The number of binary rotation words

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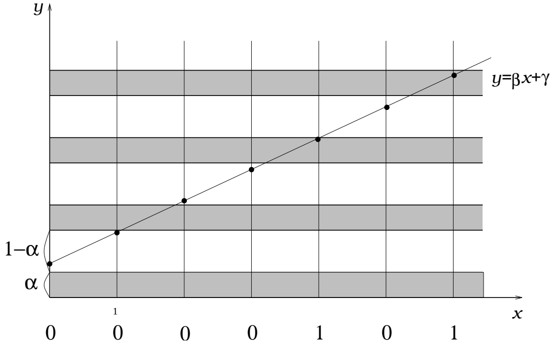
Rotation words

Formal definition : 3 constants $\alpha, \beta, \gamma \in [0,1)$, and the word

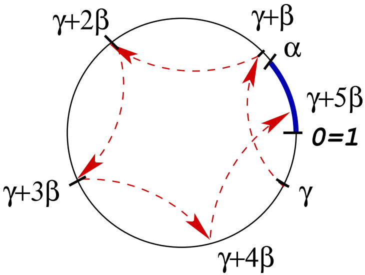
$$w = w_0 w_1 w_2 \cdots$$

is defined by

$$w_n = \begin{cases} 1 \text{ if } \{n\beta + \gamma\} < \alpha, \\ 0 \text{ otherwise.} \end{cases}$$



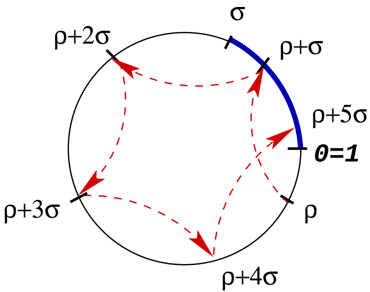
Rotation words



$w = 00000\textcolor{blue}{1}\cdots$

Particular case : Sturmian words

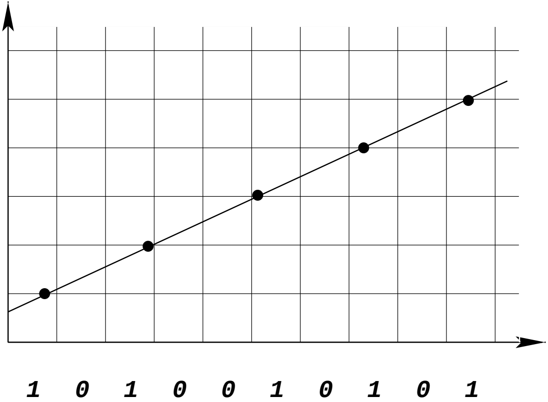
$\alpha = \beta = \sigma$



$w = 10001 \cdots$

Mechanical definition

$$y = \sigma x + \rho, \quad 0 \leq \sigma, \rho < 1.$$



$$w = w_1 w_2 \cdots$$

$$w_n = \lfloor n\sigma + \rho \rfloor - \lfloor (n-1)\sigma + \rho \rfloor \quad (\text{ or } \lceil \cdot \rceil).$$

Complexity of a word

Complexity $p(n)$ = the number of distinct factors of length n.

G. Rote (1992) : there are at most $2n$ distinct factors of length n in each rotation word.

In a Sturmian word, there are exactly $n+1$ factors of length n, and this is the minimal complexity of a non-periodic infinite word.

Complexity of a language

Question

How many Sturmian words of length n on $\{0, 1\}$ are there?

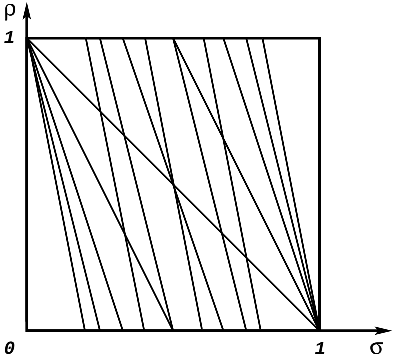
Answer :

- Lipatov, 1982;
- Mignosi, 1991;
- Berstel, Pocchiola, 1993 - geometric method.

Faces of the dual picture

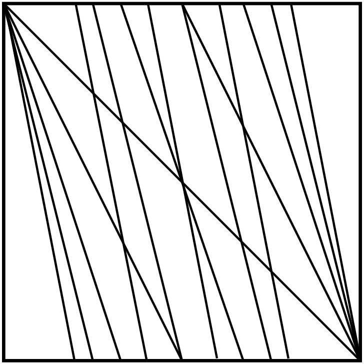
Lemma

The number of Sturmian words of length n is equal to the number of faces of the arrangement of order n.



Each line $y = \sigma x + \rho$ corresponds to the point (σ, ρ)

And now the number of faces is



$$f = e - v + 1.$$

The number of Sturmian words

Theorem (Lipatov 82, Mignosi 91, Berstel, Pocchiola 93)

The number of Sturmian words of length n is

$$1 + \sum_{p=1}^{n} \varphi(p)(n+1-p) = \frac{n^3}{\pi^2} + O(n^2 \log n).$$

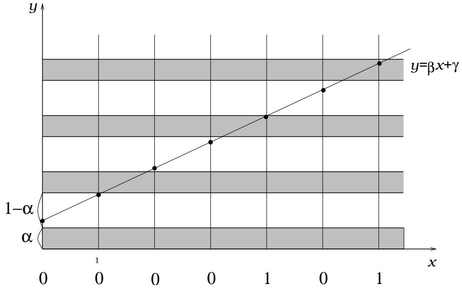
The number of rotation words with fixed interval

Question

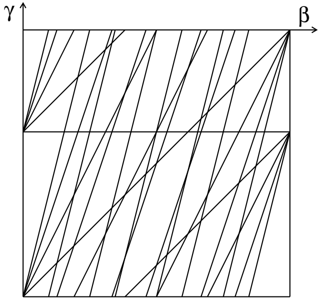
How many rotation words with a fixed interval α are there ?

Answer [Cassaigne, Frid, 2007]

We know how many the faces in the dual arrangement are. Sometimes it is sufficient to find the precise number, and sometimes it gives just an upper bound.



Dual image for rotation words

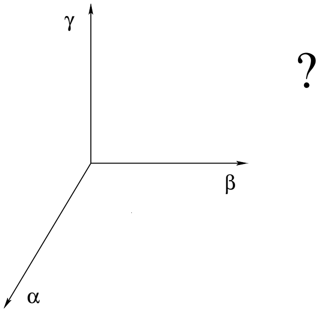


Anyway, the function grows as $O(n^3)$. A lower bound is given in [Frid, 2005].

Question

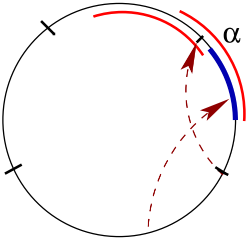
How many rotation words of length n are there ?

Parameters α, β, γ take all values from $[0, 1)$. Should we draw a 3-dimensional picture ?



Main idea

Fortunately, NO. We use another idea.



$$r_{k+1} = r_k + u_k - v_k,$$

where u, v are Sturmian word of the same slope α.

Found already by [Berstel, Vuillon 2002] (for a greater number of intervals too).

Upper bound

$$
\begin{aligned}
f(n+1) &\leq \#\{(u,v) | u, v \in S(n,\alpha), \alpha \in (0, 1/2), u \neq v\} + 2 \\
&= n(n+1) + \frac{1}{2} \sum_{p=3}^{n} \varphi(p)(n^2 - p^2 + n + p) \\
&= \frac{3n^4}{4\pi^2} + O(n^3 \log n).
\end{aligned}
$$

The number of pairs of Sturmian words of the same slope was found using the results of [Berstel, Pocchiola, 1996].

But there are pairs giving the same rotation words!

Cases

$$r_{k+1} = r_k + u_k - v_k,$$

```
u= |        w        |
v= |        w        |

r= | 0 0  . . .  0 | 0
```

or

```
r= | 1 1  . . .  1 | 1
```

Already excluded since we state $u \neq v$.

Cases

$$r_{k+1} = r_k + u_k - v_k,$$

```
u= [ s ]  1 0  [    p    ]
v= [ s ]  0 1  [    p    ]

r= 0..0 1 0 ... 0 0 0 0
```

or

```
u= [ s ]  0 1  [    p    ]
v= [ s ]  1 0  [    p    ]

r= 1..1 0 1 ... 1 1 1 1
```

$O(n^3)$ words subtracted.

Cases

$$r_{k+1} = r_k + u_k - v_k,$$

u=	···	c	1	0	c	1	0	c	1	0	···
v=	...	c	0	1	c	0	1	c	0	1	···
r=	···	0..0	1	0	0..0	1	0	0..0	1	0	···

Also $O(n^3)$ words subtracted.

So, all words appearing from several pairs are of the form

$$0^{k_1}1(0^l1)^m0^{k_2},$$

and are strongly related to *central* Sturmian words c of length $l-1$.

Precise formula

Theorem

For all $n \geq 3$, the number of rotation words of length $n+1$ is

$$f(n+1) = n^2 + 3n + 4 + \frac{1}{2}\sum_{p=3}^{n} \varphi(p)(n^2 - p^2 + n + p) - f_1(n) - 2\sum_{l=2}^{n-1} f_2(n,l),$$

where

$$f_1(n) = \begin{cases} 2\sum_{i=k}^{2k}\sum_{p=1}^{i+1}\varphi(p), & \text{if } n = 2k+1, \\ 2\sum_{i=k}^{2k-1}\sum_{p=1}^{i+1}\varphi(p) + \sum_{p=1}^{k}\varphi(p), & \text{if } n = 2k, \end{cases}$$

$$g(n,l) = n - l + 1 + (n \bmod (l+1)),$$

$$h(n,l) = \min(l+1, n-l),$$

$$f_2(n,l) = \left(\frac{1}{2}\left\lfloor \frac{n}{l+1} \right\rfloor g(n,l) - h(n,l)\right)(\varphi(l+1) - 1) + h(n,l)\left(\frac{\varphi(l+1)}{2} - 1\right).$$

Precise values

n	6	7	10	15	20	50	75	100
$f(n)$	64	112	504	2804	9442	423814	2222984	7155096
$\frac{4\pi^2 f(n)}{3n^4} \approx$	0.65	0.61	0.66	0.73	0.78	0.89	0.92	0.94

THANK YOU