

Sobolev classes of functions valued in a monotone family of Banach spaces

Nikita Evseev

joint with A. Menovschikov (Novosibirsk)

Institute for Advanced Study in Mathematics. HIT, Harbin, China.

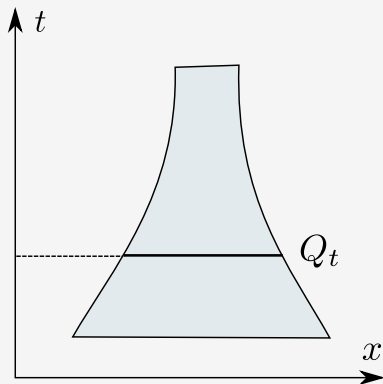
Sobolev Institute of Mathematics. Novosibirsk, Russia

Nanjing Normal University

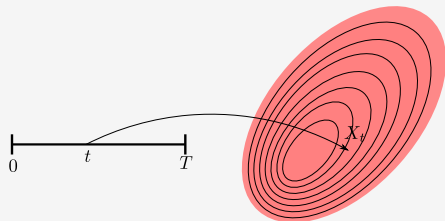
November 6, 2019

Problems in non-cylindrical domains

$$\begin{cases} \dot{u}(t, x) + A(t)u(t, x) = f(t, x) & \text{in } Q_t \\ u(0, x) = u_0(x) & \text{in } Q_0. \end{cases}$$




- $I = [0, T] \subset \mathbb{R}$ is an interval
- V – Vector space
- semi-norms $\{\|\cdot\|_t\}_{t \in [0, T]}$
- X_t is a completion $V / \ker(\|\cdot\|_t)$
- $\|v\|_{t_1} \geq \|v\|_{t_2}$ whenever $t_1 \leq t_2$.



$$f : [0, T] \rightarrow \bigcup_{t \in [0, T]} X_t, \quad f(t) \in X_t$$

L_p -direct integral of Banach spaces

$$f \in L^p(I, \{X_t\}) = \left(\int_I^\oplus X_t dt \right)_{L^p} \Leftrightarrow \left(\int_I \|f(t)\|_{X_t}^p dt \right)^{\frac{1}{p}} < \infty$$

 R. Haydon, M. Levy, Y. Raynaud, *Randomly normed spaces*, Hermann, Paris, 1991.

Examples

- $V = \text{Step}([0, T])$, $\|\cdot\|_t = \|\cdot\|_{L^p(t, 1-t)}$
- $V = \text{Step}([0, T])$, $\|\cdot\|_t = \|\cdot\|_{L^{1/t}(0,1)}$
- $V = C^\infty([0, 1])$, $\|\cdot\|_t = \|\cdot\|_{W^{1,p}(t, 1-t)}$

How to define the Sobolev space $W^{1,p}$?

Say $u \in W^{1,p}(I, \{X_t\})$ if $u \in L^p(I, \{X_t\})$ and there exists a function $g \in L^p(I, \mathbb{R})$ such that

$$\|u(t) - u(t_0)\|_t \leq \int_{t_0}^t g(s) ds.$$

(g is a p -integrable upper gradient of u)

$$\|u\|_{1,p} := \|u\|_{L^p(T, \{X_t\})} + \inf_g \|g\|_{L^p(T, \mathbb{R})},$$

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What can we say about function g ?

Is it $|u'(t)|$?

Calculus of $\{X_t\}$ -valued functions

- There are 'embedding' operators $P(t_1, t_2) : X_{t_1} \rightarrow X_{t_2}$.
- If $x_1 \in X_{t_1}$ and $x_2 \in X_{t_2}$ define

$$x_1 + x_2 := P(t_1, t_2)x_1 + x_2$$

- $P(t_1, t_2)P(t_2, t_3) = P(t_1, t_3)$

$$\left(\bigcup_{t \in I} X_t, +\right) \quad \text{– vector space (but not normed)}$$

$$\lim_{t \rightarrow t_0} f(t) = \xi \in X_{t_0} \quad \text{if} \quad \|f(t) - \xi\|_{t_0 \vee t} \rightarrow 0.$$

Calculus of $\{X_t\}$ -valued functions. Integral

- **locally integrable section:** for every compact there exist a sequence of simple sections s_k .

$$\int_I \|\chi_J(t) \cdot f(t) - s_k(t)\|_t dt \rightarrow 0 \quad \text{for } k \rightarrow \infty.$$

- **local Bochner integral:** $\int_J f dt := \lim_{k \rightarrow \infty} \int_I s_k(t) dt = x \in X_{t^*}$, where $t^* = \sup J$.
- f is a locally integrable section $\Leftrightarrow \|f(t)\|_t \in L^1_{loc}(I)$

Weak derivative

Let $f \in L^1_{loc}(I, \{X_t\})$. A function $h \in L^1_{loc}(I, \{X_t\})$ is a weak derivative of f if for any $\varphi \in C_0^\infty(I, \mathbb{R})$

$$\left\| \int_I \varphi'(t)f(t) dt + \int_I \varphi(t)h(t) dt \right\|_{t^*} = 0,$$

where $t^* = \sup\{\text{supp } \varphi(t)\}$.

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Theorem

Let $\{X_t\}$ be a monotone family of reflexive Banach spaces.

If $u \in W^{1,p}(T, \{X_t\}) \Rightarrow \exists v(t) \in L^p(T, \{X_t\})$

s. t. $\|v(t)\|_t$ — upper gradient of u and $\|v\|_{L^p(T, \mathfrak{X})} = \inf_g \|g\|_{L^p(T, \mathbb{R})}$.

Answering our question: $g(t) = \|u'(t)\|_t$

Theorem (the approach of Yu. G. Reshetnyak)

If $u : I \rightarrow \{X_t\}$ is a measurable section and :

(A) for any $v \in V$ the function $\psi_v(t) = \|u(t) - v\|_t \in W^{1,p}(I, \mathbb{R})$,

(B) the family of derivatives $\{\psi'_v(t)\}_{v \in V}$ has a majorant $\psi' \in L^p(I, \mathbb{R})$,
then $u \in W^{1,p}(I, \{X_t\})$.

The converse does not hold in general.

Example

$$V = C(0, 1), I = [0, 1].$$

semi-norms

$$\|x\|_t = \begin{cases} \sup_{(0,1)} |x(s)|, & 0 \leq t < 0.5 \\ \sup_{(0,1/2)} |x(s)|, & 0.5 \leq t \leq 1, \end{cases}$$

$$\mathfrak{D} = \{(C(0, 1), \|\cdot\|_t)\}_{t \in \mathcal{T}}.$$

function $u(t)(s) = s$ belongs to $W^{1,p}([0, 1], \mathfrak{D})$, $u'(t) = 0$. On the other hand

$$\|u(t)\|_t = \begin{cases} 1, & 0 \leq t < 0.5 \\ 0.5, & 0.5 \leq t \leq 1, \end{cases}$$

is obviously out of space $W^{1,p}(0, 1)$.

Thank you for attention!