Decomposable operators on different L_p-direct integrals

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Plan for today

1. Mixed norm Lebesgue spaces
2. Operators on L_p-direct integrals
3. Decomposable operators

① Mixed norm Lebesgue spaces

Definition

If $\Omega = S \times X$, then $\|f\|_{L^{p,q}(\Omega)} = \Big\| \|f(s,x)\|_{L^q(X)} \Big\|_{L^p(S)}$.

Let $\Omega \subset S \times X$, space $L^{p,q}(\Omega)$ consists of functions $f : \Omega \to \mathbb{R}$ s. t.

$$\|f\|_{L^{p,q}(\Omega)} = \left(\int_S \|f(s,\cdot)\|^p_{L_q(\Omega_s)} \, d\mu(s) \right)^{\frac{1}{p}} < \infty,$$

where $\Omega_s = \{x \in X \mid (s,x) \in \Omega\}$.

- non rearrangement invariant spaces
- priority of variables

Motivation. Why mixed norm?

- **Sobolev embeddings.** Let $f \in L^1(\mathbb{R}^n)$ and $\frac{\partial f}{\partial x_k} \in L^1(\mathbb{R}^n)$ then $f \in L^{1,\infty}(\mathbb{R}^{n-1} \times \mathbb{R})$ [Kolyada 2006].
- **PDE.** $W^{1,2}_{p,q}([a,b]) = \{u(t,x) \mid u, u_t, u_x, u_x x \in L^{p,q}([a,b] \times \mathbb{R}^n)\}$ [Krylov 2007]
- **Integral operators.** If a kernel $k(x,y) \in L^{q,p'}(\omega \times \Omega)$ then the operator $Kf = \int_\Omega k(x,y) f(y)\, dy$, $K : L^p(\Omega) \to L^q(\Omega)$ is bounded and $\|K\| \leq \|k(x,y)\|_{q,p'}$.

The Problem. Composition operator

$$\varphi : \Omega \to \widetilde{\Omega}$$

$$C_\varphi : L_{\tilde{p},\tilde{q}}(\widetilde{\Omega}) \to L_{p,q}(\Omega)$$

$$\varphi^* f = f \circ \varphi \quad \textit{composition operator}$$

Composition operators on mixed norm Lebesgue spaces

Theorem

A mapping $\varphi : \Omega \to \widetilde{\Omega}$, $\varphi(s,x) = (\psi(s), u(s,x))$ induces a bounded composition operator $C_\varphi : L_{\tilde{p},\tilde{q}}(\widetilde{\Omega}) \to L_{p,q}(\Omega)$ by the rule $C_\varphi f(s,x) = f(\psi(s), u(s,x))$ iff

$$J_{\psi^{-1}}^{\frac{1}{p}}(t) J_{u^{-1}}^{\frac{1}{q}}(t,y) \in L_{\frac{\tilde{p}p}{\tilde{p}-p},\frac{\tilde{q}q}{\tilde{q}-q}}(\widetilde{\Omega}).$$

A mapping $\varphi : \Omega \to \Omega$ induces a bounded composition operator $\varphi^* : L_p(\Omega') \to L_q(\Omega)$ iff

$$J_{\varphi^{-1}}(y) \in L_{\frac{pq}{p-q}}(\Omega').$$

Composition operators on mixed norm Lebesgue spaces

Theorem

A mapping $\varphi : \Omega \to \widetilde{\Omega}$, $\varphi(s,x) = (\psi(s), u(s,x))$ *induces a bounded composition operator* $C_\varphi : L_{\tilde{p},\tilde{q}}(\widetilde{\Omega}) \to L_{p,q}(\Omega)$ *by the rule* $C_\varphi f(s,x) = f(\psi(s), u(s,x))$ *iff*

$$J_{\psi^{-1}}^{\frac{1}{p}}(t) J_{u^{-1}}^{\frac{1}{q}}(t,y) \in L_{\frac{\tilde{p}p}{\tilde{p}-p},\frac{\tilde{q}q}{\tilde{q}-q}}(\widetilde{\Omega}).$$

A mapping $\varphi : \Omega \to \Omega$ induces a bounded composition operator $\varphi^* : L_p(\Omega') \to L_q(\Omega)$ iff

$$J_{\varphi^{-1}}(y) \in L_{\frac{pq}{p-q}}(\Omega').$$

However, do not have such a result for $\varphi(s,x) = (\psi(s,x), u(s,x))$

Our approach

$$L^{p,q}(\Omega) = L^p(S, L_q(\Omega_s))$$

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L_p-direct integral of Banach spaces

$$L^p(T, W_t) = \left(\int_T^{\oplus} W_t \, d\mu(t) \right)_{L_p}$$

$f : T \to \bigcup_{t\in T} W_t$ such that $t \mapsto f(t) \in (W_t, \|\cdot\|_{W_t})$.

$$\|f\|_{L^p(T,W_t)} = \left(\int_T \|f(t)\|_{W_t}^p \, d\mu(t) \right)^{\frac{1}{p}}$$

R. Haydon, M. Levy, Y. Raynaud, *Randomly normed spaces*, Hermann, Paris, 1991.

②

Operators on L_p-direct integrals

Distributed microstructure model of reactive transport in a heterogeneous porous medium

$$L_p(\Omega, W_q^l(Y_x))$$

Sebastian Meier and Michael Böhm, *A note on the construction of function spaces for distributed miscrostructure models with spatially varying cell geometry*, Int. J. Numer. Anal. Model. **5** (2008), 109–125.

Abstract parabolic partial differential equations on evolving Hilbert spaces

$$W^{1,p}([0,T], X_\omega)$$

Amal Alphonse and Charles M. Elliott and Björn Stinner, *An abstract framework for parabolic PDEs on evolving spaces*, Port. Math. (N.S.) **71** (2008), no. 1, 1–46.

Mixed operator

$(T,\mu),(S,\nu),(F,\lambda)$ — metric measure space, $F \subset S \times T$.

$P(t,s) \in \mathcal{B}(W_t, V_s)$ — operator-valued function defined on F.
$\mathcal{W} = \{w_t\}_{t\in T}$, $\mathcal{V} = \{V_s\}_{s\in S}$ — measurable families of Banach spaces.

Mixed operator

$$M_F[f](s,t) = P(s,t)[f(t)]$$

$$M_F : L^p(T,\mathcal{W}) \to L^q(F,\mathcal{V})$$

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Q. When mixed operator M_F is bounded?

Necessary conditions

$$\|M_F f\|_{L_q(F,\mathcal{V})} \leq K\|f\|_{L_p(T,\mathcal{W})}$$

Lemma (Key technology)

Suppose that operator M_F is bounded then

$$\Phi(A) = \sup_{f\in L^p(A,\mathcal{W})} \left(\frac{\|M_F f\|_{L^q(F\cap S\times A,\mathcal{V})}}{\|f\|_{L^p(A,\mathcal{W})}} \right)^{\kappa}, \quad \kappa = \frac{pq}{p-q}$$

is a bounded countable additive function on all Borel sets $A \subset T$.

$$\|M_F f\|_{L_q(F\cap S\times B(t,r),\mathcal{V})} \leq \Phi^{\frac{1}{\kappa}}(B(t,r))\|f\|_{L_p(B(t,r),\mathcal{W})}$$

$$\|M_F f\|_{L_q(F\cap S\times B,\mathcal{V})} \leq \Phi^{\frac{1}{\kappa}}(B)\|f\|_{L_p(B,\mathcal{W})}, \quad B(\mathbf{t}, r) \subset T.$$

Take $g(t)$ s. t. $\|g(t)\|_{W_t} = 1 \quad \Rightarrow \quad \|g\|_{L_p(B,\mathcal{W})} = (\mu(B))^{\frac{1}{p}}$

$$\left(\int\limits_{F\cap S\times B} \|P(s,t)[g(t)]\|_{V_s}^q \, d\lambda\right)^{\frac{1}{q}} \leq \Phi^{\frac{1}{\kappa}}(B)(\mu(B))^{\frac{1}{p}}$$

$$\|M_F f\|_{L_q(F\cap S\times B,\mathcal{V})} \leq \Phi^{\frac{1}{\kappa}}(B)\|f\|_{L_p(B,\mathcal{W})}, \quad B(\mathbf{t}, r) \subset T.$$

Take $g(t)$ s. t. $\|g(t)\|_{W_t} = 1 \quad \Rightarrow \quad \|g\|_{L_p(B,\mathcal{W})} = (\mu(B))^{\frac{1}{p}}$

$$\left(\frac{1}{\mu(B)} \int\limits_{F\cap S\times B} \|P(s,t)[g(t)]\|_{V_s}^q \, d\lambda\right)^{\frac{1}{q}} \leq \frac{\Phi^{\frac{1}{\kappa}}(B)}{(\mu^{\frac{1}{\kappa}}(B))}$$

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Take $g(t)$ s. t. $\|g(t)\|_{W_t} = 1 \quad \Rightarrow \quad \|g\|_{L_p(B,\mathcal{W})} = (\mu(B))^{\frac{1}{p}}$

$$\left(\frac{1}{\mu(B)} \int\limits_{F\cap S\times B} \|P(s,t)[g(t)]\|^q_{V_s}\, d\lambda\right)^{\frac{1}{q}} \leq \frac{\Phi^{\frac{1}{\kappa}}(B)}{(\mu^{\frac{1}{\kappa}}(B))}$$

Suppose that $\lambda << \mu \times \nu$, then $\exists\ J(s,t) = \frac{d\lambda}{d\mu\times\nu}$

$$\left(\frac{1}{\mu(B)} \int\limits_{F\cap S\times B(t,r)} \|P(s,t)[g(t)]\|^q_{V_s} J(s,t)\, d\mu\times d\nu\right)^{\frac{1}{q}} \leq \left(\frac{\Phi(B)}{\mu(B)}\right)^{\frac{1}{\kappa}}$$

$$\left(\frac{1}{\mu(B)}\int\limits_{F\cap S\times B(t,r)}\|P(s,t)[g(t)]\|^q_{V_s}J(s,t)\,d\mu\times d\nu\right)^{\frac{1}{q}}\leq\left(\frac{\Phi(B)}{\mu(B)}\right)^{\frac{1}{\kappa}}$$

Let $\mu(B)\to 0$

$$\int\limits_{S_{\mathbf{t}}}\|P(s,\mathbf{t})[g(\mathbf{t})]\|^q_{V_s}J(s,\mathbf{t})\,d\nu\leq\Phi'(\mathbf{t})\quad \text{a. e. } \mathbf{t}\in T.$$

$$\left(\frac{1}{\mu(B)}\int\limits_{F\cap S\times B(t,r)} \|P(s,t)[g(t)]\|^q_{V_s} J(s,t)\,d\mu\times d\nu\right)^{\frac{1}{q}} \leq \left(\frac{\Phi(B)}{\mu(B)}\right)^{\frac{1}{\kappa}}$$

Let $\mu(B)\to 0$

$$\int\limits_{S_{\mathbf{t}}} \|P(s,\mathbf{t})[g(\mathbf{t})]\|^q_{V_s} J(s,\mathbf{t})\,d\nu \leq \Phi'(\mathbf{t}) \quad \text{a. e. } \mathbf{t}\in T.$$

Take supremum over all $g(\mathbf{t})$, and then integrate over T

$$\int\limits_T \left(\int\limits_{S_t} \|P(s,t)\|^q J(s,t)\,d\nu\right)^{\frac{\kappa}{q}} d\mu \leq \Phi(T)$$

$$\int\limits_T \Bigg(\int\limits_{S_t} \|P(s,t)\|^q J(s,t)\, d\nu \Bigg)^{\frac{\kappa}{q}} d\mu \leq \Phi(T)$$

Theorem

Let measure λ be absolutely continuous with respect to product $\mu \times \nu$. Then the operator M_F, $p \geq q$, is bounded if and only if

$$\|P(s,t)\| \cdot J^{\frac{1}{q}}(s,t) \in L^{\kappa,q}(F).$$

Define $\lambda_T(B) = \lambda(F \cap S \times B)$, $B \subset T$.

Theorem

Let measure λ_T be absolutely continuous with respect to μ and there is a function $\rho : T \to \mathbb{R}^+$, s. t. $\|P(s,t)\| = \rho(t)$. Then the operator M_F, $p \geq q$, is bounded if and only if

$$\|P(s,t)\| \cdot J^{\frac{1}{q}}(t) \in L_\kappa(T)$$

where $J(t) = \frac{d\lambda_T}{d\mu}$ — Radon-Nikodym derivative.

Let $F = \Gamma_\psi$, where $\psi : S \to T$ is a measurable mapping.

Theorem

Let measure λ_T be absolutely continuous with respect to μ and mapping ψ is injective. Then the operator $M_F : L^p(T, \mathcal{W}) \to L^q(\Gamma_\psi, \mathcal{V})$, $p \geq q$, is bounded if and only if

$$\|P(\psi^{-1}(t), t)\| \cdot J^{\frac{1}{q}}(t) \in L_\kappa(T)$$

where $J(t) = \frac{d\lambda_T}{d\mu}$ — Radon-Nikodym derivative.

Now we consider the operator $M_\psi : L^p(T, \mathcal{W}) \to L^q(S, \mathcal{V})$, which acts be the rule $M_\psi[f](s) = P(s, \psi(s))f(\psi(s))$.

Theorem

Let measure $\nu \circ \psi^{-1}$ be absolutely continuous with respect to product μ and mapping ψ is injective. Then the operator M_ψ, $p \geq q$, is bounded if and only if

$$\|P(\psi^{-1}(t), t)\| \cdot J_{\psi^{-1}}^{\frac{1}{q}}(t) \in L_\kappa(T)$$

where $J_{\psi^{-1}}(t)$ — volume derivative.

Theorem

Let measure $\nu \circ \psi^{-1}$ be absolutely continuous with respect to μ and $c \leq \|P(s,t)\| \leq C$. Then the operator M_ψ, $p \geq q$, is bounded if and only if

$$J_{\psi^{-1}}^{\frac{1}{q}}(t) \in L_\kappa(T)$$

where $J_{\psi^{-1}}(t)$ — volume derivative.

③

Decomposable operators

Decomposable operator

A decomposable operator

$$P : L^p(T, \mathcal{W}) \to L^p(T, \mathcal{W})$$

is a map $t \mapsto P_t \in \mathcal{B}(W_t)$ and it acts by the rule

$$[Pf](t) = P_t[f(t)] \in L^p(T, \mathcal{W})$$

for all $f \in L^p(T, \mathcal{W})$.

Decomposable operator

$$P : L^p(T, \mathcal{W}) \to L^p(T, \mathcal{W}) \text{ bounded} \Leftrightarrow \operatorname*{ess\,sup}_{t\in T} \|P_t\| < \infty \; (\|P_t\| \in L^\infty)$$

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Decomposable operators on different L_p-direct integrals

$$M_\psi : L^p(T, \mathcal{W}) \to L^q(S, \mathcal{V}) \text{ bounded}$$

$$\Updownarrow$$

$$\|P(\psi^{-1}(t), t)\| \cdot J_{\psi^{-1}}^{\frac{1}{q}}(t) \in L_\kappa(T)$$

Thank you for attention!