open set  $U \subset D \setminus \Sigma_i$  satisfies*

$$\int_U \bar{\Psi}'_i(x; Q) dx \geq \Psi_i(U; Q). \tag{4}$$

PROOF. Given  $x \in D \setminus \Sigma_i$ , for every  $\varepsilon > 0$  there is a number  $\delta_0(x)$  such that

$$\frac{\Psi_i(B(x, \delta); Q)}{|B(x, \delta)|} \leq \bar{\Psi}'_i(x; Q) + \varepsilon \tag{5}$$

and

$$\overline{\Psi}'_i(x; Q) - \varepsilon \leq \frac{1}{|B(x, \delta)|} \int_{B(x, \delta)} \overline{\Psi}'_i(z; Q) dz \leq \overline{\Psi}'_i(x; Q) + \varepsilon \quad (6)$$

for all  $\delta < \delta_0(x)$ . The definition of derivative of a set function implies (5), and Lebesgue's theorem [2, Corollary 3] yields (6). Furthermore,

$$\Psi_i(B(x, \rho); Q) \leq |B(x, \delta)| \overline{\Psi}'_i(x; Q) + \varepsilon |B(x, \delta)| \leq \int_{B(x, \delta)} \overline{\Psi}'_i(z; Q) dz + 2\varepsilon |B(x, \delta)| \quad (7)$$

for all  $x \in D \setminus \Sigma_i$  and  $\delta < \delta_0(x)$ .

Consider an open set  $U \subset D \setminus \Sigma_i$  of finite measure. Choose a Vitali cover of  $U$  by the family

$$\{B(x, \delta) \mid x \in U \setminus \Sigma_i, 0 < \delta < \delta_0(x)\}$$

of balls. We can refine from it a sequence of disjoint balls  $\{B_j\}$  such that

$$\left| U \setminus \bigcup_j B_j \right| = 0 \quad \text{and} \quad |U| = \left| \bigcup_j B_j \right| = \sum_j |B_j|.$$

By Lemma 3,

$$\Psi_i(U; Q) \leq \Psi_i\left(\bigcup_j B_j; Q\right) \leq \sum_j \Psi_i(B_j; Q). \quad (8)$$

Each ball in  $\{B_j\}$  satisfies (7). Summing (7) over the balls in  $\{B_j\}$  and taking (8) into account, we obtain

$$\Psi_i(U; Q) \leq \sum_j \Psi_i(B_j; Q) \leq \int_{\bigcup_j B_j} \overline{\Psi}'_i(z; Q) dz + 2\varepsilon \sum_j |B_j| = \int_U \overline{\Psi}'_i(z; Q) dz + 2\varepsilon |U|.$$

Since  $\varepsilon$  is arbitrary,

$$\Psi_i(U; Q) \leq \int_U \overline{\Psi}'_i(z; Q) dz. \quad \square \quad (9)$$

Proposition 2 yields the corollary:

**Corollary 2.** *Every nonnegative measurable function  $u : A_i \rightarrow \mathbb{R}$  satisfies*

$$\int_{A_i} u(x) \overline{\Psi}'_i(x; Q) dx \geq \frac{1}{i} \int_{N^{-1}(i, \varphi) \cap Q} \left( \sum_{x_j \in \varphi^{-1}(y) \setminus \Sigma_i} u(x) \right) dy. \quad (10)$$

PROOF. Take the indicator function  $u(x) = \chi_V(x)$  of some Borel set  $V \subset A_i$ . Then

$$\begin{aligned} \frac{1}{i} \int_{N^{-1}(i, \varphi) \cap Q} \left( \sum_{x_j \in \varphi^{-1}(y) \setminus \Sigma_i} u(x) \right) dy &= \frac{1}{i} \int_{N^{-1}(i, \varphi) \cap Q} \left( \sum_{x_j \in V \cap \varphi^{-1}(y) \setminus \Sigma_i} 1 \right) dy \\ &= |N^{-1}(i, \varphi) \cap \varphi(V \setminus \Sigma_i) \cap Q| \leq \Psi_i(V; Q) \leq \int_{A_i} \overline{\Psi}'_i(x; Q) dx. \end{aligned}$$

Similarly we verify this inequality for step functions and then extend it to measurable functions by the standard procedure.  $\square$

Put  $Z_{i, Q} = \{x \in A_i \cap \varphi^{-1}(Q) : \overline{\Psi}'_i(x; Q) = 0\}$ . We can show that  $|\varphi(Z_{i, Q})| = 0$  for  $i < \infty$ . Indeed,

$$|\varphi(Z_{i, Q})| = \int_{\varphi(Z_{i, Q})} dy \leq \int_{N^{-1}(i, \varphi) \cap Q} \left( \sum_{x_j \in \varphi^{-1}(y) \setminus \Sigma_i} \chi_{Z_{i, Q}}(x) \right) dy \leq i \int_{Z_{i, Q}} \overline{\Psi}'_i(x; Q) dx = 0.$$

We need one more property, which follows from claim (a) of Proposition 1.

**Proposition 3.** *Every nonnegative measurable function  $g : D' \rightarrow \mathbb{R}$  satisfies*

$$\int_{A_i} g(\varphi(x)) \overline{\Psi}'_i(x; Q) dx \leq i \int_{N^{-1}(i, \varphi) \cap Q} g(y) dy. \quad (11)$$

## 2. Composition Operators

Given two domains  $D$  and  $D'$  in a Carnot group  $\mathbb{G}$ , say that a measurable mapping  $\varphi : D \rightarrow D'$  induces a bounded operator

$$\varphi^* : L_p^1(D', v) \cap C_0^\infty(D') \rightarrow L_q^1(D, u)$$

in the weighted Sobolev spaces as the composition  $\varphi^*(f) = f \circ \varphi$  whenever  $f \circ \varphi \in L_q^1(D, u)$  and

$$\|\varphi^*(f) | L_q^1(D, u)\| \leq K \|f | L_p^1(D', v)\| \tag{12}$$

for every function  $f \in L_p^1(D', v) \cap C_0^\infty(D')$ , where the constant  $K$  is independent of the choice of  $f$ .

**Theorem 2** [2, Theorem 1.2]. *Suppose that  $\varphi : D \rightarrow D'$  induces the bounded operator*

$$\varphi^* : L_p^1(D', v) \cap C^\infty(D') \rightarrow L_q^1(D, u), \quad 1 \leq q < p \leq \infty, \quad \varphi^* f = f \circ \varphi.$$

Then

$$\Phi(A') = \sup_{f \in L_p^1(A', v) \cap C_0^\infty(A')} \left( \frac{\|\varphi^* f | L_q^1(D, u)\|}{\|f | L_p^1(A', v)\|} \right)^\varkappa, \quad \varkappa = \begin{cases} \frac{pq}{p-q} & \text{for } p < \infty, \\ q & \text{for } p = \infty, \end{cases}$$

is a bounded monotone countably additive function on all bounded open sets  $A' \subset D'$ .

Theorem 2 in [1] is proved for mappings of domains in  $\mathbb{R}^n$ . A proof for Carnot groups repeats the Euclidean case verbatim.

## 3. Properties of $\varphi$

Let us describe the continuity and differentiability properties of a mapping inducing a bounded composition operator.

### 3.1. Approximate differentiability of $\varphi$ .

**Lemma 4.** *Consider a mapping  $\varphi : D \rightarrow D'$  such that for all  $\eta$  in  $C_0^\infty(D')$  the composition  $\eta \circ \varphi$  is of class  $\text{ACL}(D)$ . Then on almost all integral lines of horizontal vector fields the mapping  $\varphi : \bigcup_i \tilde{T}_i \rightarrow D'$  is continuous outside a set of zero Hausdorff-1-measure.*

Here  $\{\tilde{T}_i\}$  is an increasing sequence of sets comprising the points of positive density,  $|D \setminus \bigcup_i \tilde{T}_i| = 0$ , and  $\varphi$  is continuous on every  $\tilde{T}_i$ .

PROOF. Fix a horizontal field  $X_j$ . Assume on the contrary that there is a family  $\Gamma$  of integral lines of  $X_j$  of positive measure such that on each  $\gamma \in \Gamma$  there is a set  $s_\gamma \subset \gamma$  of positive 1-measure on which  $\varphi : \bigcup_i \tilde{T}_i \rightarrow D'$  is discontinuous.

Verify that  $S = \bigcup_{\gamma \in \Gamma} s_\gamma$  is measurable. Indeed,  $S = D \setminus A$ , where

$$A = \bigcap_{n \in \mathbb{N}} \bigcap_{m \in \mathbb{N}} \left\{ x \in \bigcup_i \tilde{T}_i \mid d(\varphi(x \exp tX_j), \varphi(x)) < \frac{1}{n} \text{ for } |t| < \frac{1}{m}, x \exp tX_j \in \bigcup_i \tilde{T}_i \right\}$$

is a measurable set since so are all sets in the braces. Fubini's theorem yields  $|S| > 0$ . Similarly we check that  $S = \bigcup_{m \in \mathbb{N}} S_m$ , where  $S_m = \{x \in s_\gamma \mid \text{osc}_l \varphi(x) > \frac{1}{m}\}$ , is a measurable set. Here  $\text{osc}_l \varphi(x)$  is the oscillation of  $\varphi : \gamma \cap \bigcup_i \tilde{T}_i \rightarrow D'$  at  $x$ . Consequently, we can choose  $m$  so that  $|S_m| > 0$ . Moreover, there is  $j$  with  $|S_m \cap \tilde{T}_j| > 0$ . Take a point  $x_0 \in S_m \cap \tilde{T}_j$  of density 1. Then every ball  $B(x_0, r)$  includes a subset of positive measure in  $S_m \cap \tilde{T}_j$ . Denote this set by  $P_r$ . Since  $\varphi$  is continuous on  $\tilde{T}_j$ , we can choose a ball  $B(x_0, r_m)$  so that  $\varphi(B(x_0, r_m) \cap S_m \cap \tilde{T}_j) \subset B(\varphi(x_0), \frac{1}{4m})$ .

Fix  $\eta \in C_0^\infty(D')$  such that  $\eta(y) = 1$  for  $y \in B(\varphi(x_0), \frac{1}{4m})$  and  $\eta(y) = 0$  for  $y \notin B(\varphi(x_0), \frac{1}{2m})$ .

Basing on the above, on each horizontal line intersecting  $P_{r_m}$  in a set of positive 1-measure we have  $\text{osc}_l \eta \circ \varphi(x) = 1$ , where  $x \in P_{r_m}$ . By the construction of  $P_{r_m}$ , the collection of these lines is of positive measure. Therefore, we arrive at a contradiction with the absolute continuity of  $\eta \circ \varphi$  on almost all lines.

Thus, on almost all horizontal lines the mapping  $\varphi : \bigcup_i \tilde{T}_i \rightarrow D'$  is continuous outside a set of zero 1-measure.  $\square$

The next lemma shows that, as we approach discontinuity points, the corresponding images tend to the boundary of the range or to infinitely distant points.

**Lemma 5.** *For almost all integral lines  $\gamma$  of horizontal vector fields the set  $\sigma_\gamma \subset \gamma$  of discontinuity points of  $\varphi$  is closed, has measure zero, and  $\varphi$  enjoys the property*

$$\lim_{\substack{x \rightarrow a \\ x \in \gamma \setminus \sigma_\gamma}} \text{dist}(\varphi(x), \partial D') = 0 \quad \text{or} \quad \lim_{\substack{x \rightarrow a \\ x \in \gamma \setminus \sigma_\gamma}} \rho(\varphi(x)) = \infty \quad (13)$$

for all  $a \in \sigma_\gamma$ .

PROOF. STEP 1. Take an integral line  $\gamma$  of a horizontal field, the set  $\sigma_\gamma \subset \gamma$  of zero 1-measure on which  $\varphi$  is discontinuous, and  $x_0 \in \sigma_\gamma$ . Verify that if for some sequence  $\{x_n\} \subset \gamma \setminus \sigma_\gamma$  the limit  $\lim_{x_n \rightarrow x_0} \varphi(x_n) = z \in D'$  exists then  $\lim_{x'_n \rightarrow x_0} \varphi(x'_n) = z$  for every sequence  $\{x'_n\} \subset \gamma \setminus \sigma_\gamma$ .

Take an everywhere dense countable set  $\{z_j\}$  of  $D'$ . Define the countable family of functions  $\eta_{z_j}^r : D' \rightarrow \mathbb{R}^+$  of class  $C_0^\infty(D')$  as

$$\eta_{z_j}^r(y) = \begin{cases} 1 & y \in B(z_j, r), \\ 0 & y \notin B(z_j, 2r), \end{cases}$$

where  $r \in \mathbb{Q}^+ = \{x \in \mathbb{Q} \mid x > 0\}$ .

For almost all integral lines  $\gamma \subset D$  of horizontal vector fields  $\varphi$  is continuous on  $\gamma$  outside a set  $\sigma_\gamma$  of zero 1-measure, while  $\eta_{z_j}^r \circ \varphi$  is absolutely continuous on  $\gamma$  for all  $j \in \mathbb{N}$  and  $r \in \mathbb{Q}^+$ . Choose a line  $\gamma$  with these properties.

Take  $x_0 \in \sigma_\gamma$  and assume that  $\lim_{x_n \rightarrow x_0} \varphi(x_n) = z \in D'$  for some sequence  $\{x_n\} \subset \gamma \setminus \sigma_\gamma$ . If there is no suitable sequence then (13) holds.

Take another sequence  $\{x'_n\} \subset \gamma \setminus \sigma_\gamma$  with  $\lim_{x'_n \rightarrow x_0} \varphi(x'_n) = z' \neq z$ . The three cases are possible:  $z' \in D'$ ,  $z' \in \partial D'$ , and  $z' = \infty$ .

Choose a point  $z_l \in \{z_j\}$  and a radius  $r_l \in \mathbb{Q}^+$  so that  $d(z, z_l) < r_l$  and  $d(z', z_l) > 2r_l$ . Then

$$\lim_{x_n \rightarrow x_0} \eta_{z_l}^{r_l} \circ \varphi(x_n) = 1 \quad \text{and} \quad \lim_{x'_n \rightarrow x_0} \eta_{z_l}^{r_l} \circ \varphi(x'_n) = 0,$$

which contradicts the continuity of  $\eta_{z_l}^{r_l} \circ \varphi$  on  $\gamma$ .

Extend  $\varphi$  by continuity as  $\varphi(x_0) = z$ . Hence, we may assume that  $x_0 \notin \sigma_\gamma$ , while all elements of  $\sigma_\gamma$  satisfy (13).

STEP 2. Verify that  $\sigma_\gamma$  is closed. Take a converging sequence  $\{x_n\} \subset \sigma_\gamma$  and show that  $\lim_{n \rightarrow \infty} x_n \in \sigma_\gamma$ . For each  $x_n$  there is a sequence  $\{x_n^k\} \subset \gamma \setminus \sigma_\gamma$  with  $\lim_{k \rightarrow \infty} x_n^k = x_n$ . By the above,

$$\lim_{k \rightarrow \infty} \text{dist}(\varphi(x_n^k), \partial D') = 0 \quad \text{or} \quad \lim_{k \rightarrow \infty} \rho(\varphi(x_n^k)) = \infty.$$

Form a sequence  $\{x_n^{k_l}\} \subset \gamma \setminus \sigma_\gamma$  such that

$$\text{dist}(\varphi(x_n^{k_l}), \partial D') < \frac{1}{l} \quad \text{or} \quad \rho(\varphi(x_n^{k_l})) > l.$$

Then  $\lim_{l \rightarrow \infty} x_n^{k_l} = x_0$ . Thus,  $x_0 \in \sigma_\gamma$  and so  $\sigma_\gamma$  is closed.  $\square$

The proofs of the next two statements (Lemmas 6 and 7) rest on the arguments of [6] (see Lemmas 19 and 20 of [6]).

**Lemma 6.** *Take  $v(y) \in L_{1,\text{loc}}(D')$  and  $u^{\frac{1}{1-p}}(x) \in L_{1,\text{loc}}(D)$ . If  $f \in \text{Lip}_{\text{loc}}(D') \cap L_p^1(D', v)$  and  $\|f \mid L_p^1(D', v)\| \leq 1$  then there is  $g_i \in L_{q,\text{loc}}(D)$  such that*

$$|\nabla_{\mathcal{L}}(f \circ \varphi)|(x) \leq g_i(x) \quad (14)$$

almost everywhere on  $A_i \cap \varphi^{-1}(Q)$ .

Here  $\text{Lip}_{\text{loc}}(D')$  is the space of locally Lipschitz functions.

PROOF. STEP 1. Fix  $y_0 \in D' \cap \varphi(A_i)$  and a ball  $B(y_0, r)$ . Consider the function  $\eta(y) = \xi(\delta_r(y_0^{-1}y))$ , where  $\xi \in C_0^\infty(\mathbb{G})$  satisfies  $\xi|_{B(0,1)} = 1$  and  $\xi|_{\mathbb{G} \setminus B(0,2)} = 0$ .

Using Theorem 2, we obtain

$$\begin{aligned} & \left( \int_{\varphi^{-1}(B(y_0, r))} |\nabla_{\mathcal{L}}(f \circ \varphi)|^q(x) u(x) dx \right)^{\frac{1}{q}} \\ & \leq \Phi(B(y_0, 2r))^{\frac{1}{\varkappa}} \left( \int_{B(y_0, 2r)} |\nabla_{\mathcal{L}}((f(y) - f(y_0))\eta(y))|^p v(y) dy \right)^{\frac{1}{p}} \\ & \leq \Phi(B(y_0, 2r))^{\frac{1}{\varkappa}} \left( \int_{B(y_0, 2r)} |\nabla_{\mathcal{L}} f|^p(y) \eta^p(y) v(y) dy \right)^{\frac{1}{p}} \\ & + \Phi(B(y_0, 2r))^{\frac{1}{\varkappa}} \left( \int_{B(y_0, 2r)} |\nabla_{\mathcal{L}} \eta|^p(y) |f(y) - f(y_0)|^p v(y) dy \right)^{\frac{1}{p}} \\ & \leq \Phi(B(y_0, 2r))^{\frac{1}{\varkappa}} (v(B(y_0, 2r)))^{\frac{1}{p}} + (c_1 r^{-1} c_2 r v(B(y_0, 2r)))^{\frac{1}{p}} \\ & = C \Phi(B(y_0, 2r))^{\frac{1}{\varkappa}} v(B(y_0, 2r))^{\frac{1}{p}}. \end{aligned}$$

Furthermore, if  $p = q$  then  $\Phi(B(y_0, 2r))^{\frac{1}{\varkappa}} = \|\varphi^*\|$ .

Finally, we arrive at the estimate

$$\int_{\varphi^{-1}(B(y_0, r))} |\nabla_{\mathcal{L}}(f \circ \varphi)|^q(x) u(x) dx \leq C_1 \Phi(B(y_0, 2r))^{\frac{q}{\varkappa}} v(B(y_0, 2r))^{\frac{q}{p}}. \quad (15)$$

STEP 2. The estimate in (15) implies that  $|\nabla_{\mathcal{L}}(f \circ \varphi)|(x) = 0$  for almost all  $x \in Z_{i,Q}$ . Indeed, since  $|\varphi(Z_i)| = 0$ , for arbitrary  $\varepsilon > 0$  we can choose a cover consisting of countably many balls  $\{B_l = B(y_l, r_l)\}$  such that the balls of doubled radii constitute a point-finite cover of  $\varphi(Z_i)$  and  $\sum_{l=1}^\infty |B(y_l, 2r_l)| < \varepsilon$ .

From (15) for  $q < p$  we have

$$\begin{aligned} \int_{Z_i} |\nabla_{\mathcal{L}}(f \circ \varphi)|^q(x) u(x) dx & \leq \sum_{l=1}^\infty \int_{\varphi^{-1}(B(y_l, r_l))} |\nabla_{\mathcal{L}}(f \circ \varphi)|^q(x) u(x) dx \\ & \leq C_1 \sum_{l=1}^\infty \Phi(B(y_l, 2r_l))^{\frac{q}{\varkappa}} v(B(y_l, 2r_l))^{\frac{q}{p}} \\ & \leq C_1 \left( \sum_{l=1}^\infty \Phi(B(y_l, 2r_l)) \right)^{\frac{q}{\varkappa}} \left( \sum_{l=1}^\infty |B(y_l, 2r_l)| \right)^{\frac{q}{p}} \leq C_2 \Phi(D')^{\frac{q}{\varkappa}} \varepsilon^{\frac{q}{p}}. \end{aligned}$$

If  $q = p$  then

$$\int_{Z_i} |\nabla_{\mathcal{L}}(f \circ \varphi)|^q(x) u(x) dx \leq C_3 \|\varphi^*\|^q \varepsilon.$$

STEP 3. Using (10) and (15), we obtain

$$\begin{aligned}
C_1 \Phi(B(y_0, 2r))^{\frac{q}{\varkappa}} V(B(y_0, 2r))^{\frac{q}{p}} &\geq \int_{\varphi^{-1}(B(y_0, r)) \cap A_i \cap \varphi^{-1}(Q)} |\nabla_{\mathcal{L}}(f \circ \varphi)|^q(x) u(x) dx \\
&= \int_{\varphi^{-1}(B(y_0, r)) \cap A_i \cap \varphi^{-1}(Q) \setminus Z_{i, Q}} \frac{|\nabla_{\mathcal{L}}(f \circ \varphi)|^q(x) \overline{\Psi}'_i(x; Q) u(x)}{\overline{\Psi}'_i(x; Q)} dx \\
&\geq \frac{1}{i} \int_{B(y_0, r) \cap \varphi(A_i \setminus Z_{i, Q}) \cap Q} \sum_{x_j \in \varphi^{-1}(y) \setminus \Sigma_{i, Q}} \left( \frac{|\nabla_{\mathcal{L}}(f \circ \varphi)|^q(x) u(x)}{\overline{\Psi}'_i(x)} \right) dy;
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\int_{B(y_0, r) \cap \varphi(A_i \setminus Z_{i, Q}) \cap Q} \sum_{x_j \in \varphi^{-1}(y) \setminus \Sigma_i} \left( \frac{|\nabla_{\mathcal{L}}(f \circ \varphi)|^q(x) u(x)}{\overline{\Psi}'_i(x; Q)} \right) dy \\
&\leq i C_1 \Phi(B(y_0, 2r))^{\frac{q}{\varkappa}} v(B(y_0, 2r))^{\frac{q}{p}}.
\end{aligned} \tag{16}$$

Divide both sides by  $|B(y_0, 2r)|$ :

$$\begin{aligned}
&\frac{1}{|B(y_0, 2r)|} \int \sum_{x_j \in \varphi^{-1}(y) \setminus \Sigma_i} \left( \frac{|\nabla_{\mathcal{L}}(f \circ \varphi)|^q(x) u(x)}{\overline{\Psi}'_i(x; Q)} \right) dy \\
&\leq i C_1 \left( \frac{\Phi(B(y_0, 2r))}{|B(y_0, 2r)|} \right)^{\frac{q}{\varkappa}} \left( \frac{v(B(y_0, 2r))}{|B(y_0, 2r)|} \right)^{\frac{q}{p}}.
\end{aligned}$$

Differentiating by Lebesgue's theorem, we infer that

$$\sum_{x_j \in \varphi^{-1}(y) \setminus \Sigma_i} \left( \frac{|\nabla_{\mathcal{L}}(f \circ \varphi)|^q(x) u(x)}{\overline{\Psi}'_i(x; Q)} \right) \leq i K \Phi'(y)^{\frac{q}{\varkappa}} v^{\frac{q}{p}}(y)$$

for almost all  $y \in \varphi(A_i \setminus Z_{i, Q})$ . Hence,

$$|\nabla_{\mathcal{L}}(f \circ \varphi)|(x) \leq g_{i, Q}(x) \quad \text{for almost all } x \in A_i, \tag{17}$$

where

$$g_{i, Q}(x) = \begin{cases} K_1 \sqrt[q]{i} \Phi'(\varphi(x))^{\frac{1}{\varkappa}} (\overline{\Psi}'_i(x; Q))^{\frac{1}{q}} u^{-\frac{1}{q}}(x) v^{\frac{1}{p}}(\varphi(x)) & \text{for } p > q, \\ K_2 \sqrt[q]{i} C (\overline{\Psi}'_i(x; Q))^{\frac{1}{q}} u^{-\frac{1}{q}}(x) v^{\frac{1}{p}}(\varphi(x)) & \text{for } p = q. \end{cases}$$

STEP 4. Verify that  $g_{i, Q}(x)$  is locally integrable. Assume that  $p > q$  and take an arbitrary ball  $B \subset D$ . Applying Hölder's inequality and Proposition 3 twice, we obtain

$$\begin{aligned}
\int_B g_{i, Q}(x) dx &= K_1 i^{\frac{1}{q}} \int_B \Phi'(\varphi(x))^{\frac{1}{\varkappa}} (\overline{\Psi}'_i(x; Q))^{\frac{1}{q}} u^{-\frac{1}{q}}(x) v^{\frac{1}{p}}(\varphi(x)) dx \\
&\leq K_1 i^{\frac{1}{q} + \frac{1}{\varkappa} + \frac{\varkappa'}{p}} \left( \int_Q \Phi'(y) dy \right)^{\frac{1}{\varkappa}} \left( \int_Q v(y) dy \right)^{\frac{1}{p}} \left( \int_B u(x)^{\frac{1}{1-q}} dx \right)^{\frac{1}{\varkappa'} - \frac{1}{p}},
\end{aligned}$$

where  $\frac{1}{\varkappa'} = 1 - \frac{1}{\varkappa}$ . In the case  $q = p$  we deduce a similar inequality. All quantities on the right-hand side are finite, and so  $g_{i, Q} \in L_{1, \text{loc}}(D)$ .  $\square$

Recall that the *approximate derivative* along a vector  $X$  at  $x$  (see [11]) is

$$\text{ap } X\varphi(x) = \text{ap } \lim_{t \rightarrow 0} \delta_t^{-1}((\varphi(x))^{-1}\varphi(x\delta_t \exp X)).$$

**Lemma 7.** *If  $\varphi$  induces the bounded composition operator  $\varphi^* : L_p^1(D', v) \cap C_0^\infty(D') \rightarrow L_q^1(D, u)$  in Sobolev spaces then  $\varphi$  has approximate partial derivatives almost everywhere in  $D$ .*

PROOF. Since the result is local, we may assume that  $D$  has finite measure. Fix  $k \in \mathbb{N}$  and a bounded set  $Q \subset \mathbb{G}$ .

Take an everywhere dense countable set of points  $\{z_j\}$  in  $D'$ . Define the countable family of functions  $d_{z_j}^r : D' \rightarrow \mathbb{R}^+$  as  $d_{z_j}^r(y) = (r - d_{z_j}(y))^+$ , where  $r \in \mathbb{Q}^+ = \{x \in \mathbb{Q} \mid x > 0\}$  and  $d_{z_j}(y) = d(z_j, y)$ . For each of these functions the pointwise equality  $\varphi^* d_{z_j}^r(x) = d_{z_j}^r \circ \varphi(x)$ , with  $r \in \mathbb{Q}^+$  and  $j \in \mathbb{N}$ , holds for all points  $x \in \tilde{T}_k$ . In addition, each of these functions satisfies the hypotheses of Lemma 6. Therefore,  $|\nabla_{\mathcal{L}}(d_{z_j}^r \circ \varphi)|(x) \leq C g_{i,Q}(x)$  for almost all  $x \in A_i$ .

Consider the foliation  $\Gamma_s$  of  $D$  induced by the horizontal vector field  $X_s$  and a line  $\gamma$  in this foliation. Almost all  $\gamma$  in  $\Gamma_s$  satisfy the following conditions:

- (1)  $\varphi$  is continuous on  $\gamma$  outside a set  $\sigma_\gamma$  of zero 1-measure (Lemma 4);
- (2) we have the pointwise inequality for measurable functions:

$$|\nabla_{\mathcal{L}}(\varphi^* d_{z_j}^r)|(t) \leq K g_{i,Q}(t), \quad r \in \mathbb{Q}^+, j \in \mathbb{N}, \text{ a.e. in } \gamma \cap \varphi^{-1}(Q),$$

and  $g_{i,Q}$  is integrable on every compact subset of  $\gamma$ ;

(3) for almost all  $x_0 \in \gamma$  the limit of  $\frac{1}{d(x_0, x)} \int_{[x_0, x]} g_{i,Q}(t) d\sigma$  as  $x \rightarrow x_0$  along  $\gamma$  exists and is equal to  $g_{i,Q}(x_0)$  (here  $[x_0, x] \subset \gamma$  is a segment of the integral line);

(4)  $A_i \cap \tilde{T}_k \cap \gamma$  is a set of full (1-dimensional) measure on  $\gamma \cap D$ ;

(5) the functions  $\varphi^* d_{z_j}^r$  are absolutely continuous on  $\gamma$  for all  $j \in \mathbb{N}$  and  $r \in \mathbb{Q}^+$ .

Fix a line  $\gamma \in \Gamma_s$  on which all five conditions hold.

Assume that  $x_0 \in A_i \cap \varphi^{-1}(Q) \cap \tilde{T}_k \cap \gamma$  is a point of positive density on  $\gamma$ , a point of continuity of the restriction  $\varphi|_\gamma$  and a point at which condition (3) holds. Put  $z = \varphi(x_0)$ . Fix a subsequence  $\{z_{j_l}\}$  of  $\{z_j\}$  converging to  $z = \varphi(x_0)$  and denote its elements by  $z_l$ . Since  $\varphi$  is continuous on  $\gamma$  at  $x_0$ , we can choose  $\delta, r$ , and  $L$  so that  $d_{z_l}^r \circ \varphi(x) \neq 0$  for all  $l \geq L$  and all points  $x \in A_i \cap \varphi^{-1}(Q) \cap B(x_0, \delta) \cap \gamma \setminus \sigma_\gamma$ .

Integrating  $C g_{i,Q}(x)$  (where  $C$  is independent of  $r$  and  $z$ ) over the part of  $\gamma$  from  $x_0$  to  $x$ , where  $x \in A_i \cap \tilde{T}_k \cap B(x_0, \delta) \cap \gamma \setminus \sigma_\gamma$ , yields

$$\begin{aligned} C \int_{[x_0, x] \cap A_i} g_{i,Q}(t) dt &\geq \int_{[x_0, x] \cap A_i} |\nabla_{\mathcal{L}}(\varphi^* d_{z_l}^r)|(t) dt \geq \left| \int_{[x_0, x] \cap A_i} \nabla_{\mathcal{L}}(\varphi^* d_{z_l}^r)(t) dt \right| \\ &\geq |d_{z_l}^r \circ \varphi(x_0) - d_{z_l}^r \circ \varphi(x)| = |r - d_{z_l}(\varphi(x_0)) - r + d_{z_l}(\varphi(x))| \\ &= |-d_{z_l}(\varphi(x_0)) + d_{z_l}(\varphi(x))| \rightarrow d_z(\varphi(x)) = d(\varphi(x_0), \varphi(x)) \quad \text{as } l \rightarrow \infty. \end{aligned} \quad (18)$$

Hence,

$$d(\varphi(x_0), \varphi(x)) \leq C \int_{[x_0, x]} g_{i,Q}(t) d\sigma \quad (19)$$

or

$$\frac{d(\varphi(x_0), \varphi(x))}{d(x_0, x)} \leq \frac{C}{d(x_0, x)} \int_{[x_0, x]} g_{i,Q}(t) d\sigma \quad (20)$$

for all  $x \in A_i \cap \varphi^{-1}(Q) \cap \tilde{T}_k \cap B(x_0, \delta) \cap \gamma \setminus \sigma_\gamma$ . Passing to the approximate limit in (20), we obtain

$$\text{ap } \overline{\lim}_{x \rightarrow x_0, x \in \gamma} \frac{d(\varphi(x_0), \varphi(x))}{d(x_0, x)} \leq C g_{i,Q}(x_0) < \infty, \quad (21)$$

which means that  $\varphi$  is approximately differentiable at  $x_0$  along  $X_s$ .

Since we picked arbitrary indices  $k$  and  $i$ , a horizontal field  $X_s$ , an integral line  $\gamma \in \Gamma_s$ , a set  $Q$ , and a point  $z_0 \in \gamma$ , the mapping  $\varphi$  is approximately differentiable almost everywhere along the horizontal lines.  $\square$

REMARK 2. Since  $\varphi$  is approximately differentiable almost everywhere along the integral lines of horizontal vector fields, its complete approximate differentiability follows [11, Theorem 3.3].

Denote by  $D\varphi$  the approximate differential of  $\varphi$  and by  $D_h\varphi$ , its horizontal part. We have  $J(x, \varphi) = \det D\varphi(x)$ .

**3.2. Piecewise absolute continuity on lines.** Recall that a mapping  $\varphi : D \rightarrow D'$  is called *absolutely continuous on a line*  $\gamma(t) : [0, 1] \rightarrow D$  whenever the line  $\varphi(\gamma(t)) : [0, 1] \rightarrow D'$  is absolutely continuous. A mapping  $\varphi$  is called *absolutely continuous on lines* ( $\varphi \in \text{ACL}(D)$ ) if for every domain  $U \subseteq D$  and every family  $\Gamma_j$  of integral lines of the basis horizontal vector field  $X_j$ , for  $j = 1, \dots, n_1$ , constituting a smooth foliation of  $U$ , the mapping  $\varphi$  is absolutely continuous on  $\gamma \cap U$  for almost all lines  $\gamma \in \Gamma_j$ .

DEFINITION 5. A mapping  $\varphi : D \rightarrow D'$  is *piecewise absolutely continuous on almost all lines* ( $\varphi \in \text{ACL}_{\text{part}}(D)$ ) whenever on almost every integral line  $\gamma$  of the horizontal field  $X_j$ , for  $j = 1, \dots, n$ , there exists an open set  $\omega_\gamma \subset \gamma$  of full measure on  $\gamma$  such that for every segment  $[\alpha, \beta] \subset \omega_\gamma$  the mapping  $\varphi$  is absolutely continuous on  $[\alpha, \beta]$  and

$$\lim_{x \rightarrow a, x \in \omega_\gamma} \text{dist}(\varphi(x), \partial D') = 0 \quad \text{or} \quad \lim_{x \rightarrow a, x \in \omega_\gamma} \rho(\varphi(x)) = \infty$$

for all  $a \in \gamma \setminus \omega_\gamma$ .

The appearance of approximate partial derivatives of  $\varphi$  enables us to use the change-of-variables formula:

**Proposition 4** [11, Corollary 5.1]. *Suppose that  $\varphi : A \rightarrow \mathbb{G}$ , where  $A \subset \mathbb{G}$  is a measurable set, has approximate partial derivatives almost everywhere in  $A$ . Then there exists a negligible set  $\Sigma_\varphi \subset A$  such that the change-of-variables formula in the Lebesgue integral takes the form*

$$\int_A f(x) |J(x, \varphi)| dx = \int_{\mathbb{G}} \left( \sum_{x \in \varphi^{-1}(y) \cap (A \setminus \Sigma_\varphi)} f(x) \right) dy \quad (22)$$

for every nonnegative measurable function  $f : A \rightarrow \mathbb{R}$ .

In particular,

$$\int_{\varphi^{-1}(F)} |J(x, \varphi)| dx = \int_F N(y, \varphi) dy \quad (23)$$

for every measurable set  $F \subset D'$ . Since  $N(y, \varphi)$  is integrable on  $F$ , this implies that  $|J(x, \varphi)|$  is integrable on  $\varphi^{-1}(F)$ .

Arguing as in Lemma 6, but using (22) instead of (10), we arrive at the following estimate.

**Lemma 8.** *If  $f \in \text{Lip}_l(D') \cap L_p^1(D', v)$  and  $\|f\|_{L_p^1(D', v)} \leq 1$  then*

$$|\nabla_{\mathcal{L}}(f \circ \varphi)|(x) \leq g(x) \quad \text{a.e. in } D, \quad (24)$$

where

$$g(x) = \begin{cases} K_1 \Phi'(\varphi(x))^{\frac{1}{p}} |J(x, \varphi)|^{\frac{1}{q}} u^{-\frac{1}{q}}(x) v^{\frac{1}{p}}(\varphi(x)) & \text{for } p > q, \\ K_2 |J(x, \varphi)|^{\frac{1}{p}} u^{-\frac{1}{p}} v^{\frac{1}{p}}(\varphi(x)) & \text{for } p = q. \end{cases}$$

Lemma 8 says nothing about the integrability of  $g(x)$ . We need local integrability on the preimage  $\varphi^{-1}(B)$  of an arbitrary ball  $B \subset D'$ . We succeeded in proving that  $g$  is locally integrable on  $\varphi^{-1}(B)$  on making the assumptions:

- (1)  $u(x)^{\frac{1}{1-q}} \in L_{1,\text{loc}}(D)$  and  $v(y)N(y, \varphi) \in L_{1,\text{loc}}(D')$  if  $q = p$ ;
  - (2)  $u(x)^{\frac{1}{1-q}} \in L_{1,\text{loc}}(D)$  and  $v(y)N(y, \varphi) \in L_{1,\text{loc}}(D')$ , as well as  $\Phi'(y)N(y, \varphi) \in L_{1,\text{loc}}(D')$ , if  $q < p$ .
- In the case  $q = p$  we justify

**Lemma 9.** *If  $\varphi$  induces the bounded composition operator*

$$\varphi^* : L_p^1(D', v) \cap C_0^\infty(D') \rightarrow L_p^1(D, u)$$

in the weighted Sobolev spaces,  $N(y, \varphi)v(y)$  is locally integrable, and  $u(x)^{\frac{1}{1-p}} \in L_{1,\text{loc}}(D)$ , then  $\varphi \in \text{ACL}_{\text{part}}(D)$ .

PROOF. As in Lemma 7, define the countable family of functions  $d_{z_j}^r : D' \rightarrow \mathbb{R}^+$  as  $d_{z_j}^r(y) = (r - d_{z_j}(y))^+$ , where  $\{z_j\}$  is an everywhere dense countable set of points in  $D'$  and  $r \in \mathbb{Q}^+ = \{x \in \mathbb{Q} \mid x > 0\}$ . For each function of this type the pointwise equality  $\varphi^* d_{z_j}^r(x) = d_{z_j}^r \circ \varphi(x)$ , with  $r \in \mathbb{Q}^+$  and  $j \in \mathbb{N}$ , holds for all points  $x \in \tilde{T}_k$  (for every  $k$ ) and the (24) holds almost everywhere in  $D$ .

Fix a ball  $B \subset D'$ . Since  $N(y, \varphi)v(y)$  is locally integrable, Proposition 4 implies that  $|J(x, \varphi)|v(\varphi(x))$  is integrable on  $\varphi^{-1}(B)$ . Take a compact set  $K \subset \varphi^{-1}(B)$ . Hölder's inequality yields

$$\int_K |J(x, \varphi)|^{\frac{1}{p}} v^{\frac{1}{p}}(\varphi(x)) u^{-\frac{1}{p}}(x) dx \leq \left( \int_K |J(x, \varphi)| v(\varphi(x)) dx \right)^{\frac{1}{p}} \left( \int_K (u(x))^{\frac{1}{1-p}} dx \right)^{\frac{p-1}{p}}.$$

The right-hand side here is bounded because  $u(x)^{\frac{1}{1-p}} \in L_{1,\text{loc}}(D)$ . Consequently,

$$g(x) = |J(x, \varphi)|^{\frac{1}{p}} v^{\frac{1}{p}}(\varphi(x)) u^{-\frac{1}{p}}(x)$$

is locally integrable on  $\varphi^{-1}(B)$ .

Consider the foliation  $\Gamma_s$  of  $D$  induced by the horizontal vector field  $X_s$ . The following hold for almost all lines  $\gamma$  in  $\Gamma_s$  intersecting  $\gamma \cap \varphi^{-1}(B)$  in a set of positive measure:

- (1)  $\varphi$  is continuous on  $\gamma$  outside some set  $\sigma_\gamma$  of zero 1-measure (Lemma 4);
- (2)  $g(x)$  is integrable on every compact subset of  $\gamma$ ;
- (3) the pointwise inequality holds for measurable functions:

$$|\nabla_{\mathcal{L}}(\varphi^* d_{z_j}^r)|(t) \leq K g(x), \quad r \in \mathbb{Q}^+, j \in \mathbb{N}, \text{ a.e. on } \gamma;$$

- (4)  $\tilde{T}_k \cap \gamma$  is a subset of full (1-dimensional) measure of  $\gamma \cap \varphi^{-1}(B)$ ;
- (5)  $\varphi^* d_{z_j}^r$  are absolutely continuous on  $\gamma$  for all  $j \in \mathbb{N}$  and  $r \in \mathbb{Q}^+$ .

Fix a line  $\gamma \in \Gamma_s$  meeting all five conditions. The set  $\gamma \cap \varphi^{-1}(B)$  is open on  $\gamma$ , and so it is a union of segments. Choose an arbitrary segment  $[\alpha, \beta] \subset \gamma \cap \varphi^{-1}(B)$ .

Take a point  $x_0 \in \tilde{T}_k \cap [\alpha, \beta]$  of positive density on  $\gamma$  at which the restriction  $\varphi|_\gamma$  is continuous. Put  $z = \varphi(x_0)$ . Fix a subsequence  $\{z_{j_l}\}$  of  $\{z_j\}$  converging to  $z = \varphi(x_0)$  and denote its elements by  $z_l$ . Since  $\varphi$  is continuous on  $\gamma$  at  $x_0$ , we can choose  $\delta$ ,  $r$ , and  $L$  so that  $d_{z_l}^r \circ \varphi(x) \neq 0$  for all  $l \geq L$  and all  $x \in B(x_0, \delta) \cap [\alpha, \beta]$ .

Integrating  $Kg(x)$  over the part of  $\gamma$  from  $x_0$  to  $x$ , where  $x \in \tilde{T}_k \cap B(x_0, \delta) \cap [\alpha, \beta]$ , yields

$$\begin{aligned} K \int_{[x_0, x]} g(x) dt &\geq \int_{[x_0, x]} |\nabla_{\mathcal{L}}(\varphi^* d_{z_l}^r)|(t) dt \geq \left| \int_{[x_0, x]} \nabla_{\mathcal{L}}(\varphi^* d_{z_l}^r)(t) dt \right| \\ &\geq |d_{z_l}^r \circ \varphi(x_0) - d_{z_l}^r \circ \varphi(x)| = |r - d_{z_l}(\varphi(x_0)) - r + d_{z_l}(\varphi(x))| \\ &= |-d_{z_l}(\varphi(x_0)) + d_{z_l}(\varphi(x))| \rightarrow d_z(\varphi(x)) = d(\varphi(x_0), \varphi(x)) \quad \text{as } l \rightarrow \infty. \end{aligned}$$

This leads to the estimate

$$d(\varphi(x_0), \varphi(x)) \leq K \int_{[x_0, x]} g(x) dt \tag{25}$$

for  $x \in \tilde{T}_k \cap B(x_0, \delta) \cap [\alpha, \beta]$ . By the absolute continuity of the integral, we deduce from this that  $\varphi$  is absolutely continuous on  $[\alpha, \beta]$ .

Since we arbitrarily picked a line  $\gamma$ , a segment  $[\alpha, \beta]$ , a vector field  $X_s$ , and a ball  $B$ , the piecewise absolute continuity is established:  $\varphi \in \text{ACL}_{\text{part}}(D)$ .  $\square$

Now the analogous claim for  $q < p$  is in order:

**Lemma 10.** *If  $\varphi$  induces the bounded composition operator*

$$\varphi^* : L_p^1(D', v) \cap C_0^\infty(D') \rightarrow L_q^1(D, u)$$

*in the weighted Sobolev spaces, the functions  $N(y, \varphi)v(y)$  and  $\Phi'(y)N(y, \varphi)$  are locally integrable, while the weight  $u(x)^{\frac{1}{1-q}}$  belongs to  $L_{1,\text{loc}}(D)$ , then  $\varphi \in \text{ACL}_{\text{part}}(D)$ .*

Here we take the function  $\Phi$  of Theorem 2.

#### 4. The Main Theorems

Let us introduce the key definitions and prove the main results of this article.

**DEFINITION 6.** A mapping  $\varphi : D \rightarrow D'$  of class  $\text{ACL}_{\text{part}}(D)$  has *finite  $(u, v)$ -weighted distortion* whenever  $D_h\varphi(x)u(x) = 0$  almost everywhere on

$$Z_v = \{x \in D \mid J(x, \varphi)v(\varphi(x)) = 0\}.$$

**REMARK 3.** It is obvious that  $Z \subset Z_v$ , where  $Z = \{x \in D \mid J(x, \varphi) = 0\}$ . If  $v > 0$  almost everywhere, and, in particular, if  $v$  satisfies Muckenhoupt's  $A_p$ -condition, then  $|Z_v \setminus Z| = 0$ . Thus, in Definition 6 we can replace  $Z_v$  with  $Z$ . Indeed, assume that  $v(\varphi(x)) = 0$ , i.e.,  $x \in \varphi^{-1}(S)$ , where  $S = \{y \mid v(y) = 0\}$ . The measure of  $S$  is zero since  $v(x) > 0$  almost everywhere. By the change-of-variables formula for the characteristic function of  $S$ ,

$$\int_D \chi_S(\varphi(x))|J(x, \varphi)| dx = \int_{\mathbb{G}} \chi_S(y)N(y, \varphi, D) dy.$$

The integral on the right-hand side vanishes since the integration is taken over a negligible set. Therefore,

$$\int_{\varphi^{-1}(S)} |J(x, \varphi)| dx = 0,$$

which implies that  $J(x, \varphi)v(\varphi(x))$  vanishes almost everywhere on  $Z$ . Finally,  $|Z_v \setminus Z| = 0$ .

**DEFINITION 7.** Define the weighted distortion function for  $\varphi$  as

$$D' \ni y \mapsto H_q^{u,v}(y) = \begin{cases} v^{-\frac{1}{p}}(y) \left( \sum_{x \in \varphi^{-1}(y) \setminus (\Sigma_\varphi \cup Z_v)} \frac{|D\varphi|^q(x)u(x)}{|J(x, \varphi)|} \right)^{\frac{1}{q}}, \\ 0 \quad \text{if } \varphi^{-1}(y) \setminus (\Sigma_\varphi \cup Z_v) = \emptyset. \end{cases} \tag{26}$$

A distortion function of this type appeared originally in [16].

**Theorem 3.** Suppose that  $N(y, \varphi)v(y)$  belongs to  $L_{1,\text{loc}}(D')$ , the weight  $u^{\frac{1}{1-p}}(x)$  belongs to  $L_{1,\text{loc}}(D)$ , and  $1 \leq p < \infty$ . If a measurable mapping  $\varphi : D \rightarrow D'$  induces the bounded composition operator

$$\varphi^* : L_p^1(D', v) \cap C_0^\infty(D') \rightarrow L_p^1(D, u)$$

in the weighted Sobolev spaces then it enjoys the properties:

- (1)  $\varphi$  is of class  $\text{ACL}_{\text{part}}(D)$ ;
- (2)  $\varphi$  has finite  $(u, v)$ -weighted distortion;
- (3) the distortion function  $H_p^{u,v}(\cdot)$  belongs to  $L_\infty(D')$ .

Furthermore,  $\|H_p^{u,v}(\cdot) \| L_\infty(D')\| \leq C\|\varphi^*\|$ .

PROOF. (1) We proved the containment in  $\text{ACL}_{\text{part}}(D)$  of Lemma 9.

(2) Let us show that  $\varphi$  has finite  $(u, v)$ -weighted distortion, i.e.,  $D_h\varphi = 0$  almost everywhere on  $Z_v$ . For every function  $f \in L_p^1(D', v) \cap C_0^\infty(D')$  we have

$$\|\varphi^* f \| L_q^1(D, u)\| \leq \|\varphi^*\| \cdot \|f \| L_p^1(D', v)\|. \quad (27)$$

Choose the cut-off  $\eta \in C_0^\infty(\mathbb{R}^N)$  equal to 1 on the Euclidean ball  $B^E(0, 1)$  and vanishing outside the ball  $B^E(0, 2)$ . Inserting  $h_j(z) = (z - y)_j \eta\left(\frac{z-y}{r}\right)$ , where  $B^E(y, 2r) \subset D'$ , into (27) yields

$$\left( \int_{\varphi^{-1}(B^E(y,r))} |D_h\varphi|^p u(x) dx \right)^{\frac{1}{p}} \leq C\|\varphi^*\| \left( \int_{B^E(y,2r)} v(y') dy' \right)^{\frac{1}{p}} \quad (28)$$

(for more detailed arguments, see the proof of Lemma 6).

Verify that

$$\int_{Z_v} |D_h\varphi|^q dx = 0.$$

By the change-of-variables formula (Proposition 4),

$$\begin{aligned} v(\varphi(Z_v \setminus \Sigma_\varphi)) &= \int_{\varphi(Z_v \setminus \Sigma_\varphi)} v(y) dy \leq \int_G \left( \sum_{x \in \varphi^{-1}(y) \cap (Z_v \setminus \Sigma_\varphi)} v(\varphi(x)) \right) dy \\ &= \int_{Z_v} v(\varphi(x)) |J(x, \varphi)| dx = 0. \end{aligned}$$

Consequently,  $v(\varphi(Z_v \setminus \Sigma_\varphi)) = 0$  and  $|\varphi(Z_v \setminus \Sigma_\varphi)| = 0$ . Fix  $\varepsilon > 0$  and an open set  $U \subset D'$  such that  $U \supset \varphi(Z_v \setminus \Sigma_\varphi)$ , while  $v(U) < \varepsilon$  and  $|U| < \varepsilon$ . By Lemma 1, choose a point-finite cover  $\{B^E(y_j, r_j)\}$  of  $U$  of multiplicity  $M$ ; this number is independent of  $U$  and satisfies  $\sum_j |B^E(y_j, r_j)| \leq M\varepsilon$ . Using (28) and Lemma 2, we obtain

$$\begin{aligned} \int_{Z_v} |D\varphi|^p(x) u(x) dx &= \int_{Z_v \setminus \Sigma_\varphi} |D\varphi|^p(x) u(x) dx \leq \sum_{j=1}^{\infty} \int_{\varphi^{-1}(B^E(y_j, r_j))} |D\varphi|^p(x) u(x) dx \\ &\leq C^p \|\varphi^*\| \sum_{j=1}^{\infty} \int_{B^E(y_j, 2r_j)} v(x) dx \leq C_2^p \|\varphi^*\| v(U) < C_2^p \|\varphi^*\| \varepsilon. \end{aligned}$$

Since  $\varepsilon > 0$  is arbitrary, while the weight function  $u(x)$  is positive almost everywhere, it follows that  $|D\varphi| = 0$  almost everywhere on  $Z_v$ . Consequently,  $\varphi$  has finite  $(u, v)$ -weighted distortion.

(3) Applying the change-of-variables formula (Proposition 4) and (28), we infer that

$$\begin{aligned} & \int_{\varphi^{-1}(B(y_0, r))} |D_h \varphi|^p u(x) dx = \int_{\varphi^{-1}(B(y_0, r)) \setminus Z_v} |D_h \varphi|^p u(x) dx \\ &= \int_{B(y_0, r)} \left( \sum_{x \in \varphi^{-1}(y') \setminus (Z_v \cup \Sigma_\varphi)} \frac{|D_h \varphi|^p(x) u(x)}{|J(x, \varphi)|} \right) dy' \leq C^p \|\varphi^*\| v(B(y_0, r)). \end{aligned} \quad (29)$$

From (29) and Theorem 1 we deduce the inequality

$$\left( \sum_{x \in \varphi^{-1}(y) \setminus (Z_v \cup \Sigma_\varphi)} \frac{|D_h \varphi|^p(x) u(x)}{|J(x, \varphi)|} \right)^{\frac{1}{p}} \leq C v^{\frac{1}{p}}(y) \|\varphi^*\|,$$

which enables us to obtain the required estimate  $\|H_p^{u,v}(\cdot) | L_{\mathcal{X}}(D')\| \leq C \|\varphi^*\|$ .  $\square$

**Theorem 4.** Assume that  $1 \leq q \leq p < \infty$ . Take  $\varphi : D \rightarrow D'$  with the properties:

- (1)  $\varphi$  is of class  $\text{ACL}_{\text{part}}(D)$ ;
- (2)  $\varphi$  has finite  $(u, v)$ -weighted distortion;
- (3) the distortion function  $H_p^{u,v}(\cdot)$  belongs to  $L_{\mathcal{X}}(D')$ , where  $\frac{1}{\mathcal{X}} = \frac{1}{q} - \frac{1}{p}$ .

Then  $\varphi$  induces the bounded composition operator

$$\varphi^* : L_p^1(D', v) \cap C_0^\infty(D') \rightarrow L_q^1(D, u)$$

in the weighted Sobolev spaces.

Furthermore,  $\|\varphi^*\| \leq \|H_p^{u,v}(\cdot) | L_{\mathcal{X}}(D')\|$ .

PROOF. Observe that  $f \circ \varphi \in \text{ACL}(D)$  for all  $f \in L_p^1(D', v) \cap C_0^\infty(D')$ . Indeed, we have  $\text{ACL}_{\text{part}}(D)$ ; hence, the following holds for every segment  $(\alpha, \beta)$  of the integral line of a horizontal vector field on which  $\varphi$  is absolutely continuous: the images  $\varphi(\alpha)$  and  $\varphi(\beta)$  of the endpoints leave the support  $\text{supp } f$  of  $f$ . Thus,  $f \circ \varphi$  is absolutely continuous on the segments of absolute continuity of  $\varphi$  and vanishes at the points where  $\varphi$  is not absolutely continuous.

Verify that  $\|\varphi^* f | L_q^1(D, u)\| \leq H_{p,q}^{u,v}(D') \|f | L_p^1(D', v)\|$  for every  $f \in L_p^1(D', v) \cap C^1(D')$ . We have

$$\begin{aligned} \|\varphi^* f | L_q^1(D, u)\| &\leq \left( \int_{D \setminus Z_v} (|\nabla_{\mathcal{L}} f|(\varphi(x)) |D_h \varphi|(x))^q u(x) dx \right)^{\frac{1}{q}} \\ &\leq \left( \int_{D'} |\nabla_{\mathcal{L}} f|^q(y) \left( \sum_{x \in \varphi^{-1}(\Sigma_\varphi \cup Z_v)} \frac{|D_h \varphi|^q(x) u(x)}{|J(x, \varphi)|} \right) dy \right)^{\frac{1}{q}} \\ &= \left( \int_{D'} |\nabla_{\mathcal{L}} f|^q(y) v^{\frac{q}{p}}(y) \left( v^{-\frac{q}{p}}(y) \sum_{x \in \varphi^{-1}(\Sigma_\varphi \cup Z_v)} \frac{|D_h \varphi|^q(x) u(x)}{|J(x, \varphi)|} \right) dy \right)^{\frac{1}{q}}. \end{aligned}$$

By Hölder's inequality for  $q < p$ ,

$$\begin{aligned} & \|\varphi^* f | L_q^1(D, u)\| \\ &\leq \left( \int_{D'} |\nabla_{\mathcal{L}} f|^p(y) v(y) dy \right)^{\frac{1}{p}} \left( \int_{D'} \left( v^{-\frac{1}{p}}(y) \left( \sum_{x \in \varphi^{-1}(\Sigma_\varphi \cup Z_v)} \frac{|D_h \varphi|^q(x) u(x)}{|J(x, \varphi)|} \right)^{\frac{1}{q}} \right)^{\mathcal{X}} dy \right)^{\frac{1}{\mathcal{X}}} \\ &= \|H_q^{u,v}(\cdot) | L_{\mathcal{X}}(D')\| \|f | L_p^1(D', v)\|. \end{aligned} \quad (30)$$

Finally,  $\|\varphi^*\| \leq H_{p,q}^{u,v}(D')$ .  $\square$

In the case  $q = p = \nu$  and  $v \circ \varphi \leq u$  the necessary conditions are sufficient.

**Theorem 5.** Consider  $v(y) \in L_{1,\text{loc}}(D')$  and  $u^{\frac{1}{1-\nu}}(x) \in L_{1,\text{loc}}(D)$  such that  $v \circ \varphi \leq u$  almost everywhere on  $D$ . A measurable mapping  $\varphi : D \rightarrow D'$  induces the bounded composition operator

$$\varphi^* : L_{\nu}^1(D', v) \cap C_0^\infty(D') \rightarrow L_{\nu}^1(D, u)$$

in the weighted Sobolev spaces if and only if

- (1)  $\varphi$  is of class  $\text{ACL}_{\text{part}}(D)$ ;
- (2)  $\varphi$  has finite  $(u, v)$ -weighted distortion;
- (3) the distortion function  $H_p^{u,v}(\cdot)$  belongs to  $L_\infty(D')$ .

Furthermore,

$$C \|H_p^{u,v}(\cdot) | L_\infty(D')\| \leq \|\varphi^*\| \leq \|H_p^{u,v}(\cdot) | L_\infty(D')\|.$$

**PROOF.** *Necessity:* The inequalities  $|J(x, \varphi)| \leq |D\varphi|^\nu(x)$  and  $v \circ \varphi \leq u$  combined with (28) guarantee that  $|J(x, \varphi)|v(\varphi(x))$  is integrable on  $\varphi^{-1}(B(y_0, r))$ , where  $B(y_0, r) \subset D'$  is an arbitrary ball. Indeed,

$$\int_{\varphi^{-1}(B(y_0, r))} |J(x, \varphi)|v(\varphi(x)) dx \leq C \int_{\varphi^{-1}(B(y_0, r))} |D_h\varphi|^\nu u(x) dx \leq C_1 v(B(y_0, r)),$$

which implies that  $N(y, \varphi)v(y) \in L_{1,\text{loc}}(D')$ . Then Theorem 3 applies.

*Sufficiency* follows from Theorem 4.  $\square$

If the Banach indicatrix  $N(y, \varphi)$  is bounded then the necessary conditions coincide with the sufficient conditions for  $1 \leq q \leq p < \infty$ .

**Corollary 3.** Take  $v(y) \in L_{1,\text{loc}}(D')$  and  $u^{\frac{1}{1-q}}(x) \in L_{1,\text{loc}}(D)$ . Assume that  $N(y, \varphi) \leq b < \infty$  almost everywhere on  $D'$ . A measurable mapping  $\varphi : D \rightarrow D'$  induces the bounded composition operator

$$\varphi^* : L_p^1(D', v) \cap C_0^\infty(D') \rightarrow L_q^1(D, u)$$

in the weighted Sobolev spaces if and only if

- (1)  $\varphi$  is of class  $\text{ACL}_{\text{part}}(D)$ ;
- (2)  $\varphi$  has finite  $(u, v)$ -weighted distortion;
- (3) the distortion function  $H_p^{u,v}(\cdot)$  belongs to  $L_\kappa(D')$ , where  $\frac{1}{\kappa} = \frac{1}{q} - \frac{1}{p}$ .

Furthermore,  $c \|H_p^{u,v}(\cdot) | L_\kappa(D')\| \leq \|\varphi^*\| \leq \|H_p^{u,v}(\cdot) | L_\kappa(D')\|$ .

Observe that if  $\varphi$  is a homeomorphism then we obtain as a corollary the statement that essentially similar to [3, Theorem A]:

**Corollary 4.** Take  $v(y) \in L_{1,\text{loc}}(D')$  and  $u^{\frac{1}{1-q}}(x) \in L_{1,\text{loc}}(D)$ . A homeomorphism  $\varphi : D \rightarrow D'$  induces the bounded composition operator

$$\varphi^* : L_p^1(D', v) \cap C_0^\infty(D') \rightarrow L_q^1(D, u)$$

in the weighted Sobolev spaces if and only if  $\varphi \in \text{ACL}(D)$  and the weighted distortion function

$$K_p^{u,v}(x) = \inf\{K(x) : |D_h\varphi|(x)u^{\frac{1}{q}}(x) \leq K(x)|J(x, \varphi)|^{\frac{1}{p}}v^{\frac{1}{p}}(\varphi(x))\}$$

belongs to  $L_\kappa(D)$ . The norm of the operator  $\|\varphi^*\|$  is equivalent to  $\|K_p^{u,v}(x) | L_\kappa(D)\|$ .

Let us state one more claim, in which bounded composition operators in Sobolev spaces are associated to quasiconformal mappings.

**Corollary 5.** Take two weight functions  $u$  and  $v$  of the Muckenhoupt class  $A_\nu$ . Assume that  $\varphi : D \rightarrow D'$  is a homeomorphism and  $v \circ \varphi(x) \leq u(x)$  for almost all  $x \in D$ . The mapping  $\varphi$  is quasiconformal if and only if the composition operator

$$\varphi^* : L_\nu^1(D', v) \cap C_0^\infty(D') \rightarrow L_\nu^1(D, u)$$

is bounded.

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