Siberian Mathematical Journal, Vol. 55, *No.* 5, *pp.* 817–848, 2014
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ISOMORPHISMS OF SOBOLEV SPACES ON CARNOT GROUPS AND QUASI-ISOMETRIC MAPPINGS

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Abstract: We study the properties of the mappings on a Carnot group which induce, via the change-of-variables rule, the isomorphisms of Sobolev spaces with the summability exponent different from the Hausdorff dimension of the group.

DOI: 10.1134/S0037446614050048

Keywords: superposition operator, Sobolev space, quasi-isometry, Carnot group

1. Introduction

The study of superposition operators on Sobolev spaces stems from the classical article of Sobolev [1] (see [2] which includes the detailed history and bibliography). A new impetus for this topic came up while solving the Reshetnyak problem on the description of all isomorphisms φ^* of the homogeneous Sobolev space L^1_n generated by a quasiconformal mapping φ of the Euclidean space $\mathbb{R}^n$ as $\varphi^*(u) = u \circ \varphi$. This problem was posed in 1968 at the first Donetsk colloquium on the theory of quasiconformal mappings. As shown in [3], these are precisely the lattice isomorphisms of L^1_n spaces. It is natural to consider the approach to the Reshetnyak problem which was proposed in [3] in the context of prior results (see [4] for instance): the theorems of Banach, Stone, Eilenberg, Arens and Kelley, Hewitt, Gelfand and Kolmogorov yielded the conditions on various structures of the space $C(S)$ of continuous functions whose isomorphism determines the topological space S up to homeomorphisms. In particular, note the result of Stone by which $C(S)$ determines S as a lattice ordered group. On the other hand, Nakai [5] and Lewis [6] established that the isomorphism of Royden algebras is equivalent to the quasiconformal equivalence of the domains. Selecting two structures in the homogeneous Sobolev space L^1_n, the vector lattice and the seminormed space, we see the situation that is close to the work of Stone in the algebraic sense and to that of Nakai in the metric sense. This view on the problem is the most natural enabling us to reconstruct the mapping from minimal data, prove the continuity of the mapping, and establish its metric properties.

The following question arose in the framework of the approach of [3] to the Reshetnyak problem: *Which metric and analytical properties has the measurable mapping φ that induces the isomorphism φ^* where $\varphi^*(f) = f \circ \varphi$ for $f \in L^1_n$?* Changing the function space, each time we arrive at a new problem. The Sobolev spaces W^1_p with $p > n$ are considered in [7]; the homogeneous Besov space $b^l_p(\mathbb{R}^n)$ with $n > 1$ and $lp = n$, in [8] for $p = n + 1$ and in [9] for $p > n + 1$; the Sobolev spaces W^1_p with $n - 1 < p < n$, in [10]; the space of Riesz and Bessel potentials, in [11]; the three-index scales of Nikol′skiĭ–Besov and Lizorkin–Triebel spaces (and their anisotropic analogs), in [12], the Sobolev spaces W^1_p on domains in higher-dimensional Euclidean spaces with $1 \le p < \infty$ and $p \ne n$, in [13] (with a new proof as compared to [7, 11]). Multiplier theory is applied in [14] to the problem of changing variables in Sobolev spaces. The properties of bounded superposition operators on Besov spaces are studied, apart from [9], also in [15] and [16]. The quasiconformal equivalence of Lizorkin–Triebel classes is investigated in [17].

The authors were partially supported by a Grant of the Government of the Russian Federation (Agreement No. 14.B25.31.0029).

†) To dear Yuriĭ Grigor′evich Reshetnyak on the occasion of his 85th birthday.

Novosibirsk. Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 55, No. 5, pp. 1001–1039, September–October, 2014. Original article submitted December 20, 2013.

We can infer from [8–14] that if φ^* is an isomorphism then, depending on the relation between the smoothness exponent, summability exponent, and the dimension of the space, the mapping is quasiconformal or quasi-isometric in the metric of the domain adequate to the geometry of the function space.

This article can be regarded as a natural development of the methods and results of [13]. The problem under study here is to find some necessary and sufficient conditions on the measurable mapping φ so that φ induces, via the change-of-variables rule, the isomorphism φ^* of nonholonomic Sobolev spaces with generalized horizontal first derivatives on domains of Carnot groups provided that the summability exponent differs from the Hausdorff dimension of the group (in [13] this problem is solved in the Euclidean space).

Given two domains D and D' in a Carnot group $\mathbb{G}$, say that a measurable mapping $\varphi : D \to D'$ is *of class* IL^1_p whenever φ induces the superposition operator of the Sobolev spaces

$$\varphi^* : L^1_p(D') \cap C^\infty(D') \to L^1_p(D), \quad \varphi^*(f) = f \circ \varphi, \quad f \in L^1_p(D') \cap C^\infty(D'), \tag{1}$$

such that

(1) every $f \in L^1_p(D') \cap C^\infty(D')$ satisfies

$$K^{-1}\|f \mid L^1_p(D')\| \le \|\varphi^*(f) \mid L^1_p(D)\| \le K\|f \mid L^1_p(D')\|, \tag{2}$$

where the constant K is independent of the choice of f;

(2) the image $\varphi^*\big(L^1_p(D') \cap C^\infty(D')\big)$ is everywhere dense in $L^1_p(D)$.

Observe that condition 2 is independent of condition 1. Indeed, consider the mapping $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ acting as $\varphi(x_1, x_2) = (|x_1|, x_2)$. It is obvious that condition 1 is fulfilled. On the other hand, the image $\varphi^*(L^1_p(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2))$ consists of the functions even with respect to the axis $x_2 = 0$, and so the image cannot be dense in $L^1_p(\mathbb{R}^2)$.

We state the main result as follows:

Theorem 1 [18]. *Assume that $p \ge 1$ and $p \ne \nu$. Consider two domains D and D' in a Carnot group $\mathbb{G}$ whose Hausdorff dimension is ν. A measurable mapping $\varphi : D \to D'$ is of class IL^1_p if and only if φ coincides almost everywhere with some quasi-isometry $\Phi : D \to \Phi(D)$ such that the Sobolev spaces $L^1_p(\Phi(D))$ and $L^1_p(D')$ are $(1,p)$-equivalent.*

Remark 1. 1. By the two-sided estimate (2), (1) is a monomorphism.

2. We will show in Lemma 10 that (1) extends by continuity to an isomorphism of the Sobolev space L^1_p, and the extension is also a superposition operator in a certain sense.

Let us define the main concepts of use in the statement of this theorem.

Definition 1. A homeomorphism $\Phi : D \to D'$ between two open sets is called a *quasi-isometry* whenever

$$\varlimsup_{y \to x} \frac{d(\Phi(y), \Phi(x))}{d(y,x)} \le M, \quad \varlimsup_{y \to z} \frac{d(\Phi^{-1}(y), \Phi^{-1}(z))}{d(y,z)} \le M \tag{3}$$

for all $x \in D$ and $z \in D'$, where M is a constant independent of the choice of $x \in D$ and $z \in D'$, and d stands for the Carnot–Carathéodory distance on $\mathbb{G}$.

Definition 2. Two open sets D_1 and D_2 are called $(1,p)$-*equivalent* whenever the restriction operators

$$r_i : L^1_p(D_i) \to L^1_p(D_1 \cap D_2), \quad r_i(f) = f|_{D_1 \cap D_2}, \quad \text{where } f \in L^1_p(D_i),$$

are isomorphisms.

The properties of $(1,p)$-equivalent domains of Euclidean spaces are studied in [19] and of Carnot groups, in [20].

Remark 2. Theorem 1 has the corollary: If a mapping $\varphi : D \to D'$ is a quasi-isometry then $f \circ \varphi \in L^1_p(D)$ for every $f \in L^1_p(D')$; while φ^*, induced as the superposition, is an isomorphism of Sobolev spaces.

This article is organized as follows: Sections 2 and 3 contain the main definitions and auxiliary results. In Section 4 we show that it is possible to define the superposition operator (1) on the whole space $L_p^1(D')$ preserving properties (1) and (2). Moreover, the resulting superposition operator $\varphi^* : L_p^1(D') \to L_p^1(D)$ is an isomorphism between the Sobolev spaces $L_p^1(D')$ and $L_p^1(D)$. Then in Section 5 we prove the main result of this article.

The proof of Theorem 1 splits into setting the two main cases. The first case $p > \nu$ is simpler. In essence, it reduces to the situation of a bijective measurable mapping φ and relies on the fact that the capacity of two points x and y in $L_p^1(\mathbb{G})$ is comparable with $d(x,y)^{\nu-p}$. Then φ^* is an isomorphism if and only if $M^{-1}d(x,y) \leq d(\varphi(x),\varphi(y)) \leq Md(x,y)$ for sufficiently close points $x, y \in D$ chosen in a special everywhere dense subset of D. The latter implies (3) (for the details, see the proof of Theorem 4).

The second case $1 \leq p < \nu$ treated in Lemma 22 is considerably more delicate. Some lengthy argument precedes Lemma 22; which amounts to removing at each step a negligible set while aiming to obtain eventually a restricted domain $\operatorname{Dom}_6 \varphi \subset D$ of the measurable mapping φ, with $|D \setminus \operatorname{Dom}_6 \varphi| = 0$, on which φ enjoys a series of remarkable properties, like bijectivity, Luzin's $\mathscr{N}$-property, and Luzin's $\mathscr{N}^{-1}$-property. These properties enable us to prove that φ is approximatively differentiable along horizontal vector fields. The latter is fundamental for applying analytical methods to study φ. It turns out that the direct mapping φ is approximatively differentiable and its approximative differential $D\varphi(x)$ and the Jacobian $J(x,\varphi) = \det D\varphi(x)$ satisfy

$$|D\varphi|(x) \leq L < \infty, \quad |J(x,\varphi)| \geq \alpha_1 > 0 \quad \text{almost everywhere in } D.$$

Here we prove a similar relation for the approximative differential $D\psi(y)$ of the inverse mapping $\psi = \varphi^{-1}$:

$$|D\psi|(y) \leq L' < \infty, \quad |J(y,\psi)| \geq \alpha > 0 \quad \text{almost everywhere in } D'.$$

Using these relations and the conditions on φ^*, we reduce the investigation of the metric properties of φ to the first case. This reduction enables us to prove that φ coincides almost everywhere on D with some quasi-isometry $\Phi : D \to D'$.

Certain methods we use to prove the main results generalize classical approaches, like the approximation of functions in a Sobolev space by smooth functions. On the other hand, in some cases we need the new methods that base on the properties of a metric measure space. The main difficulty we face is to remove correctly a negligible set from the domain and show that the restricted mapping is approximatively differentiable almost everywhere along the horizontal vector fields. In the proof we also apply the results and methods of [20–25].

In the case $p = \nu$ we have

Theorem 2 [18]. *Given two domains D and D' in a Carnot group $\mathbb{G}$ of Hausdorff dimension ν, a measurable mapping $\varphi : D \to D'$ is of class IL_ν^1 if and only if φ coincides almost everywhere with a quasiconformal mapping $\Phi : D \setminus \{x_0\} \to \mathbb{G}$ such that the Sobolev spaces $L_\nu^1(\Phi(D))$ and $L_\nu^1(D')$ are $(1,\nu)$-equivalent, where $x_0 \in \overline{\mathbb{G}}$ is some point (here $\overline{\mathbb{G}}$ is the one-point compactification of $\mathbb{G}$).*

Theorem 2 is established in [26] by some statements obtained in this article for $p \leq \nu$.

2. Sobolev Spaces on a Carnot Group

Definition 3. A *Carnot group* $\mathbb{G}$ is a connected simply-connected nilpotent Lie group whose Lie algebra $\mathscr{G}$ is graded; i.e., $\mathscr{G} = V_1 \oplus \cdots \oplus V_m$, where $\dim V_1 = n_1 \geq 2$, so that $[V_1, V_k] = V_{k+1}$ for $1 \leq k \leq m-1$ and $[V_1, V_m] = 0$. Put

$$N = \sum_{i=1}^{m} \dim V_i.$$

Identify $g \in \mathbb{G}$ with $x \in \mathbb{R}^N$ via the exponential mapping $\exp\bigl(\sum x_{ij}X_{ij}\bigr)$. The numbers x_{ij} are called the *coordinates of the first kind* of $g \in \mathbb{G}$ for $1 \leq i \leq m$ and $1 \leq j \leq n_i = \dim V_i$. For brevity, we denote the coordinates that correspond to the horizontal subspace by avoiding double indices: $x_j = x_{1j}$ for $1 \le j \le n_i$. Thus, $\mathbb{G}$ carries some global coordinate system that we use to identify the points of $\mathbb{G}$ with points in $\mathbb{R}^N$. Denote by g_h the elements of the group whose all coordinates but $x_1, \dots, x_{n_1}$ vanish.

In the coordinates of the first kind, the mapping

$$\delta_t : (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_m) \mapsto (t\bar{x}_1, t^2\bar{x}_2, \dots, t^m\bar{x}_m), \quad t > 0,$$

determines a one-parameter family of dilations. Here $\bar{x}_i \in \mathbb{R}^{n_i}$.

The left-invariant vector fields $X_i = X_{i,1}$, for $i = 1, \dots, n_1$, constituting the standard basis of the subbundle V_1, are called *horizontal*.

Fix on $\mathbb{G}$ the homogeneous norm (see [22] for instance)

$$\rho(x) = \left(\sum_{i=1}^{m} |\bar{x}_i|^{2m!/i} \right)^{1/2m!},$$

where $|\bar{x}_i|$ is the Euclidean norm on V_i. As for every homogeneous norm, [27] has the properties:

(1) $\rho(x) = 0 \Leftrightarrow x = 0$,

(2) $\rho(x^{-1}) = \rho(x)$,

(3) $\rho(\delta_\lambda(x)) = \lambda\rho(x)$,

(4) $\rho(xy) \le c(\rho(x) + \rho(y))$,

where c is some constant independent of $x, y \in \mathbb{G}$; the homogeneous norm determines the homogeneous quasimetric $\rho(x, y) = \rho(x^{-1}y)$ for two points $x, y \in \mathbb{G}$.

An absolutely continuous piecewise smooth curve $\gamma : [a, b] \to \mathbb{G}$ whose tangent vector lies in V_1 is called a *horizontal curve*.

Definition 4. The *Carnot–Carathéodory metric* $d(x, y)$ on $\mathbb{G}$ is the sharp lower bound of the lengths of all horizontal curves connecting the points x and y.

Denote the distance to the origin by $d(x) = d(0, x)$.

We can show that $d(x, y)$ and $\rho(x, y)$ are equivalent (see the arguments in Lemma 1.4 and Proposition 1.5 of [27]); i.e.,

$$c\rho(x, y) \le d(x, y) \le C\rho(x, y)$$

for all points $x, y \in \mathbb{G}$ with some constants $0 < c \le C < \infty$.

Take $g \in \mathbb{G}$. It is not difficult to show that g_h satisfy

$$d(g_h) = \rho(g_h) = \sqrt{x_1^2 + \dots + x_{n_1}^2} \tag{4}$$

and

$$d(g_h) \le d(g). \tag{5}$$

Indeed, take $g_h = (x_1, \dots, x_{n_1}, 0, \dots, 0)$. The curve $\gamma(t) = (tx_1, \dots, tx_{n_1}, 0, \dots, 0)$, for $t \in [0, 1]$, is horizontal, while $\gamma(0) = 0$ and $\gamma(1) = g_h$. Since $\gamma(t)$ is the straight line segment connecting 0 and g_h, while the metric on the horizontal plane coincides with the Euclidean metric; therefore, $\gamma(t)$ is of minimal length. Thus, $d(g_h)$ coincides with the length of this segment: $d(g_h) = \sqrt{x_1^2 + \dots + x_{n_1}^2}$. For every horizontal curve $\gamma(t) = (\gamma_1(t), \dots, \gamma_2(t), \gamma_{21}(t), \dots, \gamma_{mn_m}(t))$ connecting 0 and g, we have

(1) $\mathrm{Pr}_h\,\gamma(0) = 0$ and $\mathrm{Pr}_h\,\gamma(1) = g_h$;

(2) the length of $\mathrm{Pr}_h\,\gamma$ equals that of γ,

where $\mathrm{Pr}_h\,\gamma(t) = (\gamma_1(t), \dots, \gamma_2(t), 0, \dots, 0)$ is the projection of the curve onto the horizontal subspace; On the other hand, $d(g_h)$ is the length of the segment connecting 0 and g_h, and so it is at most the length of an arbitrary curve connecting these points; i.e., $d(g_h) \le d(g)$.

The Hausdorff dimension of the group $\mathbb{G}$ equals $\nu = n_1 + 2n_2 + 3n_3 + \dots + mn_m$, where $n_i = \dim V_i$.

Take a Carnot group $\mathbb{G}$ with a one-parameter group of dilations δ and an open set D in $\mathbb{G}$. Define the $L_p(D)$ space of p-summable functions, $p \in [1, \infty)$, as the collection of measurable Lebesgue functions with the finite norm

$$\|f \mid L_p(D)\| = \Big(\int\limits_D |f(x)|^p \, dx \Big)^{1/p} < \infty.$$

Here dx is the Lebesgue measure on $\mathbb{R}^N$ normalized so that the ball of unit radius (with respect to the quasimetric ρ) have measure 1. A locally summable function $v_i : D \mapsto \mathbb{R}$ is called the *generalized derivative of f along the vector field X_i*, for $i = 1, \dots, n_1$, whenever

$$\int\limits_D v_i \psi \, dx = - \int\limits_D f X_i \psi \, dx$$

for every compactly supported function $\psi \in C_0^\infty(D)$.

The homogeneous Sobolev space $L_p^1(D)$ consists of all locally summable functions of finite seminorm

$$\big\| f \mid L_p^1(D) \big\| = \| \nabla_{\mathscr{L}} f \mid L_p(D) \|,$$

where $\nabla_{\mathscr{L}} f(x) = (X_1 f(x), \dots, X_{n_1} f(x))$ is the generalized subgradient of f at $x \in D$ which uses only the derivatives along the horizontal fields.

The Sobolev space $W_p^1(D)$ consists of all locally summable functions of finite norm

$$\big\| f \mid W_p^1(D) \big\| = \| f \mid L_p(D) \| + \| \nabla_{\mathscr{L}} f \mid L_p(D) \|.$$

Say that f is *of class* $W_{p,\mathrm{loc}}^1(D)$ whenever $f \in W_p^1(V)$ for every bounded subdomain $V \subset D$ with $\overline{V} \subset D$.

Reshetnyak proposed in [28] some approach to defining the Sobolev classes of functions with values in a metric space. Consider a complete metric space $(\mathbb{X}, r)$ and an open set D in a Carnot group $\mathbb{G}$. Say that a mapping $\varphi : D \to \mathbb{X}$ is *of class* $W_{p,\mathrm{loc}}^1(D; \mathbb{X})$ whenever φ meets the conditions:

(A) for every $z \in \mathbb{X}$ the function $[\varphi]_z : x \in D \mapsto r(\varphi(x), z)$ is of class $W_{p,\mathrm{loc}}^1(D)$;

(B) the family of subgradients $(\nabla_{\mathscr{L}}[\varphi]_z)_{z \in \mathbb{X}}$ has a majorant in $L_{p,\mathrm{loc}}(D)$; i.e., there exists $g \in L_{p,\mathrm{loc}}(D)$ independent of z with $|\nabla_{\mathscr{L}}[\varphi]_z(x)| \le g(x)$ for almost all $x \in D$.

If $\mathbb{X} = \mathbb{G}'$ is another Carnot group with one-parameter group δ' of dilations, distance ρ', and so forth; then we obtain the definition of Sobolev-class mapping of distinct Carnot groups and denote this class by $W_{p,\mathrm{loc}}^1(D; \mathbb{G}')$. In this case it is convenient to use an equivalent description of a Sobolev-class mapping (see [22] for instance): a mapping $\varphi : D \mapsto \mathbb{G}'$ belongs to $W_{p,\mathrm{loc}}^1(D; \mathbb{G}')$ if and only if we can change φ on a negligible set so that

(a) the function $D \ni x \mapsto [\varphi]_z(x) = \rho'(\varphi(x), z)$ belongs to $L_{p,\mathrm{loc}}(D)$ for every point $z \in G'$;

(b) the mapping $\varphi : D \to \mathbb{G}'$ is absolutely continuous on the integral lines of the horizontal vector fields; i.e., given a bounded open set U with $\overline{U} \subset D$ and the foliations Γ_j of U defined by the left-invariant vector fields X_j, for $j = 1, \dots, n_1$, the mapping φ is absolutely continuous on $\gamma \cap U \in \Gamma_j$ with respect to the one-dimensional Hausdorff measure for $d\tau$-almost all $\gamma \in \Gamma_j$ (here γ is the left translation of the curve $\exp tX_j$, while the measure $d\tau$ on the foliation Γ_j equals the contraction $i(X_j)$ of X_j and the bi-invariant volume form dx);

(c) the derivative $X_j \varphi(x) = \lim_{t \to 0} \delta_{t^{-1}}(\varphi(x)^{-1} \varphi(\exp tX_j))$ exists and belongs to $V_1'(\varphi(x))$ almost everywhere on the open set D and, moreover, $|X_j(\varphi)| \in L_{p,\mathrm{loc}}(D)$ for all j.

If $\varphi : D \mapsto \mathbb{G}'$ satisfies only conditions (a) and (b) then we say that φ is *of class $ACL(D)$*. This mapping possesses the derivatives $X_j \varphi \in V_1'$ along the vector fields X_j, for $j = 1, \dots, n_1$, almost everywhere in D (see [29]).

The matrix $(X_j \varphi_k(x))$, for $j, k = 1, \dots, n_1$, called the (*formal*) *horizontal differential* of φ at x, determines the linear operator $D_h \varphi : V_1 \mapsto V_1'$ from the horizontal space V_1 into the horizontal space V_1'

for almost all x (see [29]), and $|D_h\varphi|$ stands for the norm of this operator. It is shown in [21, 22] that the linear operator $D_h\varphi : V_1 \mapsto V_1'$ extends to a homomorphism $D\varphi : \mathscr{G} \mapsto \mathscr{G}'$ of the Lie algebra $\mathscr{G}$ into the Lie algebra $\mathscr{G}'$. In the case $\mathbb{G} = \mathbb{G}'$ the Jacobian $J(x, \varphi) = \det D\varphi(x)$ amounts to the determinant of the matrix $D\varphi(x)$.

Recall the definitions of locally Lipschitz and bi-Lipschitz mappings.

Definition 5. A mapping $\varphi : U \to \mathbb{G}$ is called *locally Lipschitz* (*bi-Lipschitz*) whenever for every point $x \in U$ there are a neighborhood V with $V \subset U$ and a constant L_V so that

$$d(\varphi(y), \varphi(z)) \le L_V d(y, z) \quad (L_V^{-1} d(y, z) \le d(\varphi(y), \varphi(z)) \le L_V d(y, z)) \tag{6}$$

for all $y, z \in V$. If we can take U as V then $\varphi : U \to \mathbb{G}$ is *Lipschitz* (*bi-Lipschitz*) on U.

Denote by $\mathrm{Lip}_{\mathrm{loc}}(D)$ the space of locally Lipschitz functions $f : D \to \mathbb{R}$. Observe that the space $\mathrm{Lip}_{\mathrm{loc}}(D)$ of locally Lipschitz functions coincides with the intersection $C(D) \cap W^1_{\infty,\mathrm{loc}}(D)$ (see [30] for instance).

Lemma 1. *Fix $x_0 \in \mathbb{G}$ and put $g(x) = d(x_0, x)$. Then $|\nabla_{\mathscr{L}} g(x)| = 1$ almost everywhere in $\mathbb{G}$.*

Proof. Split the proof into three steps:

1. We have

$$|g(x) - g(y)| = |d(x_0, x) - d(x_0, y)| \le d(x, y).$$

Thus, g is a Lipschitz function. Hence, we deduce that

$$\frac{|g(x) - g(y)|}{d(x, y)} \le 1 \quad \text{for all } x \ne y. \tag{7}$$

2. Given $x \in \mathbb{G}$, the shortest curve γ connecting x and x_0 exists. Take a point y on γ. Then $|g(x) - g(y)| = d(x, y)$, and so

$$\frac{|g(x) - g(y)|}{d(x, y)} = 1 \quad \text{for } y \in \gamma. \tag{8}$$

From (7) and (8) we obtain

$$\varlimsup_{y \to x \in \mathbb{G}} \frac{|g(y) - g(x)|}{d(x, y)} = 1. \tag{9}$$

3. Since g is a Lipschitz function, g is $\mathscr{P}$-differentiable almost everywhere in the sense of Pansu [29] (see a different proof in [22]). Consequently, at a $\mathscr{P}$-differentiability point x of g we have

$$\lim_{d(x,y) \to 0} \frac{g(y) - g(x) - \nabla_{\mathscr{L}} g(x) \cdot (x^{-1}y)_h}{d(x, y)} = 0, \tag{10}$$

where $\nabla_{\mathscr{L}} g(x) \cdot (x^{-1}y)_h = \sum_{i=1}^{n_1} X_i g(x)(x^{-1}y)_{1i}$.

Put $z = x^{-1}y$. By (9) and (10),

$$\varlimsup_{d(z) \to 0} \frac{|\nabla_{\mathscr{L}} g(x) \cdot z_h|}{d(z)} = 1. \tag{11}$$

Then

$$1 = \varlimsup_{d(z) \to 0} \frac{|\nabla_{\mathscr{L}} g(x) \cdot z_h|}{d(z)} \ge \varlimsup_{z = z_h, d(z) \to 0} \frac{|\nabla_{\mathscr{L}} g(x) \cdot z_h|}{d(z)} = \varlimsup_{d(z_h) \to 0} \frac{|\nabla_{\mathscr{L}} g(x) \cdot z_h|}{d(z_h)}. \tag{12}$$

On the other hand, (5) yields

$$\varlimsup_{d(z_h) \to 0} \frac{|\nabla_{\mathscr{L}} g(x) \cdot z_h|}{d(z_h)} = \varlimsup_{d(z) \to 0} \frac{|\nabla_{\mathscr{L}} g(x) \cdot z_h|}{d(z_h)} \ge \varlimsup_{d(z) \to 0} \frac{|\nabla_{\mathscr{L}} g(x) \cdot z_h|}{d(z)} = 1. \tag{13}$$

Therefore,

$$\overline{\lim_{d(z)\to 0}} \frac{|\nabla_{\mathscr{L}} g(x)\cdot z_h|}{d(z)} = \overline{\lim_{d(z_h)\to 0}} \frac{|\nabla_{\mathscr{L}} f(x)\cdot z_h|}{d(z_h)} = 1. \tag{14}$$

Since $d(z_h)=\rho(z_h)$ (see (4)), we infer that

$$1 = \overline{\lim_{\rho(z_h)\to 0}} \left|\nabla_{\mathscr{L}} g(x)\cdot \frac{z_h}{\rho(z_h)}\right| = \sup_{y_h\in\mathbb{R}^{n_1},\,\rho(y_h)=1} |\nabla_{\mathscr{L}} g(x)\cdot y_h|,$$

where $y_h=\frac{z_h}{\rho(z_h)}$. Thus, we arrive at the equality $|\nabla_{\mathscr{L}} g(x)|=1$. $\square$

Remark 3. The claim of Lemma 1 holds also for the function

$$g(x)=d(x,F)=\inf_{y\in F} d(x,y)$$

in the form: $|\nabla_{\mathscr{L}} g(x)|=1$ almost everywhere in $\mathbb{G}\setminus F$. Here g is the distance from the point x to a fixed set F.

2.1. Approximation by smooth functions. The arguments in this subsection rest largely on [27, 31]. Assume that $\varphi\in C_0^\infty(\mathbb{G})$ with $\operatorname{supp}\varphi\subset B(0,1)$ and

$$\int\limits_{\mathbb{G}} \varphi(x)\,dx = a. \tag{15}$$

Take $u\in L_p^1(D)$ with $u(x)=0$ for all $x\in\mathbb{G}\setminus D$. Consider the family of averages

$$u_\varepsilon(x)=\frac{1}{\varepsilon^\nu}\int\limits_{\mathbb{G}} \varphi(\delta_{\varepsilon^{-1}} xy^{-1})u(y)\,dy. \tag{16}$$

The function φ is called the *averaging kernel*, while ε is the *averaging radius*.

Proposition 1. *If $u\in L_p(D)$ and $p\in[1,\infty)$ then*
(1) $u_\varepsilon\in C^\infty(\mathbb{G})$;
(2) $u_\varepsilon\to au$ in $L_p(D)$.

Proof. 1. The functions $\varphi(x)$ and $\tau_y(x)=xy^{-1}$ belong to $C^\infty(\mathbb{G})$. Therefore, $u_\varepsilon\in C^\infty(\mathbb{G})$ by the theorem on differentiation under the integral sign.

2. Continuous compactly supported functions are dense in $L_p(\mathbb{G})$; hence,

$$\int\limits_{\mathbb{G}} |v(xz^{-1})-v(x))|^p\,dx \to 0 \quad \text{as } z\to e \tag{17}$$

for every $v\in L_p(\mathbb{G})$, where e is the unit element of $\mathbb{G}$.

Put $\varphi_\varepsilon(x)=\frac{1}{\varepsilon^\nu}\varphi(\delta_{\varepsilon^{-1}}x)$. Furthermore,

$$\begin{aligned}
|u_\varepsilon(x)-au(x)| &= \left|\int\limits_{\mathbb{G}} \varphi_\varepsilon(xy^{-1})u(y)\,dy - u(x)\int\limits_{\mathbb{G}} \varphi_\varepsilon(xy^{-1})\,dy\right| \\
&\le \int\limits_{\mathbb{G}} |\varphi_\varepsilon(xy^{-1})||u(y)-u(x)|\,dy \\
&\le \Bigg(\int\limits_{\mathbb{G}} |\varphi_\varepsilon(xy^{-1})|\,dy\Bigg)^{\frac{1}{p'}} \Bigg(\int\limits_{\mathbb{G}} |\varphi_\varepsilon(xy^{-1})||u(y)-u(x)|^p dy\Bigg)^{\frac{1}{p}} \\
&= K\Bigg(\int\limits_{\mathbb{G}} |\varphi_\varepsilon(z)|\cdot|u(xz^{-1})-u(x)|^p dz\Bigg)^{\frac{1}{p}},
\end{aligned}$$

where K is a positive constant. We have

$$\|u_\varepsilon - au \mid L_p(\mathbb{G})\|^p = \int\limits_{\mathbb{G}} |u_\varepsilon(x) - au(x)|^p\,dx$$

$$\leq K \int\limits_{\mathbb{G}} \int\limits_{\mathbb{G}} |\varphi_\varepsilon(z)||u(xz^{-1}) - u(x)|^p\,dz\,dx = K \int\limits_{\mathbb{G}} |\varphi_\varepsilon(z)| \int\limits_{\mathbb{G}} |u(xz^{-1}) - u(x)|^p\,dx\,dz$$

$$= K \int\limits_{\mathbb{G}} |\varphi(h)| \int\limits_{\mathbb{G}} |u(x(\delta_\varepsilon h)^{-1}) - u(x)|^p\,dx\,dh \to 0 \quad \text{as } \varepsilon \to 0.$$

In the penultimate integral we put $h = \delta_{\varepsilon^{-1}} z$. Thus, $u_\varepsilon \to au$ as $\varepsilon \to 0$ in $L_p(\mathbb{G})$. □

Remark 4. In particular, if $a = 1$ then $u_\varepsilon \to u$ as $\varepsilon \to 0$ in $L_p(\mathbb{G})$.

Proposition 2. *Take $u \in L^1_p(D)$ and $a = 1$ (i.e., $\int_{\mathbb{G}} \varphi(x)\,dx = 1$) and assume that $V \Subset D$ is a compactly embedded domain.*[1] *Then*

$$X_i u_\varepsilon \to X_i u \quad \text{in } L_p(V). \tag{18}$$

Proof. Convolution on a Carnot group need not be commutative; therefore, the arguments applicable in $\mathbb{R}^n$ may fail on a Carnot group. Our proof of this proposition on a Carnot group relies on more refined methods. As shown in [32, Lemma 2.1; 33], there exist functions $\chi_{ij} \in C_0^\infty(B(0,1))$ such that $\int_G \chi_{ij}(x)\,dx = \delta_{ij}$ and for all $x \in V$ and $\varepsilon < \operatorname{dist}(V, \partial D)$ we have

$$X_i u_\varepsilon(x) = \sum_{j=1}^N (X_j u) * \chi_{ij,\varepsilon}(x),$$

where $\chi_{ij,\varepsilon}(x) = \frac{1}{\varepsilon^\nu}\chi_{ij}(\delta_{\varepsilon^{-1}} x)$. Proposition 1 yields $(X_j u) * \chi_{ij,\varepsilon} \to \delta_{ij} X_j u$ as $\varepsilon \to 0$ in $L_p(V)$. Finally,

$$\|X_i u_\varepsilon - X_i u \mid L_p(V)\| = \left\| \sum_{j=1}^N (X_j u) * \chi_{ij,\varepsilon} - \sum_{j=1}^N \delta_{ij} X_j u \mid L_p(V) \right\|$$

$$\leq \sum_{j=1}^N \|(X_j u) * \chi_{ij,\varepsilon} - \delta_{ij} X_j u \mid L_p(V)\| \to 0$$

as $\varepsilon \to 0$. □

Proposition 2 enables us to establish that smooth functions are dense in $L^1_p(D)$.

Lemma 2. *The space $L^1_p(D) \cap C^\infty(D)$ is dense in $L^1_p(D)$. If $f \in L^1_p(D)$ is a locally Lipschitz function then there exists a sequence of functions $f_l \in L^1_p(D) \cap C^\infty(D)$, for $l \in \mathbb{N}$, converging to f locally uniformly and in $L^1_p(D)$.*

Proof. This proof rests on the arguments of [31, Theorem 1].

Let $u \in L^1_p(D)$. Consider a locally finite covering[2] $\{B_k\}_{k \geq 1}$ of the domain D by balls $B_k \subset D$ and the partition $\{\psi_k\}_{k \geq 1}$ of unity subordinate to this covering. Take a decreasing and vanishing sequence $\{\rho_k\}$ of positive numbers such that the sequence of balls $\{(1 + \rho_k)B_k\}$ also forms a locally finite covering of D. Denote by w_k the average of $u_k = \psi_k u$ with the averaging radius $\rho_k r_k$, where r_k is the radius of B_k. It is easy to see that $w = \sum_k w_k$ belongs to $C^\infty(D)$ since the sum is locally finite (each point has

[1] In other words, V is a bounded domain with $\overline{V} \subset D$.

[2] Each point $x \in D$ has a neighborhood $U \subset D$ intersecting only finitely many balls of the covering $\{B_k\}$.

a neighborhood on which only finitely many functions w_k are nonzero). Take $\varepsilon \in (0, \frac{1}{2})$ arbitrarily. By Propositions 1 and 2, we can choose ρ_k so that

$$\|u_k - w_k \mid L_p^1(D)\| \le \varepsilon^k. \tag{19}$$

The equality $u = \sum_k u_k$ holds on every bounded domain V with $\overline{V} \subset D$. Consequently,

$$\|u - w \mid L_p^1(V)\| \le \sum_k \|u_k - w_k \mid L_p^1(V)\| \le \frac{\varepsilon}{1-\varepsilon}. \tag{20}$$

Hence, given $u \in L_p^1(D)$ and $\varepsilon \in \left(0, \frac{1}{2}\right)$, there is $w \in L_p^1(D) \cap C^\infty(D)$ such that $\|u - w \mid L_p^1(D)\| \le \varepsilon$. $\square$

The properties are immediate from Lemma 2.

Remark 5. If $f \in L_p^1(D)$ then there is a sequence of smooth functions $f_n \in L_p^1(D) \cap C^\infty(D)$ converging to f almost everywhere in D, and if $p > \nu$ then we can choose a sequence converging locally uniformly in D.

Proof. We can take as this sequence the functions $\{w_k\}$ of Lemma 2. $\square$

Moreover, Lemma 2 yields

Corollary 1. *$L_p^1(D) \cap \mathrm{Lip}_{\mathrm{loc}}(D)$ is dense in $L_p^1(D)$, where $D \subset \mathbb{G}$ is a domain.*

2.2. Poincaré inequalities and John domains. Recall that a curve $\gamma : [a, b] \to \mathbb{G}$ is *rectifiable* whenever

$$\sup_P \sum_{i=1}^{k_P} d(\gamma(x_{i-1}), \gamma(x_i)) < \infty,$$

where the supremum is over all partitions $P = \{a = x_0 < x_1 < \dots < x_{k_P} = b\}$. Given two points $x, y \in \mathbb{G}$, the *shortest curve* is a horizontal curve of minimal length connecting these points.

Definition 6. A domain $\Omega \subset \mathbb{G}$ is called a $J(\alpha, \beta)$ *John domain* (briefly, $\Omega \in J(\alpha, \beta)$), with $0 < \alpha \le \beta$, whenever there is $x_0 \in \Omega$ such that we can connect each $x \in \Omega$ to x_0 by a rectifiable curve γ lying in Ω and satisfying the conditions: Given the natural parametrization $s \in [0, l]$ of γ, we have $l \le \beta$ and

$$\gamma(0) = x, \quad \gamma(l) = x_0, \quad \mathrm{dist}(\gamma(s), \partial\Omega) \ge \frac{\alpha s}{l} \text{ for all } s \in [0, l]. \tag{21}$$

Remark 6. It is easy to verify that the ball $B(x, r)$ in the Carnot–Carathéodory metric is a $J(r, r)$ John domain, where the center (x) is the selected point.

Lemma 3. *Consider an arbitrary domain D in $\mathbb{G}$ and take two balls B_0 and B_1 in D. Then there is a John domain $\Omega \in J(\alpha, \beta)$ with $\Omega \subset D$, with some parameters α and β depending on D, B_0, and B_1 which includes both balls.*

Proof. Denote by x_0 and x_1 the centers of these balls and by r_0 and r_1, their radii. Construct a rectifiable curve connecting x_0 and x_1.

To this end, firstly consider an arbitrary continuous curve K in D connecting x_0 and x_1, i.e., a continuous mapping $K : [0, 1] \to D$ with $K(0) = x_0$ and $K(1) = x_1$. This curve exists since D is a connected open set. (Note that K is not necessarily rectifiable.)

The collection of balls $\left\{B(K(t)), \frac{1}{2}\,\mathrm{dist}(K(t), \partial D)\right\}_{t \in [0,1]}$ constitutes a covering of the compact set $K([0, 1])$. We can refine from this covering a finite subcovering $B(\xi_1, \rho_1), \dots, B(\xi_m, \rho_m)$, where $\xi_j = K(\tau_j)$ and $\tau_j \in [0, 1]$ with $\tau_1 < \tau_2 < \dots < \tau_m$.

Take the ball $B(\xi_l, \rho_l)$ containing x_0 with the greatest index. If $x_1 \in B(\xi_l, \rho_l)$ then the curve γ composed of the shortest curves connecting ξ_l to x_0 and x_1 is rectifiable and its length $|\gamma|$ is at most $2\rho_l$.

Otherwise, there is the maximal value t_1 of $t \in (0, 1)$ such that $v_1 = K(t_1) \in \partial B(\xi_l, \rho_l)$, while $K(t) \notin \overline{B(\xi_l, \rho_l)}$ for all $t \in (t_1, 1]$. Then there is a curve $\gamma_1 \subset \overline{B(\xi_l, \rho_l)} \subset D$ composed of the shortest curves connecting ξ_l to x_0 and v_1. The length of γ_1 is at most $2\rho_l$.

In turn, v_1 belongs to some ball $B(\xi_k, \rho_k)$, where $l < k \leq m$ is the maximal index of the ball containing v_1. If $x_1 \in B(\xi_k, \rho_k)$ then prolong γ_1 with the shortest curve connecting ξ_k to v_1 and x_1. The length of the resulting curve γ is at most $2(\rho_l + \rho_k)$.

Otherwise, there is the maximal value t_2 of $t \in (t_1, 1)$ such that $v_2 = K(t_2) \in \partial B(\xi_k, \rho_k)$, while $K(t) \notin \overline{B(\xi_k, \rho_k)}$ for all $t \in (t_2, 1]$. Then prolong γ_1 with a new curve $\gamma_2 \subset \overline{B(\xi_k, \rho_k)} \subset D$ composed of the shortest curves connecting ξ_k to v_1 and v_2. Since the length of γ_2 is at most $2\rho_k$, the length of $\gamma_1 \cup \gamma_2$ is at most $2(\rho_l + \rho_k)$.

Continuing this process, after finitely many steps (at most m) we obtain some rectifiable curve $\Gamma = \gamma_1 \cup \gamma_2 \dots$ in D of length at most $2\sum_{k=1}^{m} \rho_k$.

Thus, Γ is a rectifiable curve and connects the centers of B_0 and B_1: $\Gamma(0) = x_0$ and $\Gamma(L) = x_1$ (we assume that Γ is naturally parametrized and L is its length).

Denote by $\delta = \operatorname{dist}(\Gamma, \partial D)$ the distance from Γ to the boundary of D. Consider the domain $\Omega = B_0 \cup B_1 \cup \bigcup_{x\in\Gamma} B(x, \delta)$ consisting of the balls B_0 and B_1 as well as all radius δ balls centered at the points of Γ. Put $\alpha = \min\{\operatorname{dist}(\Gamma, \partial\Omega), r_0, r_1\}$ (or, which is the same, $\alpha = \min\{\delta, r_0, r_1\}$) and $\beta = L + r_0 + r_1 + \delta$.

Verify that Ω is a $J(\alpha, \beta)$ John domain with the selected point x_0. Take $x \in \Omega$. If $x \in B_0$ then conditions (21) hold automatically: we choose as γ the shortest curve connecting x and x_0. Then $l = |\gamma| < r_0 < \beta$, and

$$\operatorname{dist}(\gamma(s), \partial\Omega) \geq \operatorname{dist}(\gamma(s), \partial B_0) \geq s = \frac{sl}{l} \geq \frac{sr_0}{l} \geq \frac{\alpha s}{l} \tag{22}$$

for all $s \in [0, l]$. If $x \in B_1$ then put $\gamma = \gamma_1 \cup \Gamma$, where γ_1 is the shortest curve connecting x and x_1. Put $l_1 = |\gamma_1| < r_1$. Then $l = |\gamma| = l_1 + L < r_1 + L < \beta$, while $\operatorname{dist}(\gamma(s), \partial\Omega) \geq \frac{\alpha s}{l_1} \geq \frac{\alpha s}{l}$ for $s \in [0, l_1]$ and $\operatorname{dist}(\gamma(s), \partial\Omega) \geq \alpha \geq \frac{\alpha s}{l}$ for $s \in [l_1, l_1 + L]$. Consequently, $\operatorname{dist}(\gamma(s), \partial\Omega) \geq \frac{\alpha s}{l}$ for all $s \in [0, l]$.

Suppose that x lies outside both B_0 and B_1. Then $x \in B(\xi, r)$, where ξ is a point on Γ and $r < \delta$. Take as γ the curve consisting of the shortest curve connecting x and ξ and the part of Γ from ξ to x_0. As in the previous two cases, $l_1 = |\gamma_1| < \delta$ and $l = |\gamma| < \delta + L < \beta$, while

$$\operatorname{dist}(\gamma(s), \partial\Omega) \geq \frac{\alpha s}{l_1} \geq \frac{\alpha s}{l} \text{ for } s \in [0, l_1], \quad \operatorname{dist}(\gamma(s), \partial\Omega) \geq \alpha \geq \frac{\alpha s}{l} \text{ for } s \in [l_1, l].$$

Therefore, $\operatorname{dist}(\gamma(s), \partial\Omega) \geq \frac{\alpha s}{l}$ for all $s \in [0, l]$.

Thus, $\Omega \subset D$ is a John domain including the given balls B_0 and B_1. □

Let us present the Poincaré inequality for John domains (see [34]).

Proposition 3 [34, Theorem 4]. *Assume that $p < \nu$ and $p \leq q \leq \frac{\nu p}{\nu - p}$. If U is a $J(\alpha, \beta)$ John domain then every $u \in W_p^1(U)$ satisfies*

$$\|u - c_u \mid L_q(U)\| \leq C\left(\frac{\alpha}{\beta}\right)^{\nu} \operatorname{diam}(U)^{1 - \frac{\nu}{p} + \frac{\nu}{q}} \|\nabla_{\mathscr{L}} u \mid L_p(U)\|, \tag{23}$$

where c_u and C are constants; furthermore, $C > 0$ is independent of u, U, α, and β.

Below we need the following version of Poincaré's inequality (see also [24]).

Lemma 4. *Consider a $J(\alpha, \beta)$ John domain U and a subset $F \subset U$ of positive measure, $|F| > 0$. Then all $u(x) \in W_p^1(U)$ with $p \leq q \leq \frac{\nu p}{\nu - p}$ and $p < \nu$ such that $u|_F = 0$ obey the inequality*

$$\left(\int_U |u(x)|^q \, dx\right)^{\frac{1}{q}} \leq \frac{|U|^{\frac{1}{q}}}{|F|^{\frac{1}{q}}} C\left(\frac{\alpha}{\beta}\right)^{\nu} \operatorname{diam}(U)^{1 - \frac{\nu}{p} + \frac{\nu}{q}} \left(\int_U |\nabla_{\mathscr{L}} u(x)|^p \, dx\right)^{\frac{1}{p}}. \tag{24}$$

Proof. Consider an arbitrary function $u(x) \in W_p^1(U)$ such that $u|_F = 0$, where the subset $F \subset U$ is of positive measure. Put $M = \|u \mid L_q(U)\| > 0$. Then

$$|F| = \int_U (\chi_F(x))^q \, dx \leq \int_U \left|1 - \frac{u(x)|U|^{\frac{1}{q}}}{M}\right|^q dx \leq \frac{|U|}{M^q} \int_U \left|\frac{M}{|U|^{\frac{1}{q}}} - u(x)\right|^q dx.$$

Consequently,

$$M^q|F| \le |U| \|(M|U|^{-\frac{1}{q}} - u) \mid L_q(U)\|^q. \tag{25}$$

The constant c_u of Proposition 3 satisfies

$$|M|U|^{-\frac{1}{q}} - c_u| = ||U|^{-\frac{1}{q}}\|u \mid L_q(U)\| - |U|^{-\frac{1}{q}}\|c_u \mid L_q(U)\|| \le |U|^{-\frac{1}{q}}\|u - c_u \mid L_q(U)\|.$$

Poincaré's inequality (23) yields

$$\begin{aligned}\|(M|U|^{-\frac{1}{q}} - u) \mid L_q(U)\| &\le \|(M|U|^{-\frac{1}{q}} - c_u) \mid L_q(U)\| + \|u - c_u \mid L_q(U)\| \\ &\le 2\|u - c_u \mid L_q(U)\| \le 2C\left(\frac{\alpha}{\beta}\right)^\nu \operatorname{diam}(U)^{1-\frac{\nu}{p}+\frac{\nu}{q}}\|\nabla_{\mathscr{L}} u \mid L_p(U)\|.\end{aligned}$$

Applying (25), we obtain

$$\|u \mid L_q(U)\| \le \frac{|U|^{\frac{1}{q}}}{|F|^{\frac{1}{q}}} 2C\left(\frac{\alpha}{\beta}\right)^\nu \operatorname{diam}(U)^{1-\frac{\nu}{p}+\frac{\nu}{q}}\|\nabla_{\mathscr{L}} u \mid L_p(U)\|.$$

The proof of the lemma is complete. □

Lemma 5. *Assume that $p > \nu$ and $f \in C(D) \cap L^1_p(D)$ satisfies $f(x_0) = 0$ and $f(x_1) = 1$ for some $x_0, x_1 \in B \subset D$. Then*

$$\frac{1}{d(x_0, x_1)^{1-\nu/p}} \le K\|f \mid L^1_p(D)\|. \tag{26}$$

Proof. We need Poincaré's inequality

$$\|u - c_u \mid L_\infty(B)\| \le C \operatorname{diam}(B)^{1-\frac{\nu}{p}}\|\nabla_{\mathscr{L}} u \mid L_p(B)\| \tag{27}$$

of [34, Theorem 4]. Fix a ball B with $x_0, x_1 \in B$. Then

$$\begin{aligned}|f(x_0) - f(x_1)| &\le |f(x_0) - c_f| + |f(x_1) - c_f| \le 2\|f - c_f \mid L_\infty(B)\| \\ &\le Kd(x_0, x_1)^{1-\nu/p}\|\nabla_{\mathscr{L}} f \mid L_p(B)\| \le Kd(x_0, x_1)^{1-\nu/p}\|\nabla_{\mathscr{L}} f \mid L_p(D)\|,\end{aligned} \tag{28}$$

where K is some constant. In our case $|f(x_0) - f(x_1)| = 1$, and consequently

$$\frac{1}{d(x_0, x_1)^{1-\nu/p}} \le K\|f \mid L^1_p(D)\|. \quad \square \tag{29}$$

3. The Set Function

Assume that $D \subset \mathbb{G}$ is an open subset. Consider the collection $\Theta(D)$ of open subsets of D such that
(1) $B \in \Theta(D)$ for every ball $B \subset D$;
(2) $\bigcup_{i=1}^n U_i \in \Theta(D)$ for all disjoint sets $U_1, \dots, U_n \in \Theta(D)$, where $n \in \mathbb{N}$.

Remark 7. Among these collections we have the minimal (all possible unions of finitely many open balls with disjoint closures) and the maximal (all open subsets of D).

Definition 7. A mapping $\Phi : \Theta(D) \mapsto [0, \infty]$ is called a *finitely quasiadditive set function* whenever
(1) for every $x \in D$ there exists δ with $0 < \delta < \operatorname{dist}(x, \partial D)$ such that $0 \le \Phi(B_\delta(x)) < \infty$;
(2) $\sum_{i=1}^k \Phi(U_i) \le \Phi(U)$ for every tuple of disjoint open sets $U_1, \dots, U_k \subset U$, where $U, U_i \in \Theta(D)$ for $i = 1, \dots, k$.

Every finitely quasiadditive function is also countably quasiadditive.

In the definition of set function, if instead of the second condition we require that

$$\sum_{i=1}^{k} \Phi(U_i) = \Phi(U)$$

for every finite tuple $U_i \in \Theta(D)$, with $i = 1, \dots, k$, of disjoint open sets then this function is called *finitely additive*. If this equality extends to countable collections then the function is called *countably additive*.

DEFINITION 8. A quasiadditive function Φ is called *monotone* whenever $\Phi(U_1) \le \Phi(U_2)$ for $U_1, U_2 \in \Theta(D)$ with $U_1 \subset U_2$.

The upper and lower derivatives of a quasiadditive function on the collection $\Theta(D)$ of open subsets are defined as

$$\overline{\Phi}'(x) = \lim_{h\to 0} \sup_{\delta<h} \frac{\Phi(B_\delta)}{\mu(B_\delta)}, \quad \underline{\Phi}'(x) = \lim_{h\to 0} \inf_{\delta<h} \frac{\Phi(B_\delta)}{\mu(B_\delta)}.$$

Here the supremum and infimum are over all open balls B_δ of radius $\delta < h$ containing the point x. If at some point x the upper and lower derivatives coincide, $\overline{\Phi}'(x) = \underline{\Phi}'(x)$, then their common value is called the *derivative* $\Phi'(x)$ *of the set function* Φ.

For every quasiadditive set function we have

Proposition 4 [25]. *Consider a Carnot group* $\mathbb{G}$ *and a quasiadditive set function* Φ *on some system* $\Theta(D)$ *of open subsets of a domain* $D \subset \mathbb{G}$*. Then*

(a) *a finite derivative*

$$\Phi'(x) = \lim_{\delta\to 0} \frac{\Phi(B_\delta)}{|B_\delta|}$$

exists at almost every point $x \in D$*;*

(b) $\Phi'(x)$ *is a measurable function;*

(c) *for every open set* $U \in \Theta(D)$ *we have*

$$\int_U \Phi'(x)\, dx \le \Phi(U).$$

The use of set functions enables us to prove the following Lebesgue-type theorem.

Theorem 3 [25]. *Consider a Carnot group* $\mathbb{G}$ *and a domain* D *in* $\mathbb{G}$*. If* f *belongs to* $L_{1,\mathrm{loc}}(D)$ *then for almost all* $x \in D$ *we have*

$$\lim_{\delta\to 0,\, x\in B_\delta} \frac{1}{|B_\delta|} \int_{B_\delta} |f(y) - f(x)|\, dy = 0.$$

4. The Superposition Operator

Henceforth $D, D' \subset \mathbb{G}$ are connected domains and $\varphi : D \to D'$ is a mapping of class IL^1_p.

Recall the definition of convergence in the seminormed space $L^1_p(D)$.

DEFINITION 9. Say that a sequence of functions $\{f_n\} \in L^1_p(D)$ *converges to* $f \in L^1_p(D)$ *in* $L^1_p(D)$, and write $f_n \to f$, whenever

$$\| f_n - f \mid L^1_p(D) \| \to 0 \quad \text{as } n \to \infty.$$

Observe that if $f_n \to f$ in $L^1_p(D)$ then also $f_n \to f + C$, where $C \in \mathbb{R}$ is an arbitrary constant.

Lemma 6. *Take $f \in L_p^1(D')$ and $f_n \in L_p^1(D') \cap C^\infty(D')$ with $f_n \to f$ in $L_p^1(D')$ as $n \to \infty$ and assume the conditions:*

(1) $f \circ \varphi$ is defined and takes finite values almost everywhere on D;

(2) $f_n \circ \varphi(x) \to f \circ \varphi(x)$ as $n \to \infty$ for almost all $x \in D$.

Then

(1) $f \circ \varphi \in L_p^1(D)$;

(2) $K^{-1}\|f \mid L_p^1(D')\| \le \|f \circ \varphi \mid L_p^1(D)\| \le K\|f \mid L_p^1(D')\|$.

Proof. Step 1. To begin with, consider the case $|f| < C$ (applying cutoffs, we may assume that $|f_n| < 2C$ for all $n \in \mathbb{N}$).

Put $g_n(x) = f_n \circ \varphi(x)$. By Lebesgue's dominated convergence theorem, the sequence $\{g_n\}$ converges to $f \circ \varphi$ in $L_1(B)$ on every ball $B \subset D$. Moreover, the sequence of gradients $\{\nabla_{\mathscr{L}} g_n\}$ is fundamental in $L_p(D)$ and so it has a limit in $L_p(D)$. These properties guarantee the existence of generalized derivatives of the locally summable function $f \circ \varphi$, and these derivatives lie in $L_p(D)$. Consequently, $f \circ \varphi \in L_p^1(D)$.

Step 2. Take an arbitrary function f in $L_p^1(D')$ which we may assume positive since $f = f^+ - f^-$, while $f^+, f^- \in L_p^1(D')$.

Fix a ball $B_0 \subset D$. We may suppose that $f \circ \varphi = 0$ on some set $F \subset B_0$ of positive measure. Indeed, $\{x \in B_0 : f(\varphi(x)) - k_0 \le 0\}$ is of positive measure for some $k_0 \in \mathbb{N}$. Then instead of f we can consider the function $\max\{f(y) - k_0, 0\}$ and instead of $\{f_n\}$, the function $\max\{f_n(y) - k_0, 0\}$. We have

$$f_n \circ \varphi \to 0 \quad \text{almost everywhere on } F \text{ as } n \to \infty. \tag{30}$$

Now consider some nonnegative function $f \in L_p^1(D')$ such that $F = \{x \in B_0 : f \circ \varphi(x) = 0\}$ is of positive measure.

Define the monotone sequence of functions $u_m = g_m \circ \varphi$, where $g_m = \min\{f, m\}$. Since g_m is bounded, Step 1 implies that $u_m \in L_p^1(D)$. In addition, $u_m \to f \circ \varphi$ almost everywhere as $m \to \infty$. Indeed, for almost all $x \in D$ there is m with $f(\varphi(x)) < m$. Then $u_k(x) = f(\varphi(x))$ for all $k > m$.

Given an arbitrary ball $B \subset D$, choose a John domain $U \supset B \cup B_0$ and apply Poincaré's inequality (24) to u_m. For $q = p$ we infer that

$$\begin{aligned}\int\limits_U |u_m(x)|^p\,dx &\le \frac{\operatorname{diam} U}{|F|} Cr^p \int\limits_U |\nabla_{\mathscr{L}} u_m(x)|^p\,dx \\ = \frac{\operatorname{diam} U}{|F|} Cr^p \int\limits_U |\nabla_{\mathscr{L}}(g_m \circ \varphi(x))|^p\,dx &= \frac{\operatorname{diam} U}{|F|} Cr^p \|\varphi^* g_m \mid L_p^1(D)\|^p \\ &\le K \frac{\operatorname{diam} U}{|F|} Cr^p \|g_m \mid L_p^1(D')\|^p \le C_2 \|f \mid L_p^1(D')\|^p.\end{aligned} \tag{31}$$

In the third inequality we use the results of Step 1. Thus, $u_m \in L_p(B)$. Since the monotone increasing functions $u_m = g_m \circ \varphi$ converge on B to $f \circ \varphi$, the Beppo Levi theorem yields $f \circ \varphi \in L_p(B)$. Since the ball $B \subset D$ is arbitrary, the superposition $f \circ \varphi$ is locally summable on D. Observe also that the sequence of gradients $\nabla_{\mathscr{L}} u_m$ is fundamental in $L_p(D)$ since so is the sequence of gradients $\nabla_{\mathscr{L}} g_m$. Indeed,

$$\|\nabla_{\mathscr{L}} g_l - \nabla_{\mathscr{L}} g_m \mid L_p(D')\| \le \int\limits_{\{x \in D' : f(x) \ge m\}} |\nabla_{\mathscr{L}} f|^p\,dx \quad \text{for } l > m.$$

Consequently, $f \circ \varphi \in L_p^1(D)$. Thus, the lemma is justified even in the case that f is not bounded. □

Lemma 7. *If $p > \nu$ then for every $f \in L_p^1(D')$ we have*

(1) $\tilde{f} \circ \varphi \in L_p^1(D)$,

(2) $K^{-1}\|f \mid L_p^1(D')\| \le \|\tilde{f} \circ \varphi \mid L_p^1(D)\| \le K\|f \mid L_p^1(D')\|$,

where $\tilde{f}$ is a continuous representative of f.

Proof. Take $f \in L^1_p(D')$ and a continuous representative $\tilde f$ of f. Then there is a sequence $f_n \in C^\infty(D') \cap L^1_p(D')$, $n \in \mathbb{N}$, such that $f_n \to \tilde f$ in $L^1_p(D')$ and $f_n(x) \to \tilde f(x)$ for all $x \in D'$ (see Remark 5). Observe that the superposition of the continuous function $\tilde f$ with φ is defined almost everywhere on D. Then

$$f_n \circ \varphi(x) \to \tilde f \circ \varphi(x) \quad \text{as } n \to \infty \text{ for almost all } x \in D. \tag{32}$$

Thus, $\tilde f$ meets the conditions of Lemma 6. □

Proposition 5 [24, Lemma 1]. *Assume that $\varphi : D \to D'$ induces the bounded operator*

$$\varphi^* : L^1_p(D') \cap C^\infty(D') \to L^1_q(D), \quad 1 \le q < p \le \infty, \quad \varphi^* f = f \circ \varphi.$$

Then

$$\Phi(A') = \sup_{f \in \overset{\circ}{L}{}^1_p(A') \cap C^\infty(A')} \left(\frac{\|\varphi^* f \mid L^1_q(D)\|}{\|f \mid \overset{\circ}{L}{}^1_p(A')\|} \right)^{\varkappa}, \qquad \text{where } \varkappa = \begin{cases} \frac{pq}{p-q} & \text{for } p \le \infty, \\ q & \text{for } p = \infty, \end{cases}$$

is a bounded countably additive function on a bounded open subset $A' \subset D'$.

Lemma 14 (see below) yields

Corollary 2 [24, Corollary 1]. *The additive set function of Proposition 5 is absolutely continuous.*

Remark 8. In [24] the claims of Corollary 2 and Proposition 5 are proved in $\mathbb{R}^n$. By analogy with [25], the proofs carry over to a Carnot group with obvious modifications.

Lemma 8 [24, Theorem 4]. *If $\varphi : D \to D'$ induces a bounded monomorphism*

$$\varphi^* : L^1_p(D') \cap C^\infty_0(D') \to L^1_p(D), \quad 1 \le p \le \nu,$$

then φ enjoys Luzin's $\mathscr{N}^{-1}$-property.

Proof. Establish firstly that the preimage of every open set is of positive measure. Take an open set $U \subset D'$ and verify that $|\varphi^{-1}(U)| > 0$. Suppose that this fails; i.e., $|\varphi^{-1}(U)| = 0$. Since U is an open set, we can choose a ball $B = B(y_0, r)$ such that $2B = B(y_0, 2r) \subset U$. Take a function $f \in C^\infty_0(D')$ with $f = 1$ on B and $f = 0$ outside $2B$. Hence, $f \not\equiv 0$. On the other hand, $\varphi^* f = 0$ almost everywhere on D. Consequently, $\varphi^* f = 0$, which is false since φ^* is a monomorphism. Therefore, the preimage of an open set cannot be negligible.

Fix the cutoff $\eta \in C^\infty_0(\mathbb{G})$ equal to 1 on $B(0,1)$ and vanishing outside the ball $B(0,2)$. Consider a ball $B = B(y_0, r) \subset D'$ with $B(y_0, 2r) \subset D'$ and the function $f(y) = \eta\big(\delta_{r^{-1}}(y_0^{-1} y)\big) \in L^1_p(D')$, whose seminorm satisfies

$$\big\|f \mid L^1_p(D')\big\| \le C r^{\frac{1}{p} - \frac{1}{\nu}}. \tag{33}$$

Case $p < \nu$. Since φ^* is bounded, $\big\|\varphi^* f \mid L^1_p(D)\big\| \le \|\varphi^*\| \times \big\|f \mid L^1_p(D')\big\|$, we infer from (33) that

$$\big\|\varphi^* f \mid L^1_p(D)\big\| \le \|\varphi^*\| \big\|f \mid L^1_p(D')\big\| \le C_1 \|\varphi^*\| |B|^{\frac{1}{p} - \frac{1}{\nu}}. \tag{34}$$

Fix an open set $U \subset D'$. Choose a covering of D by countably many balls $Q_0, Q_1, Q_2, \dots$ with $Q_l \subset D$ and $\bigcup_l Q_l = D$. Take a negligible set $E \subset D'$ at a positive distance from U. By Luzin's theorem, there is a compact set $T \subset \varphi^{-1}(U)$ of positive measure such that φ is continuous on T, and so $\varphi(T)$ is a compact set as well. In addition, we may assume that $T \subset Q_0$. Consider an open set $V \subset D'$ of finite measure (moreover, of arbitrarily small measure, for instance $|V| = \varepsilon$) with $V \supset E$ and $V \cap \varphi(T) = \varnothing$. Assume that the collections $\{B(y_i, r_i)\} \subset V$ and $\{B(y_i, 2r_i)\} \subset V$ constitute coverings of V with finite multiplicity. The functions $f_i(y) = \eta\big(\delta_{r_i^{-1}}\big(y_i^{-1} y\big)\big)$ satisfy $\varphi^* f_i = 1$ on $\varphi^{-1}(B(y_i, r_i))$ and $\varphi^* f_i = 0$ outside $\varphi^{-1}(B(y_i, 2r_i))$. In particular, $\varphi^* f_i = 0$ on T. In this case (34) becomes

$$\big\|\varphi^* f_i \mid L^1_p(D)\big\| \le C_1 \|\varphi^*\| |B(y_i, r_i)|^{\frac{1}{p} - \frac{1}{\nu}}. \tag{35}$$

In addition, we also have an estimate for the measure of $\varphi^{-1}(B(y_i, r_i))$:

$$|\varphi^{-1}(B(y_i, r_i))| \leq \int\limits_D |\varphi^* f_i|^q \, dx, \tag{36}$$

where $1 \leq q$. Given a ball Q_j in the covering $\{Q_l\}$, by Lemma 3 there is a John domain $\Omega \subset D$ (with $\Omega \in J(\alpha, \beta)$) such that $Q_j \subset \Omega$ and $Q_0 \subset \Omega$. Then we use Poincaré's inequality (24),

$$\biggl(\int\limits_\Omega |g|^{p^*} \, dx\biggr)^{\frac{1}{p^*}} \leq C_2 (\operatorname{diam}(\Omega))^{\frac{\nu}{p^*}} \biggl(\int\limits_\Omega |\nabla_{\mathscr{L}} g|^p \, dx\biggr)^{\frac{1}{p}}, \tag{37}$$

where $q = p^* = \frac{p\nu}{\nu - p}$ and $g \in L^1_p(D)$ is an arbitrary function vanishing on the set $T \subset Q_0 \subset \Omega$ of positive measure, while the constant C_2 is

$$C_2 = \frac{C}{|T|^{\frac{1}{p^*}}} \Bigl(\frac{\alpha}{\beta}\Bigr)^\nu.$$

Insert $g = \varphi^* f_i$ into (37) and apply (35) and (36). We arrive at the chain of inequalities

$$|\varphi^{-1}(B(y_i, r_i)) \cap Q_j|^{\frac{1}{p} - \frac{1}{\nu}} \leq \biggl(\int\limits_\Omega |\varphi^* f_i|^{p^*} \, dx\biggr)^{\frac{1}{p^*}}$$

$$\leq C_2 (\operatorname{diam}(\Omega))^{\frac{\nu}{p^*}} \biggl(\int\limits_\Omega |\nabla_{\mathscr{L}} \varphi^* f_i|^p \, dx\biggr)^{\frac{1}{p}} \leq C_2 (\operatorname{diam}(\Omega))^{\frac{\nu}{p^*}} C_1 \|\varphi^*\| \cdot |B(y_i, r_i)|^{\frac{1}{p} - \frac{1}{\nu}},$$

whence

$$|\varphi^{-1}(B(y_i, r_i)) \cap Q_j| \leq C_4 |B(y_i, r_i)|. \tag{38}$$

Since $\{B(y_i, r_i)\}$ is a covering of V with finite multiplicity, we conclude that

$$\sum_i |\varphi^{-1}(B(y_i, r_i)) \cap Q_j| \leq C_4 \sum_i |B(y_i, r_i)| \leq C_4 \zeta_\nu |V|, \tag{39}$$

where the constant ζ_ν is the multiplicity of the covering. Since $\varphi^{-1}(B(y_i, r_i)) \cap Q_j$ cover $\varphi^{-1}(E) \cap Q_j$, while V is of arbitrarily small measure, we infer that $|\varphi^{-1}(E) \cap Q_j| = 0$ for every ball Q_j in the countable collection $\{Q_l\}$. Consequently, $|\varphi^{-1}(E)| = 0$.

If a negligible set $E \subset D' \setminus U$ satisfies $\operatorname{dist}(E, U) = 0$ then we can exhaust it as $E = \bigcup E_k$, where $E_k = \bigl\{y \in E : \operatorname{dist}(y, U) \geq \frac{1}{k}\bigr\}$. Then $\varphi^{-1}(E) \subset \bigcup \varphi^{-1}(E_k)$. Since $|\varphi^{-1}(E_k)| = 0$, it follows that $|\varphi^{-1}(E)| = 0$. Thus, for all negligible sets $E \subset D' \setminus U$ the preimage $\varphi^{-1}(E)$ is negligible. In other words, if $|E| = 0$ then $|\varphi^{-1}(E \setminus \overline{U})| = 0$.

Take another open set U_1 at a positive distance from U. We can express every $E \subset D'$ as $E = (E \setminus U) \cup (E \setminus U_1)$. If E is negligible then $|\varphi^{-1}(E \setminus U)| = 0$ and $|\varphi^{-1}(E \setminus U_1)| = 0$; consequently, $|\varphi^{-1}(E)| \leq |\varphi^{-1}(E \setminus U)| + |\varphi^{-1}(E \setminus U_1)| = 0$.

Case $p = \nu$. Fix an arbitrary ball $B_1 \subset D$. If $g \in L^1_\nu(B_1)$ then $g \in L^1_q(B_1)$, where $q < \nu$. Consider the bounded operator

$$\varphi^*_B : L^1_\nu(D') \to L^1_q(B_1)$$

defined as $\varphi^*_B f = \varphi^* f|_{B_1}$. In this case (34) becomes

$$\bigl\|\varphi^*_B f \mid L^1_p(B_1)\bigr\| \leq C_1 \Phi(2B)^{\frac{1}{\varkappa}}, \tag{40}$$

where the function Φ is from Proposition 5.

Take a negligible set E and an open set $V \supset E$ of measure $\varepsilon > 0$. Choose a collection of balls $\{B_i\}$ such that $2B_i \subset V$ and both collections $\{B_i\}$ and $\{2B_i\}$ are coverings of V with finite multiplicity. By analogy with the argument in the case $p < \nu$, using Lemma 14 (see below), we obtain

$$|\varphi^{-1}(E) \cap B_1| \le \sum_i |\varphi^{-1}(B_i)| \le C \sum_i \Phi(2B_i) \le C\zeta_N \Phi(V).$$

Applying the absolute continuity of Φ (Corollary 2), we deduce that $|B_1 \cap \varphi^{-1}(E)| = 0$ for every negligible set $E \in D'$. Since the ball B_1 is arbitrary, the claim follows.

Finally, the preimage of an arbitrary negligible set is also negligible; i.e., $\varphi : D \to D'$ enjoys Luzin's $\mathscr{N}^{-1}$-property. $\square$

Lemma 9. *If $p \le \nu$ then every $f \in L_p^1(D')$ satisfies*

(1) *$f \circ \varphi \in L_p^1(D)$;*

(2) *$K^{-1}\|f \mid L_p^1(D')\| \le \|f \circ \varphi \mid L_p^1(D)\| \le K\|f \mid L_p^1(D')\|$.*

Proof. Take $f \in L_p^1(D')$ and a sequence $\{f_n\}$ of functions in $C^\infty(D') \cap L_p^1(D')$ such that $\|f - f_n \mid L_p^1(D')\| \to 0$ and $f_n \to f$ almost everywhere on D' (see Remark 5). Since (1) is a monomorphism (see Remark 1), φ enjoys Luzin's $\mathscr{N}^{-1}$-property (Lemma 8). Consequently, $f \circ \varphi$ is defined almost everywhere on D and $f_n \circ \varphi \to f \circ \varphi$ almost everywhere on D. Then Lemma 6 implies both claims (1) and (2). $\square$

Lemma 10. *If $\varphi \in IL_p^1$ then $\varphi* : L_p^1(D') \cap C^\infty(D') \to L_p^1(D)$ extends by continuity to the operator $\widetilde{\varphi*} : L_p^1(D') \to L_p^1(D)$ and enjoys the properties:*

(1) *we can evaluate $\widetilde{\varphi*} : L_p^1(D') \to L_p^1(D)$ on a given class $[f] \in L_p^1(D')$ as*

$$\widetilde{\varphi^*}([f]) = \begin{cases} f \circ \varphi & \text{for } p \le \nu,\ f \text{ is an arbitrary representative of class } [f], \\ \tilde{f} \circ \varphi & \text{for } p > \nu,\ \tilde{f} \text{ is a continuous representative of class } [f]; \end{cases}$$

(2) *$K^{-1}\|f \mid L_p^1(D')\| \le \|\widetilde{\varphi^*}(f) \mid L_p^1(D)\| \le K\|f \mid L_p^1(D')\|$;*

(3) *$\widetilde{\varphi^*} : L_p^1(D') \to L_p^1(D)$ is an isomorphism.*

Proof. Since $L_p^1(D') \cap C^\infty(D')$ is dense in $L_p^1(D')$, the operator $\varphi* : L_p^1(D') \cap C^\infty(D') \to L_p^1(D)$ extends by continuity to $L_p^1(D')$. Indeed, take $f \in L_p^1(D')$ and choose a sequence $f_n \in L_p^1(D') \cap C^\infty(D')$ with $f_n \to f$ in $L_p^1(D')$. Then the sequence $\varphi^* f_n$ converges in $L_p^1(D)$. On the other hand, we may assume that the same sequence converges pointwise. Basing on Lemma 6, it is natural to put $\lim_{n\to\infty} \varphi^* f_n = f \circ \varphi$ since the superposition $f \circ \varphi$ is defined almost everywhere for $p \le \nu$ (for $p > \nu$ we should consider a continuous representative $\tilde{f} \in L_p^1(D')$).

By Lemmas 7 and 9, every $f \in L_p^1(D)$ satisfies $f \circ \varphi \in L_p^1(D)$ for $p \le \nu$ and $\tilde{f} \circ \varphi \in L_p^1(D)$ for $p > \nu$, whence we obtain claims (1) and (2). Property (2) implies that $\widetilde{\varphi^*}$ is an isomorphism. $\square$

Assume henceforth that φ^* is defined on $L_p^1(D')$.

5. The Main Theorems

The following definition of quasi-isometric homeomorphism is equivalent to Definition 1.

Definition 10. A homeomorphism $\Phi : D \to D'$, where $D, D' \subset \mathbb{G}$, in the nonholonomic Sobolev class $W^1_{1,\mathrm{loc}}(D, \mathbb{G})$ is called a *quasi-isometry* whenever

$$|D\Phi(x)| \le M \quad \text{and} \quad 0 < \alpha \le |\det D\Phi(x)| \tag{41}$$

for almost all $x \in D$, where the constants M and α are independent of x.

Using Hadamard's inequality $|\det D\Phi(x)| \le |D\Phi(x)|^\nu$, we infer from (41) that

$$L^{-1} \le |D\Phi(x)| \le L, \tag{42}$$

where the constant L satisfies $L^{-1} \le \alpha^{\frac{1}{\nu}}$ and $L \ge M$.

Observe also that Definition 10 is equivalent to the definition in [35].

Definition 11. Consider an open set U in a Carnot group $\mathbb{G}$ and a homeomorphism $\Phi : U \to \mathbb{G}$ of the Sobolev class $W^1_{1,\mathrm{loc}}(U, \mathbb{G})$. A mapping Φ is a quasi-isometry whenever the Jacobian $J(x, \Phi)$ keeps the sign on U and $L^{-1}|\xi| \le |D\Phi(x)\xi| \le L|\xi|$ for all $\xi \in V_1$ and almost all $x \in U$, where $L \ge 1$.

The following lemma provides a link between quasi-isometric and locally bi-Lipschitz mappings (see Definition 5).

Lemma 11 [35, Lemma 1]. *A homeomorphism $\Phi : D \to D'$ is a quasi-isometry if and only if Φ is a locally bi-Lipschitz mapping with the same bi-Lipschitz constant.*

Lemma 12. *Given two open sets $D, D' \subset \mathbb{G}$ with $|D| < \infty$ and a measurable mapping $\varphi : D \to D'$ defined almost everywhere on D, there is an increasing sequence $\{T_k\} \subset \operatorname{Dom}\varphi \subset D$ of compact sets such that φ is continuous on each T_k and $\left|D \setminus \bigcup_k T_k\right| = 0$.*

Proof. By Luzin's theorem, there is a compact set $P_1 \subset \operatorname{Dom}\varphi$ such that φ is continuous on P_1 and $|\operatorname{Dom}\varphi \setminus P_1| < 1$. Similarly, there is a compact set $P_2 \subset \operatorname{Dom}\varphi \setminus P_1$ such that φ is continuous on P_2 and $|(\operatorname{Dom}\varphi \setminus P_1) \setminus P_2| < \frac{1}{2}$, and so on. This yields a sequence of sets $\{P_i\}$. Put $T_k = \bigcup_1^k P_i$; then $T_k \subset T_{k+1} \subset \operatorname{Dom}\varphi$. The mapping φ is continuous on each T_k since T_k amounts to a finite union of disjoint compact sets $P_1, \dots, P_k$, on each of which φ is continuous. In addition, $|D \setminus T_k| = |\operatorname{Dom}\varphi \setminus T_k| < \frac{1}{k}$ for every $k \in \mathbb{N}$. Consequently, $\left|D \setminus \bigcup_k T_k\right| = 0$. $\square$

Thus, we may assume that the domain of φ is

$$\operatorname{Dom}_1 \varphi = \bigcup_k T_k. \tag{43}$$

Remark 9. We can choose $\widetilde{T}_k \subset T_k$ (where T_k are the sets of Lemma 12) to contain only the points of nonzero density. Then φ is continuous on each $\widetilde{T}_k$ and $\left|D \setminus \bigcup_k \widetilde{T}_k\right| = 0$.

Proof. Indeed, assume that $\widetilde{T}_k$ is the set of points of nonzero density of T_k. Then $|T_k \setminus \widetilde{T}_k| = 0$, $\widetilde{T}_{k+1} \supset \widetilde{T}_k$, and $\left|D \setminus \bigcup_k \widetilde{T}_k\right| = 0$. $\square$

Remark 10. Lemma 12 is also valid in the case of an unbounded domain D.

Proof. We can express D as a countable union of disjoint sets: $D = \bigcup_i D_i$, where $D_1 = D \cap B(0, 1)$, $D_2 = (D \cap B(0, 2)) \setminus \overline{D}_1$, and so on. By Lemma 12, for each D_i there is an increasing (with respect to the index k) sequence $T^i_k \subset \operatorname{Dom}\varphi$ of compact sets such that $\left|D^i \setminus \bigcup_k T^i_k\right| = 0$. Putting $T_1 = T^1_1$, $T_2 = T^1_2 \cup T^2_2$, $T_3 = T^1_3 \cup T^2_3 \cup T^3_3$, and so on, we obtain an increasing sequence $T_k \subset \operatorname{Dom}\varphi$ of compact sets. In particular, $\left|D^i \setminus \bigcup_k T_k\right| = 0$ since $\bigcup_k T^i_k \subset \bigcup_k T_k$. Then $D \setminus \bigcup_k T_k = \bigcup_i \left(D^i \setminus \bigcup_k T_k\right)$. Consequently, $\left|D \setminus \bigcup_k T_k\right| = 0$, as a countable union of negligible sets. $\square$

Below we need the obvious properties of continuous functions.

Proposition 6. (1) *Consider two continuous functions f and g on a set T consisting of points of positive density. If $f = g$ almost everywhere on T then $f = g$ everywhere on T.*

(2) *Every continuous function $f : D \to \mathbb{R}$ is uniquely determined by its values on a dense subset T of D.*

5.1. The case $p > \nu$.

Lemma 13. *Assume that $p > \nu$ and consider a measurable mapping $\varphi : D \to D'$ (where D and D' are domains in $\mathbb{G}$). Suppose that*

(1) *φ is continuous on some set $T \subset D$ and all points of T are points of positive density in $\mathbb{G}$;*

(2) *for every locally Lipschitz function f with compact support in D' we have $f \circ \varphi \in L^1_p(D)$ and*

$$\left\|f \circ \varphi \mid L^1_p(D)\right\| \le C\left\|f \mid L^1_p(D')\right\|. \tag{44}$$

Then, given two distinct points $x_0, x_1 \in T$ with $B(x_1, d(x_0, x_1)) \subset D$, we have

$$d(\varphi(x_0), \varphi(x_1)) \leq c d(x_0, x_1). \tag{45}$$

If condition (2) *is replaced with*

(2′) *for every locally Lipschitz function g with compact support in D there exists a function $f \in L^1_p(D')$ such that $g = f \circ \varphi$ and*

$$\|f \mid L^1_p(D')\| \leq C' \|f \circ \varphi \mid L^1_p(D)\|; \tag{46}$$

then, given two points $x_0, x_1 \in T$ with $B(\varphi(x_1), d(\varphi(x_0), \varphi(x_1))) \subset D'$ and $\varphi(x_0) \neq \varphi(x_1)$, we have

$$d(x_0, x_1) \leq c' d(\varphi(x_0), \varphi(x_1)). \tag{47}$$

Here the constants c and c' depend only on ν, p, C, and C'.

Proof. If $\varphi(x_0) = \varphi(x_1)$ then (45) is obvious.

Assume that $\varphi(x_0) \neq \varphi(x_1)$ and conditions (1) and (2) hold. Consider the function

$$f(y) = \left(1 - \frac{d(y, \varphi(x_1))}{d(\varphi(x_0), \varphi(x_1))}\right)^+, \tag{48}$$

where $(\cdot)^+$ stands for the positive part of a number. Observe that $f(\varphi(x_0)) = 0$ and $f(\varphi(x_1)) = 1$. Lemma 1 implies the estimates

$$\|f \mid L^1_p(D')\| \leq \|f \mid L^1_p(\mathbb{G})\| \leq \frac{\varkappa}{d(\varphi(x_0), \varphi(x_1))^{1-\nu/p}}, \tag{49}$$

where the constant $\varkappa$ depends on the measure of the unit ball.

The support of f lies in the ball $B(\varphi(x_1), d(\varphi(x_1), \varphi(x_0)))$. Take a continuous representative g of $f \circ \varphi$. By Lemma 10, the continuous function g coincides with $f \circ \varphi$ almost everywhere on D. But on the set T consisting of the points of positive density the mapping φ is continuous. Therefore, $g|_T(x) = f \circ \varphi|_T(x)$ for all $x \in T$. Consequently, $g(x_0) = 0$ and $g(x_1) = 1$. By Lemma 5, every continuous function $g \in L^1_p(D)$ with $g(x_0) = 0$ and $g(x_1) = 1$, $x_0, x_1 \in B \cap T \subset D$, satisfies

$$\frac{1}{d(x_0, x_1)^{1-\nu/p}} \leq K \|g \mid L^1_p(D)\|. \tag{50}$$

Recalling that $g = f \circ \varphi$, where f is defined in (48), and using (50), (49), and the inequality $\|g \mid L^1_p(D)\| \leq C \|f \mid L^1_p(D')\|$, we obtain

$$\frac{1}{d(x_0, x_1)^{1-\nu/p}} \leq K \|g \mid L^1_p(D)\| \leq KC \|f \mid L^1_p(D')\| \leq \frac{KC\varkappa}{d(\varphi(x_0), \varphi(x_1))^{1-\nu/p}}. \tag{51}$$

Hence,

$$d(\varphi(x_0), \varphi(x_1)) \leq c d(x_0, x_1), \tag{52}$$

where the constant c depends only on ν, p, and C.

Assume conditions (1) and (2′). Consider

$$g(x) = \left(1 - \frac{d(x, x_1)}{d(x_0, x_1)}\right)^+. \tag{53}$$

Observe again that $g(x_0) = 0$ and $g(x_1) = 1$, as well as (Lemma 1)

$$\|g \mid L^1_p(D)\| \leq \frac{\varkappa}{d(x_0, x_1)^{1-\nu/p}}. \tag{54}$$

Take $f \in L^1_p(D')$ such that $g = f \circ \varphi$ (we may assume that f is continuous). Since φ is continuous on T, we have $f \circ \varphi|_T(x) = g|_T(x)$ for all $x \in T$ (Proposition 6). Then $f(\varphi(x_0)) = 0$ and $f(\varphi(x_1)) = 1$, and Lemma 5 yields

$$\frac{1}{p(\varphi(x_0), \varphi(x_1))^{1-\nu/p}} \le K' \| f \mid L^1_p(D') \|. \tag{55}$$

Since $\| f \mid L^1_p(D') \| \le C' \| g \mid L^1_p(D) \|$, we deduce from (54) and (55) that

$$\frac{1}{d(\varphi(x_0), \varphi(x_1))^{1-\nu/p}} \le K' \| f \mid L^1_p(D') \| \le K'C' \| g \mid L^1_p(D) \| \le \frac{K'C'\varkappa}{d(x_0, x_1)^{1-\nu/p}}. \tag{56}$$

Hence,

$$d(x_0, x_1) \le c' d(\varphi(x_0), \varphi(x_1)), \tag{57}$$

where the constant c' depends only on ν, p, and C'. $\square$

Theorem 4. *Assume that $p > \nu$ and take two domains $D, D' \subset \mathbb{G}$. If a measurable mapping $\varphi : D \to D'$ is such that every bounded $f \in L^1_p(D')$ satisfies the conditions*

(1) $\tilde{f} \circ \varphi \in L^1_p(D)$,

(2) $K^{-1} \| f \mid L^1_p(D') \| \le \| \tilde{f} \circ \varphi \mid L^1_p(D) \| \le K \| f \mid L^1_p(D') \|$,

where $\tilde{f}$ is a continuous representative of f and K is a positive constant, then φ coincides almost everywhere with some quasi-isometry.

Proof. By Lemma 12 together with Remarks 9 and 10, there is an increasing sequence of sets $\{\widetilde{T}_k\} \subset D$ such that φ is continuous on each $\widetilde{T}_k$ and $|D \setminus \bigcup_k \widetilde{T}_k| = 0$, where T_k contains only points of nonzero density. Then we may assume that the domain of φ is the set

$$\operatorname{Dom}_2 \varphi = \bigcup_k \widetilde{T}_k. \tag{58}$$

Given two distinct points $x_0, x_1 \in \operatorname{Dom}_2 \varphi$, we have $x_0, x_1 \in \widetilde{T}_k$ for some index k. Verify firstly that $\varphi(x_0) \ne \varphi(x_1)$ for two distinct points. Suppose that, on the contrary, $\varphi(x_0) = \varphi(x_1)$, and consider a continuous function $f \in L^1_p(D)$ with $f(x_0) \ne f(x_1)$ (which obviously exists since x_0 and x_1 are at a positive distance). By Lemma 10, there is a function $g \in L^1_p(D')$ with $f = \varphi^* g$. Lemma 7 implies that $f = \varphi^* g = \tilde{g} \circ \varphi$ almost everywhere on $\widetilde{T}_k$, where $\tilde{g}$ is a continuous representative of g. The mapping φ is continuous on $\widetilde{T}_k$; therefore, $\tilde{g} \circ \varphi$ is also continuous on $\widetilde{T}_k$, and so $f|_{\widetilde{T}_k}(x) = g \circ \varphi|_{\widetilde{T}_k}(x)$ for all $x \in \widetilde{T}_k$, since the points of $\widetilde{T}_k$ are of positive density (Proposition 6). But then $f(x_0) = \tilde{g}(\varphi(x_0)) = \tilde{g}(\varphi(x_1)) = f(x_1)$; i.e., $f(x_0) = f(x_1)$, which contradicts the choice of f. Thus, $\varphi(x_0) \ne \varphi(x_1)$ for $x_0 \ne x_1$.

In addition, condition (1) of Lemma 13 holds if we take $\widetilde{T}_k$ as T.

Since $\varphi^* : L^1_p(D') \to L^1_p(D)$ is a bounded operator, every Lipschitz function f with compact support satisfies $f \circ \varphi \in L^1_p(D)$ and

$$\| f \circ \varphi \mid L^1_p(D) \| \le C \| f \mid L^1_p(D') \|.$$

Therefore, condition (2) of Lemma 13 holds. Consequently,

$$d(\varphi(x_0), \varphi(x_1)) \le c d(x_0, x_1)$$

provided that $B(x_0, d(x_0, x_1)) \subset D$, $x_0, x_1 \in \operatorname{Dom}_2 \varphi$.

Let us justify the inequality inverse to (52). Since the operator $\varphi^* : L^1_p(D') \to L^1_p(D)$ is an isomorphism, for every Lipschitz function g with compact support in D there exists $f \in L^1_p(D')$ satisfying $g = f \circ \varphi$ and

$$\| f \mid L^1_p(D') \| \le C' \| f \circ \varphi \mid L^1_p(D) \|.$$

In other words, condition ($2'$) of Lemma 13 holds. Hence,

$$d(x_0, x_1) \le c' d(\varphi(x_0), \varphi(x_1)) \tag{59}$$

provided that $B(\varphi(x_0), d(\varphi(x_0), \varphi(x_1))) \subset D'$, $x_0, x_1 \in \operatorname{Dom}_2 \varphi$. From (52) and (59) we infer that

$$\frac{1}{c'} d(x_0, x_1) \le d(\varphi(x_0), \varphi(x_1)) \le c d(x_0, x_1) \tag{60}$$

for sufficiently close points $x_0, x_1 \in \operatorname{Dom}_2 \varphi$. Indeed, the left relation holds provided that $d(x_0, x_1) < c^{-1} d(\varphi(x_0), \partial D')$ since in this case $d(\varphi(x_0), \varphi(x_1)) \le c d(x_0, x_1) < d(\varphi(x_0), \partial D')$, and so

$$B(\varphi(x_0), d(\varphi(x_0), \varphi(x_1))) \subset D'.$$

Observe that (60) holds whenever

$$d(x_0, x_1) < r_{x_0} = \min(d(x_0, \partial D), c^{-1} d(\varphi(x_0), \partial D')).$$

Choose a positive number $\rho_{x_0} < r_{x_0}$ so that

$$d(x, y) < d(x, \partial D) \quad \text{and} \quad d(x, y) < c^{-1} d(\varphi(x), \partial D')$$

for all points $x, y \in B(x_0, \rho_{x_0}) \cap \operatorname{Dom}_2 \varphi$. If the first inequality holds then it suffices to choose ρ_{x_0} such that $0 < \rho_{x_0} < r_{x_0}/3$. Indeed, in this case we obtain

$$d(x, y) \le d(x, x_0) + d(x_0, y) \le 2\rho_{x_0} < r_{x_0} - \rho_{x_0} < d(x, \partial D).$$

To ensure the second inequality, fix arbitrarily $0 < \rho_{x_0} < r_{x_0}/3$ such that

$$\operatorname{diam} \varphi(B(x_0, \rho_{x_0})) < c^{-1} c' \operatorname{dist}(\varphi(B(x_0, \rho_{x_0})), \partial D').$$

We can do this since $\operatorname{diam} \varphi(B(x_0, \rho))$ decreases as ρ does, while $\operatorname{dist}(\varphi(B(x_0, \rho)), \partial D')$ increases as ρ decreases. With this choice of ρ_{x_0}, we obtain

$$d(\varphi(x), \varphi(y)) \le c d(x, y) \le c c'^{-1} \operatorname{diam} \varphi(B(x_0, \rho_0)) < \operatorname{dist}(\varphi(B(x_0, \rho_{x_0})), \partial D') \le d(\varphi(x), \partial D').$$

Therefore, for all points $x, y \in B(x_0, \rho_{x_0}) \cap \operatorname{Dom}_2 \varphi \subset D$ we have $B(\varphi(x), d(\varphi(x), \varphi(y))) \subset D'$, and so,

$$\frac{1}{c'} d(x, y) \le d(\varphi(x), \varphi(y)) \le c d(x, y).$$

Put $D_1 = \bigcup_{x_0 \in \operatorname{Dom}_2 \varphi} B(x_0, \rho_{x_0})$. Then $D_1 \subset D$ and $|D \setminus D_1| = 0$. In addition, $\varphi : \operatorname{Dom}_2 \varphi \to D'$ extends by continuity to a locally bi-Lipschitz mapping $\Phi : D_1 \to D' \subset \mathbb{G}$. Observe that $\Phi : D_1 \to D' \subset \mathbb{G}$ is injective. Suppose that, on the contrary, $\Phi(x) = \Phi(y)$ for some distinct points $x, y \in D_1$. Then there are balls $B(x, r)$ and $B(y, r)$ with disjoint closures, and $\Phi(B(x, r)) \cap \Phi(B(y, r))$ is an open set. It is obvious that we cannot obtain an arbitrary function $f \in L^1_p(D)$ vanishing on $B(x, r)$ equal to 1 on $B(y, r)$ as the image $\varphi^*(g)$ of some continuous function $g \in L^1_p(D')$ since $\Phi(D_1 \setminus \operatorname{Dom}_2 \varphi)$ is negligible. Consequently, $\Phi : D_1 \to D' \subset \mathbb{G}$ is a homeomorphism.

Consider a collection $Q_L = B(e, L)$, $L \in \mathbb{N}$, of balls and a smooth compactly supported function

$$\eta_L(y) = \begin{cases} 1, & y \in Q_L, \\ 0, & y \notin Q_{L+1}. \end{cases}$$

Take $x \in D \setminus D_1$ and suppose that there exists a sequence $\{x_n \in D_1\}$ with $\lim_{n\to\infty} x_n = x$, while the sequence of images $\{\Phi(x_n)\}$ is bounded: there is a number L such that $\{\Phi(x_n)\} \subset Q_{L-1}$ for all $n \in \mathbb{N}$. Then it is impossible that for some sequence $\{y_n \in D_1\}$ with $\lim_{n\to\infty} y_n = x$ the sequence of images $\{\Phi(y_n)\}$ is unbounded. Indeed, the function $\varphi^*(\eta_L)$ is continuous and at the points of $\operatorname{Dom}_2 \varphi$

coincides with the superposition $\eta_L \circ \varphi$. Since $\mathrm{Dom}_2\,\varphi$ is dense in D_1, it follows that the continuous function $\varphi^*(\eta_L)$ equals the superposition $\eta_L \circ \Phi$ at all points of D_1. Hence, $\varphi^*(\eta_L)(x_n) = 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \varphi^*(\eta_L)(y_n) = 0$, which contradicts the continuity of $\varphi^*(\eta_L)$.

Consider continuous functions $y_i \cdot \eta_L \in L^1_p(\mathbb{G})$. Then $\varphi^*(y_i \cdot \eta_L)(x) \in L^1_p(D)$ is a continuous function coinciding pointwise with $\Phi_i(x) \cdot \eta_L(\Phi(x))$ on D_1. The condition $\Phi_i(x_n) = \Phi_i(x_n) \cdot \eta_L(\Phi(x_n))$ for $\Phi(x_n) \in Q_L$ implies that the sequence $\Phi(x_n)$ has a limit. The mapping $\Phi : D_1 \to \mathbb{G}$ has a limit as $y \to x$ in D_1 by Heine's criterion. Putting the value of $\Phi(x)$ equal to this limit, we obtain the continuous extension of Φ at x.

It remains to consider the set of points

$$F = \{x \in D \setminus D_1 : d(\varphi(x_n)) \to \infty \text{ for every sequence } x_n \to x,\ x_n \in D_1\}$$

which is closed. Suppose that $F \neq \varnothing$. Choose a point $x \in D_1$ and a ball $B(x,\rho) \subset D_1$ with $\overline{B(x,\rho)} \cap F \neq \varnothing$. Take $y \in \overline{B(x,\rho)} \cap F$. There is a curve $\gamma \subset B(x, d(x,y))$ of finite length connecting x and y. The image $\Phi(\gamma)$ is also of finite length (see the proof of Lemma 1 in [35]). Consequently, there exists a sequence $x_n \in D_1$ such that the sequence $\Phi(x_n)$ is bounded. The latter contradicts the definition of F. Therefore, $F = \varnothing$.

Thus, we have constructed a mapping $\Phi : D \to \mathbb{G}$ which is obviously continuous. Let us show that Φ is open. Fix $x \in D \setminus D_1$ and put $z = \Phi(x)$. The hypotheses of the theorem imply (for the details, see the proof of Lemma 22 below) that $\Phi : D \to \mathbb{G}$ is of Sobolev class $W^1_{\nu,\mathrm{loc}}(D)$ and induces the bounded superposition operator $\Phi^* : L^1_\nu(\mathbb{G}) \to L^1_\nu(D)$. Hence, we infer that the preimage $\Phi^{-1}(z)$ has $(1,\nu)$-capacity zero, and so the Hausdorff ν-measure zero (for the details, see [26]). Hence, we deduce in the standard fashion that $\Phi(x) \notin \Phi(S(x,r))$ for some ball $B(x,r) \subset D$.

If $\Phi(x)$ belongs to a bounded component of the complement $\mathbb{G} \setminus \Phi(S(x,r))$ then there are points $y \in B(x,r) \cap D_1$ whose images $\Phi(y)$ lie in the same connected component, and $\Phi(y)$ is $\mathscr{P}$-differentiable at y with nondegenerate $\mathscr{P}$-differential. Then the degree of Φ at $\Phi(y)$ is nonzero. This implies that the image $\Phi(B(x,r))$ is a neighborhood of $\Phi(x)$. Thus, in this case Φ is an open mapping.

It remains to exclude the possibility that $\Phi(x)$ belongs to an unbounded component of the complement $\mathbb{G} \setminus \Phi(S(x,r))$. If this happens then similarly there are points $y \in B(x,r) \cap D_1$ for which the image $\Phi(y)$ belongs to an unbounded connected component and $\Phi(y)$ is $\mathscr{P}$-differentiable at y with nondegenerate $\mathscr{P}$-differential. Then the degree of Φ at $\Phi(y)$ is nonzero, which is obviously impossible.

Since $\Phi : D \to \mathbb{G}$ is open, we deduce that Φ is injective by analogy with the injectivity of $\Phi : D_1 \to \mathbb{G}$ obtained above. Consequently, $\Phi : D \to \mathbb{G}$ is a quasi-isometric homeomorphism in accordance with Definition 10.

The proof of Theorem 4 is complete.

5.2. The case $p < \nu$.

Lemma 14 [25, Lemma 10]. *If a monotone countably additive function Φ is defined on open subsets of an open set $D \subset \mathbb{G}$ then for every open set $U \subset D$ with $U \neq \mathbb{G}$ there exists a sequence $\{B_j\}$ of Euclidean balls such that*

(1) *the families $\{B_j\}$ and $\{2B_j\}$ constitute coverings of U with finite multiplicity;*
(2) $\sum_{j=1}^{\infty} \Phi(2B_j) \le \zeta_N \Phi(U)$, *where ζ_N is a constant depending only on the dimension N.*

Lemma 15. *Assume that $p \le \nu$ and consider $D, D' \subset \mathbb{G}$ and a mapping $\varphi : D \to D'$ of class IL^1_p. Then we can remove from the domain of φ a negligible set so that the property holds on the new domain* $\mathrm{Dom}_3\,\varphi$*: given two balls $B_1, B_2 \subset D$ with $\overline{B}_1 \cap \overline{B}_2 = \varnothing$, the intersection of their images is negligible:*

$$|\varphi(B_1 \cap \mathrm{Dom}_3\,\varphi) \cap \varphi(B_2 \cap \mathrm{Dom}_3\,\varphi)| = 0.$$

Proof. Split the proof of the lemma into several steps:

Step 1. Take a countable everywhere dense subset $\{z_l\}$, $l \in \mathbb{N}$, of $\operatorname{Dom}\varphi$ and a collection of closed balls $\{\overline{B}_{ml} = \overline{B}(z_l, \frac{1}{m})\} \subset D$, where $m \in \mathbb{N}$. Take two disjoint balls $\overline{B}_{ml}$ and $\overline{B}_{ks}$ in this collection and a locally Lipschitz function $f \in L^1_p(D)$ satisfying

$$f_{mlks}(x) = \begin{cases} 0, & \text{if } x \in \overline{B}_{ml}, \\ 1, & \text{if } x \in \overline{B}_{ks}. \end{cases} \tag{61}$$

Lemma 10 yields a function $g_{mlks} \in L^1_p(D')$ such that $\varphi^* g_{mlks}(x) = f_{mlks}(x)$ everywhere in D with the exception of some negligible set Σ_{mlks}. The union $\Sigma = \bigcup \Sigma_{mlks}$ over all possible indices m, l, k, and s is negligible. Remove Σ from the domain of φ. Denote this restricted domain by $\operatorname{Dom}_0 \varphi$. For all points $x \in \operatorname{Dom}_0 \varphi$ we have $\varphi^* g_{mlks}(x) = f_{mlks}(x)$, where m, l, k, and s are all possible indices.

Step 2. By Lemma 12, we can remove from $\operatorname{Dom}_0 \varphi$, which is the domain of φ, a negligible set and obtain the restricted domain $\operatorname{Dom}_1 \varphi \subset \operatorname{Dom}_0 \varphi \subset D$ enjoying the properties

$$|D \setminus \operatorname{Dom}_1 \varphi| = 0 \quad \text{and} \quad \operatorname{Dom}_1 \varphi = \bigcup_k T_k, \tag{62}$$

where T_k is the increasing sequence of compact sets of Remark 10.

Step 3. Put $F_{ml} = \overline{B}_{ml} \cap \operatorname{Dom}_1 \varphi$ and observe that F_{ml} are Borel sets of positive measure. Consider all possible pairs of sets $F_{m_i l_i}, F_{m_j l_j} \subset \operatorname{Dom}_1 \varphi$ (for brevity, we denote them by F_i and F_j) such that

(a) the closed balls $\overline{B}_{m_i l_i}$ and $\overline{B}_{m_j l_j}$ are disjoint;

(b) $|\varphi(F_i) \cap \varphi(F_j)| > 0$ (i.e., the intersection of the images is of positive measure).

As $\varphi(F_i)$ and $\varphi(F_j)$ are Borel sets, they are measurable.

Put $E_{ij} = \varphi^{-1}(\varphi(F_i) \cap \varphi(F_j))$. Consider the two main cases:

(1) both $E_{ij} \cap F_i$ and $E_{ij} \cap F_j$ are of positive measure;

(2) either $E_{ij} \cap F_i$ or $E_{ij} \cap F_j$ is negligible.

Verify that case 1 contradicts the properties of φ. The inclusions $F_i \subset \overline{B}_{m_i l_i}$ and $F_j \subset \overline{B}_{m_j l_j}$ yield (the function used below are defined in (61)):

(c) on the one hand, $g_{m_i l_i m_j l_j}(x) = 0$ at all points F_i since $g_{m_i l_i m_j l_j}(y) = f_{m_i l_i m_j l_j}(x)$ for all $y = \varphi(x) \in F_i$ and $f_{m_i l_i m_j l_j}(x) = 0$ for all $x \in F_i$;

(d) on the other hand, $g_{m_i l_i m_j l_j}(x) = 1$ at all points F_j since $g_{m_i l_i m_j l_j}(y) = f_{m_i l_i m_j l_j}(x)$ for almost all $y = \varphi(x) \in F_j$, and $f_{m_i l_i m_j l_j}(x) = 1$ for $x \in F_j$.

Hence, case 1 is impossible. Thus, for $F_i, F_j \subset \operatorname{Dom}_1 \varphi$ we have either $|\varphi(F_i) \cap \varphi(F_j)| = 0$, which contradicts (b), or case 2.

Step 4. Proceed to case 2. Put $E_0 = \bigcup (E_{ij} \cap F_j)$, where the union is over all indices i and j satisfying $|E_{ij} \cap F_j| = 0$. We have $|E_0| = 0$.

If F_i and F_j lie in disjoint closed balls then

$$|\varphi(F_i \setminus E_0) \cap \varphi(F_j \setminus E_0)| = 0.$$

Removing from the domain of φ the negligible set E_0, assume now that the domain is

$$\operatorname{Dom}_3 \varphi = \operatorname{Dom}_1 \varphi \setminus E_0. \tag{63}$$

Take two balls $B_1, B_2 \subset D$ with $\overline{B}_1 \cap \overline{B}_2 = \varnothing$. Since B_1 and B_2 are at a positive distance, we can choose two tuples $\{F_i\}$ and $\{F_j\}$ with $B_1 \cap \operatorname{Dom}_3 \varphi = \bigcup_i F_i \setminus E_0$ and $B_2 \cap \operatorname{Dom}_3 \varphi = \bigcup_i F_j \setminus E_0$. Since

$$\varphi(B_1 \cap \operatorname{Dom}_3 \varphi) = \bigcup_i \varphi(F_i \setminus E_0) \quad \text{and} \quad \varphi(B_2 \cap \operatorname{Dom}_3 \varphi) = \bigcup_j \varphi(F_j \setminus E_0),$$

we obtain

$$|\varphi(B_1 \cap \operatorname{Dom}_3 \varphi) \cap \varphi(B_2 \cap \operatorname{Dom}_3 \varphi)| = \Big| \bigcup_i \varphi(F_i \setminus E_0) \cap \bigcup_j \varphi(F_j \setminus E_0) \Big|$$

$$\leq \sum_{i,j} |\varphi(F_i \setminus E_0) \cap \varphi(F_j \setminus E_0)|.$$

All terms in the last sum vanish; consequently,

$$|\varphi(B_1 \cap \mathrm{Dom}_3\,\varphi) \cap \varphi(B_2 \cap \mathrm{Dom}_3\,\varphi)| = 0.$$

Thus, the proof of Lemma 15 is complete. $\square$

Fix a ball $Q \subset \mathbb{G}$ and define the set function

$$\Psi_Q : B \mapsto |\varphi(B \cap \mathrm{Dom}_3\,\varphi) \cap Q|; \tag{64}$$

i.e., Ψ_Q assigns to each ball $B \subset D$ the measure of the intersection of its image with Q. By Lemma 15, the function Ψ_Q enjoys the additivity property $\Psi_Q(B_1 \cup B_2) = \Psi_Q(B_1) + \Psi_Q(B_2)$ for all balls $B_1, B_2 \subset D$ with $\overline{B}_1 \cap \overline{B}_2 = \varnothing$. Using this property and arguing as in the proof of Theorem 3 of [25] (see Proposition 4), we can show that the derivative

$$\Psi'_Q(x) = \lim_{r \to 0} \frac{\Psi_Q(B(x,r))}{|B(x,r)|} \tag{65}$$

is defined and finite for almost all $x \in D$ while satisfying the inequality

$$\int\limits_U \Psi'_Q(x)\,dx \le \Psi_Q(U), \tag{66}$$

where U is a finite union of balls with disjoint closures. Denote by Σ_Ψ the negligible set on which the derivative Ψ'_Q is either undefined or equals ∞. Then a finite derivative Ψ'_Q exists at all points of the complement $D \setminus \Sigma_\Psi$.

Lemma 16. *The mapping φ enjoys Luzin's $\mathscr{N}$-property on the complement $D \setminus \Sigma_\Psi$.*

Proof. Put $A_k = \{x \in D \setminus \Sigma_\Psi : \Psi'_Q(x) < k\}$. Then $D \setminus \Sigma_\Psi = \bigcup_k A_k$. Take a negligible set $E_k \subset A_k$. We may assume that E_k is bounded. For every $\varepsilon > 0$ there exists an open set $U_\varepsilon \supset E_k$ with $|U_\varepsilon| < \varepsilon$. The definition of A_k and (65) imply that for each $x \in E_k$ there is a number $r_x > 0$ with $B(x,r) \subset U_\varepsilon$ and $|\varphi(B(x,r))| < k|B(x,r)|$ for every number $0 < r < r_x$. By Vitali's lemma, we can extract from the family of balls $\mathscr{B} = \{B(x,r) : x \in \Sigma_k, B(x,r) \subset U_\varepsilon,\ 0 < r < r_\varepsilon\}$ a countable family $\{B_j\}$ such that the conditions hold: $\overline{B}_i \cap \overline{B}_j = \varnothing$ for $i \ne j$ and $E_k \subset \bigcup_j cB_j$, where c is a constant depending only on ν. Moreover, $cB(x,r) \subset U_\varepsilon$ and $|\varphi(cB(x,r))| < k|cB(x,r)|$. Then

$$|\varphi(E_k)| \le \sum_{j=1}^{\infty} |\varphi(cB_j)| < k \sum_{j=1}^{\infty} |cB_j| \le ck \sum_{j=1}^{\infty} |B_j| \le ck|U_\varepsilon| < ck\varepsilon, \tag{67}$$

whence $|\varphi(E_k)| = 0$, because $\varepsilon > 0$ is arbitrary. For every negligible set $E \subset D \setminus \Sigma_\Psi$ we also have $|\varphi(E)| = 0$ since $E = \bigcup_k E \cap A_k$.

Therefore, $\varphi : D \setminus \Sigma_\Psi \to D'$ enjoys Luzin's $\mathscr{N}$-property. $\square$

Since $|\Sigma_\Psi| = 0$, we may assume that the domain of φ is

$$\mathrm{Dom}_4\,\varphi = \mathrm{Dom}_3\,\varphi \setminus \Sigma_\Psi. \tag{68}$$

Now we can generalize the claim of Lemma 15.

Lemma 17. *Assume that $\varphi : \mathrm{Dom}_4\,\varphi \to D'$ satisfies the hypotheses of Lemma* 15*. If $A_1, A_2 \subset \mathrm{Dom}_4\,\varphi$ are two disjoint measurable sets of positive measure then $|\varphi(A_1) \cap \varphi(A_2)| = 0$.*

Proof. The proof of Lemma 15 implies that if $K_1, K_2 \subset D$ are two compact sets of positive measure with $K_1 \cap K_2 = \varnothing$ then $|\varphi(K_1 \cap \mathrm{Dom}_4\,\varphi) \cap \varphi(K_2) \cap \mathrm{Dom}_4\,\varphi| = 0$. Indeed, take two finite coverings $\{B_i^1\}$ and $\{B_j^2\}$ of K_1 and K_2 such that the balls of the first covering lie at positive distances from the balls of the second covering. Then

$$|\varphi(K_1 \cap \mathrm{Dom}_4\,\varphi) \cap \varphi(K_2 \cap \mathrm{Dom}_4\,\varphi)| \le \sum_{i,j} \bigl|\varphi\bigl(B_i^1 \cap \mathrm{Dom}_4\,\varphi\bigr) \cap \varphi\bigl(B_j^2 \cap \mathrm{Dom}_4\,\varphi\bigr)\bigr| = 0.$$

Take two disjoint sets $A_1, A_2 \subset \mathrm{Dom}_4\,\varphi$ of positive measure. We can exhaust each of them by compact sets: $\big|A_1 \setminus \bigcup K_i^1\big| = 0$ and $\big|A_2 \setminus \bigcup K_j^2\big| = 0$. Since φ enjoys Luzin's $\mathscr{N}$-property, we have $\big|\varphi\big(A_l \setminus \bigcup K_i^l\big)\big| = 0$, $l = 1, 2$, whence $|\varphi(A_1) \cap \varphi(A_2)| = 0$. $\square$

Remark 11. By Lemma 17, the set function Ψ_Q enjoys the usual additivity. In other words, for two disjoint open sets A_1 and A_2 we have

$$\begin{aligned}\Psi_Q(A_1 \cup A_2) &= |\varphi(A_1 \cup A_2) \cap Q| = |(\varphi(A_1) \cup \varphi(A_2)) \cap Q| \\ &= |(\varphi(A_1)) \cap Q| + |(\varphi(\varphi(A_2)) \cap Q| - |(\varphi(A_1) \cap \varphi(A_2)) \cap Q| \\ &= |(\varphi(A_1)) \cap Q| + |(\varphi(\varphi(A_2)) \cap Q| = \Psi_Q(A_1) + \Psi_Q(A_2). \end{aligned} \tag{69}$$

Remark 12. Refer to a mapping satisfying the conclusion of Lemma 17 as *almost everywhere injective.*

Let us verify now that we can remove from the image $\varphi(\mathrm{Dom}_4\,\varphi)$ a negligible set so that the inverse mapping φ^{-1} could be defined.

Proposition 7. *φ is injective outside some negligible set.*

Proof. Take a countable everywhere dense subset $\{z_i\}$ of $\mathrm{Dom}_4\,\varphi$. Consider a collection $\{B_{ij} = B(z_i, \rho_j)\}$ of balls with decreasing radii ($\rho_j \to 0$ as $j \to \infty$). We obtain the countable collection of sets $V_{ij} = B_{ij} \cap \mathrm{Dom}_4\,\varphi$. Preserve only those tuples of indices i_1, j_1, i_2, and j_2 that the sets of the form $V_{i_1 j_1}$ and $V_{i_2 j_2}$ are disjoint ($V_{i_1 j_1} \cap V_{i_2 j_2} = \varnothing$). Put $F_{kl} = \varphi(V_{i_k j_k}) \cap \varphi(V_{i_l j_l})$. Since φ is almost everywhere injective, we have $|F_{kl}| = 0$. Put $\Sigma = \bigcup_{kl} F_{kl}$. Then $|\Sigma| = 0$ and $|\varphi^{-1}(\Sigma)| = 0$ (the latter because φ enjoys Luzin's $\mathscr{N}^{-1}$-property).

Put

$$\mathrm{Dom}_5\,\varphi = \mathrm{Dom}_4\,\varphi \setminus \varphi^{-1}(\Sigma) \tag{70}$$

($\mathrm{Dom}\,\varphi^{-1} = \varphi(\mathrm{Dom}_5\,\varphi)$). Then $\varphi : \mathrm{Dom}_5\,\varphi \to \mathrm{Dom}\,\varphi^{-1}$ is bijective. Indeed, suppose that there exist distinct $x_1, x_2 \in \mathrm{Dom}_5\,\varphi$ with $\varphi(x_1) = \varphi(x_2)$. There are two disjoint sets V_1 and V_2 of positive measure which contain x_1 and x_2 respectively. On the other hand, the definition of $\mathrm{Dom}_5\,\varphi$ yields $\varphi(V_1 \cap \mathrm{Dom}_5\,\varphi) \cap \varphi(V_2 \cap \mathrm{Dom}_5\,\varphi) = \varnothing$. This implies that $\varphi(x_1) \neq \varphi(x_2)$. $\square$

By Lemma 17, we can define Ψ_Q on the open subsets of D (by analogy with (64)). It enjoys the usual additivity: $\Psi_Q(A_1 \cup A_2) = \Psi_Q(A_1) + \Psi_Q(A_2)$ for the open sets $A_1, A_2 \subset D$ with $A_1 \cap A_2 = \varnothing$.

Proposition 8. *Assume that $\varphi : D \to D'$ enjoys Luzin's $\mathscr{N}$-property and is almost everywhere injective. Then the change-of-variables rule is valid for every summable function $f : D' \cap Q \to \mathbb{R}$:*

$$\int\limits_D f \circ \varphi(x) J_{\varphi,Q}(x)\, dx = \int\limits_{D' \cap Q} f(y)\, dy, \tag{71}$$

where $J_{\varphi,Q}(x) = \Psi'_Q(x)$.

Proof. Firstly, let us establish the absolute continuity of Ψ_Q (see (75) below).

For all $x \in D \setminus \Sigma_0$, where Σ_0 is a negligible set (on which the derivative Ψ'_Q fails to exist or is infinite), we have the property: for every $\varepsilon > 0$ there is a number $\rho_0(x)$ such that

$$\Psi'_Q(x) - \varepsilon \le \frac{\Psi_Q(B(x,\rho))}{|B(x,\rho)|} \le \Psi'_Q(x) + \varepsilon, \tag{72}$$

$$\Psi'_Q(x) - \varepsilon \le \frac{1}{|B(x,\rho)|} \int\limits_{B(x,\rho)} \Psi'_Q(y)\, dy \le \Psi'_Q(x) + \varepsilon \tag{73}$$

for all $\rho < \rho_0(x)$. Indeed, (72) follows from the definition of the derivative of a set function (Proposition 4), while (73) follows from the Lebesgue-type theorem (see Theorem 3).

We have

$$\Psi_Q(B(x,\rho)) \le |B(x,\rho)|\Psi'_Q(x) + \varepsilon|B(x,\rho)| \le \int\limits_{B(x,\rho)} \Psi'_Q(y)\,dy + 2\varepsilon|B(x,\rho)| \tag{74}$$

for all $x \in D \setminus \Sigma_0$ and all $\rho < \rho_0(x)$.

Take an open set $U \subset D$ of finite measure and a Vitali covering $\mathscr{B}$ of $U \setminus \Sigma_0$ by a family of balls $\{B(x,\rho) \mid x \in U \setminus \Sigma_0,\ 0 < \rho < \rho_0(x)\}$. We can extract from $\mathscr{B}$ a sequence of disjoint balls $\{B_j\}$ with $\bigl|U \setminus \bigcup_j B_j\bigr| = 0$. Then $|U| = \bigl|\bigcup_j B_j\bigr| = \sum_j |B_j|$. Applying Luzin's $\mathscr{N}$-property to φ, we obtain $\Psi_Q(U) = \Psi_Q\bigl(\bigcup_j B_j\bigr) = \sum_j \Psi_Q(B_j)$.

For each ball in $\{B_j\}$ we have (74). Summing (74) over the balls in $\{B_j\}$, we obtain

$$\Psi_Q(U) = \sum_j \Psi_Q(B_j) \le \int\limits_{\bigcup_j B_j} \Psi'_Q(y)\,dy + 2\varepsilon \sum_j |B_j| = \int\limits_U \Psi'_Q(y)\,dy + 2\varepsilon|U|.$$

Since ε is arbitrary,

$$\Psi_Q(U) \le \int\limits_U \Psi'_Q(y)\,dy. \tag{75}$$

This and the inverse inequality (66) ensure that

$$\Psi_Q(U) = \int\limits_U \Psi'_Q(y)\,dy. \tag{76}$$

Put $J_{\varphi,Q}(x) = \Psi'_Q(x)$. Now (76) implies the change-of-variables rule for a step function s:

$$\int\limits_D s \circ \varphi(x) J_{\varphi,Q}(x)\,dx = \int\limits_{D' \cap Q} s(y)\,dy. \tag{77}$$

Using the standard procedure, we can extend this formula to $f \in L_1(D' \cap Q)$. $\square$

Remark 13. Put $Z = \{x \in D \cap \varphi^{-1}(Q) : J_{\varphi,Q}(x) = 0\}$. In accordance with the change-of-variables rule (71), $|\varphi(Z)| = 0$. Since φ enjoys Luzin's $\mathscr{N}^{-1}$-property, it follows that $|Z| = 0$.

Consider now a family Γ of horizontal curves. The elements $\gamma \in \Gamma$ are the integral curves of a horizontal field V. Denote by f_s the flow corresponding to this field. Then $\gamma \in \Gamma$ is of the form $\gamma(s) = f_s(p)$, where p lies on the surface S transversal to V, while the parameter s belongs to an interval $I \subset \mathbb{R}$. Equip the family Γ with a measure $d\gamma$ satisfying

$$c_0|B|^{\frac{\nu-1}{\nu}} \le \int\limits_{\gamma\in\Gamma,\gamma\cap B(x,r)\ne\varnothing} d\gamma \le c_1|B|^{\frac{\nu-1}{\nu}}$$

for sufficiently small balls $B = B(x,r) \subset \mathbb{G}$, where the constants c_0 and c_1 are independent of $B(x,r)$. For the family defined by a vector field V, we can obtain $d\gamma$ as the contraction $i(V)$ of V with the bi-invariant volume form dx. If $\mathscr{J}_{f_s}$ is the Jacobian of the flow f_s then

$$f_s^* i(V)dx = \mathscr{J}_{f_s} i(V)\,dx$$

or

$$f_s^*(\mathscr{J}_{f_{-s}} i(V)dx) = i(V)\,dx.$$

We can identify the tangent vector to the one-parameter family of curves γ_t passing through a point $p\exp tX$ with the tangent vector X at p. The flow f_s carries X to $(f_s)_*X$. Consequently, the form $\mathscr{J}_{f_{-s}} i(V)\,dx$ determines the measure $d\gamma$ on Γ. In the case that V is a left-invariant horizontal vector field, the flow f_s is the right translation by $\exp sV$. Since dx is a bi-invariant form, it follows that $\mathscr{J}_{f_s} = 1$. Using the left invariance and the homogeneity of dilation, we find that

$$\int\limits_{\gamma\cap B(x,r)\ne\varnothing} d\gamma = c|B(x,r)|^{\frac{\nu-1}{\nu}}\|V\|,$$

where $\|V\|$ is the length of the horizontal tangent vector V.

Lemma 18. *On almost all integral curves of horizontal vector fields, $\varphi : \bigcup_i \widetilde{T}_i \cap \operatorname{Dom}_5 \varphi \to D'$ is a continuous mapping outside a set of Hausdorff 1-measure zero.*

Proof. Fix a horizontal field X_j. Assume on the contrary that there is a family of integral curves Γ of the field X_j of positive measure such that on each curve $\gamma \in \Gamma$ there exists a set $s_\gamma \subset \gamma$ of positive 1-measure on which the mapping $\varphi : \bigcup_i \widetilde{T}_i \cap \operatorname{Dom}_5 \varphi \to D'$ is discontinuous.

Put $S = \bigcup_{\gamma\in\Gamma} s_\gamma$ and verify that S is measurable. Indeed, $S = D \setminus A$, where

$$A = \bigcap_{n\in\mathbb{N}} \bigcap_{m\in\mathbb{N}} \Big\{ x \in \bigcup_i \widetilde{T}_i \cap \operatorname{Dom}_5 \varphi \mid \\ d(\varphi(x \exp tX_j), \varphi(x)) < \frac{1}{n} \text{ for } |t| < \frac{1}{m},\ \exp tX_j \in \bigcup_i \widetilde{T}_i \cap \operatorname{Dom}_5 \varphi \Big\},$$

is a measurable set since so is every set in the braces. Fubini's theorem yields $|S| > 0$. Similarly we verify that $S = \bigcup_{m\in\mathbb{N}} S_m$, where $S_m = \{x \in s_\gamma \mid \operatorname{osc}_l \varphi(x) > \frac{1}{m}\}$ is a measurable set. Here $\operatorname{osc}_l \varphi(x)$ is the oscillation of $\varphi : \gamma \cap \bigcup_i \widetilde{T}_i \cap \operatorname{Dom}_5 \varphi \to D'$ at x. Consequently, we can choose m so that $|S_m| > 0$. In addition, there is j such that $|S_m \cap \widetilde{T}_j| > 0$. Take a point $x_0 \in S_m \cap \widetilde{T}_j$ of density 1. Then an arbitrary ball $B(x_0, r)$ includes a subset of $S_m \cap \widetilde{T}_j$ of positive measure, which we denote by P_r. Since φ is continuous on $\widetilde{T}_j$, we can choose the ball $B(x_0, r_m)$ so that $\varphi(B(x_0, r_m) \cap S_m \cap \widetilde{T}_j) \subset B(\varphi(x_0), \frac{1}{4m})$.

Fix $\eta \in C_0^\infty(D')$ with $\eta(y) = 1$ for $y \in B(\varphi(x_0), \frac{1}{4m})$ and $\eta(y) = 0$ for $y \notin B(\varphi(x_0), \frac{1}{2m})$. The superposition $\eta \circ \varphi$ belongs to $L_p^1(D)$. Consequently, we can change $\eta \circ \varphi$ on a negligible set so that $\eta \circ \varphi$ becomes an absolutely continuous function on almost all lines (i.e., is of class ACL). This modification preserves the mapping $\varphi : \bigcup_i \widetilde{T}_i \cap \operatorname{Dom}_5 \varphi \to D'$; therefore, $\varphi^*\eta(x) = \eta \circ \eta(x)$ always holds for all $x \in \widetilde{T}_j$.

By the above, on each horizontal curve intersecting P_{r_m} along a set of positive 1-measure we have $\operatorname{osc}_l \eta \circ \varphi(x) = 1$, where $x \in P_{r_m}$. By the construction of P_{r_m}, the collection of these curves is of positive measure. Thus, we arrive at a contradiction with the absolute continuity of $\eta \circ \varphi$ on almost all lines.

Consequently, on almost all horizontal curves the mapping $\varphi : \bigcup_i \widetilde{T}_i \cap \operatorname{Dom}_5 \varphi \to D'$ is continuous outside a set of 1-measure zero. $\square$

Lemma 19. *If $u \in \operatorname{Lip}_l(D') \cap L_p^1(D')$ and $\|u \mid L_p^1(D')\| \le 1$, where $p \le \nu$, then*

$$|\nabla_{\mathscr{L}}(u \circ \varphi)|(x) \le K J_{\varphi,Q}^{\frac{1}{p}}(x) \tag{78}$$

almost everywhere on $D \cap \varphi^{-1}(Q)$, where K is a constant independent of Q.

Proof. Fix $y_0 \in D' \cap Q$ and a ball $B(y_0, r)$ with $B(y_0, r) \subset D' \cap Q$. Consider the function $\eta(y) = \xi\big(\delta_r\big(y_0^{-1} y\big)\big)$, where $\xi \in C_0^\infty(\mathbb{G})$ is a function satisfying $\xi|_{B(0,1)} = 1$ and $\xi|_{\mathbb{G}\setminus B(0,2)} = 0$.

Since φ^* is a bounded operator, it follows in particular that

$$\big\|\varphi^*(u - u(y_0)) \mid L_p^1(D)\big\| \le \|\varphi^*\| \big\|(u - u(y_0)) \mid L_p^1(D')\big\|.$$

Then

$$\int\limits_{\varphi^{-1}(B(y_0,r))} |\nabla_{\mathscr{L}}(u \circ \varphi)|^p(x)\,dx \le \|\varphi^*\|^p \int\limits_{B(y_0,2r)} |\nabla_{\mathscr{L}}((u - u(y_0))\eta)|^p\,dy \\ \le c_0\|\varphi^*\|^p \int\limits_{B(y_0,2r)} |\nabla_{\mathscr{L}} u|^p \eta^p(y)\,dy + c_0\|\varphi^*\|^p \int\limits_{B(y_0,2r)} |\nabla_{\mathscr{L}}\eta|^p |u - u(y_0)|^p\,dy \\ \le c_0\|\varphi^*\|^p(|B(y_0, 2r)| + (c_1 r^{-1} c_2 r |B(y_0, 2r)|)) = C\|\varphi^*\|^p |B(y_0, 2r)|. \tag{79}$$

Thus, we arrive at

$$\int\limits_{\varphi^{-1}(B(0,r))} |\nabla_{\mathscr{L}}(u\circ\varphi)|^p(x)\,dx \le C\|\varphi^*\|^p|B(y_0,2r)|. \tag{80}$$

Using (71), (80), and Remark 13, which implies that $J_{\varphi,Q}(x)\neq 0$ on $\varphi^{-1}(B(y_0,r))\setminus Z$, we obtain

$$\int\limits_{\varphi^{-1}(B(y_0,r))} |\nabla_{\mathscr{L}}(u\circ\varphi)|^p(x)\,dx = \int\limits_{\varphi^{-1}(B(y_0,r))\setminus Z} \frac{|\nabla_{\mathscr{L}}(u\circ\varphi)|^p(x)J_{\varphi,Q}(x)}{J_{\varphi,Q}(x)}\,dx$$
$$= \int\limits_{B(y_0,r)\cap Q\setminus\varphi(Z)} \left(\frac{|\nabla_{\mathscr{L}}(u\circ\varphi)|^p(x)}{J_{\varphi,Q}(x)}\right)_{\varphi(x)=y} dy \le C\|\varphi^*\|^p|B(y_0,2r)|. \tag{81}$$

Differentiating (81) by the Lebesgue-type Theorem 3, we deduce that

$$\frac{|\nabla_{\mathscr{L}}(u\circ\varphi)|^p(x)}{J_{\varphi,Q}(x)}\bigg|_{\varphi(x)=y} \le C\|\varphi^*\|^p$$

for almost all $y\in D'\cap Q$. Hence, we have

$$|\nabla_{\mathscr{L}}(u\circ\varphi)|(x) \le KJ^{\frac{1}{p}}_{\varphi,Q}(x) \quad \text{for almost all } x\in D\cap\varphi^{-1}(Q) \tag{82}$$

since φ is injective outside a negligible set (Proposition 7) and enjoys Luzin's $\mathscr{N}^{-1}$-property. □

Lemma 20. *Assume that $p\in[1,\infty)$ and $p\le\nu$. Consider $D,D'\subset\mathbb{G}$. If a mapping $\varphi: D\to D'$ is of class IL^1_p then φ is approximatively differentiable almost everywhere along horizontal curves.*

Proof. Since the result is local, we may assume that D is of finite measure.

Fix $k\in\mathbb{N}$ and choose a ball $Q\subset\mathbb{G}$ with $Q\supset\varphi(T_k\cap\operatorname{Dom}_5\varphi)$.

Take a countable everywhere dense set $\{z_j\}$ of points of D'. Define a countable family $d^r_{z_j}: D'\to\mathbb{R}^+$ of functions: $d^r_{z_j}(y)=(r-d_{z_j}(y))^+$, where $r\in\mathbb{Q}^+=\{x\in\mathbb{Q}\mid x>0\}$ (and $d_{z_j}(y)=d(z_j,y)$). Each of these functions satisfies pointwise the equality $\varphi^*d^r_{z_j}(x)=d^r_{z_j}\circ\varphi(x)$, with $r\in\mathbb{Q}^+$ and $j\in\mathbb{N}$, for all points $x\in\widetilde{T}_k$. (In this proof $\widetilde{T}_k$ is the collection of points of density 1 of the set $T_k\cap\operatorname{Dom}_5\varphi$, see similar arguments in the proof of Theorem 4.)

In addition, each of these functions satisfies the hypotheses of Lemma 19. Therefore,

$$|\nabla_{\mathscr{L}}(d^r_{z_j}\circ\varphi)|(x)\le CJ^{\frac{1}{p}}_{\varphi,Q}(x)$$

for almost all $x\in\operatorname{Dom}_5\varphi\cap\varphi^{-1}(Q)$.

Consider the foliation Γ_j of D generated by the horizontal vector field X_j and a curve γ of this foliation. Almost all curves γ in Γ_j satisfy the conditions:

(1) φ is continuous on γ outside some set σ_γ of 1-measure zero (Lemma 18);

(2) for the measurable functions we have the pointwise inequality

$$|\nabla_{\mathscr{L}}\big(\varphi^*d^r_{z_j}\big)|(t)\le KJ^{\frac{1}{p}}_{\varphi,Q}(t),\quad r\in\mathbb{Q}^+,\ j\in\mathbb{N},\ \text{almost everywhere on }\gamma,$$

and the function $J_{\varphi,Q}$ is integrable on γ;

(3) for almost all $x_0\in\gamma$ the ratio

$$\frac{1}{d(x_0,x)}\int\limits_{[x_0,x]} J^{\frac{1}{p}}_{\varphi,Q}(t)\,d\sigma$$

has a finite limit as $x \to x_0$ along γ equal to $J^{\frac{1}{p}}_{\varphi,Q}(x_0)$, where $[x_0, x] \subset \gamma$ is a segment of the integral line;

(4) $\mathrm{Dom}_5\, \varphi \cap \gamma$ is a set of full (1-dimensional) measure on $\gamma \cap D$;

(5) inequality (78) is valid for all functions $\{d^r_{z_j}\}$ simultaneously almost everywhere on γ;

(6) the functions $\varphi^* d^r_{z_j}$ are absolutely continuous on γ for all $j \in \mathbb{N}$ and $r \in \mathbb{Q}^+$.

Fix a curve $\gamma \in \Gamma_j$ satisfying all six conditions.

Take a point $x_0 \in \widetilde{T}_k \cap \gamma$ of positive density on γ which is a continuity point of the restriction $\varphi|_\gamma$ and satisfies condition 3. Put $z = \varphi(x_0)$. Fix a subsequence $\{z_{j_l}\}$ of points in $\{z_j\}$ converging to $z = \varphi(x_0)$. Henceforth we denote the elements of this subsequence by z_l. Since φ is continuous on γ at x_0, we can choose three numbers δ, r, and L so that $\varphi(B(x_0, \delta) \cap \gamma \setminus \sigma_\gamma) \subset Q$ and $d^r_{z_l} \circ \varphi(x) \neq 0$ for all $l \geq L$ and all points $x \in B(x_0, \delta) \cap \gamma \setminus \sigma_\gamma$.

Integrating $CJ^{\frac{1}{p}}_{\varphi,Q}(x)$, where C is independent of r and z, over the part of γ from x_0 to x, where $x \in \widetilde{T}_k \cap B(x_0, \delta) \cap \gamma \setminus \sigma_\gamma$, we infer that

$$\begin{aligned} C \int\limits_{[x_0,x]} J^{\frac{1}{p}}_{\varphi,Q}(t)\,dt &\geq \int\limits_{[x_0,x]} |\nabla_{\mathscr{L}}(\varphi^* d^r_{z_j})|(t)\,dt \\ &\geq |d^r_{z_l} \circ \varphi(x_0) - d^r_{z_l} \circ \varphi(x)| = |r - d_{z_l}(\varphi(x_0)) - r + d_{z_l}(\varphi(x))| \\ &= |-d_{z_l}(\varphi(x_0)) + d_{z_l}(\varphi(x))| \to d_z(\varphi(x)) = d(\varphi(x_0), \varphi(x)) \quad \text{as } l \to \infty. \end{aligned} \tag{83}$$

Thus,

$$d(\varphi(x_0), \varphi(x)) \leq C_1 \int\limits_{[x_0,x]} J^{\frac{1}{p}}_{\varphi,Q}(x)\,d\sigma \tag{84}$$

or

$$\frac{d(\varphi(x_0), \varphi(x))}{d(x_0, x)} \leq \frac{C}{d(x_0, x)} \int\limits_{[x_0,x]} J^{\frac{1}{p}}_{\varphi,Q}(x)\,d\sigma \tag{85}$$

for all $x \in \widetilde{T}_k \cap B(x_0, \delta) \cap \gamma \setminus \sigma_\gamma$. Passing to the approximative limit in (85), we obtain

$$\operatorname{ap} \varlimsup_{x \to x_0, x \in \gamma} \frac{d(\varphi(x_0), \varphi(x))}{d(x_0, x)} \leq C_1 J^{\frac{1}{p}}_{\varphi,Q}(x_0) < \infty, \tag{86}$$

which establishes the approximative differentiability of φ at x_0 along the vector field X_j.

Since we chose the horizontal field X_j, the integral curve $\gamma \in \Gamma_j$, and the point $z_0 \in \gamma$ arbitrarily, the mapping φ is approximatively differentiable along horizontal curves almost everywhere. $\square$

Remark 14. The approximative differentiability of φ almost everywhere along the integral curves of horizontal vector fields implies the complete approximative differentiability [22, Theorem 3.3], and so the hypotheses of the following statement.

Proposition 9 [21, Theorem 3; 22, Theorem 3.3]. *If $D \subset \mathbb{G}$ and*

$$\operatorname{ap} \varlimsup_{x \to a} \frac{d(\varphi(a), \varphi(x))}{d(a, x)} < \infty$$

for all $a \in D$, then we can express D as a countable union $D = \bigcup_i E_i$ so that $\varphi \in \mathrm{Lip}(E_i)$.

Remark 15. Each set E_i of Proposition 9 is included into the countable union of the sets

$$F_{ki} = \Big\{z : d(\varphi(x), \varphi(z)) \geq \frac{1}{k} d(x, z),\ x \in E_i \cap B\Big(z, \frac{1}{k}\Big)\Big\};$$

i.e., $E_i \subset \bigcup_k F_{ki}$ (see [21]). Then we can express D as a countable union, $D = \bigcup_j D_j$, so that φ is a bi-Lipschitz mapping on each D_j. In addition, we may assume that D_j consists of points of density 1.

Taking Remark 15 into account, we may assume that the domain of φ is

$$\mathrm{Dom}_6\,\varphi = \bigcup_j E_j \cap \mathrm{Dom}_5\,\varphi, \tag{87}$$

and φ is a bi-Lipschitz mapping on $E_j \cap \mathrm{Dom}_5\,\varphi$.

Denote by $D\varphi$ the approximative differential of φ and by $D_h\varphi$, its horizontal part.

Lemma 21. *Assume that φ satisfies the hypotheses of Theorem 1, while $\psi = \varphi^{-1}$ is that of the proof of Proposition 7. Then*

$$|D\varphi|(x) < L, \quad |J(x,\varphi)| > \alpha_1 \quad \textit{and} \quad |D\psi|(y) < L', \quad |J(y,\psi)| > \alpha \tag{88}$$

for almost all $x \in D$ and almost all $y \in \varphi(\mathrm{Dom}_6\,\varphi)$, where $J(x,\varphi) = \det D\varphi(x)$.

Observe that $J(x,\varphi) = J_\varphi(x) = \lim_{r\to 0} \frac{|\varphi(B(x,r))|}{|B(x,r)|}$ almost everywhere in D.

Proof. Let us show that the mapping

$$\psi : \varphi(\mathrm{Dom}_6\,\varphi) \to \mathrm{Dom}_6\,\varphi$$

inverse to φ induces by the superposition rule the operator $\varphi^{*-1} : L^1_p(D) \to L^1_p(D')$. For $g \in L^1_p(D)$ there is a function $f \in L^1_p(D')$ with $g = \varphi^* f$ (Lemma 10). On the other hand, $f \circ \varphi = \varphi^* f$ on $\mathrm{Dom}_6\,\varphi$ by Lemma 9. Therefore, $(\psi^* \circ \varphi^*) f_n(y) = f_n \circ \varphi \circ \psi(y) = f_n(y)$ almost everywhere on D', i.e., $\psi^*(\varphi^* f) = f$ and $\psi^* = \varphi^{*-1}$.

Given a Lipschitz function $f \in L^1_p(D')$, we have $\nabla_{\mathscr L}\varphi^* f = \nabla_{\mathscr L}(f\circ\varphi) = D_h\varphi^T\nabla_{\mathscr L} f$. Inserting into (82) a Lipschitz function η with $|\nabla_{\mathscr L}\eta|(y) = 1$ and taking the relation

$$K J^{\frac{1}{p}}(x,\varphi) \ge |\nabla_{\mathscr L}(\eta\circ\varphi)|(x) = \bigl|D_h\varphi^T\nabla_{\mathscr L}\eta\bigr|(x)$$

into account, we estimate the norm of the operator:

$$|D_h\varphi|^p(x) \le K_1 J(x,\varphi) \quad \text{almost everywhere on } D. \tag{89}$$

Similarly,

$$|D_h\psi|^p(y) \le K_2 J(y,\psi) \quad \text{almost everywhere on } \varphi(\mathrm{Dom}_6\,\varphi). \tag{90}$$

This implies that

$$CK_2^p \ge \frac{|D_h\psi|^p}{|J(y,\psi)|} = \left(\frac{|D_h\psi|^\nu}{|J(y,\psi)|}\right)^{\frac{p}{\nu}} \frac{1}{|J(y,\psi)|^{1-\frac{p}{\nu}}} \ge \frac{1}{|J(y,\psi)|^{1-\frac{p}{\nu}}}. \tag{91}$$

Consequently, $|J(y,\psi)| > \alpha$, and so $|J(x,\varphi)| < \beta$. Arguing similarly, we deduce from (89) that $|J(x,\varphi)| > \alpha_1$ and $|J(y,\psi)| < \beta_1$. Finally,

$$|D\varphi|(x) < L, \quad |J(x,\varphi)| > \alpha_1 \quad \text{and} \quad |D\psi|(y) < L', \quad |J(y,\psi)| > \alpha$$

for almost all $x \in D$ and almost all $y \in \varphi(\mathrm{Dom}_6\,\varphi)$. □

Lemma 22. *Assume that $p < \nu$ and that $\varphi : D \to D'$ is a mapping of class IL^1_p. Then φ coincides with a quasi-isometric homeomorphism almost everywhere.*

Proof. Fix $q > \nu$. Given a Lipschitz function f with compact support in D', we have $f\circ\varphi \in L^1_p(D)$ and $|\nabla_{\mathscr L}(f\circ\varphi)|(x) \le |\nabla_{\mathscr L} f||D_h\varphi|(x)$. Hence, taking (88) and the change-of-variables rule (71) into account, we obtain

$$\int\limits_D |\nabla_{\mathscr L} f\circ\varphi|^q(x)\,dx \le C\int\limits_D |\nabla_{\mathscr L} f|^q(\varphi(x))|D\varphi|^q(x)\,dx \le CL^q\int\limits_D |\nabla_{\mathscr L} f|^q(\varphi(x))\,dx$$
$$\le \frac{L^q C}{\alpha}\int\limits_D |\nabla_{\mathscr L} f|^q(\varphi(x))|J(x,\varphi)|\,dx = \widetilde{C}\int\limits_{\varphi(D)} |\nabla_{\mathscr L} f|^q(y)\,dy. \tag{92}$$

Therefore, $\|f \circ \varphi \mid L^1_q(D)\| \leq C\|f \mid L^1_q(D')\|$ for $q > \nu$. Consequently, condition (2) of Lemma 13 holds.

Take a Lipschitz function g with compact support in D. By Lemma 10, there is $f \in L^1_p(D')$ with $g = f \circ \varphi$. Furthermore,

$$\nabla_{\mathscr{L}} g(x) = D_h\varphi^T(x)\nabla_{\mathscr{L}} f(\varphi(x)),$$

and so

$$\nabla_{\mathscr{L}} f(\varphi(x)) = (D_h\varphi^T)^{-1}(x)\nabla_{\mathscr{L}} g(x).$$

Since $|(D_h\varphi^T)^{-1}| = |(D_h\varphi)^{-1}| < L$, it follows that $f \in L^1_q(D')$ and

$$\|f \mid L^1_q(D')\| \leq C'\|f \circ \varphi \mid L^1_q(D)\|.$$

Therefore, condition (2′) of Lemma 13 holds (with $q > \nu$ instead of p).

In addition, condition (1) of Lemma 13 holds, where we choose T_k as T.

Lemmas 13 and 11 imply that φ coincides with a quasi-isometric homeomorphism almost everywhere. □

5.3. Proof of Theorem 1. We can now give a proof of the main result of this article.

Proof of Theorem 1. *Sufficiency:* We may assume that $\varphi : D \to D'$ is a locally bi-Lipschitz mapping and enjoys Luzin's $\mathscr{N}$-property and $\mathscr{N}^{-1}$-property. Given a function $f \in L^1_p(D') \cap C^\infty(D')$, the superposition $f \circ \varphi$ is absolutely continuous on almost all integral lines of horizontal vector fields since it is a locally Lipschitz function. Moreover, $\nabla_{\mathscr{L}}(f \circ \varphi) = D_h\varphi^T(x)\nabla_{\mathscr{L}} f(\varphi(x))$ [36, p. 263], where $D_h\varphi(x) = \{X_i\varphi_j(x)\}$ for $i, j = 1, \dots, n_1$ is the horizontal part of the $\mathscr{P}$-differential. Hence,

$$\int_D |\nabla_{\mathscr{L}}(f \circ \varphi)|^p\,dx = \int_D |D_h\varphi^T(x)\nabla_{\mathscr{L}} f(\varphi(x))|^p\,dx$$
$$\leq \int_D |D_h\varphi^T(x)|^p|\nabla_{\mathscr{L}} f(\varphi(x))|^p\,dx \leq M^p\int_D |\nabla_{\mathscr{L}} f|^p(\varphi(x))\,dx$$
$$= M^p\int_{D'} \frac{|\nabla_{\mathscr{L}} f|^p(y)}{J(\varphi^{-1}(y),\varphi)}\,dy \leq \frac{M^p}{\alpha}\int_{D'} |\nabla_{\mathscr{L}} f|^p(y)\,dy,$$

where in the second and third inequalities we use the quasi-isometry property (41), and in the second equality apply the change-of-variables rule (71). By Lemma 6, the resulting inequality holds for all $f \in L^1_p(D')$; i.e.,

$$\|\varphi^*(f) \mid L^1_p(D)\| \leq K_1\|f \mid L^1_p(D')\|. \tag{93}$$

The mapping $\psi = \varphi^{-1}$ is also a quasi-isometry. Then, for $g \in L^1_p(D)$,

$$\|\psi^*(g) \mid L^1_p(D')\| \leq K_2\|g \mid L^1_p(D)\|. \tag{94}$$

Observe that $\psi^*(f \circ \varphi) = f$ holds for $f \in L^1_p(D') \cap C^\infty(D')$. Consequently, (94) becomes

$$K_2^{-1}\|f \mid L^1_p(D')\| \leq \|\varphi^*(f) \mid L^1_p(D)\|.$$

Therefore,

$$K^{-1}\|f \mid L^1_p(D')\| \leq \|\varphi^*(f) \mid L^1_p(D)\| \leq K\|f \mid L^1_p(D')\|,$$

where the constant K depends only on the properties of φ.

Verify that $\varphi^*\big(L^1_p(D') \cap C^\infty(D')\big)$ is everywhere dense in $L^1_p(D)$. Take $g \in L^1_p(D)$. There is a sequence $g_n \in L^1_p(D) \cap C^\infty(D)$ with $\|g - g_n \mid L^1_p(D)\| \to 0$. On the other hand, the two-sided estimate yields $g_n \circ \varphi^{-1} \in L^1_p(D')$. Hence, there is a sequence $f_{nk} \in L^1_p(D') \cap C^\infty(D')$ with

$$\|g_n \circ \varphi^{-1} - f_{nk} \mid L^1_p(D')\| \to 0$$

as $k \to \infty$. Then for some sequence of positive integers l_n we have $\varphi^* f_{nl_n} \in \varphi^*\big(L^1_p(D') \cap C^\infty(D')\big)$ and $\big\| g - \varphi^* f_{nl_n} \mid L^1_p(D)\big\| \to 0$ as $n \to \infty$.

Necessity: We established the existence of a quasi-isometry Φ in Theorem 4 in the case $p > \nu$ and in Lemma 22 in the case $p < \nu$. By the above proof, the superposition operator $\Phi^* : L^1_p(\Phi(D)) \to L^1_p(D)$ is an isomorphism. Hence, we have the isomorphism $\varphi^{*-1} \circ \Phi^* : L^1_p(\Phi(D)) \to L^1_p(D')$ with $\varphi^{*-1} \circ \Phi^*(f)(x) = f(x)$ for all points $x \in \Phi(D)) \cap D'$, where $f \in L^1_p(\Phi(D))$ is an arbitrary function. By analogy with [19, Theorem 3.1] and [20, Proposition 6.10], we can obtain the properties:

(1) $|\Phi(D)\Delta D'| = 0$;

(2) for every ball $B \subset D'$ the set $B \setminus \Phi(D)\Delta D'$ is connected.

Let us show that under these assumptions the operator of restriction $r : L^1_p(D') \to L^1_p(\Phi(D) \cap D')$ is an isomorphism. If this fails then there exists a nonzero function $f \in L^1_p(D')$ with $f \equiv 0$ on $\Phi(D) \cap D'$. By properties 1 and 2, this function vanishes identically on D' since $\nabla_{\mathscr{L}} f = 0$ almost everywhere in D', while the complement $D' \setminus \Phi(D)\Delta D'$ is a locally connected set. $\square$

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