

# On minimal factorizations of words as products of palindromes

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September 13, 2012, JM 2012, Louvain, Belgium

# Decompositions to palindromes

A palindrome :  $w = \tilde{w}$ .

$\lambda, a, abbaaabba$

Decomposition to palindromes :

$$\underbrace{abba} \underbrace{bab} \underbrace{a} = \underbrace{a} \underbrace{bb} \underbrace{ababa}$$

$$|abbababa|_{\text{pal}} = 3$$

## Question

What are infinite words  $w$  such that for some constant  $C$ , we have

$$|u|_{\text{pal}} \leq C$$

for all factors (version : prefixes) of  $w$  ?

## Lemma

*If  $w$  is periodic, then  $|u|_{\text{pal}}$  is bounded for factors of  $w$  if and only if the period of  $w$  is of the form  $p_1 p_2$ , where  $p_1, p_2$  are palindromes.*

$$\underbrace{p_1 p_2 p_1 p_2 p_1 p_2 p_1}_{p_1 p_2} p_2 \dots$$

On the other hand, for  $w = 123123123 \dots$ ,  $|u|_{\text{pal}} = |u|$ .

# Non-periodic words ?

## Conjecture

If  $|u|_{\text{pal}}$  is bounded for all factors (version : prefixes) of  $w$ , then  $w$  is ultimately periodic.

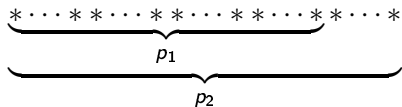
Just try it!

A  $k$ -power is a word of the form  $u^k$ ; a word is  $k$ -power-free if it does not contain  $k$ -powers.

### Theorem

*If a word is  $k$ -power-free for some  $k$ , then  $|u|_{\text{pal}}$  is unbounded for its prefixes  $u$ .*

## Sketch of the proof



If the word is  $k$ -power-free, then

$$\frac{|p_2|}{|p_1|} > 1 + \frac{1}{k-1}.$$

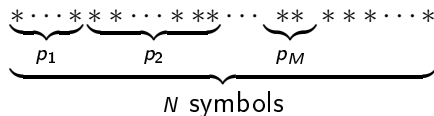
So, the number of palindromes of length at most  $N$  starting from any given point is at most

$$2 + \log_{1+1/(k-1)} N.$$

# Sketch of the proof

So, the number of partitions of prefixes of length  $\leq N$  to at most  $M$  palindromes is at most

$$(2 + \log_{1+1/(k-1)} N)^M < N \text{ for } N \text{ large enough.}$$





The result is applicable to

- the Thue-Morse word (3-power-free)
- the Fibonacci word (4-power-free)

For a square-free word, the proof says that there is a prefix of length  $\leq 65$  not decomposable to 2 palindromes. In fact,  $\leq 5$  is enough.

The result is NOT applicable to

- the Sierpinski word *aba bbb aba bbbbbbbb aba bbb aba...* ;
- some of Sturmian words.

A  $k$ -run is the *longest* possible power  $\geq k$ .

01 01000 01000 01000 01000 01000 01000 0100 1  
5-run

$k$ -runs may intersect, one of them can be covered by another.  
A symbol of  $w$  can be covered by several  $k$ -runs (and even by an infinite number of them).

# $(k, l)$ -condition

## Definition

An infinite word is said to satisfy the  $(k, l)$ -condition if each its symbol is covered by at most  $l$  of  $k$ -runs.

## Example

The Sierpinski word

*aba bbb aba bbbbbbbbbb aba bbb aba b<sup>27</sup> a...*

satisfies the  $(3, 1)$ -condition.

### Theorem

*For each  $P > 0$  in each non-periodic infinite word satisfying the  $(k, l)$ -condition for some  $k$  and  $l$  there exists a factor not decomposable to  $P$  palindromes.*

This statement covers e. g. the Sierpinski word,  
but does not cover arbitrary Sturmian words.

???

## Questions

What about the general case?

What about prefixes?

## Definition

Kellendonk, Lenz, Savinien, 2011 A word is *privileged* if it is

- either  $\lambda$ , or  $a \in \Sigma$ ;
- or the complete first return of a shorter privileged word.

## Example

00101100 is privileged but not a palindrome

1231321 is a palindrome but not privileged

# Statement for privileged words

## Theorem

*For each  $P > 0$  in each  $k$ -power-free word there exists a prefix not decomposable to  $P$  privileged words.*

Similar statements for other regularities in words are possible.

THANK YOU

And we also thank Michelangelo Bucci and Alessandro De Luca for useful discussions