Vector-valued Sobolev spaces based on Banach function spaces

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Prague online January 21, 2021

Introduction

Let $\Omega \subset \mathbb{R}^n$, V be a Banach space, and $X(\Omega)$ be a Banach function space.

$$\boxed{W^1X(\Omega; V)}$$

from metric analysis:

R^1X, N^1X, ...

Köthe-Bochner function space

- **Banach function space** $X(\Omega)$ is a set of functions $u : \Omega \to \mathbb{R}$ s.t.
  1) $|f| \leq g$ and $g \in X(\Omega) \Rightarrow f \in X(\Omega)$ and $\|f\|_{X(\Omega)} \leq \|g\|_{X(\Omega)}$
  2) $0 \leq f_n \nearrow f$ a.e., $\Rightarrow \|f_n\|_{X(\Omega)} \nearrow \|f\|_{X(\Omega)}$
  3) $\chi_A \in X(\Omega)$, $|A| < \infty$
  4) $\|f\|_{L^1(A)} \leq C_A \|f \cdot \chi_A\|_{X(\Omega)}$, $f \in X(\Omega)$, $|A| < \infty$

  Example: $L^p(\Omega)$
- V is a Banach space. We define $X(\Omega; V)$ as a set of measurable (in the Bochner sense) $u : \Omega \to V$ s.t. $\|u(\cdot)\|_V \in X(\Omega)$

L. Pick, A. Kufner, O. John, S. Fučík, Function spaces. Volume 1. 2nd revised and extended ed., 2nd Edition, Vol. 14, Berlin: de Gruyter, 2013.

P.-K. Lin, Köthe-Bochner function spaces., Boston, MA: Birkhäuser, 2004.

the Radon-Nikodým property (RNP)

V has the Radon-Nikodým property if the Radon-Nikodým theorem holds for vector measures. However, for our purposes we make use of equivalent descriptions for this property:

Example: L^p, $1<p<\infty$

Theorem

For any Banach space V, the following assertions are equivalent:

- *V has the Radon-Nikodým property;*
- *every locally Lipschitz continuous function $f : \mathbb{R} \to V$ is differentiable almost everywhere.*
- *every locally absolutely continuous function $f : \mathbb{R} \to V$ is differentiable almost everywhere;*

T. Hytönen, J. van Neerven, M. Veraar, L. Weis, Analysis in Banach spaces. Volume I. Martingales and Littlewood-Paley theory., Vol. 63, Cham: Springer, 2016.

SOBOLEV SPACE

$u \in L^1_{loc}(\Omega; V)$. A function $v \in L^1_{loc}(\Omega; V)$ is said to be a weak partial derivative of u if $\int_\Omega \frac{\partial \varphi}{\partial x_j}(x)u(x)\,dx = -\int_\Omega \varphi(x)v(x)\,dx$ for all $\varphi \in C_0^\infty(\Omega)$.

$$W^1X(\Omega; V) = \{u \in X(\Omega; V) \mid \nabla u \in X(\Omega; V)\}$$

THE MEYERS-SERRIN THEOREM (W. FARKAŞ, 1995)

If the norm $\|\cdot\|_X$ is **absolutely continuous** and has **the translation inequality property** then $C^\infty(\Omega; V) \cap W^1X(\Omega; V)$ is dense in $W^1X(\Omega; V)$.

abs. cont: $\|f \cdot \chi_{A_n}\|_X \to 0$ when $A_n \to \emptyset$

t.i.p.: $\tau_h u = u(\cdot + h)$, $\|\tau_h u\|_X \leq \|u\|_X$ • rearrangement invariant $\Rightarrow$ t.i.p.

Reshetnyak-Sobolev space

$R^1X(\Omega; V)$ is a class of functions $u \in X(\Omega; V)$ s.t.
(A) for every $v^* \in V^*$, $\|v^*\| \leq 1$, we have $\langle v^*, u(\cdot)\rangle \in W^1X(\Omega)$;
(B) there is a non-negative $g \in X(\Omega)$ s.t.

$$|\nabla\langle v^*, u\rangle| \leq g \quad \text{a.e. on } \Omega$$

for every $v^* \in V^*$ with $\|v^*\| \leq 1$.

Yu. Reshetnyak (1997) provided this definition for metric settings.

Question

$$R^1X(\Omega; V) = W^1X(\Omega; V) \text{ ?}$$

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Theorem (P. Hajłasz and J. Tyson, 2008)

If $\Omega \subset \mathbb{R}^n$ is open, $V = Y^*$ is dual to a separable Banach space, then $R^{1,p}(\Omega; V) = W^{1,p}(\Omega; V)$ and $\|f\|_{R^{1,p}} \leq \|f\|_{W^{1,p}} \leq \sqrt{n}\|f\|_{R^{1,p}}$.

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separable dual $\Rightarrow$ RNP

Theorem

1) If $u \in W^1X(\Omega; V)$, then $u \in R^1X(\Omega; V)$.
2) If V has the RNP and $u \in R^1X(\Omega; V)$, then $u \in W^1X(\Omega; V)$.

Proof.

1) straightforward
2) • $u \in R^1X \Rightarrow \forall j=1..n \ \exists \tilde{u}$ absolutel continuous on lines $\parallel x_j$

• RNP $\Rightarrow \exists$ derivatives $\Rightarrow u \in W^1X$.

□

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Theorem (I. Caamaño, J. A. Jaramillo, Á. Prieto, and A. Ruiz, 2020)

$R^{1,p}(\Omega; V) = W^{1,p}(\Omega; V)$ *if, and only if, the space V has the RNP.*

NEWTONIAN SPACE

$N^1X(\Omega; V)$ consists of all functions $u \in X(\Omega; V)$ for which there is a non-negative Borel function $\rho \in X(\Omega)$ such that

$$\|u(\gamma(0)) - u(\gamma(l_\gamma))\|_V \leq \int_\gamma \rho\, ds$$

for ${}_X$-a.e. curve γ in Ω.

$N^{1,p}(\Omega; M)$ - J. Heinonen, P. Koskela, N. Shanmugalingam, J. T. Tyson (2017)
$N^1X(\Omega; M)$ - L. Malý (2013, 2016)

THEOREM (SCALAR CASE V=R)

1)If $u \in N^1X(\Omega)$, then $u \in W^1X(\Omega)$.
2) Suppose norm $\|\cdot\|_X$ is absolutely continuous and has the translation inequality property. If $u \in W^1X(\Omega)$, then there is a representative $\tilde{u} \in N^1X(\Omega)$.

Theorem (vector case)

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RNP

$$N^1X \subset R^1X \subset W^1X$$

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$\|\cdot\|_X$ a.c. & t.i.p.

Description via difference quotients

Theorem

1) Assume $u \in W^{1,p}(\Omega)$, $1 \le p < \infty$. *Then for every* $\omega \Subset \Omega$, *we have*

$$\|u(\cdot + h) - u(\cdot)\|_{L^p(\omega)} \le \|\nabla u\|_{L^p(\Omega)} |h|$$

for all $0 < |h| < dist(\omega, \partial\Omega)$.
2) If $u \in L^p(\Omega)$, $1 < p < \infty$, *and there is a constant* c *such that*

$$\|u(\cdot + h) - u(\cdot)\|_{L^p(\omega)} \le c|h|$$

whenever $0 < |h| < dist(\omega, \partial\Omega)$, *then* $u \in W^{1,p}(\Omega)$ *and* $\|\nabla u\|_{L^p(\Omega)} \le c$.

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whenever $0 < |h| < dist(\omega, \partial\Omega)$, then $u \in W^{1,p}(\Omega)$ and $\|\nabla u\|_{L^p(\Omega)} \le c$.

Theorem (W. Arendt, M. Kreuter, 2018)

Let $1 < p \le \infty$. A Banach space V has the RNP if and only if the Difference Quotient Criterion characterizes the space $W^{1,p}(\Omega; V)$.

Description via difference quotients

Theorem (scalar case V=R)

Let $X(\Omega)$ have the RNP. If $u \in X(\Omega)$ and there is a constant $C \in [0, \infty)$ such that

$$\|\tau_{te_j} u - \tau_{se_j} u\|_{X(\omega)} \leq C|t - s|, \quad j \in 1, \dots, n \tag{1}$$

then $u \in W^1X(\Omega)$ and $\|\nabla u\|_{X(\Omega)} \leq nC$.

Description via difference quotients

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then $u \in W^1X(\Omega)$ and $\|\nabla u\|_{X(\Omega)} \leq nC$.

Theorem (vector case)

Let $X(\Omega)$ have the RNP. If $u \in X(\Omega; V)$ and (1) holds, then $u \in R^1X(\Omega; V)$.

Theorem

1) Suppose norm $\|\cdot\|_X$ is absolutely continuous and has the translation inequality property. If $u \in W^1X(\Omega; V)$, then

$$\|\tau_{te_j} u - \tau_{se_j} u\|_{X(\omega;V)} \leq \|\partial_j u\|_{X(\Omega;V)}|t-s|, \quad j \in 1, \dots, n \tag{2}$$

2) Suppose X and V have the RNP. If $u \in X(\Omega; V)$ and there is a constant $C \in [0, \infty)$ such that

$$\|\tau_{te_j} u - \tau_{se_j} u\|_{X(\omega;V)} \leq C|t-s|, \quad j \in 1, \dots, n \tag{3}$$

then $u \in W^1X(\Omega; V)$ and $\|\nabla u\|_{X(\Omega)} \leq nC$.

Theorem

1) Suppose norm $\|\cdot\|_X$ is absolutely continuous and has the translation inequality property. If $u \in W^1X(\Omega; V)$, then

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2) Suppose X and V have the RNP. If $u \in X(\Omega; V)$ and there is a constant $C \in [0,\infty)$ such that

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then $u \in W^1X(\Omega; V)$ and $\|\nabla u\|_{X(\Omega)} \leq nC$.

Open problem

If the difference quotient criterion (1) characterizes the space $W^1X(\Omega)$, then the Banach function space $X(\Omega)$ has the Radon-Nikodým property.

∘ holds for L^p

A maximal function characterization

Theorem

Let $X(\Omega)$ have the RNP, and norm $\|\cdot\|_{X(\Omega)}$ have the translation inequality property. If $u \in X(\Omega; V)$ and there is a non-negative function $h \in X(\Omega)$ such that

$$\|u(x) - u(y)\|_V \leq |x - y|(h(x) + h(y)), \qquad \text{a.e. on } \Omega, \tag{4}$$

then $u \in R^1X(\Omega; V)$ and $\|g\|_{X(\Omega)} \leq 2n\|h\|_{X(\Omega)}$, where g is some Reshetnyak upper gradient of u.

In the assumptions of the above theorem suppose that V has the RNP. Then it follows that $u \in W^1X(\Omega; V)$ and $\|\nabla u\|_{X(\Omega)} \leq 2n\|h\|_{X(\Omega)}$.

Theorem

Let $\Omega \subset \mathbb{R}^n$, V be a Banach space, $X(\Omega)$ be a Banach function space such that the Hardy-Littlewood maximal operator M is bounded in $X(\Omega)$. If $u \in W^1X(\Omega; V)$, then

$$\|u(x) - u(y)\|_V \leq C|x-y|\big(M(|\nabla u|)(x) + M(|\nabla u|)(y)\big)$$

holds for some constant C and almost all $x, y \in \Omega$ with $B(x, 3|x-y|) \subset \Omega$.

P. Jain, A. Molchanova, M. Singh, and S. Vodopyanov (2020) obtained the above result for the real-valued case.

Mapping theorems

$$f \circ u \in N^1X(\Omega; Z) \text{ whenever } u \in N^1X(\Omega; V)$$

Let $f : V \to Z$ be Lipschitz continuous and $f(0) = 0$ if $|\Omega| = \infty$.

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Let $f : V \to Z$ be Lipschitz continuous and $f(0) = 0$ if $|\Omega| = \infty$.

Theorem

Z has the RNP $\Rightarrow f \circ u \in W^1X(\Omega; Z)$ for any $u \in W^1X(\Omega; V)$.

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Theorem

If for any Lipschitz function $f : V \to Z$ $f \circ u \in W^1X(\Omega; Z)$ whenever $u \in W^1X(\Omega; V)$, then Z has the RNP.

Embedding

Theorem

$$W^1X(\Omega) \hookrightarrow Y(\Omega) \Rightarrow W^1X(\Omega; V) \hookrightarrow Y(\Omega; V)$$

Proof: $\|u\|_V \in W^1X(\Omega) \Rightarrow \|u\|_V \in Y(\Omega) \Rightarrow u \in Y(\Omega)$ ($\|\cdot\|_V : V \to \mathbb{R}$ Lipschitz)

$\|u\|_{Y(\Omega;V)} = \| \|u\|_V \|_{Y(\Omega)} \le C \| \|u\|_V \|_{W^1X(\Omega)} \le C\sqrt{n} \|u\|_{W^1X(\Omega;V)}$.

Conclusions

- Properties $W^1X(\Omega;V)$
- Characterization without derivatives

$$W^1X(\Omega;\{V_\omega\}), \quad u(\omega)\in V_\omega$$

(example: evalution PDEs)


Thank you!