ON THE REALIZATION OF LATTICE-NORMED SPACES

A. E. Gutman

UDC 517.98

Introduction

The theory of lattice-normed spaces is at present an actively developing area of functional analysis. Arising from the theory of vector lattices it has acquired independence while preserving a certain continuity with it. One can regard a lattice-normed space as a generalization of the concept of a vector lattice, an approach that is supported by the explicit correspondence of many facts of the theories of vector lattices and lattice-normed spaces.

On of the most important results of the study of vector lattices was their realization as spaces of continuous functions on compact sets (cf. [1, Ch. V]). An analog of this result in the theory of lattice-normed spaces would seem to be no less desirable. The space $C(Q, X)$ of continuous functions on a compact set Q with values in a Banach space X is a lattice-normed space. But not all lattice-normed spaces have such a simple structure, and a realization of the form $C(Q, X)$ is not universal. An escape from this situation was provided by the representation of a lattice-normed space as a space of functions assuming values not in a fixed Banach space, but in a bundle of Banach spaces over a compact set Q. In other words, the functions u from such a lattice-normed space assume values $u(q)$ in a Banach space $\mathcal{X}(q)$ that is different for each point $q \in Q$. (We note that the idea of representing different objects as bundles is not new in analysis. Evidence for this assertion can be found, for example, in [2], which in turn refers to other works in this area.) In the theory of lattice-normed spaces the first step in this direction was taken in [3], where it was established that an arbitrary lattice-normed space can be represented as a space of equivalence classes of almost global sections of some Banach bundle. But such a realization has a number of deficiencies (cf. Sec. 1.5). In particular it is not unique.

In the present article we develop the result of [3] just noted, and as an application of the realization constructed we study the structure of the linear operators of a lattice-normed space that preserve disjointness. Let us discuss this in more detail. In § 1 we give a list of the notation used and basic definitions. In § 2 we study the concept of a complete Banach bundle and construct a realization of an arbitrary lattice-normed space as a space of maximal sections of some complete Banach bundle (unique for a given lattice-normed space). In § 3 we establish criteria for an operator on section spaces to possess a certain multiplicative representation that is a generalization of the composition of a change of variable and multiplication by a scalar-valued function (such transformations are also called *weighted shift operators*). We note that the analogs of the results of § 3 for the case of a vector lattice can be found in [4] and [5].

The author is deeply grateful to A. G. Kusraev for attention to this work.

§ 1. Preliminary information

In the main we adhere to the terminology and notation of [1, 3, 6, 7]. All vector spaces in this paper are assumed to be real.

The basic concepts of the theory of lattice-normed spaces and Banach-Kantorovich spaces are discussed and rather thoroughly studied in [3] and [7]. We refer the reader to [3] for explanation of the concepts of lattice-normed spaces and Banach-Kantorovich spaces, disjoint completeness, and maximal extension of a lattice-normed space majorized by an operator. We note that instead of the terms *bo-convergence* and *br-convergence* used in [3] we shall use the terms *o-convergence* and *r-convergence* respectively. The vector norms will be denoted by $|\cdot|$. Two lattice-normed spaces $\mathcal{U}$ and $\mathcal{V}$ with E- and F-valued norms are called *isomorphic* if there exist a linear bijection $\varphi : \mathcal{U} \to \mathcal{V}$ and a linear and order isomorphism $\psi : E \to F$ such that $|\varphi(u)| = \psi(|u|)$ for all $u \in \mathcal{U}$. Here the isomorphism φ will be said to be *associated with* ψ.

Novosibirsk. Translated from *Sibirskii Matematicheskii Zhurnal*, Vol. 32, No. 2, pp. 41–54, March-April, 1991. Original article submitted March 23, 1989.

 0037-4466/91/3202-0210 $12.50 © 1991 Plenum Publishing Corporation

If in addition $E = F$ and $\psi = \mathrm{id}_E$, we shall call φ an *E-isomorphism* and the lattice-normed spaces $\mathcal{U}$ and $\mathcal{V}$ *E-isomorphic*. The order projections in the norming K-space are identified with the corresponding projections on the components of the lattice-normed space, as was done in [7], for example. The word *projection* always means *order projection*.

A family of elements of a lattice-normed space is called *disjoint* if its elements are pairwise disjoint. The sum $\sum u_\xi$ of a disjoint family (u_ξ) is taken to mean its o-sum, i.e., the o-limit of the net of sums of finite subfamilies. An element u of a lattice-normed space is called a *fragment* of the element v (and denoted $u \sqsubset v$) if $u = \pi v$ for some projection π. The symbol $\langle u \rangle$ denotes the projection on the component $\{|u|\}^{\perp\perp}$, where $\perp$ is the relation of disjointness. If $\mathcal{U}$ is a subset of a Banach-Kantorovich space, we denote by $d\mathcal{U}$ (resp. $d_{\mathrm{fin}}\mathcal{U}$) the set of sums of all (resp. all finite) disjoint families of elements of $\mathcal{U}$.

A *realization* of a K-space E is defined to be an order-dense ideal $\hat{E}$ of the K-space $C_\infty(Q)$ linearly and order-isomorphic to E (more precisely, an isomorphism of E onto $\hat{E}$), where Q is the Stone compact boolean algebra of ordinal projections of the K-space E.

If Q is an extremally disconnected compact set, then $\mathcal{B}(Q)$ denotes the boolean algebra of its open-closed subsets and $\mathcal{B}(q)$ the set of open-closed neighborhoods of the point $q \in Q$. The symbol χ_U denotes the characteristic function of the set $U \in \mathcal{B}(Q)$, and $\langle U \rangle$ denotes the projection in the K-space $C_\infty(Q)$ on the component $\{\chi_U\}^{\perp\perp}$. The function χ_Q is denoted by 1. If the norming K-space of the lattice-normed space $\mathcal{U}$ is an order-dense ideal of $C_\infty(Q)$, then the *support* of an element $u \in \mathcal{U}$ is defined to be the set $\operatorname{supp} u := \operatorname{cl}\{q \in Q : |u|(q) \neq 0\}$, where cl is the closure operator in a topological space. We shall say that a certain assertion holds *for almost all* $q \in Q$ or *almost everywhere* on Q if it holds for all the elements of some comeager set Q, i.e., a subset whose complement is a set of first Baire category.

Let S be a set and c a convergence on S. The subset $S_0 \subset S$ will be called *c-dense* in S if for each element $s \in S$ there exists a net $\{s_\alpha) \subset S_0$ that is c-convergent to s (we shall write $s_\alpha \xrightarrow{c} s$). Let S_1 and S_2 be sets with convergences c_1 and c_2 respectively. The mapping $T : S_1 \to S_2$ is called *c_1c_2-continuous* (resp. *sequentially c_1c_2-continuous*) if for any net (resp. sequence) $(s_\alpha) \subset S_1$ the convergence $s_\alpha \xrightarrow{c_1} s$ in S_1 always implies the convergence $Ts_\alpha \xrightarrow{c_2} Ts$ in S_2. If the convergences c_1 and c_2 have the same notation c, then a (sequentially) c_1c_2-continuous mapping is called (sequentially) *c-continuous*.

If X and Y are Banach spaces, we denote by $\mathcal{L}(X, Y)$ the Banach space of bounded linear transformations from X into Y.

One of the basic objects with which we shall work in the present article is a Banach bundle over an extremally disconnected compact set. Banach bundles were studied in [8] as a special case of bundles of topological vector spaces. Some of the properties of Banach bundles were studied in [3] from the point of view of the theory of Banach-Kantorovich spaces. To complete the picture we give the basic definitions here.

1.1. Let $\mathcal{X}$ and Q be topological spaces and $\mathrm{pr} : \mathcal{X} \to Q$ a continuous surjective mapping. Then the pair $(\mathcal{X}, \mathrm{pr})$ is called a *bundle over* Q. The preimage $\mathrm{pr}^{-1}(q)$ is called the *fiber* of $\mathcal{X}$ at the point $q \in Q$ and denoted $\mathcal{X}(q)$. A continuous mapping u from a nonempty subset $U \subset Q$ into the space $\mathcal{X}$ is called a *section* (over Q) if $u(q) \in \mathcal{X}(q)$ for each point $q \in U$ (and we use the notation $u \in C_U(\mathcal{X})$). If the set $U \subset Q$ is a) open, b) comeager, or c) equal to Q, the section u is called respectively a) *local* (and this fact is denoted $u \in C_0(\mathcal{X})$), b) *almost global* ($u \in \tilde{C}(\mathcal{X})$), or c) *global* ($u \in C(\mathcal{X})$).

Assume that $\mathcal{X}$ and Q are topological spaces and that we have the mappings

$$\begin{aligned}
&\mathrm{pr} : \mathcal{X} \to Q, \\
&+ : (x_1, x_2) \in \mathcal{X}^2_{\mathrm{pr}} \mapsto x_1 + x_2 \in \mathcal{X}, \\
&* : (\lambda, x) \in \mathbf{R} \times \mathcal{X} \mapsto \lambda x \in \mathcal{X}, \\
&\|\cdot\| : x \in \mathcal{X} \mapsto \|x\| \in \mathbf{R},
\end{aligned}$$

where $\mathcal{X}^2_{\mathrm{pr}} = \{(x_1, x_2) \in \mathcal{X}^2 : \mathrm{pr}(x_1) = \mathrm{pr}(x_2)\}$. The quintuple $(\mathcal{X}, \mathrm{pr}, +, *, \|\cdot\|)$ is called a *Banach bundle over* Q if the following conditions are met:

(a) the pair $(\mathcal{X}, \mathrm{pr})$ is a bundle;

(b) the mappings $+$ and $*$ are continuous (in the Tychonoff product topology), and $+[\mathcal{X}(q)^2] \subset \mathcal{X}(q)$ and $*[\mathbf{R} \times \mathcal{X}(q)] \subset \mathcal{X}(q)$ for all $q \in Q$;

(c) for each point $q \in Q$ the fiber $\mathcal{X}(q)$ is a Banach space when endowed with the operations $+|_{\mathcal{X}(q)^2}$, $*|_{\mathbf{R}\times\mathcal{X}(q)}$, and $\|\cdot\|_{\mathcal{X}(q)}$, and in addition the function $O : q \in Q \mapsto O(q) \in \mathcal{X}$, where $O(q)$ is the zero element of the Banach space $\mathcal{X}(q)$, is continuous (it is called the *zero section*);

(d) for all $x \in \mathcal{X}$ and $\varepsilon > 0$ there exists $u \in C_0(\mathcal{X})$ such that $\|x - u(\mathrm{pr}\,(x))\| < \varepsilon$, i.e., the set $\{u(q) : u \in C_0(\mathcal{X}),\ q \in \operatorname{dom} u\}$ is dense in $\mathcal{X}(q)$ for all $q \in Q$;

(e) the set of "tubes" $\{T(u,\varepsilon) : u \in C_0(\mathcal{X}),\ \varepsilon > 0\}$, where $T(u,\varepsilon) = \{x \in \mathcal{X} : \mathrm{pr}\,(x) \in \operatorname{dom} u,\ \|x - u(\mathrm{pr}\,(x))\| < \varepsilon\}$ forms a base for the topology of the space $\mathcal{X}$.

We shall write simply $\mathcal{X}$ instead of $(\mathcal{X}, \mathrm{pr}\,, +, *, \|\cdot\|)$ and use subscripts for precision where necessary: $\mathrm{pr}_{\mathcal{X}}$, $+_{\mathcal{X}}$, and the like. We remark that the norm topology in the fiber $\mathcal{X}(q)$ of the Banach bundle $\mathcal{X}$ coincides with the topology induced in $\mathcal{X}(q)$ from $\mathcal{X}$. If u is a section, then we shall write $u(q) = 0$ for short instead of $u(q) = O(q)$.

1.2. In the case when the topological space Q is locally paracompact the set of axioms (a)–(e) in the definition of a Banach bundle can be significantly weakened (cf. [8, 3.2]). It turns out in this case that axiom (d) holds in a stronger form: for each point $q \in Q$ the set $\{u(q) : u \in C(\mathcal{X})\}$ coincides with $\mathcal{X}(q)$.

1.3. Let $\mathcal{X}_0$ and $\mathcal{X}$ be Banach bundles over Q. The bundle $\mathcal{X}_0$ is called a *Banach subbundle of* $\mathcal{X}$ if

(a) $\mathcal{X}_0$ is a topological subspace of $\mathcal{X}$,

(b) $\mathrm{pr}_{\mathcal{X}_0} = (\mathrm{pr}_{\mathcal{X}})|_{\mathcal{X}_0}$,

(c) $\mathcal{X}_0(q)$ is a Banach subspace of $\mathcal{X}(q)$ for all $q \in Q$.

1.4. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach bundles over P and Q respectively. The mapping $H : \mathcal{X} \to \mathcal{Y}$ will be called a *fiberwise mapping* if

$$(\forall x_1,\ x_2 \in \mathcal{X})\ \big(\mathrm{pr}\,(x_1) = \mathrm{pr}\,(x_2) \Rightarrow \mathrm{pr}\,(H(x_1)) = \mathrm{pr}\,(H(x_2))\big).$$

In this situation we shall denote by τ_H the function mapping P into Q according to the rule $\tau_H(p) = \mathrm{pr}\,(H(x))$ for $p \in P$ and $x \in \mathcal{X}$ such that $\mathrm{pr}\,(x) = p$, i.e., $\tau_H = \mathrm{pr}_{\mathcal{Y}} \circ H \circ \mathrm{pr}_{\mathcal{X}}^{-1}$.

The Banach bundles $\mathcal{X}$ and $\mathcal{Y}$ are called *isomorphic* if there exists a fiberwise homeomorphism $H : \mathcal{X} \to \mathcal{Y}$ such that $\tau_H : P \to Q$ is also a homeomorphism and for each point $p \in P$ the restriction $H|_{\mathcal{X}(p)}$ is an isomorphism (i.e., a linear isometry) of the Banach space $\mathcal{X}(p)$ onto $\mathcal{Y}(\tau_H(p))$. In this case H is called an *isomorphism of* $\mathcal{X}$ *onto* $\mathcal{Y}$. Banach bundles $\mathcal{X}$ and $\mathcal{Y}$ over the same space Q are called *Q-isomorphic* if there exists an isomorphism $H : \mathcal{X} \to \mathcal{Y}$ such that $\tau_H = \mathrm{id}_Q$. When this happens, H is called a *Q-isomorphism of* $\mathcal{X}$ *onto* $\mathcal{Y}$.

1.5. Let $\mathcal{X}$ be a Banach bundle over an extremally disconnected compact set Q. The sections u_1, $u_2 \in \tilde{C}(\mathcal{X})$ are called *equivalent* (and we write $u_1 \sim u_2$) if they coincide on $\operatorname{dom} u_1 \cap \operatorname{dom} u_2$. The quotient set $\tilde{C}(\mathcal{X})/\sim$ becomes a lattice-normed space with $C_\infty(Q)$-valued norm in a natural way if $u_1^\sim + u_2^\sim$ and $\lambda u^\sim$ are defined as the equivalence classes $(q \in \operatorname{dom} u_1 \cap \operatorname{dom} u_2 \mapsto u_1(q) + u_2(q))^\sim$ and $(q \in \operatorname{dom} u \mapsto \lambda u(q))^\sim$ respectively and the norm of the class $u^\sim$ is defined as the (unique) function $|u^\sim| \in C_\infty(Q)$ that coincides almost everywhere with the function $\|\cdot\| \circ u$ for some (and hence every) element $u \in u^\sim$. We set $|u| := |u^\sim|$ for every almost global section $u \in \tilde{C}(\mathcal{X})$. In addition the introduction of such a norm makes the vector space $C(\mathcal{X})$ into a lattice-normed space with a $C(Q)$-valued norm. We note that the space $\tilde{C}(\mathcal{X})/\sim$ is a direct generalization of the space $C_\infty(Q, X)$ first introduced in [10, 11].

Now consider an order-dense ideal E in $C_\infty(Q)$. The symbol $E[\mathcal{X}]$ denotes the set $\{u \in \tilde{C}(\mathcal{X}) : |u| \in E\}$. Then taking account of the operations introduced above, we find that the quotient space $E[\mathcal{X}]/\sim$ is a lattice-normed space with E-valued norm (cf. [3, 5.4]). This lattice-normed space is very important, since, as shown in [3], every lattice-normed space is isomorphic to some $E[\mathcal{X}]/\sim$. But working with the space $E[\mathcal{X}]/\sim$ is somewhat complicated by the fact that its elements are not individual functions but equivalence classes of them. Moreover complications may arise in connection with the difference between the functions $\|\cdot\| \circ u$ and

$|u|$. We note also that the Banach bundle $\mathcal{X}$ that represents an arbitrary lattice-normed space as $E[\mathcal{X}]/\sim$ is in general not unique (up to isomorphism). Among other things that it does the technique of complete Banach bundles developed below removes all these difficulties.

§ 2. Complete Banach bundles and the realization of Banach-Kantorovich spaces

2.1. A Banach bundle $\mathcal{X}$ over Q is called a (continuously) *normed Banach bundle* if the mapping $\|\cdot\| : \mathcal{X} \to \mathbf{R}$ is continuous. This last property of Banach bundles was studied to some extent in [8, 15.11]. We add that a Banach bundle over an extremally disconnected compact space Q is a normed Banach bundle if for each point $q \in Q$ the set

$$\{u(q) : u \in \tilde{C}(\mathcal{X}), \text{ the mapping } \|\cdot\| \circ u \text{ is continuous at } q\}$$

is dense in $\mathcal{X}(q)$.

In what follows we shall study only normed Banach bundles over extremally disconnected compact sets, although many of the propositions given below hold for larger classes of Banach bundles.

Throughout the following Q is an extremally disconnected compact set.

2.2. Let $\mathcal{X}$ be a normed Banach bundle over Q. The section u of the bundle $\mathcal{X}$ is called *bounded* if the set $\{\|u(q)\| : q \in \operatorname{dom} u\}$ is bounded.

The following properties of a bundle $\mathcal{X}$ are equivalent:

(1) *each almost global bounded section of $\mathcal{X}$ can be extended to a global section*;*

(2) *every section $u \in \tilde{C}(\mathcal{X})$ can be extended to a section $\bar{u} \in C_{\operatorname{dom}|u|}(\mathcal{X})$;*

(3) *the lattice-normed space $C(\mathcal{X})$ is disjointly complete;*

(4) *$C(\mathcal{X})$ is a Banach-Kantorovich space.*

◄ (4) $\Rightarrow$ (1). We take an arbitrary bounded section $u \in \tilde{C}(\mathcal{X})$. For a fixed natural number n we construct a disjoint family $(u_\beta) \subset C(\mathcal{X})$ such that $\left|\sum u_\beta - u\right| \leq 1/n$ (here $\sum u_\beta$ is computed in the maximal extension of the lattice-normed space $C(\mathcal{X})$). We use transfinite induction. Assume that for some ordinal α we have constructed a disjoint family $(u_\beta)_{\beta<\alpha} \subset C(\mathcal{X})$ satisfying the relation $\left\langle \sum\limits_{\beta<\alpha} u_\beta \right\rangle \left| \sum\limits_{\beta<\alpha} u_\beta - u\right| \leq 1/n$. (The one-element family $\{0\}$ satisfies this relation, providing a basis for the induction.) If $\operatorname{supp} \sum\limits_{\beta<\alpha} u_\beta = \operatorname{supp} u$, then the family $(u_\beta)_{\beta<\alpha}$ is the one sought. Otherwise we choose an arbitrary point $q \in \left(\operatorname{supp} u \setminus \operatorname{supp} \sum\limits_{\beta<\alpha} u_\beta\right) \cap \{q \in \operatorname{dom} u : u(q) \neq 0\}$ and we take a section $\tilde{u}_\alpha \in C(\mathcal{X})$ for which $\tilde{u}_\alpha(q) = u(q)$. Then there exists a neighborhood $U \in \mathcal{B}(q)$ that is disjoint from the set $\operatorname{supp} \sum\limits_{\beta<\alpha} u_\beta$, such that $\|\tilde{u}_\alpha(q) - u(q)\| \leq 1/n$ for all $q \in U$. Setting $u_\alpha := \langle U\rangle \tilde{u}_\alpha = u_\alpha|_U \cup 0|_{Q\setminus U}$, we see that the family $(u_\beta)_{\beta\leq\alpha}$ is disjoint and satisfies the relation $\left\langle \sum\limits_{\beta<\alpha} u_\beta \right\rangle \left| \sum\limits_{\beta<\alpha} u_\beta - u\right| \leq 1/n$. In addition $\left\langle \sum\limits_{\beta<\alpha} u_\beta \right\rangle > \left\langle \sum\limits_{\beta<\alpha} u_\beta \right\rangle$. This last inequality makes it possible to conclude from cardinality considerations that the induction process for constructing the family (u_β) must terminate at some step. (From now on we shall omit the detailed discussion of such constructions.)

It follows from the boundedness of the section u that the family (u_β) constructed is bounded. Therefore $v_n := \sum u_\beta \in C(\mathcal{X})$ by the o-completeness of $C(\mathcal{X})$. It is easy to see that the sequence (v_n) is r-Cauchy and consequently has an r-limit $v \in C(\mathcal{X})$, which is obviously the desired extension of the section u.

---

* We note that in this case every bounded section of $\mathcal{X}$ defined on an everywhere dense subset of Q can be extended to a global section.

(1) $\Rightarrow$ (2). For an arbitrary section $u \in \tilde{C}(\mathfrak{X})$ and natural number n we set $U_n := \operatorname{cl}\{q \in Q : |u|(q) < n\}$ and $u_n := \langle U_n \rangle u$. Since $\bigcup_{n=1}^{\infty} U_n = \operatorname{dom}|u|$, the section $\bar{u}$ that coincides with $\bar{u}_n$ on U_n for each n is the desired extension of the section u.

The implication (2) $\Rightarrow$ (3) is obvious, and (4) follows from (3) by the r-completeness of $C(\mathfrak{X})$ (cf. [3, 3.3]). ▶

If one of the equivalent conditions (1)–(4) holds, we shall call $\mathfrak{X}$ a *complete Banach bundle*. We remark that the domain of definition of each section $u \in \tilde{C}(\mathfrak{X})$ is contained in $\operatorname{dom}|u|$, and consequently the section $\bar{u}$ that occurs in property (2) has the largest domain of definition of all sections equivalent to u. Thus it is an extension of any section of the class $u^{\sim}$. This section $\bar{u}$ will be denoted $\operatorname{ext} u$ or, more precisely, $\operatorname{ext}_{\mathfrak{X}} u$. The study of the sections $\operatorname{ext} u$ as canonical representatives of the classes $u^{\sim}$ enables us to simplify our work with the space $E[\mathfrak{X}]/\sim$ in the case of a complete Banach bundle $\mathfrak{X}$. It will be shown below (cf. Sec. 2.6) that similar canonical representatives (with maximal domain of definition) can also be exhibited in the case of an arbitrary normed Banach bundle.

Let $\mathfrak{X}$ be a complete Banach bundle over Q. A section $u \in \tilde{C}(\mathfrak{X})$ will be called *maximal* if $\operatorname{ext} u = u$. The set $\operatorname{ext}[\tilde{C}(\mathfrak{X})]$ of all maximal sections, which we shall denote $C_\infty(\mathfrak{X})$, has a natural Banach-Kantorovich space structure Q-isomorphic to $\tilde{C}(\mathfrak{X})/\sim$. If E is an order-dense ideal in $C_\infty(Q)$, we shall use the symbol $E(\mathfrak{X})$ to denote the Banach-Kantorovich space $\{u \in C_\infty(\mathfrak{X}) : |u| \in E\}$ with E-valued norm. It is obvious that the lowering of the operator ext realizes an E-isomorphism of $E[\mathfrak{X}]/\sim$ and $E(\mathfrak{X})$. We note that the equality $|u| = \|\cdot\| \circ u$ holds for all $u \in C_\infty(\mathfrak{X})$.

2.3. The following lemma gives a method of constructing Banach bundles that will be used repeatedly in what follows.

Lemma. *Let $\mathcal{U}$ be a vector space equipped with a Hausdorff multinorm $(\|\cdot\|_q)_{q\in Q}$ such that the function $q \mapsto \|u\|_q$ is continuous for each $u \in \mathcal{U}$. Then there exist a normed Banach bundle $\mathfrak{X}$ over Q and a monomorphism $u \in \mathcal{U} \mapsto \hat{u} \in C(\mathfrak{X})$ such that for each point $q \in Q$ the set $\{\hat{u}(q) : u \in \mathcal{U}\}$ is dense in $\mathfrak{X}(q)$ and $\|\hat{u}(q)\| = \|u\|_q$ for all $u \in \mathcal{U}$. Such a Banach bundle $\mathfrak{X}$ is unique up to a Q-isomorphism and will be denoted $\mathcal{U}(Q)$. If the following conditions hold:*

(a) *for any $u \in \mathcal{U}$ and $U \in \mathcal{B}(Q)$ there exists an element $u_0 \in \mathcal{U}$ such that $\|u_0 - u\|_q = 0$ for $q \in U$ and $\|u_0\|_q = 0$ for $q \notin U$;*

(b) *$\mathcal{U}$ is complete in the norm $\|u\| = \sup\{\|u\|_q : q \in Q\}$,*

then $\{\hat{u} : u \in \mathcal{U}\} = C_\infty(\mathfrak{X})$. We shall identify $u \in \mathcal{U}$ and $\hat{u} \in C_\infty(\mathfrak{X})$.

◀ We define the fiber $\mathfrak{X}(q)$ of the future normed Banach bundle as the completion of the normed space $(\mathcal{U}, \|\cdot\|_q)/\ker\|\cdot\|_q$ and we set $\mathfrak{X} := \bigcup_{q\in Q} \mathfrak{X}(q) \times \{q\}$. The operations pr, $+$, $*$, and $\|\cdot\|$ are introduced in the natural way. For each element $u \in \mathcal{U}$ we set $\hat{u} : q \in Q \mapsto (u + \ker\|\cdot\|_q, q) \in \mathfrak{X}(q) \times \{q\}$. We endow $\mathfrak{X}$ with the topology having as a base of open sets

$$\{U(u,\varepsilon) : U \text{ is an open subset of } Q,\ u \in \mathcal{U},\ \varepsilon > 0\},$$

where $U(u,\varepsilon) = \{(x,q) \in \mathfrak{X} : q \in U,\ \|\hat{u}(q) - (x,q)\| < \varepsilon\}$. It can be shown that $\mathfrak{X}$ thereby becomes a normed Banach bundle. It is not difficult to verify that property (a), which means that the set $\hat{\mathcal{U}} := \{\hat{u} : u \in \mathcal{U}\}$ is closed under the operation of taking fragments, implies that $\hat{\mathcal{U}}$ is dense in $C(\mathfrak{X})$ with the norm topology, and then the completeness of $\hat{\mathcal{U}}$ guarantees the equality $\hat{\mathcal{U}} = C(\mathfrak{X})$. The uniqueness of $\mathfrak{X}$ is proved rather simply and evidently requires no explanation. ▶

2.4. Theorem. *For any Banach-Kantorovich space $\mathcal{U}$ with an E-valued norm and any realization $\hat{E} \subset C_\infty(Q)$ of the K-space E there exists a complete Banach bundle $\mathfrak{X}$ over Q, unique up to a Q-isomorphism, such that the Banach-Kantorovich space $\hat{E}(\mathfrak{X})$ is isomorphic to $\mathcal{U}$.*

◀ We set $\mathcal{V} := \{v \in m\mathcal{U} : |v|\hat{} \in C(Q)\}$, where $m\mathcal{U}$ is a maximal extension of $\mathcal{U}$ and $e \in E \mapsto \hat{e} \in \hat{E}$ is a realization of E. We endow $\mathcal{V}$ with a multinorm by setting $\|v\|_q := |v|\hat{}(q)$ for all $v \in \mathcal{V}$ and $q \in Q$. Then the Banach bundle $\mathfrak{X} := \mathcal{V}(Q)$ is the one sought. ▶

It is natural to call the space $\hat{E}(\mathfrak{X})$ (more precisely, an isomorphism of $\mathfrak{U}$ onto $\hat{E}(\mathfrak{X})$) a *realization* of $\mathfrak{U}$ and $\mathfrak{X}$ a *realization bundle for* $\mathfrak{U}$. More precisely the complete Banach bundle $\mathfrak{X}$ is a realization bundle for $\mathfrak{U}$ if there exists a realization $\hat{E} \subset C_\infty(Q)$ of the K-space E such that the Banach-Kantorovich space $\hat{E}(\mathfrak{X})$ is isomorphic to $\mathfrak{U}$. It is obvious that this is equivalent to the statement that the Banach-Kantorovich space $C_\infty(\mathfrak{X})$ is isomorphic to a maximal extension of $\mathfrak{U}$.

2.5. We shall say that the normed Banach bundle $\mathfrak{X}$ over Q is *densely imbedded* in the normed Banach bundle $\mathfrak{Y}$ over Q if $\mathfrak{X}$ is Q-isomorphic to a Banach subbundle of $\mathfrak{Y}$ that is dense in $\mathfrak{Y}$.

Let E be an order-dense ideal in $C_\infty(Q)$ and X a Banach space. Since the lattice-normed space $E(X)$ (cf. [3, 6.1]) is naturally identified with $E(X \times Q)$ (cf. 2.6), where $X \times Q$ is a normed Banach bundle with "constant" fiber, the question arises: how is the bundle $X \times Q$ connected with a realization complete Banach bundle $\mathfrak{X}$ for $E(X)$? It turns out that $X \times Q$ is densely imbedded in $\mathfrak{X}$. Moreover the following theorem holds.

Theorem. *Every normed Banach bundle $\mathfrak{X}$ over Q is densely imbedded in some complete Banach bundle $\bar{\mathfrak{X}}$ over Q, and the bundle $\bar{\mathfrak{X}}$ is unique up to a Q-isomorphism.*

◀ Let $\mathfrak{X}$ be a realization bundle for $\tilde{C}(\mathfrak{X})/\sim$ and φ a $C_\infty(Q)$-isomorphism of $\tilde{C}(\mathfrak{X})/\sim$ onto $C_\infty(\bar{\mathfrak{X}})$. For each $x \in \mathfrak{X}$ we set $H(x) := \varphi(u^\sim)(q)$, where $u \in \tilde{C}(\mathfrak{X})$ is such that $u(q) = x$. It is not difficult to verify that the mapping H is well-defined and that it provides an isomorphism of $\mathfrak{X}$ onto a dense Banach subbundle of $\bar{\mathfrak{X}}$. ▶

It is natural to call the bundle $\bar{\mathfrak{X}}$ the *completion* of $\mathfrak{X}$. Identifying $x \in \mathfrak{X}$ and $H(x) \in \bar{\mathfrak{X}}$, we shall assume that $\mathfrak{X} \subset \bar{\mathfrak{X}}$. In justification of the term *completion* we give the following proposition without proof.

Let $\mathfrak{X}_0$ be a Banach subbundle of the complete Banach bundle $\mathfrak{X}$. Then

(a) *$\mathfrak{X}_0$ is complete $\Leftrightarrow$ $\mathfrak{X}_0$ is closed in $\mathfrak{X}$,*

(b) *the completion of $\mathfrak{X}_0$ is the smallest closed Banach subbundle of $\mathfrak{X}$ containing $\mathfrak{X}_0$, and is in fact the closure of $\mathfrak{X}_0$ in $\mathfrak{X}$.*

2.6. The theorem of the preceding section enables us to introduce canonical representatives in the equivalence classes of the almost global sections of an arbitrary normed Banach bundle, as was done in Sec. 2.2 for a complete Banach bundle. We take a normed Banach bundle $\mathfrak{X}$ and its completion $\bar{\mathfrak{X}}$. For an arbitrary section $u \in \tilde{C}(\mathfrak{X})$ we set $\bar{u} := \mathrm{ext}_{\bar{\mathfrak{X}}} u \in C_\infty(\bar{\mathfrak{X}})$. It is easy to see that the section $\mathrm{ext}_{\mathfrak{X}} u := \bar{u}|_{\mathfrak{X}} \in \tilde{C}(\mathfrak{X})$ will have maximal domain of definition in the class $u^\sim$. In complete agreement with Sec. 2.2 we introduce the concept of a maximal section for the normed Banach bundle $\mathfrak{X}$, as well as the Banach-Kantorovich spaces $C_\infty(\mathfrak{X})$ and $E(\mathfrak{X})$.

2.7. We shall say that a subset $\mathcal{V}_0$ of a lattice-normed space $\mathcal{V}$ is *properly r-dense in* $\mathcal{V}$ if for any element $v \in \mathcal{V}$ there exists a sequence $(v_n) \subset \mathcal{V}_0$ that is r-convergent to v with regulator $|v|$.

Consider a normed Banach bundle $\mathfrak{X}$ over Q. We define the *image* of a subset $\mathfrak{U} \subset C_\infty(\mathfrak{X})$ to be $\bigcup_{u \in \mathfrak{U}} \mathrm{im}\, u = \bigcup_{q \in Q} \mathfrak{U}(q)$, where $\mathfrak{U}(q) := \{u(q) : u \in \mathfrak{U},\ q \in \mathrm{dom}\, u\}$.

Theorem. *The following assertions are equivalent:*

(1) *the image of $\mathfrak{U}$ is dense in $\mathfrak{X}$;*

(2) *every global section of $\mathfrak{X}$ passes through $\bigcup_{q \in Q} \mathrm{cl}\, \mathfrak{U}(q)$ almost everywhere;*

(3) *for some (and hence every) order-dense ideal $E \subset C_\infty(Q)$ every section of $E(\mathfrak{X})$ passes through $\bigcup_{q \in Q} \mathrm{cl}\, \mathfrak{U}(q)$ almost everywhere.*

(4) *the uniform closure of the set $d\mathfrak{U}$ contains $E(\mathfrak{X})$ for some (and hence every) order-dense ideal $E \subset C_\infty(Q)$;*

(5) *$d\mathfrak{U} \cap E(\mathfrak{X})$ is r-dense (and hence properly r-dense) in $E(\mathfrak{X})$ for some (and hence every) order-dense ideal $E \subset C_\infty(Q)$;*

(6) *$d_{\mathrm{fin}}\mathfrak{U} \cap E(\mathfrak{X})$ is dense with respect to convergence almost everywhere (and hence o-dense) in $E(\mathfrak{X})$ for some (and hence every) order-dense ideal $E \subset C_\infty(Q)$).*

◀ We denote the weak form of (3) by (3′) and the strong form by (3″), and we do the same with (4)–(6). The following implications are obvious or easily verified: (6″) ⇒ (6′) ⇒ (3′), (5″) ⇒ (5′) ⇒ (3′), (4″) ⇒ (4′) ⇒ (1), (3″) ⇒ (3′) ⇒ (1), (3″) ⇒ (2) ⇒ (1). To complete the chain of implications it suffices to prove (1) ⇒ (5″) ⇒ (6″) ⇒ (3″) and (1) ⇒ (4″).

(1) ⇒ (5″). We fix an arbitrary section $u \in C_\infty(\mathfrak{X})$. As was done in Sec. 2.2 using transfinite induction, we can construct for each natural number n a family of sections $(u_\beta) \subset \mathfrak{U}$ and a disjoint family of projections (π_β) such that $\left|\sum \pi_\beta u_\beta - u\right| \leq (1/n)|u|$.

The implication (1) ⇒ (4″) is proved in a completely analogous manner.

(5″) ⇒ (6″). We take an arbitrary order-dense ideal E and choose $u \in E(\mathfrak{X})$. On the set $\mathcal{V} := d_{\text{fin}}\mathfrak{U} \cap E(\mathfrak{X})$ we introduce a pre-ordering: $v_1 \leq v_2 \Leftrightarrow |v_1 - u| \geq |v_2 - u|$. Taking the quotient space of $\mathcal{V}$ with respect to the equivalence relation $v_1 \sim v_2 \Leftrightarrow |v_1 - u| = |v_2 - u|$, we arrive at an ordered set $A := \mathcal{V}/\sim$ filtered upward. Choosing one element v_a in each class $\alpha \in A$, we obtain a net $(v_\alpha)_{\alpha\in A}$ that o-converges to u.

(6″) ⇒ (3″). We take an arbitrary $u \in C_\infty(\mathfrak{X})$ and let the net $(u_\alpha) \subset d_{\text{fin}}\mathfrak{U}$ be o-convergent to u. Obviously $u_\alpha(q) \in \mathfrak{U}(q)$ for all $q \in \operatorname{dom} u_\alpha$ such that $u_\alpha(q) \neq 0$. Since $u_\alpha(q) \leftrightarrow u(q)$ almost everywhere, we thus have $u(q) \in \operatorname{cl}\mathfrak{U}(q)$ for almost every $q \in \operatorname{supp} u$. It now remains only to remark that $O(q) \in \operatorname{cl}\mathfrak{U}(q)$ for almost every $q \in Q$. ▶

2.8. To conclude this section we note some curious corollaries of the last theorem. Consider a lattice-normed space $\mathfrak{U}$ and let $\bar{\mathfrak{U}}$ be the o-completion of it constructed in [3]. Then *for any element* $\bar{u} \in \bar{\mathfrak{U}}$ *there exists a net* $(u_\alpha) \subset \mathfrak{U}$ *that is o-convergent to* $\bar{u}$. This was not certain previously (cf., for example, [6, 4.1.8(b)]). Moreover $d\mathfrak{U}$ is r-dense in $\bar{\mathfrak{U}}$ and consequently the third stage of the construction of $\bar{\mathfrak{U}}$ mentioned in [3, 3.9] is unnecessary. Finally, the theorem on the o-completion of a lattice-normed space [3, 3.10] is simplified and acquires the following form: *for an arbitrary lattice-normed space* $\mathfrak{U}$ *there exists a complete Banach bundle* $\bar{\mathfrak{U}}$, *unique up to isomorphism, that contains* $\mathfrak{U}$ *as an o-dense lattice-normed subspace.*

§ 3. The multiplicative representation of linear operators

Throughout this section P and Q are extremally disconnected compact sets.

3.1. Consider a complete Banach bundle $\mathfrak{X}$ and a normed Banach bundle $\mathcal{Y}$ over the same space Q. If U and V are subsets of Q, $h : q \in U \mapsto h(q) \in \mathcal{L}(\mathfrak{X}(q)), \mathcal{Y}(q))$, and $u \in C_V(\mathfrak{X})$, the symbol $h \otimes u$ will denote the mapping $q \in U \cap V \mapsto h(q)u(q) \in \mathcal{Y}(q)$.

We denote by $\mathcal{H}$ the set of all mappings $h : q \in Q \mapsto h(q) \in \mathcal{L}(\mathfrak{X}(q), \mathcal{Y}(q))$ such that the function $h \otimes u$ is continuous for any $u \in C(\mathfrak{X})$. We endow the set $\mathcal{H}$ with the structure of a vector space and equip it with a multinorm by setting $\|h\|_q := \|h(q)\|$ (the operator norm) for all $h \in \mathcal{H}$ and $q \in Q$. We shall show that the function $q \in Q \mapsto \|h\|_q$ is continuous.

◀ Since $\|h(q)\| = \sup\{|hu|(q) : |u| \leq 1\}$, where $|hu| := \|\cdot\|\circ(h \otimes u)$, the set $\{|hu| : |u| \leq 1\}$ is bounded in $C_\infty(Q)$. We denote its least upper bound by $|h|$. Obviously $\|\cdot\|\circ h \leq |h|$. It suffices to prove the opposite inequality. Based on the fact that $\|h(q)\| = |h|(q)$ almost everywhere, for each $0 < \varepsilon < 1$ we can construct a family $(u_\xi^\varepsilon) \subset C(\mathfrak{X})$ and a partition of unity (π_ξ^ε) in the boolean algebra of projections such that for all ξ

$$|u_\xi^\varepsilon| \leq 1 \text{ and } \pi_\xi^\varepsilon |hu_\xi^\varepsilon| \geq \pi_\xi^\varepsilon \varepsilon |h|.$$

Setting $u_\varepsilon := \sum_\xi \pi_\xi^\varepsilon u_\xi^\varepsilon$, we see that $u_\varepsilon \in C(\mathfrak{X})$ by the completeness of $\mathfrak{X}$ and $|hu_\varepsilon| \geq \varepsilon|h|$. Therefore for each point $q \in Q$

$$\|h(q)\| \geq \sup_{0<\varepsilon<1} |hu_\varepsilon|(q) \geq \sup_{0<\varepsilon<1} \varepsilon|h|(q) = |h|(q). \text{ ▶}$$

Thus $\mathcal{H}$ satisfies the hypotheses of Lemma 2.3 and we are justified in considering the normed Banach bundle $\mathcal{H}(Q)$, which we shall denote $\mathcal{L}(\mathfrak{X}, \mathcal{Y})$. Here $\mathcal{H} = C(\mathcal{L}(\mathfrak{X}, \mathcal{Y}))$, and for each point $q \in Q$ the fiber $\mathcal{L}(\mathfrak{X}, \mathcal{Y})(q)$ is a Banach subspace of $\mathcal{L}(\mathfrak{X}(q), \mathcal{Y}(q))$. We introduce the notation $C_\infty(\mathfrak{X}, \mathcal{Y})$ for $C_\infty(\mathcal{L}(\mathfrak{X}, \mathcal{Y}))$. In addition if $h \in C_\infty(\mathfrak{X}, \mathcal{Y})$ and $u \in C_\infty(\mathfrak{X})$, then the maximal section $\operatorname{ext}(h \otimes u) \in C_\infty(\mathcal{Y})$ will be denoted by hu. It is not difficult to verify that the following properties hold for the bundle $\mathcal{L}(\mathfrak{X}, \mathcal{Y})$.

In addition if $h \in C_\infty(\mathfrak{X}, \mathfrak{Y})$ and $u \in C_\infty(\mathfrak{X})$, then the maximal section $\operatorname{ext}(h \otimes u) \in C_\infty(\mathfrak{Y})$ will be denoted by hu. It is not difficult to verify that the following properties hold for the bundle $\mathcal{L}(\mathfrak{X}, \mathfrak{Y})$.

(a) *If $\mathfrak{Y}$ is complete, then $\mathcal{L}(\mathfrak{X}, \mathfrak{Y})$ is complete.*

(b) *If $L_a \to L$ in $\mathcal{L}(\mathfrak{X}, \mathfrak{Y})$, then $L_a x_a \to Lx$ for any $x_a, x \in \mathfrak{X}$ such that $\operatorname{pr}(x_a) = \operatorname{pr}(L_a)$, $\operatorname{pr}(x) = \operatorname{pr}(L)$ and $x_a \to x$.*

(c) *Let Ω be a comeager subset of Q and $h : q \in \Omega \mapsto h(q) \in \mathcal{L}(\mathfrak{X}(q), \mathfrak{Y}(q))$. Then $h \in \tilde{C}(\mathcal{L}(\mathfrak{X}, \mathfrak{Y}))$ if and only if the mapping $h \otimes u$ is continuous for any section $u \in C(\mathfrak{X})$.*

(d) *If $h \in C_\infty(\mathfrak{X}, \mathfrak{Y})$, then $|h| = \sup\{|hu| : |u| \leq 1\} = \sup\{|hu| : |u| \sqsubset 1\}$.*

3.2. Consider a normed Banach bundle $\mathfrak{X}$ over P, and suppose given a continuous mapping $\sigma : Q_0 \to P$, where $Q_0 \in \mathcal{B}(Q)$. The set

$$\mathfrak{X}^\sigma := \Big(\bigcup_{q \in Q_0} \mathfrak{X}(\sigma(q)) \times \{q\}\Big) \cup (\{0\} \times (Q \setminus Q_0))$$

becomes a normed Banach bundle in a natural way, where $\operatorname{pr}_{\mathfrak{X}^\sigma}(x, q) = q$, and the topology is defined by the relation $(x_a, q_a) \to (x, a)$ in $\mathfrak{X}^\sigma$ if and only if $x_a \to x$ in $\mathfrak{X}$ and $q_a \to q$ in Q. For convenience of notation we shall identify the fiber $\mathfrak{X}(\sigma(q))$ of the bundle $\mathcal{H}$ with the fiber $\mathfrak{X}(\sigma(q)) \times \{q\}$ of the bundle $\mathfrak{X}^\sigma$ for all $q \in Q_0$. We denote the completion of $\mathfrak{X}^\sigma$ by $\sigma^*\mathfrak{X}$.

3.3. Assume that $\mathfrak{X}$ is a normed Banach bundle over P. For an arbitrary maximal section $u \in C_\infty(\mathfrak{X})$, we define a function w by the formula

$$w(q) := \begin{cases} u(\sigma(q)) \in \mathfrak{X}^\sigma(q), & q \in Q_0 \cap \sigma^{-1}[\operatorname{dom} u], \\ (0, q) \in \mathfrak{X}^\sigma(q), & q \in Q \setminus Q_0, \\ \text{undefined}, & \text{for other } q \in Q_0. \end{cases}$$

It is clear that if $\sigma^{-1}[\operatorname{dom} u]$ is a comeager subset of Q_0, then $w \in C_\infty(\sigma^*\mathfrak{X})$. In this case we shall say that the *shift operator* $\sigma^*_{\mathfrak{X}}$ is defined on the element u ($u \in \operatorname{dom} \sigma^*_{\mathfrak{X}}$), and denote the function w by $\sigma^*_{\mathfrak{X}} u$. We denote the shift operator $\sigma^*_{\mathbf{R} \times P}$ mapping the order-dense ideal $\operatorname{dom} \sigma^* \subset C_\infty(P)$ into $C_\infty(Q)$ by the symbol σ^*. It is clear that $u \in \operatorname{dom} \sigma^*_{\mathfrak{X}}$ if and only if $|u| \in \operatorname{dom} \sigma^*$ and $|\sigma^*_{\mathfrak{X}} u| = \sigma^*|u|$.

3.4. Let E be an order-dense ideal in $C_\infty(P)$, F an order-dense ideal in $C_\infty(Q)$, and $\mathfrak{X}$ and $\mathfrak{Y}$ complete Banach bundles over P and Q respectively. The operator $T : E(\mathfrak{X}) \to F(\mathfrak{Y})$ will be called *multiplicative* (or a *weighted shift operator*) if there exist an open-closed subset $Q_0 \subset Q$, a continuous mapping $\sigma : Q_0 \to P$, and a maximal section $h \in C_\infty(\sigma^*\mathfrak{X}, \mathfrak{Y})$ such that $E(\mathfrak{X}) \subset \operatorname{dom} \sigma^*_{\mathfrak{X}}$ and the equality $Tu = h\sigma^*_{\mathfrak{X}} u$ holds for all $u \in E(\mathfrak{X})$. In this situation we shall write $T = h\sigma^*_{\mathfrak{X}}$ and call this formula (more precisely, the pair (h, σ)) a *multiplicative representation* of T.

We shall call the multiplicative representation $T = h\sigma^*_{\mathfrak{X}}$ *canonical* if $\operatorname{dom} \sigma = \operatorname{supp} h$. Obviously every multiplicative representation $T = h\sigma^*_{\mathfrak{X}}$ can be converted into a canonical representation by replacing σ by $\sigma\big|_V$ and h by $\langle V \rangle h$, where $V = \operatorname{dom} \sigma \cap \operatorname{supp} h$.

3.5. Let the linear operator $T : E(\mathfrak{X}) \to F(\mathfrak{Y})$ preserve disjointness, i.e., let it map disjoint elements u_1 and u_2 into disjoint elements Tu_1 and Tu_2. For every set $U \in \mathcal{B}(P)$ we set $b(U) := \operatorname{cl} \cup \{\operatorname{supp} T\langle U \rangle u : u \in E(\mathfrak{X})\} \in \mathcal{B}(Q)$. Then $b : \mathcal{B}(P) \to \mathcal{B}(b(P))$ is a boolean homomorphism with $T\langle U \rangle u = \langle b(U) \rangle Tu$ for all $U \in \mathcal{B}(P)$ and $u \in E(\mathfrak{X})$. According to [9, Ch. 1, § 11] there exists a unique continuous mapping $\sigma : b(P) \to P$ such that $b(U) = \sigma^{-1}[U]$ for each $U \in \mathcal{B}(P)$. We shall denote it by the symbol σ_T and call it the *shift of the operator* T.

3.6. Theorem. *The canonical multiplicative representation $T = h\sigma^*_{\mathfrak{X}}$ of the operator $T : E(\mathfrak{X}) \to F(\mathfrak{Y})$ is unique. Here $\sigma = \sigma_T$. (Analogously the section h will be denoted h_T and called the* weight *of the operator T.*

◀ Assume that $T = h\sigma^*_{\mathfrak{X}} = \tilde{h}\tilde{\sigma}^*_{\mathfrak{X}}$ are both canonical multiplicative representations of T. It is easy to see that $\operatorname{dom} \sigma = \operatorname{dom} \tilde{\sigma}$, and that $\langle \operatorname{dom} \sigma \rangle = \sup\{\langle Tu \rangle : u \in E(\mathfrak{X})\}$. Let us choose arbitrarily a set

$U \in \mathcal{B}(P)$ and a section $u \in E(\mathfrak{X})$. We have $\langle T\langle U\rangle u\rangle = \langle h\sigma^*_{\mathfrak{X}}\langle U\rangle u\rangle = \langle\sigma^{-1}[U]\rangle\circ\langle h\sigma^*_{\mathfrak{X}}u\rangle = \langle\sigma^{-1}[U]\rangle\langle Tu\rangle$, and consequently $\langle\sigma^{-1}[U]\rangle\langle Tu\rangle = \langle\tilde{\sigma}^{-1}[U]\rangle\circ\langle Tu\rangle$. Taking the supremum over all $u \in E(\mathfrak{X})$ in these inequalities, we find that $\sigma^{-1}[U] = \tilde{\sigma}^{-1}[U] = \sigma_T^{-1}[U]$, and by the arbitrariness of U we arrive at the equality $\sigma = \tilde{\sigma} = \sigma_T$.

Thus $h\sigma^*_{\mathfrak{X}}u = \tilde{h}\sigma^*_{\mathfrak{X}}u$ for all $u \in E(\mathfrak{X})$. Since the image of the set of sections $\sigma^*_{\mathfrak{X}}[E(\mathfrak{X})]$ is dense in $\sigma^*\mathfrak{X}$, by Theorem 2.7 $hw = \tilde{h}w$ for all $w \in C(\sigma^*\mathfrak{X})$; hence $h = \tilde{h}$. ▶

3.7. Suppose we are given Banach-Kantorovich spaces $\mathcal{U}$ and $\mathcal{V}$ and some realizations of them $u \in \mathcal{U} \mapsto \hat{u} \in \hat{\mathcal{U}}$ and $v \in \mathcal{V} \mapsto \hat{v} \in \hat{\mathcal{V}}$. We define the *realization of the operator* $T : \mathcal{U} \to \mathcal{V}$ to be the operator $\hat{T} : \hat{\mathcal{U}} \to \hat{\mathcal{V}}$ acting according to the rule $\hat{T}\hat{u} = (Tu)\hat{}$. The operator $T : \mathcal{U} \to \mathcal{V}$ will be called *multiplicatively realizable* if some realization of it is multiplicative.

We remark that a multiplicatively realizable operator $T : E(\mathfrak{X}) \to F(\mathfrak{Y})$ may itself be non-multiplicative, i.e., multiplicativeness in general depends on the realization of the operator.

3.8. We call the operator $T : \mathcal{U} \to \mathcal{V}$ *r-semicontinous* if the equality $\inf|Tu_n| = 0$ holds for any sequence $(u_n) \subset \mathcal{U}$ that is r-convergent to zero.

Theorem. *Assume that the linear operator $T : E(\mathfrak{X}) \to F(\mathfrak{Y})$ preserves disjointness, and let σ be a shift of it. then T is multiplicative if and only if the following three conditions hold:*

(a) $E \subset \operatorname{dom}\sigma^*$;

(b) *T is r-semicontinuous;*

(c) *the set $\{|Tu| : |u| \sqsubset 1\}$ is bounded in $C_\infty(Q)$.*

◀ By condition (b) the boundedness of the set $\{|Tu| : |u| \leq 1\}$ in $C_\infty(Q)$ follows from (c). We denote the supremum of this set by f. We then set $\mathcal{D} := \{p \in P : (\exists e \in E)(e(p) \neq 0)\}$. It is easy to see that $\sigma^{-1}[\mathcal{D}]$ is a comeager subset of $\operatorname{dom}\sigma$.

1^0. We shall show that the following implication holds for all points q_0 in the set $\Omega := \sigma^{-1}[\mathcal{D}] \cap \operatorname{dom} f$ and for each $u \in E(\mathfrak{X})$:

$$u(\sigma(q_0)) = 0 \Rightarrow Tu(q_0) = 0.$$

Indeed, assume that $u(\sigma(q_0)) = 0$ but $Tu(q_0) \neq 0$. Then there exists a neighborhood $V_0 \in \mathcal{B}(q_0) \in \mathcal{D}$ and a number $\lambda > 0$ such that

$$|Tu|(q) \geq \lambda \tag{3.1}$$

for all $q \in V_0$. Since $\sigma(q_0) \in \mathcal{D}$, there exists a neighborhood $U_0 \in \mathcal{B}(\sigma(q_0))$ such that $\chi_{U_0} \in E$. For each natural number n we set $U_n := \operatorname{cl}\{p \in P : |u|(p) < 1/n\} \cap U_0$, $\Delta_n := U_n \setminus U_{n+1}$, $u_n := \sum_{i=1}^{n}\langle\Delta_i\rangle u$, $u_0 := \langle U_1\rangle u$. Then $\bar{u} := \sum_{n=1}^{\infty} n\langle\Delta_n\rangle u \in E(\mathfrak{X})$, since $|\bar{u}| \leq \chi_{U_0}$. Since $|u_n - u_0| \leq (1/n)\chi_{U_0}$, it follows that $u_n \xrightarrow{r} u_0$, and by the r-semicontinuity of T

$$\inf|T(u_n - u_0)| = 0. \tag{3.2}$$

Let $V \in \mathcal{B}(q_0)$ be a neighborhood such that $f(q) \leq f(q_0) + 1$ for all $q \in V$. We choose a natural number k such that $1/k \leq \lambda/2(f(q_0) + 1)$. We shall show that the inequality

$$|Tu_n|(q) \leq \lambda/2 \tag{3.3}$$

holds for all n at any point $q \in V \cap \sigma^{-1}[U_k]$. Indeed if $\sigma(q) \neq \operatorname{supp} u_n$, then relation (3.3) holds. Otherwise $\sigma(q) \in \Delta_m$ for some $m \in [k, n]$ and then

$$\begin{aligned}|Tu_n|(q) = |T\langle\Delta_m\rangle u|(q) &= (1/m)|Tm\langle\Delta_m\rangle u|(q) = (1/m)|T\bar{u}|(q) \\ &\leq (1/m)f(q) \leq (1/k)f(q) \leq (\lambda/2)(f(q_0)+1)f(q) \leq (\lambda/2).\end{aligned}$$

It follows from (3.1) and (3.3) that for each n and all $q \in V_0 \cap V \cap \sigma^{-1}[U_k]$ we have

$$|T(u_0 - u_n)|(q) \geq |Tu_0|(q) - |Tu_n|(q) \geq \lambda - \lambda/2 = \lambda/2,$$

contradicting Eq. (3.2).

2^0. For each point $q \in \Omega$ and any $x \in \mathfrak{X}(\sigma(q))$ we set $h_0(q)x := Tu(q)$, where $u \in E(\mathfrak{X})$ is a section such that $u(\sigma(q)) = x$. Since $|Tu|(q) \leq \|x\| f(q) < \infty$, it follows that $Tu(q)$ is defined. Moreover, by 1^0 above the definition of $h_0(q)x$ is independent of the choice of u. Thus a mapping $h_0 : q \in \Omega \mapsto h_0(q) \in \mathcal{L}(\mathfrak{X}^\sigma(q), \mathfrak{Y}(q))$ is defined such that $Tu(q) = h_0(q)u(\sigma(q))$ for all $u \in E(\mathfrak{X})$ and $q \in \Omega \cap \sigma^{-1}[\operatorname{dom} u]$. It turns out that h_0 has a unique "extension" to a maximal section $h \in C_\infty(\sigma^*\mathfrak{X}, \mathfrak{Y})$ such that $h_0(q) \subset h(q)$ for all $q \in \Omega$. ▶

3.9. Condition (c) in Theorem 3.8 cannot be omitted. Moreover the following proposition is true.

Theorem. *There exist a Banach space X (regarded as the space of maximal sections $E(\mathfrak{X})$, where $\mathfrak{X} = X \times \{0\}$, $E = \mathbf{R} \times \{0\}$, an extremally disconnected compact set Q, and a linear operator $T : X \to C_\infty(Q)$ (and then obviously T preserves disjointness and $E \subset \operatorname{dom} \sigma_T^*$) such that*

(a) *T is sectionally r-continuous;*

(b) *there exists a subset $X_0 \subset X$ such that the closure of the linear span of X_0 coincides with X, and the set $\{|Tx| : x \in X_0\}$ is bounded in $C_\infty(Q)$;*

but the operator T is nevertheless not multiplicative or even multiplicatively realizable.

We omit the proof of this fact due to its cumbersomeness.

3.10. An operator $T : \mathcal{U} \to \mathcal{V}$ that preserves disjointness (where $\mathcal{U}$ and $\mathcal{V}$ are Banach-Kantorovich spaces) will be called *normally defined* if for some realization of it $\hat{T} : E(\mathfrak{X}) \to F(\mathfrak{Y})$ the inclusion $E \subset \operatorname{dom} \sigma_{\hat{T}}^*$ holds. We note that every sequentially o-continuous operator that preserves disjointness (and even an operator that merely satisfies the equality $T \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} Tu_n$ for any disjoint sequence (u_n)), is normally defined. Moreover the inclusion $E \subset \operatorname{dom} \sigma_{\hat{T}}^*$ then holds for any realization $\hat{T}$ of the operator.

We shall say that the net $(u_\alpha) \subset \mathcal{U}$ is *$\bar{o}$-convergent* (resp. *$\bar{r}$-convergent*) to $u \in \mathcal{U}$ if (u_α) is o-convergent (resp. r-convergent) to u in a maximal extension $m\mathcal{U}$. A *principal ideal* of the Banach-Kantorovich space $\mathcal{U}$ with E-valued norm is a subspace $\{u \in \mathcal{U} : (\exists \lambda > 0)(|u| \leq \lambda e)\}$, defined by an arbitrary element $e \in E^+$.

Theorem. *Let $\mathcal{U}$ and $\mathcal{V}$ be Banach-Kantorovich spaces with E- and F-valued norms respectively, and assume that the linear operator $T : \mathcal{U} \to \mathcal{V}$ preserves disjointness. Then the following assertions are equivalent:*

(1) *T is r-semicontinuous and for any net $(u_\alpha)_{\alpha \in A} \subset \mathcal{U}$ that is r-convergent to zero there exists an index $\bar{\alpha} \in A$ such that the set $\{|Tu_\alpha| : \alpha \geq \bar{\alpha}\}$ is bounded in mF;*

(2) *T is r-$\bar{o}$-continuous;*

(3) *T is r-$\bar{r}$-continuous;*

(4) *T is majorized as an operator from $\mathcal{U}$ into $m\mathcal{V}$;*

(5) *for any principal ideal $\mathcal{U}_0 \subset \mathcal{U}$ the operator $T|_{\mathcal{U}_0}$ is multiplicatively realizable.*

Moreover if one of conditions (1)–(5) holds and the operator T is normally defined, then it is itself multiplicatively realizable and any realization of it $\hat{T} : \hat{E}(\mathfrak{X}) \to \hat{F}(\mathfrak{Y})$ satisfying the condition $\hat{E} \subset \operatorname{dom} \sigma_{\hat{T}}^*$ is multiplicative.

◀ The implications (5) ⇒ (4) ⇒ (3) ⇒ (2) ⇒ (1) are obvious.

1^0. We shall prove that if the operator T is normally defined, then condition (1) implies that T is multiplicatively realizable. To do this, as follows from 3.8, it suffices to establish that the set $\{|\hat{T}u| : u \in \hat{E}(\mathfrak{X}), |u| \sqsubset 1\}$ is bounded in $m\hat{F}$, where $\hat{T} : \hat{E}(\mathfrak{X}) \to \hat{F}(\mathfrak{Y})$ is a realization of the operator T such that $\hat{E} \subset \operatorname{dom} \sigma_{\hat{T}}^*$. We denote the set $\{u \in \hat{E}(\mathfrak{X}) : |u| \sqsubset 1\}$ by $\mathcal{W}$ and take an arbitrary net $(w_\alpha)_{\alpha \in A} \subset \mathcal{W}$ such that $\{w_\alpha : \alpha \in A\} = \mathcal{W}$. We endow the product $\mathbf{N} \times A$ (where $\mathbf{N}$ is the ordered set of natural numbers) with the "lexicographic" order:

$$(n, \alpha) < (n', \alpha') \Leftrightarrow n < n' \vee (n = n' \,\&\, \alpha < \alpha').$$

Since the net $u_{(n,\alpha)} := (1/n)w_\alpha$ is r-convergent to zero (we may assume that $1 \in \hat{E} \subset C_\infty(P)$ because the set $\sigma_{\hat{T}}^{-1}\big[\cup\{U \in \mathcal{B}(P) : \chi_U \in \hat{E}\}\big] = \sigma_{\hat{T}}^{-1}\big[\{p \in P : (\exists e \in E)e(p) \neq 0)\}\big]$ is dense in $\operatorname{dom}\sigma_{\hat{T}}$), it follows by condition (1) that there is an index $(\bar{n}, \bar{\alpha}) \in \mathbf{N} \times A$ such that the set $\{|Tu_{(n,\alpha)}| : (n, \alpha) \geq (\bar{n}, \bar{\alpha})\}$ is bounded in $m\hat{F}$. In particular the following sets are bounded:

$$\{|Tu_{(\bar{n}+1,\alpha)}| : \alpha \in A\} = \frac{1}{\bar{n}+1}\{|Tw_\alpha| : \alpha \in A\} = \frac{1}{\bar{n}+1}\{|Tw| : w \in W\}.$$

2^0. By 1^0 it is obvious that (1) $\Rightarrow$ (5), since for any principal ideal $\mathcal{U}_0 \subset \mathcal{U}$ the operator $T\big|_{\mathcal{U}_0}$ is normally defined. ►

We note that if the operator $T : E(\mathfrak{X}) \to F(\mathcal{Y})$ is multiplicative and $T = h\sigma_{\mathfrak{X}}^*$ is a canonical multiplicative representation of it, then $|T| = |h|\sigma^*$ and $|h| = \sup\limits_{|u|\leq 1} |Tu| = \sup\limits_{|u|\sqsubset 1} |Tu|$.

Remark. Abramovich [4] gives criteria for multiplicative realizability of an operator acting on vector lattices. If the lattices are o-complete (and consequently are Banach-Kantorovich spaces), then the result of [4] is a special case of this last theorem. (The extra strength of assumption (1) in comparison with its analog in [4] is easily removed.)

3.11. As follows from Theorem 3.10, if a Banach-Kantorovich space $\mathcal{V}$ is extended (i.e., coincides with $m\mathcal{V}$), then the multiplicative realizability of the operator $T : \mathcal{U} \to \mathcal{V}$ guarantees that it can be majorized. Otherwise this cannot be asserted. Indeed, suppose s_0 is the vector space of sequences $x : \mathbf{N} \to \mathbf{R}$ that converge to zero and s the space of all sequences. Endowing s_0 with the uniform norm, we obtain a Banach space (hence a Banach-Kantorovich space) that we denote X. Endowing s_0 and s with coordinatewise order, we obtain K-spaces (hence Banach-Kantorovich spaces), which we denote by F_0 and F respectively (F_0 is an order-dense ideal of F). Then the identity id_{s_0} will be majorized as an operator from X into F, but not as an operator from X into F_0.

3.12. To conclude this section we note a situation in which the list of conditions for multiplicativeness of an operator can be simplified.

Theorem. *Suppose a linear operator $T : E(\mathfrak{X}) \to F(\mathcal{Y})$ preserving disjointness is such that a shift of it σ is injective. Then T is multiplicative if and only if the following conditions hold:*

(a) $E \subset \operatorname{dom}\sigma^*$;

(b) *T is r-semicontinuous.*

◄ According to Theorem 3.8 it suffices to show that the set $\{|Tu| : |u| \sqsubset 1\}$ is bounded in $C_\infty(Q)$. Suppose the contrary. Without loss of generality we can assume that $1 \in E$ and $\sup\{|Tu|(q) : |u| \sqsubset 1\} = \infty$ for all q in some comeager subset $\Omega \subset Q$. We fix an arbitrary natural number n and construct a section $u_n \in E(\mathfrak{X})$ such that $|u| \sqsubset 1$ and $|Tu_n| \geq n$. (Then $(1/n)u_n \xrightarrow{r} 0$, but $\inf\left|T\frac{1}{n}u_n\right| \geq 1$, contradicting condition (b)). In the construction we use transfinite induction.

1^0. At the zeroth step we set $\bar{u}_0 := O \in E(\mathfrak{X})$.

2^0. Let α be a nonlimit ordinal and assume that the section $\bar{u}_{\alpha-1} \in E(\mathfrak{X})$ has been constructed so that $|\bar{u}_{\alpha-1}| \sqsubset 1$ and $|T\bar{u}_{\alpha-1}|(q) \geq n$ for all $q \in \sigma^{-1}[\operatorname{supp}\bar{u}_{\alpha-1}]$. If $\sigma^{-1}[\operatorname{supp}\bar{u}_{\alpha-1}] = Q$, then the section $\bar{u}_{\alpha-1}$ is the one desired. In the opposite case we choose an arbitrary $q_0 \in (Q \setminus \sigma^{-1}[\operatorname{supp}\bar{u}_{\alpha-1}) \cap \Omega$. Since $q_0 \in \Omega$, we have $u \in E(\mathfrak{X})$ such that $|u| \sqsubset 1$ and $|Tu|(q_0) > n$. The fact that σ is injective enables us to find a neighborhood $U \in \mathcal{B}(\sigma(q_0))$ such that $|Tu|(q) > n$ for all $q \in \sigma^{-1}[U]$. We set $\bar{u}_\alpha := \bar{u}_{\alpha-1} + \langle U \setminus \operatorname{supp}\bar{u}_{\alpha-1}\rangle u$. It is clear that $|\bar{u}_\alpha| \sqsubset 1$ and $|T\bar{u}_\alpha|(q) \geq n$ for all $q \in \sigma^{-1}[\operatorname{supp}\bar{u}_\alpha]$, and $\operatorname{supp}\bar{u}_\alpha$ is strictly larger than $\bar{u}_{\alpha-1}$.

3^0. Let α be a limit ordinal, and assume that for each $\beta < \alpha$ a section $\bar{u}_\beta$ has been constructed so that $|\bar{u}_\beta \sqsubset 1$ and $|T\bar{u}_\beta|(q) \geq n$ for all $q \in \sigma^{-1}[\operatorname{supp} \bar{u}_\beta]$, and that $\bar{u}_{\beta_1} \sqsubset \bar{u}_{\beta_2}$ for $\beta_1 < \beta_2 < \alpha$. Then setting $\bar{u}_\alpha := o\text{-}\lim_{\beta<\alpha} \bar{u}_\beta$, we see that $|\bar{u}_\alpha| \sqsubset 1$ and $\operatorname{supp} \bar{u}_\alpha = \operatorname{cl} \bigcup_{\beta<\alpha} \operatorname{supp} \bar{u}_\beta$. Moreover, since σ is injective

$$\sigma^{-1}[\operatorname{supp} \bar{u}_\alpha] = \operatorname{cl} \bigcup_{\beta<\alpha} \sigma^{-1}[\operatorname{supp} \bar{u}_\beta],$$

and consequently $|T\bar{u}_\alpha|(q) \geq n$ for all $q \in \sigma^{-1}[\operatorname{supp} \bar{u}_\alpha]$.

By the constant increase of $\operatorname{supp} \bar{u}_\alpha$ we are able to conclude from cardinality considerations that the inductive construction terminates at some step. ▶

3.13. Let $\mathfrak{U}$ and $\mathcal{V}$ be Banach-Kantorovich spaces with E- and F-valued norms respectively, and let E and F be ideals in the same K-space G. The operator $T : \mathfrak{U} \to \mathcal{V}$ is called *nonexpanding* if $|Tu|$ belongs to the component $\{|u|\}^{\perp\perp}$ of the K-space G for any $u \in \mathfrak{U}$. The operator $T : \mathfrak{U} \to \mathcal{V}$ is said to be a *weight operator* if some realization of it $\hat{T} : \hat{E}(\mathfrak{X}) \to \hat{F}(\mathfrak{Y})$ acts according to the rule

$$\hat{T}u = hu \qquad (u \in \hat{E}(\mathfrak{X})),$$

where $h \in C_\infty(\mathfrak{X}, \mathfrak{Y})$.

The following proposition is an immediate consequence of Theorem 3.12.

Every nonexpanding r-semicontinuous linear operator is a weight operator and conversely.

Literature Cited

1. B. Z. Vulikh, *Introduction to the Theory of Partially Ordered Spaces*, Wolters-Noordhoff, Groningen (1967).
2. K. Hoffman and K. Keimel, "Sheaf-theoretical concepts in analysis: bundles and sheaves of Banach C(X)-modules," in: *Applications of Sheaves* (Lecture Notes in Mathematics, No. 753), Springer, Berlin (1979), pp. 415–442.
3. A. G. Kusraev and V. Z. Strizhevskii, "Lattice-normed spaces and majorized operators," in: *Studies in Global Geometry and Mathematical Analysis* [in Russian], Nauka, Novosibirsk (1987), pp. 132–158.
4. Yu. A. Abramovich, "Multiplicative representation of disjointness-preserving operators," *Nederl. Akad Indag. Math.*, **45**, No. 3, 265–279 (1983).
5. Yu. A. Abramovich, A. I. Veksler, and A. V. Koldunov, "Operators that preserve disjointness, their continuity and multiplicative representation," in: *Linear Operators and their Applications* [in Russian], Leningrad (1981), pp. 13–34.
6. A. G. Kusraev, *Vector Duality and its Applications* [in Russian], Nauka, Novosibirsk (1985).
7. A. G. Kusraev, "Linear operators in lattice-normed spaces," in: *Studies in Global Geometry and Mathematical Analysis* [in Russian], Nauka, Novosibirsk (1987), pp. 84–123.
8. G. Gierz, *Bundles of topological vector spaces and their duality* (Lecture Notes in Mathematics, No. 955), Springer, Berlin (1982).
9. R. Sikorski, *Boolean Algebras*, Springer, New York (1969).
10. A. V. Bukhvalov, "Spaces of vector-valued functions and tensor products," *Sib. Mat. Zh.*, **13**, No. 6, 1229–1238 (1972).
11. A. V. Bukhvalov, "The order structure of spaces of measurable vector-valued functions," in: *Qualitative and Approximate Methods of Studying Operator Equations* [in Russian], Yaroslavl (1981), pp. 18–29.