# Composition operators on weighted Sobolev spaces

*when changing variables is safe*

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**Abstract**

We study mappings inducing a bounded composition operator on weighted Sobolev spaces on Carnot groups. An analytical description of these mappings is given in terms of integrability of weighted distortion function. For the special cases we prove that mapping which generates a bounded composition operator is partially absolutely continuous on almost all horizontal lines.

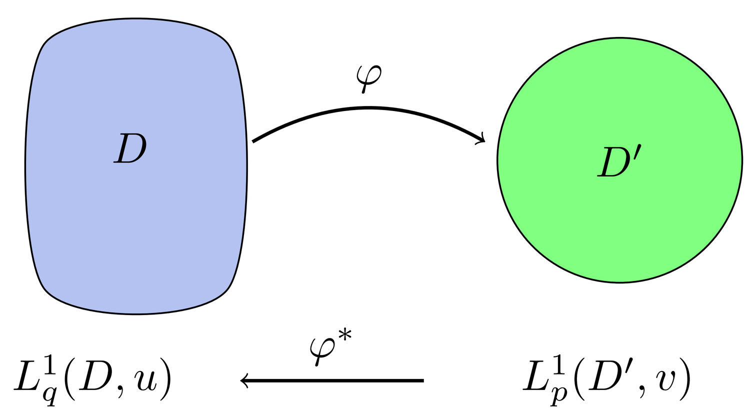
## Introduction

Let $f(y) \in L_p^1(D', v)$ be a function of weighted Sobolev class. One wants to change variables $f(\varphi(x))$ in such away that the following properties be held

1. the composition $f \circ \varphi$ **stays** in weighted Sobolev class $L_q^1(D, u)$,
2. the norm is **under control**: $\|f \circ \varphi\|_{L_q^1(D,u)} \leq C\|f\|_{L_p^1(D',v)}$.

Two properties above actually says that the following composition operator is bounded

$$\varphi^* : L_p^1(D', v) \cap C_0^\infty(D') \to L_q^1(D, u), \quad \varphi^*(f) = f \circ \varphi, \quad f \in L_p^1(D', v) \cap C_0^\infty(D').$$




In the framework of this approach to geometric function theory, the following problem arises: **what analytic and geometric properties are possessed by a mapping $\varphi$ that induces a bounded operator of weighted Sobolev spaces by the composition rule?**

## Definitions

We deal with **Carnot group** to develop a wider approaches which work in metric spaces (not necessary Euclidean one). If you are not familiar with it just skip and assume that $\mathbb{G} = \mathbb{R}^n$.

**Definition 1.** *A Carnot group $\mathbb{G}$ is a connected, simply connected, stratified nilpotent Lie group. This means that the Lie algebra $\mathfrak{g}$ of the group $\mathbb{G}$ admits a nilpotent stratification:*

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_m,$$

*and $[V_1, V_j] = V_{j+1}$ for $j = 1, \dots, m-1$, whereas $[V_1, V_m] = \{0\}$. Space $V_1$ is called the horizontal subspace and $X_1, X_2, \dots, X_n$ its orthonormal basis.*

*The metric of CarnotCarathodory is defined as follows*

$$d(x, y) = \inf_{\gamma} \int_0^1 |\dot{\gamma}(t)|\, dt, \quad \gamma \text{ abs. c. and } \dot{\gamma} \in V_1(\gamma(t)) \text{ for a. e. } t.$$

*The Hausdorff dimension* $\nu = \sum_1^m j \dim V_j$.

**Definition 2.** *A locally summable function $f : D \to \mathbb{R}$ ($D$ is a domain in $\mathbb{G}$) belongs to* ***weighted Sobolev space*** *$L_p^1(D, w)$ if it has weak derivatives $X_1 f$, $X_2 f, \dots, X_n f$ in the Sobolev sense and a finite semi-norm*

$$\|f \mid L_p^1(D, w)\| = \Big( \int_D |\nabla_{\mathcal{L}} f|^p(x) w(x)\, dx \Big)^{\frac{1}{p}},$$

*where*

$$\nabla_{\mathcal{L}} f = (X_1 f, X_2 f, \dots, X_n f)$$

*is a generalized subgradient of $f$.*

**Definition 3.** *A mapping $\varphi : D \to D'$ is said to be partially absolutely continuous on lines ($\varphi \in ACL_{\text{part}}(D)$), if for almost all integral curves $\gamma$ of horizontal vector field $X_j$ ($j = 1, \dots, n$) there is an open subset $\omega_\gamma \subset \gamma$ of full measure such that for any segment $[\alpha, \beta] \subset \omega_\gamma$ the mapping $\varphi$ is absolutely continuous on $[\alpha, \beta]$,*

$$\lim_{\substack{x \to a \\ x \in \omega_\gamma}} \operatorname{dist}(\varphi(x), \partial D') = 0 \quad \text{or} \quad \lim_{\substack{x \to a \\ x \in \omega_\gamma}} \rho(\varphi(x)) = \infty,$$

*for all $a \in \gamma \setminus \omega_\gamma$.*

**Definition 4.** *A mapping $\varphi$ has a finite $(u, v)$-weighted distortion if its horizontal differential $D_h\varphi(x) = 0$ almost everywhere on the set $Z_v = \{x \in D | J(x, \varphi) v(\varphi(x)) = 0\}$.*

**Definition 5.** *A* weighted distortion function *for mapping $\varphi$ is defined as follows*

$$D' \ni y \mapsto H_q^{u,v}(y) = \begin{cases} v^{-\frac{1}{p}}(y) \left( \sum\limits_{x \in \varphi^{-1}(y) \setminus (\Sigma_\varphi \cup Z_v)} \frac{|D\varphi|^q(x) u(x)}{|J(x,\varphi)|} \right)^{\frac{1}{q}}, \\ 0, \quad \text{if } \varphi^{-1}(y) \setminus (\Sigma_\varphi \cup Z_v) = \emptyset. \end{cases}$$

Change of Variables

$$\int_A f(x) |J(x, \varphi)|\, dx = \int_{\mathbb{G}} \Bigg( \sum_{x \in \varphi^{-1}(y) \cap (A \setminus \Sigma_\varphi)} f(x) \Bigg)\, dy.$$

## Sufficiency

**Theorem 1.** *Let $1 \leq q \leq p < \infty$. If a mapping $\varphi : D \to D'$ has the following properties:*
*1) is of the class $ACL_{\text{part}}(D)$,*
*2) has a finite $(u, v)$-weighted distortion,*
*3) the distortion function $H_p^{u,v}(\cdot) \in L_\varkappa(D')$, where $\frac{1}{\varkappa} = \frac{1}{q} - \frac{1}{p}$.*
*Then the mapping $\varphi$ induces a bounded composition operator on weighted Sobolev spaces*

$$\varphi^* : L_p^1(D', v) \cap C_0^\infty(D') \to L_q^1(D, u).$$

*Moreover* $\|\varphi^*\| \leq \|H_p^{u,v}(\cdot) \mid L_\kappa(D')\|$.

*Proof.* 1) If $f \in L_p^1(D', v) \cap C_0^\infty(D')$ then $f \circ \varphi \in ACL(D)$.
2)

$$\|\varphi^* f \mid L_q^1(D, u)\| \leq \Big( \int_{D \setminus Z_v} \Big( |\nabla_{\mathcal{L}} f|(\varphi(x)) |D_h\varphi|(x) \Big)^q u(x)\, dx \Big)^{\frac{1}{q}}$$

$$\leq \Big( \int_{D'} |\nabla_{\mathcal{L}} f|^q(y) \Big( \sum_{x \in \varphi^{-1} \setminus (\Sigma_\varphi \cup Z_v)} \frac{|D_h\varphi|^q(x) u(x)}{|J(x, \varphi)|} \Big)\, dy \Big)^{\frac{1}{q}}$$

$$= \Big( \int_{D'} |\nabla_{\mathcal{L}} f|^q(y) v^{\frac{q}{p}}(y) \Big( v^{-\frac{q}{p}}(y) \sum_{x \in \varphi^{-1} \setminus (\Sigma_\varphi \cup Z_v)} \frac{|D_h\varphi|^q(x) u(x)}{|J(x, \varphi)|} \Big)\, dy \Big)^{\frac{1}{q}}$$

3) Hlder's inequality:

$$\leq \Big( \int_{D'} |\nabla_{\mathcal{L}} f|^p(y) v(y)\, dy \Big)^{\frac{1}{p}} \Bigg( \int_{D'} \Big( v^{-\frac{1}{p}}(y) \Big( \sum_{x \in \varphi^{-1} \setminus (\Sigma_\varphi \cup Z_v)} \frac{|D_h\varphi|^q(x) u(x)}{|J(x,\varphi)|} \Big)^{\frac{1}{q}} \Big)^{\varkappa} dy \Bigg)^{\frac{1}{\varkappa}} \qquad \square$$

## Necessity

Obtaining necessary conditions is rather hard problem. Starting from almost nothing one should get features of continuity and differentiability for the mapping $\varphi$. The necessary conditions are formulated in terms of integrability of the Banach indicatrix $N(y, \varphi) = \#\{x \in D \mid \varphi(x) = y\}$ (it is the number of pre-images of $y \in D'$.)

**Theorem 2.** *Let $N(y, \varphi) v(y) \in L_{1,\text{loc}}(D')$, $u^{\frac{1}{1-p}}(x) \in L_{1,\text{loc}}(D)$, $1 \leq p < \infty$. If measurable mapping $\varphi : D \to D'$ induces a bounded composition operator on weighted Sobolev spaces $\varphi^* : L_p^1(D', v) \cap C_0^\infty(D') \to L_p^1(D, u)$. Then the mapping $\varphi$ has the following properties:*
*1) is of the class $ACL_{\text{part}}(D)$,*
*2) has a finite $(u, v)$-weighted distortion,*
*3) the distortion function $H_p^{u,v}(\cdot) \in L_\infty(D')$.*
*Moreover $\|H_p^{u,v}(\cdot) \mid L_\infty(D')\| \leq C\|\varphi^*\|$.*

Sketch of proof.

- Luzin $\mathcal{N}$-property
- analogous to the **change of variables formula** (inequality)
- **continuity** on almost all integral curves of horizontal vector fields
- existence of **approximate partial derivatives**
- partially absolutely continuity on almost all horizontal lines ($ACL_{\text{part}}$)

## Equivalence

In the case $q = p = \nu$ and $v \circ \varphi \leq u$ necessary conditions are also sufficient:

**Theorem 3.** *Let $v(y) \in L_{1,\text{loc}}(D')$, $u^{\frac{1}{1-p}}(x) \in L_{1,\text{loc}}(D)$ and $v \circ \varphi \leq u$ a. e. on $D$. A measurable mapping $\varphi : D \to D'$ induces a bounded composition operator on weighted Sobolev spaces $\varphi^* : L_p^1(D', v) \cap C_0^\infty(D') \to L_p^1(D, u)$ if and only if the mapping $\varphi$ has the following properties:*
*1) is of the class $ACL_{\text{part}}(D)$,*
*2) has a finite $(u, v)$-weighted distortion,*
*3) the distortion function $H_p^{u,v}(\cdot) \in L_\infty(D')$.*
*Moreover*

$$C\|H_p^{u,v}(\cdot) \mid L_\infty(D')\| \leq \|\varphi^*\| \leq \|H_p^{u,v}(\cdot) \mid L_\infty(D')\|.$$

## Consequences

**Corollary 1.** *Let $v(y) \in L_{1,\text{loc}}(D')$ $u^{\frac{1}{1-q}}(x) \in L_{1,\text{loc}}(D)$. Homeomorphism $\varphi : D \to D'$ induces bounded composition operator $\varphi^* : L_p^1(D', v) \cap C_0^\infty(D') \to L_q^1(D, u)$ if and only if $\varphi \in ACL(D)$ and distortion function $K_p^{u,v}(x) = \inf\{K(x) : |D_h\varphi|(x) u^{\frac{1}{q}}(x) \leq K(x)|J(x, \varphi)|^{\frac{1}{p}} v^{\frac{1}{p}}(\varphi(x))\}$ is in $L_\varkappa(D)$. The norm $\|\varphi^*\|$ is equivalent $\|K_p^{u,v}(x) \mid L_\varkappa(D)\|$.*

**Corollary 2.** *Let $u, v$ be weights of Muckenhoupt $A_\nu$ class, $\varphi : D \to D'$ — homeomorphism and inequality $v \circ \varphi(x) \leq u(x)$ holds for a. e. $x \in D$. Then Mapping $\varphi$ is quasiconformal if and only if composition operator $\varphi^* : L_\nu^1(D', v) \cap C_0^\infty(D') \to L_\nu^1(D, u)$ is bounded.*

Function $w$ is *a weight in Muckenhoupt's $A_p$-class* ($w \in A_p$), if $w$ is locally integrable function in $\mathbb{G}$ such that

$$\sup_{B \subset \mathbb{G}} \Big( \frac{1}{|B|} \int_B w\, dx \Big) \Big( \frac{1}{|B|} \int_B w^{\frac{1}{1-p}}\, dx \Big)^{p-1} = c_{w,p} < \infty,$$

were the supremum is taken over all balls $B$ in $\mathbb{G}$. In particular if $w \in A_p$ then functions from $L_p^1(D, w)$ can be approximated by smooth functions from $C^\infty(D) \cap L_p^1(D, w)$.

## References

[1] N. A. Evseev. Composition operators on weighted sobolev spaces on a carnot group. *Siberian Mathematical Journal*, 56(6), 2015.

[2] S. K. Ukhlov, A. Vodopyanov. Mappings associated with weighted Sobolev spaces. In *Complex analysis and dynamical systems III*, volume 455 of *Contemp. Math.*, pages 369–382. Amer. Math. Soc., Providence, RI, 2008.