# On minimal factorizations of words as products of palindromes

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# Decompositions to palindromes

A palindrome : $w = \tilde{w}$.

$\lambda, a, abbaaabba$

Decomposition to palindromes :

$$\underbrace{abba}\underbrace{bab}\underbrace{a} = \underbrace{a}\underbrace{bb}\underbrace{ababa}$$

$$|abbababa|_{\mathrm{pal}} = 3$$

### Question

What are infinite words $w$ such that for some constant $C$, we have

$$|u|_{\mathrm{pal}} \leq C$$

for all factors (version : prefixes) of $w$ ?

# Partial answer

**Lemma**

*If $w$ is periodic, then $|u|_{\mathrm{pal}}$ is bounded for factors of $w$ if and only if the period of $w$ is of the form $p_1p_2$, where $p_1$, $p_2$ are palindromes.*

$$\underbrace{p_1\ p_2\ p_1\ p_2\ p_1\ p_2\ p_1}\underbrace{p_2\cdots}$$

On the other hand, for $w = 123123123\cdots$, $|u|_{\mathrm{pal}} = |u|$.

# Non-periodic words ?

**Conjecture**

If $|u|_{\mathrm{pal}}$ is bounded for all factors (version : prefixes) of $w$, then $w$ is ultimately periodic.

Just try it !

# $k$-power-free case

A $k$-power is a word of the form $u^k$ ; a word is $k$-power-free if it does not contain $k$-powers.

**Theorem**

*If a word is k-power-free for some $k$, then $|u|_{\text{pal}}$ is unbounded for its prefixes $u$.*

# Sketch of the proof

$$\underbrace{\underbrace{* \cdots * * \cdots * * \cdots * * \cdots *}_{p_1} * \cdots *}_{p_2}$$

If the word is $k$-power-free, then

$$\frac{|p_2|}{|p_1|} > 1 + \frac{1}{k-1}.$$

So, the number of palindromes of length at most $N$ starting from any given point is at most

$$2 + \log_{1+1/(k-1)} N.$$

# Sketch of the proof

So, the number of partitions of prefixes of length $\leq N$ to at most $M$ palindromes is at most

$$(2 + \log_{1+1/(k-1)} N)^M < N \text{ for } N \text{ large enough.}$$

$$\underbrace{\underbrace{* \cdots *}_{p_1}\underbrace{* * \cdots * **}_{p_2} \cdots \underbrace{**}_{p_M} * * * \cdots *}_{N \text{ symbols}}$$

# Applicability

The result is applicable to

- the Thue-Morse word (3-power-free)
- the Fibonacci word (4-power-free)

For a square-free word, the proof says that there is a prefix of length $\leq 65$ not decomposable to 2 palindromes. In fact, $\leq 5$ is enough.

The result is NOT applicable to

- the Sierpinski word $aba\ bbb\ aba\ bbbbbbbbb\ aba\ bbb\ aba \cdots$ ;
- some of Sturmian words.

# $k$-runs

A $k$-run is the *longest* possible power $\geq k$.

$$01\,\underbrace{01000\ 01000\ 01000\ 01000\ 01000\ 0100}_{5-\mathrm{run}}\,1$$

$k$-runs may intersect, one of them can be covered by another.
A symbol of $w$ can be covered by several $k$-runs (and even by an infinite number of them).

# $(k, l)$-condition

## Definition

An infinite word is said to satisfy the $(k, l)$-condition if each its symbol is covered by at most $l$ of $k$-runs.

## Example

The Sierpinski word

$$aba\ bbb\ aba\ bbbbbbbbb\ aba\ bbb\ aba\ b^{27} a \cdots$$

satisfies the $(3, 1)$-condition.

# More general statement

**Theorem**

*For each $P > 0$ in each non-periodic infinite word satisfying the $(k, l)$-condition for some $k$ and $l$ there exists a factor not decomposable to $P$ palindromes.*

This statement covers e. g. the Sierpinski word,

but does not cover arbitrary Sturmian words.

### Questions

What about the general case ?

What about prefixes ?

# Privileged words

## Definition

Kellendonk, Lenz, Savinien, 2011 A word is *privileged* if it is

- either $\lambda$, or $a \in \Sigma$;
- or the complete first return of a shorter privileged word.

## Example

00101100 is privileged but not a palindrome
1231321 is a palindrome but not privileged

# Statement for privileged words

**Theorem**

*For each $P > 0$ in each k-power-free word there exists a prefix not decomposable to $P$ privileged words.*

Similar statements for other regularities in words are possible.

THANK YOU

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