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MATHEMATICS

Isomorphisms of Sobolev Spaces on Carnot Groups and Metric Properties of Mappings[1]

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Abstract—We study metric properties of measurable mappings on a Carnot group inducing via the change-of-variable formula an isomorphism of Sobolev spaces. We prove that such a mapping can be redefined on a set of measure zero to be quasiconformal or quasi-isometric depending on a relation between the Hausdorff dimension of the group and a summability exponent of the Sobolev space.

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This work can be regarded as a natural continuation of the studies started in [1–5] and deals with the following problem: what metric and analytic properties does a measurable mapping φ inducing an isomorphism φ^* by the rule $\varphi^*(f) = f \circ \varphi, f \in L_p^1$ have. In the mentioned papers there were obtained various proofs of the following assertion: if the composition operator φ^* is an isomorphism then $\mathbb{G}$ is quasiconformal or quasi-isometric in the metric of a domain, adequate to the geometry of function space, in dependence of the relations between the smoothness, the summability and Hausdorff dimension of the domain.

The goal of this article consists in describing the solution of a similar question for measurable mappings on a Carnot group inducing an isomorphism of horizontal Sobolev spaces. In the paper [6] we provide a detailed history of this subject, an exhaustive bibliography and a partial solution to the problem.

1. A Carnot group $\mathbb{G}$ is a connected, simply connected stratified nilpotent Lie group. It means that Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ admits the decomposition into direct sum of vector subspaces: $\mathfrak{g} = V_1 \oplus \ldots \oplus V_m$ such that $[V_1, V_j] = V_{j+1}$ for $j = 1, 2, \ldots, m-1$ and $[V_1, V_m] = \{0\}$. Let $n_i = \dim V_i$. Below, we use the following notation $n = n_1$. Let $X_1, \ldots, X_n$ be vector fields constituting an orthonormal basis of the horizontal subspace V_1.

An absolutely continuous piecewise smooth curve $\gamma: [a, b] \to \mathbb{G}$ whose tangent vector $\dot\gamma(t)$ is in V_1 for a.e. $t \in [a, b]$, is called a horizontal curve. The length of a horizontal curve $\gamma: [a, b] \to \mathbb{G}$ is defined by the formula $l(\gamma) = \int\limits_a^b |\dot\gamma(t)|\, dt$ ($|\dot\gamma(t)|$ is the length of the tangent vector).

Carnot-Carathéodory metric $d(x, y)$ on a group $\mathbb{G}$ is defined as the infimum of the lengths over all horizontal curves with endpoints x and y. The Hausdorff dimension of the group $\mathbb{G}$ equals $\nu = n_1 + 2n_2 + 3n_3 + \ldots + mn_m$.

2. Sobolev classes on a Carnot group. Let D be an open set on a Carnot group $\mathbb{G}$. The Sobolev space $L_p^1(D)$ consists of all locally summable functions f: $D \to \mathbb{R}$ whose generalized derivatives $X_i f \in L_p(D)$, $i = 1, 2, \ldots n$. A seminorm in $L_p^1(D)$ is defined as

$$\|f \,|\, L_p^1(D)\| = \|\nabla_{\mathcal{L}} f \,|\, L_p(D)\| = \left(\int\limits_D |\nabla_{\mathcal{L}} f(x)|^p dx\right)^{\frac{1}{p}},$$

where $\nabla_{\mathcal{L}} f(x) = (X_1 f(x), \ldots, X_n f(x))$ is the generalized subgradient of a function f at the point $x \in D$ and $|\nabla_{\mathcal{L}} f(x)| = \sqrt{(X_1 f(x))^2 + \ldots + (X_n f(x))^2}$.

A mapping $\varphi: D \to \mathbb{G}$ belongs to the Sobolev class $W_{p,\mathrm{loc}}^1(D)$ if and only if it can changed on a negligible set in such a way that

(1) for every $z \in \mathbb{G}$ the function $[\varphi]_z$: $D \ni x \mapsto d(\varphi(x), z)$ belongs to $L_{p,\mathrm{loc}}(D)$;

(2) the mapping $\varphi: D \to \mathbb{G}$ is absolutely continuous on almost all integral lines of the horizontal vector fields X_j, $j = 1, 2, \ldots, n$ ($\varphi \in ACL(D)$);

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(3) the derivative $X_j\varphi(x) = \lim_{t\to 0}\delta_{t^{-1}}(\varphi(x)^{-1}\varphi(\exp tX_j))$ belongs to $V_1(\varphi(x))$ a.e., moreover $|X_j\varphi| \in L_{p,\mathrm{loc}}(D)$ for all j.

Recall that a mapping $\varphi: D \to \mathbb{G}$ is called absolutely continuous on almost all integral lines of the horizontal vector fields X_j, $j = 1, 2, \ldots, n$, if for every open set $U \Subset D$, and fibration Γ_j determined by the left-invariant vector field X_j ($j = 1, 2, \ldots, n$), the mapping is absolutely continuous on $\gamma \cap U$ with respect to one-dimensional Hausdorff measure for $d\gamma$-almost all $\gamma \in \Gamma_j$. For such mapping the derivative $X_j\varphi$ ($j = 1, 2, \ldots, n$) exists a.e. in D (see various proofs of this fact in [7–9]).

The symbol $D\varphi$ denotes the approximative differential of a mapping φ [10], whereas $D_h\varphi: V_1 \to V_1$ is the horizontal part of this differential. For the Jacobian $\det D\varphi$ of a mapping φ we use the notation $J(x, \varphi)$.

Definition 1. A homeomorphism $\Phi: D \to D'$ belonging to $W^1_{\nu,\mathrm{loc}}(D)$, is called quasiconformal if there is a constant K such that $|D\Phi(x)|^{\nu} \le K|J(x, \Phi)|$ a.e. in D.

Definition 2. A homeomorphism $\Phi: D \to D'$ of $W^1_{1,\mathrm{loc}}(D)$ is called quasi-isometric, if $|D\Phi(x)| \le M$ and $0 < \alpha \le |J(x, \Phi)|$ for almost all $x \in D$, constants M and α are independent of x.

Definition 3. Two open sets D_1 and D_2 are said to be $(1, p)$-equivalent if the restriction operators r_i: $L^1_p(D_1 \cup D_2) \to L^1_p(D_i)$, $r_i(f) = f|_{D_i}$, $f \in L^1_p(D_1 \cup D_2)$, are isomorphisms.

Properties of $(1, p)$-equivalent domains are studied in [11] for the Euclidean spaces, and in [12] for the Carnot groups.

3. Capacity in the space $L^1_{\nu,F}(D)$. Let $F \subset D$ be a closed set of positive measure without isolated points. Consider a family of functions

$$L^1_{\nu,F}(D)$$
$$= \{u \in (L^1_{\nu}(D)\colon u(x) = 0 \text{ for almost all } x \in F)\}.$$

Note that $L^1_{\nu,F}(D)$ is a closed subspace of $L^1_{\nu}(D)$ and a normed space with the following norm $\|u \mid L^1_{\nu,F}(D)\| = \|u|L^1_{\nu}(D)\|$. One could prove that $L^1_{\nu,F}(D)$ is a Banach space. Denote $D_F = D\setminus F$.

A capacity $\mathrm{Cap}(K;\ L^1_{\nu,F}(D))$ of a compact $K \subset D_F$ in the space $L^1_{\nu,F}(D)$ is the value $\mathrm{Cap}(K;\ L^1_{\nu,F}(D)) = \inf\|g \mid L^1_{\nu,F}(D)\|^{\nu}$, where the infimum is taken over all continuous functions $g \in L^1_{\nu,F}(D)$ such that $g \ge 1$ on K.

For an arbitrary set $E \subset D_F$ its interior capacity equals $\underline{\mathrm{Cap}}(E;\ L^1_{\nu,F}(D)) = \sup\{\mathrm{Cap}(K;\ L^1_{\nu,F}(D))\colon K \subset E,\ K \text{ is compact}\}$, while its outer capacity equals $\overline{\mathrm{Cap}}(E;\ L^1_{\nu,F}(D)) = \inf\{\underline{\mathrm{Cap}}(U;\ L^1_{\nu,F}(D))\colon E \subset U,\ U \subset D_F \text{ is open}\}$. A set E is called measurable with respect to capacity if $\underline{\mathrm{Cap}}(E;\ L^1_{\nu,F}(D)) = \overline{\mathrm{Cap}}(E;\ L^1_{\nu,F}(D))$.

Note [4] that the capacity in the space $L^1_{\nu,F}(D)$ is a Choquet capacity [13].

We say that a certain property holds quasieverywhere, if it holds everywhere except of a set of zero capacity.

Definition 4. A function f, defined quasieverywhere on D_F, is called quasicontinuous if for every $\varepsilon > 0$ there is an open set $U_\varepsilon \subset D_F$ such that $\overline{\mathrm{Cap}}(U_\varepsilon;\ L^1_{\nu,F}(D)) < \varepsilon$ and the restriction of the function f to $D_F\setminus U_\varepsilon$ is continuous.

(For more details with respect to other functional spaces see [3, 4, 12].)

4. The composition operator and mappings of the class IL^1_p. In [6] we have introduced the main object of the research: the class IL^1_p of mappings on a Carnot group.

Definition 5. Let D, D' be domains on a Carnot group $\mathbb{G}$. A measurable mapping $\varphi: D \to D'$ belongs to IL^1_p, $p \in [1, \infty]$, whenever φ induces the composition operator of Sobolev spaces

$$\varphi^*\colon L^1_p(D') \cap C^\infty(D') \to L^1_p(D), \quad \varphi^*(f) = f\circ\varphi, \quad f \in L^1_p(D') \cap C^\infty(D'), \tag{1}$$

such that

(1) following inequalities $K^{-1}\|f|L^1_p(D')\| \le \|\varphi^*(f)|L^1_p(D)\| \le K\|f|L^1_p(D')\|$ hold for every function $f \in L^1_p(D') \cap C^\infty(D')$ where the constant K is independent of the choice of f;

(2) the image $\varphi^*(L^1_p(D') \cap C^\infty(D'))$ is dense in $L^1_p(D)$.

In [6] it is shown that condition 2 is independent of condition 1.

The mappings IL^1_p possess the following properties [6].

Proposition 1. (1) *The domain of a mapping* φ *could be reduced to a set* $T = \bigcup_k T_k$, $|D\setminus T| = 0$, *where* $\{T_k\}$ *is an increasing sequence of bounded sets of positive measure, which contain the points of nonzero density only*;

(2) *the mapping* φ *is continuous on every* T_k;

(3) *the mapping enjoys Lusin's* $\mathcal{N}$*-property and* $\mathcal{N}^{-1}$*-property on* T;

(4) *the mapping* $\varphi: T \to D'$ *is injective*;

(5) *the image* $\varphi(T)$ *is dense in* D' *and* $|D'\setminus\varphi(T)| = 0$.

The operator (1) extends onto $L_p^1(D)$ and keeps properties of a composition operator:

Lemma 1 [6, Lemma 10]. *Let a measurable mapping* φ: $D \to D'$ *belongs to* IL_p^1. *Then the operator* φ^*: $L_p^1(D') \cap C^\infty(D') \to L_p^1(D)$ *extends by the continuity to the operator* $\tilde{\varphi}^*: L_p^1(D') \to L_p^1(D)$ *enjoying the following properties*:

(1) *the value of the operator* $\tilde{\varphi}^*: L_p^1(D') \to L_p^1(D)$ *on a given class* $[f] \in L_p^1(D')$ *can be found as follows*:

$$\tilde{\varphi}^*([f]) = \begin{cases} f \circ \varphi \text{ for } p \le \nu, & f \text{ is an arbitrary} \\ \text{representative of class } [f], & \\ \tilde{f} \circ \varphi \text{ for } p > \nu, & \tilde{f} \text{ is the continuos} \\ \text{representative of class } [f], & \end{cases}$$

(2) $K^{-1}\|f \mid L_p^1(D')\| \le \|\tilde{\varphi}^*(f) | L_p^1(D)\| \le K\|f | L_p^1(D')\|$;

(3) $\tilde{\varphi}^*: L_p^1(D') \to L_p^1(D)$ *is an isomorphism.*

5. The case $p \neq \nu$. A complete description of measurable mappings on a Carnot group inducing isomorphisms of Sobolev spaces L_p^1 in the sense of definition 5 provided $p \neq \nu$ is given in [6].

Theorem 1 [6, Theorem 1]. *Let* $p \ge 1$, $p \ne \nu$, *and* D, D' *be domains on a Carnot group* $\mathbb{G}$ (ν *is the Hausdorff dimension of* $\mathbb{G}$). *A measurable mapping* φ: $D \to D'$ *belongs to* IL_p^1 *if and only if* φ *coincides with some quasi-isometry* Φ: $D \to \Phi(D)$ *almost everywhere for which domains* $\Phi(D)$ *and* D' *are* $(1, p)$*-equivalent.*

6. The case $p = \nu$. The main result provided $p = \nu$ is the following.

Theorem 2. *Let* D, D' *be domains on a Carnot group* $\mathbb{G}$, ν *be the Hausdorff dimension of the group* $\mathbb{G}$. *A measurable mapping* φ: $D \to D'$ *belongs to* IL_ν^1 *if and only if* φ *coincides with a quasiconformal mapping* Φ: $D\backslash\{x_0\} \to \mathbb{G}$ *almost everywhere such that the domains* $\Phi(D\backslash\{x_0\})$ *and* D' *are* $(1, \nu)$*-equivalent, where a point* $x_0 \in \overline{\mathbb{G}}$ ($\overline{\mathbb{G}}$ *is the one-point compactification of* $\mathbb{G}$).

The proof of the sufficiency is similar to the corresponding part of the proof of [6, Theorem 1]. The proof of the necessity is splitted into several assertions formulated below.

Lemma 2. *There are a set* $S_\varphi \subset D_F$ *of measure zero and a quasicontinuous mapping* φ_0: $D_F\backslash S_\varphi \to \overline{D'_F}$ *such that* φ_0 *coincides with* φ *a.e. on* D_F. *The mapping* φ_0 *satisfies the following estimate*

$$\begin{aligned} &\overline{\mathrm{Cap}}(\varphi_0(B_j) \cap D'_F;\, L^1_{\nu,\varphi(F)}(D')) \\ &\le K^{-\nu}\,\overline{\mathrm{Cap}}(B_j;\, L^1_{\nu,F}(D)) \end{aligned} \tag{2}$$

for any ball

$$B_j \Subset D_F \tag{3}$$

from a countable system, which form a base of topology of D_F. *Here* $D'_F = D'\backslash\varphi(F)$.

From [14, Proposition 5] we get the following result.

Proposition 2. (1) *The mapping* φ_0 *is continuous on* $S(x, r)$ *for a.e.* $r \in (0, \mathrm{dist}(x, \partial D_F))$.

(2) *The mapping* φ_0 *is continuous on almost all integral lines of horizontal vector fields: for any ball* $B(x, r) \subset D_F$ *and almost all integral lines* $\gamma \subset B(x, r)$ *of a horizontal vector field* X_i, $i = 1, 2, \ldots, n$, *the mapping* φ_0 *is continuous on integral line* γ.

Let $x \in T \cap D_F$ (T is a set from Proposition 1). Denote

$$\hat{B}(x, r) = \left\{ \bigcup_{\rho \in (0, r)} S(x, \rho) \mid \text{the mapping } \varphi_0\colon S(x, \rho) \to \mathbb{G} \text{ is continuous} \right\} \subset D_F. \tag{4}$$

Lemma 2 is a crucial tool in the proof of the following statement.

Lemma 3. *Let* $\{r_k\}$ *be a sequence of positive numbers converging to zero as* $k \to \infty$. *Assume further that* $x \in D_F$ *and there is a sequence* $u_k \in \hat{B}(x, r_k) \cap D_F$ *for which* $\varphi_0(u_k) \to y \in D'_F$ *as* $k \to \infty$ *where* y *is some point. Then the images* $\varphi_0(\hat{B}(x, r_k))$ *shrink to the point* $y \subset D'_F$ $k \to \infty$:

$$\{y\} = \bigcap_{k \in \mathbb{N}} \overline{\varphi_0(\hat{B}(x, r_k))} \in D'_F. \tag{5}$$

Proof. Obviously (5) is equivalent to the following:

$$\sup_{z \in \hat{B}(x, r_k) \cap D_F} d(\varphi_0(z), y) \to 0 \tag{6}$$

as $k \to \infty$. To the contrary, suppose (6) is not valid. Then there are a number $\vartheta > 0$ and a sequence of radii $\kappa_k \in (0, r_k)\backslash\sigma_{x, r_k}$, where the measure of the set $\sigma_{x, r_k} \subset (0, r_k)$ equals zero, such that

$$\begin{aligned} &\mathrm{diam}(\{y\} \cup \varphi_0 S(x, \kappa_k)) \\ &= \sup_{z \in S(x, \kappa_k) \cap D_F} d(\varphi_0(z), y) \ge \vartheta, \quad k \in \mathbb{N}. \end{aligned} \tag{7}$$

Since $u_k \in \hat{B}(x, r_k) \cap D_F$, then $u_k \in S(x, \tau_k)$, where $\tau_k \in (0, r_k)\backslash\sigma_{x, r_k}$, and $\tau_k \to 0$ as $k \to \infty$.

It is clear that collection (3) could be chosen in such a way that every ball $B(x, r_k)$ under k large enough would be contained in a ball $B_j = B(x_j, \rho_j)$ from (3) such that $x_j \in B(x, r_k)$ and $\rho_j > 2r_k$ (the last inequality implies the inclusion $B(x, r_k) \subset B(x_j, \rho_j)$); moreover, $\rho_j \to 0$ as $k \to \infty$ (it means that decreasing r_k to zero implies decreasing ρ_j to zero).

For any $k \in \mathbb{N}$, consider a continuous curve $\gamma_k \subset B(x_j, \rho_j)$ with endpoints in a ball $B(x, \min(\tau_k, \kappa_k))$ and in the complement $B(x_j, \rho_j)\setminus B(x, r_k)$. Assume the mapping is well defined and continuous on this curve (the existence of a curve with these properties can be proved with help of Proposition 2).

By symbol K_k denote the following compact set $S(x, \tau_k) \cup S(x, \kappa_k) \cup \gamma_k$. We have the inclusion $K_k \subset B(x_j, \rho_j)$. The compact is connected, and the mapping φ_0: $K_k \to \overline{D'_F}$ is continuous.

By the choice of the compact K_k and balls $B_j = B(x_j, \rho_j)$ satisfying relation (2), we have the chain of inequalities below:

$$\begin{aligned} &\overline{\mathrm{Cap}}(\varphi_0(K_k) \cap D'_F;\, L^1_{\nu,\varphi}(D')) \\ &\le \overline{\mathrm{Cap}}(\varphi_0(\hat{B}(x, r_k)) \cap D'_F;\, L^1_{\nu,\varphi(F)}(D')) \\ &\le \overline{\mathrm{Cap}}(\varphi_0(B(x_j, \rho_j)) \cap D'_F;\, L^1_{\nu,\varphi(F)}(D')) \qquad (8) \\ &\le K^{-\nu}\overline{\mathrm{Cap}}(B(x_j, \rho_j);\, L^1_{\nu,F}(D)) = O\left(\left(\ln\frac{2}{\rho_j}\right)^{1-\nu}\right) = o(1) \end{aligned}$$

as $k \to \infty$. From (8) we get $\overline{\mathrm{Cap}}\,(\varphi_0(K_k) \cap D'_F;$ $L^1_{\nu,\varphi(F)}(D')) \to 0$ as $k \to \infty$. Well known properties of capacity imply diam $\varphi_0(K_k) \to 0$ as $k \to \infty$. With the condition $\varphi_0(u_k) \to y \in D'_F$ as $k \to \infty$ we obtain $\mathrm{diam}(\{y\} \cap \varphi_0(K_k)) \to 0$ as $k \to \infty$. The last contradicts (7), because $S(x, \kappa_k) \subset K_k$. Lemma 3 is thereby proved.

The lemma above is a statement and plays the principal role in the proof of continuity of φ_0.

Let $x \in T \cap D_F$. For $\rho > 0$ small enough we find a number $\delta_{x,\rho} > 0$ such that the inclusion $\varphi_0(\hat{B}(x, \delta_{x,\rho})) \subset B(\varphi_0(x), \rho) \subset D'_F$ holds. A point $y \in B(x, \delta_{x,\rho})$ either belongs to the intersection $T \cap D_F$, or does not. In the first case we have $\lim\limits_{z \to y,\, z \in \hat{B}(y, \delta_1)} \varphi_0(z) = \varphi_0(y)$, δ_1 is a positive number small enough. Otherwise the value of mapping φ_0 in y is not defined, but there is a limit lim $\lim\limits_{z \to y,\, z \in \hat{B}(y, \delta_2)} \varphi_0(z) \in D'_F$ which we take as $\varphi_0(y)$ (again δ_2 is a positive number small enough).

Proposition 3. *The mapping*

$$\varphi_0\colon B(x, \delta_{x,\rho}) \to \overline{B(\varphi_0(x), \rho)} \subset D'_F$$

is continuous in the points of $x \in T \cap D_F$.

Proposition 4. *Mappings*

$$\varphi_0\colon B(x, \delta_{x,\rho}) \to \overline{B(\varphi_0(x), \rho)}\ ,$$

and

$$\varphi_0\colon B(y, \delta_{y,\rho}) \to \overline{B(\varphi_0(x), \rho)}\,,$$

coincide on a non empty intersection $B(x, \delta_{x,\rho}) \subset B(y, \delta_{y,\rho})$ *provided* $x, y \in T \cap D_F$.

For points $x \in T \cap D_F$ we consider the collection of balls $B(x, \delta_{x,\rho}) \subset D_F$. On the following open set

$$U = \bigcup_{x \in T \cap D_F} B(x, \delta_{x,\rho})$$

by means of Proposition 4 one can define correctly a continuous mapping which we denote by $\tilde{\varphi}_0$, wherein $U \subset D_F$ and $|D_F\setminus U| = 0$.

It is obvious that the mapping $\tilde{\varphi}_0\colon U \to D'_F$ is an extension of the mapping φ_0: $T \cap D_F \to D'_F$ by the continuity on the set U. Since $T \cap D_F$ is dense in U this extension is unique.

Proposition 5. *The mapping* $\tilde{\varphi}_0\colon U \to D'_F$ *is an homeomorphism.*

Lemma 4. *Let* $D, D' \subset \mathbb{G}$. *If a mapping* $\varphi\colon D \to D'$ *belongs to* IL^1_ν *then* $\tilde{\varphi}_0 \in W^1_{\nu,\mathrm{loc}}(U)$.

Denote $\tilde{\varphi}_0(U) = V$. From [15] we obtain

Proposition 6. *The mapping* $\tilde{\varphi}_0\colon U \to V$ *is quasiconformal.*

Denote $S = D_F\setminus U$. Let $x \in S$. Then there are two cases:

(1) there exists $r_0 > 0$ such that $\overline{\tilde{\varphi}_0(\hat{B}(x, r))} \subset D'_F$ whenever $r < r_0$;

(2) $\tilde{\varphi}_0(S(x, r_k)) \cap \partial D'_F \ne \phi$ for some sequence $r_k \to 0$.

If the case (1) holds one can define the value of the mapping $\tilde{\varphi}_0$ at the point x assuming

$$\tilde{\varphi}_0(x) = \bigcap_{r \to 0} \overline{\tilde{\varphi}_0(\hat{B}(x, r))} \in D_F.$$

This approach allows to define the value of the mapping $\tilde{\varphi}_0$ not only at the point x, but also in points of some ball $B(x, \delta_{x,\rho})$. The mapping $\tilde{\varphi}_0\colon B(x, \delta_{x,\rho}) \to D'_F$ is continuous on the ball $B(x, \delta_{x,\rho})$ (as in Proposition 3). Thus domains U and V can be extended, while the set S can be shrunk.

Assume further that for every point $x \in S$ the case (2) holds. In such a way there is a sequence $\{x_k \in U\}$ converging to x such that $\tilde{\varphi}_0(x_k) \to \partial D'$ provided $k \to \infty$. This justifies the following result.

Lemma 5. *For any ball* $B(x, r) \subset D_F\,(B(y, r) \subset D'_F)$ *centered at* $x \in S$ $(x \in \varphi_0(S))$, *an intersection* $B(x, r) \cap U\,(B(y, r) \cap V)$ *is a connected set.*

Proposition 7. *The mapping* $\tilde{\varphi}_0\colon U \to V$ *can be extended by the continuity to points* $x \in S$ *for which there is a sequence* $\{x_n \in U\}$ *converging to* x *such that the sequence of images* $\tilde{\varphi}_0(x_n)$ *converges to a point* $z \in \partial D'_F$. *The extended mapping is injective.*

Lemma 6. *There exists at most one point* $x_{\mathrm{inv}} \in S$ *such that for any subsequence* $\{x_n\} \subset U$ *converging to* x_{inv}, *we* *have* $d(\tilde{\varphi}_0(x_n)) \to \infty$ *as* $n \to \infty$ (*the case of inversion transformation*).

Proposition 8. *The mapping* $\tilde{\varphi}_0: D_F \backslash \{x_{\text{inv}}\} \to \mathbb{G}$ *is quasiconformal.*

Choosing another closed set $F_1 \subset D_F$ of positive measure without isolated points, we obtain quasiconformality of the original mapping $\tilde{\varphi}_0: D \backslash \{x_{\text{inv}}\} \to \mathbb{G}$.

For a Euclidean space Theorem 2 was proved in [1] by a different method, assuming that D' is a bounded domain.

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