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MEASURABILITY OF THE BANACH INDICATRIX

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Abstract. We establish the measurability of the Banach indicatrix for a measurable mapping in a geometrically doubling metric space. This is a generalization of a known result for continuous transformations in Euclidean space. A system of dyadic cubes in metric space is employed to construct a sequence of measurable functions converging to the indicatrix, and we partly follow Banach's original proof.

1. Introduction. Given metric measure spaces X, Y, let $f : X \to Y$ be a measurable mapping and $A \subset X$. The *Banach indicatrix* (multiplicity function) is defined as $N(y, f, A) = \#\{x \in A \mid f(x) = y\}$, the number of elements of $f^{-1}(y)$ in A (possibly ∞). In case $A = X$ write $N(y, f) = N(y, f, X)$. The question we consider is as follows: *is the function $N(y, f, A)$ measurable?*

Let us briefly discuss some results and examples. The measurability of the multiplicity function for a continuous function $f : [a, b] \to \mathbb{R}$ was proved by S. Banach in [B, Théorème 1.1], whereas [B, Théorème 1.2] states that $\int_a^b N(y, f)\, dy$ is equal to the total variation $\mathrm{TV}(f, [a, b])$. Together Théorèmes 1.1 and 1.2 are named the Banach indicatrix theorem (see [N, pp. 225–227], [Bo, p. 406], [L, pp. 66–72], [BC, pp. 177–178]). There are further generalizations of this result—see for example [Ša, St, Ł] and the bibliography therein.

The Banach indicatrix plays a role in the change of variables formula

$$\int_A (u \circ f)|J(x, f)|\, dx = \int_{\mathbb{R}^n} u(y) N(y, f, A)\, dy.$$

In [H] the formula was obtained under minimal assumptions: the a.e. existence of approximate partial derivatives. In particular, the measurability of $N(y, f, A)$ was proved.

In [RR, IV.1.2] the multiplicity function of a continuous transform was studied in detail. See also [GR, p. 272] for further investigation. The treatment in the setting of metric spaces is given in [F, 2.10.10–15].

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This note aims to show the measurability of the Banach indicatrix for an arbitrary measurable mapping (Theorem 3.1). An application of this result appears in [E]. The proof of Lemma 3.3 is based upon ideas of the original proof of [B, Théorème 1.1].

2. Assumptions. Though we intend to prove the assertion in a most general setting, some requirements on the spaces involved are assumed.

Let (X, d_X, μ_X) be a complete, separable metric space with measure. Additionally X is supposed to be *geometrically doubling*: there is a constant $\lambda \in \mathbb{N}$ such that every ball $B(x,r) = \{z \in X \mid d_X(x,z) < r\}$ can be covered by at most λ balls $B(y, r/2)$ of half the radius. The measure μ_X is a Borel regular measure such that each ball has finite measure. Assume (Y, d_Y, μ_Y) is a separable metric measurable space.

A mapping $f : X \to Y$ is said to be μ_X-*measurable* if it is defined μ_X-almost everywhere on X and $f^{-1}(E)$ is μ_X-measurable whenever E is an open subset of Y [F, 2.3.2].

A system of *dyadic cubes* is exploited, which is a family

$$\{Q^k_\alpha \mid k \in \mathbb{Z},\, \alpha \in \mathcal{A}_k \subset \mathbb{N}\}$$

of Borel sets together with parameters $\delta \in (0,1)$, $0 < c \le C < \infty$ and centres $\{x^k_\alpha\}$, meeting the following properties:

(1) if $l \ge k$ then either $Q^l_\beta \subset Q^k_\alpha$ or $Q^l_\beta \cap Q^k_\alpha = \emptyset$;
(2) for each $k \in \mathbb{Z}$, $X = \bigcup_{\alpha \in \mathcal{A}_k} Q^k_\alpha$ is a disjoint union;
(3) $B(x^k_\alpha, c\delta^k) \subset Q^k_\alpha \subset B(x^k_\alpha, C\delta^k)$;
(4) if $l \ge k$ and $Q^l_\beta \subset Q^k_\alpha$ then $B(x^l_\beta, C\delta^l) \subset B(x^k_\alpha, C\delta^k)$.

This specific dyadic system in doubling quasi-metric spaces was constructed in [HK] and generalizes the dyadic cubes in Euclidean space.

3. Establishing measurability. The main result of this paper is formulated as follows.

THEOREM 3.1. *Let $f : X \to Y$ be a μ_X-measurable mapping, and $A \subset X$ be a Borel set. Then f can be redefined on a set of μ_X-measure zero in such a way that the Banach indicatrix $N(y, f, A)$ is a μ_Y-measurable function.*

Before proceeding with the proof of the theorem we mention an example where the indicatrix of a measurable function is not measurable, illustrating why it is sometimes necessary to redefine the function on a null set.

EXAMPLE 3.2. Let $C \subset \mathbb{R}$ denote the Cantor set and $V \subset \mathbb{R}$ denote the Vitali non-measurable set. There is a bijection $f : C \to V$. Define $\tilde{f}(x) = f(x)$ if $x \in C$ and 0 otherwise. Then $\tilde{f}$ is measurable, while $N(y, f)$ is not (as the characteristic function of the non-measurable set V).

The proof of Theorem 3.1 is split into two lemmas.

LEMMA 3.3. *Let $A \subset X$ be a Borel set and $f : X \to Y$ a μ_X-measurable mapping with the following property: the image $f(B)$ is μ_Y-measurable whenever $B \subset A$ is a Borel set. Then $N(y, f, A)$ is a μ_Y-measurable function.*

Proof. Take a system $\{Q_\alpha^k\}$ of dyadic cubes on X, and define the family of functions $L_\alpha^k(y) = \chi_{f(Q_\alpha^k \cap A)}(y)$. The functions $L_\alpha^k(y)$ are non-negative and μ_Y-measurable (as the characteristic functions of μ_Y-measurable sets $f(Q_\alpha^k \cap A)$). Therefore the sum $N_k(y) = \sum_{\alpha \in \mathcal{A}_k} L_\alpha^k(y)$ is also measurable. Thus the sequence $\{N_k(y)\}$ of measurable functions is non-decreasing and the pointwise limit $N^*(y) = \lim_{k\to\infty} N_k(y)$ exists and is a μ_Y-measurable function.

Note that $N_k(y)$ is the number of the sets $Q_\alpha^k \cap A$ in which the function f attains the value y at least once. So for each k, $N(y, f, A) \geq N_k(y)$ and $N(y, f, A) \geq N^*(y)$.

We prove the reverse inequality. Let q be an integer such that $N(y, f, A) \geq q$. Then there exist q different points $x_1, \dots, x_q \subset A$ such that $f(x_j) = y$. If k is so large that $x_1, \dots, x_q$ are in disjoint cubes $\{Q_{\alpha_j}^k\}$, $j = 1, \dots, q$, then $N_k(y) \geq q$. This shows $N(y, f, A) \leq N^*(y)$ and $N^*(y) = N(y, f, A)$. ∎

LEMMA 3.4 (see also [VE, Lemma 12]). *Let $f : X \to Y$ be a μ_X-measurable mapping. Then there is an increasing sequence of closed sets $T_j \subset X$ such that f is continuous on every T_j and $\mu_X(X \setminus \bigcup_j T_j) = 0$.*

Proof. Let $\{Q_\alpha\}$ be a collection of dyadic cubes of one generation such that

$$X = \bigcup_\alpha Q_\alpha \quad \text{is a disjoint union.}$$

By Luzin's theorem [F, 2.3.5], for each α there is a closed set $C_\alpha^1 \subset Q_\alpha$ such that f is continuous on C_α^1 and $\mu_X(Q_\alpha \setminus C_\alpha^1) < 1$. Similarly f continuous on $C_\alpha^2 \subset Q_\alpha \setminus C_\alpha^1$ and $\mu_X((Q_\alpha \setminus C_\alpha^1) \setminus C_\alpha^2) < 1/2$ and so on. This yields a sequence $\{C_\alpha^j\}$ of closed sets.

Set $P_\alpha^j = \bigcup_{i=1}^j C_\alpha^i$. Then $P_\alpha^j \subset P_\alpha^{j+1}$ and the mapping f is continuous on each P_α^j. Furthermore $\mu_X(Q_\alpha \setminus P_\alpha^j) < 1/j$, and hence $\mu_X(Q_\alpha \setminus \bigcup_k P_\alpha^j) = 0$.

Now defining $T_j = \bigcup_\alpha P_\alpha^j$, we get an increasing sequence of closed sets. In particular, $\mu_X(Q_\alpha \setminus \bigcup_j T_j) = 0$ since $\bigcup_j P_\alpha^j \subset \bigcup_j T_j$. Then $X \setminus \bigcup_{j=1}^\infty T_j = \bigcup_\alpha (Q_\alpha \setminus \bigcup_{j=1}^\infty T_j)$. Consequently, $X \setminus \bigcup_j T_j$ is of μ_X-measure zero since it is a countable union of negligible sets. ∎

Proof of Theorem 3.1. Let $\{T_j\}$ be a sequence of closed sets from Lemma 3.4. Observe that the image of each Borel set $B \subset T_j$ is μ_Y-measurable since f is continuous on T_j [F, 2.2.13]. This puts us in a position to apply Lemma 3.3 to deduce that $N(y, f, A \cap T_j)$ is a μ_Y-measurable function. The sequence $N(y, f, A \cap T_j)$ is non-decreasing, and hence

$$N\Big(y, f, A \cap \bigcup_j T_j\Big) = \lim_{j\to\infty} N(y, f, A \cap T_j)$$

is a μ_Y-measurable function.

Take a point $y_0 \in Y$ and redefine $f(x) = y_0$ for $x \in X \setminus \bigcup_j T_j$. ∎

REMARK 3.5. Note that Theorem 3.1 requires that A be a Borel set. On the other hand, one can prove an analogous assertion for a measurable set A, but assuming that the mapping f has the Luzin $\mathcal{N}$-property (because in this case the continuous image of every measurable set is measurable and Lemma 3.3 is applicable).

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