Siberian Advances in Mathematics
1993, v.3, N4, 8-40

BANACH BUNDLES IN THE THEORY OF LATTICE-NORMED SPACES. II

MEASURABLE BANACH BUNDLES

A.E.Gutman

Abstract

The notions of measurable Banach bundle and lifting in a quotient space of measurable sections are introduced and discussed. The question is studied of representing lattice-normed spaces as those of measurable sections of Banach bundles.

Key words and phrases: measure space, extremally disconnected compactum, lifting, vector lattice, lattice-normed space, continuous and measurable Banach bundle.

The present article continues the paper [2] and is the second part of the intended article on Banach bundles in the theory of LNSs. References to Chapter 0 (Sections 0.1–0.5), Chapter 1, and Chapter 2 relate to [2]. Chapter 0 of the present article contains the necessary definitions and some preliminary information about the objects considered in the sequel. It continues Chapter 0 of [2] and, thus, starts with Section 0.6. In addition, the article includes the new two chapters, 3 and 4. In Chapter 3, we develop the theory of measurable Banach bundles by transferring Daniel's scheme to the case of sections. Note that this idea is not new (N.Dinculeanu has proposed it as early as 1966), but the author is not aware of any publications that realize the approach. In the same chapter, we introduce and discuss the notion of lifting in a space of measurable sections and present the results of applying the theory of complete Banach bundles to the study of measurable sections. Finally, Chapter 4 contains applications of previous chapters to various spaces of continuous and measurable vector-functions.

In the third part of the intended article (to appear in SIBAM, 1994, v.4, N1), we will apply the results of the first and second parts to studying disjointness-preserving operators.

0. PREREQUISITES

Among the main objects considered in the article, there are measure spaces, liftings, and vector lattices of measurable functions. The present chapter contains preliminary information on the objects listed. See [2] for Sections 0.1 – 0.5.

Received April, 1993

0.6. Measure spaces

In this section, we give some definitions and notation concerning measure spaces. In particular, we present the notion and the basic properties of lifting in quotient spaces of measurable functions and sets.

The two Boolean algebras are most frequently the focus of attention in the present article; these are the algebra Clop(Q) of clopen subsets of an extremally disconnected compactum Q, and the algebra $B(\Omega)$ of equivalence classes of measurable subsets of a measure space Ω. The first algebra was treated in Section 0.2. In this section, we describe the properties of the algebra $B(\Omega)$ and present the formulas for calculating the bounds of its subsets. By the Stone–Ogasawara theorem, Clop(Q) is a general form of complete Boolean algebra. Therefore, in most cases, the algebra $B(\Omega)$ is isomorphic to Clop(Q) for a suitable compactum Q. Existence of a lifting in $B(\Omega)$ enables us to express the connection between the algebras $B(\Omega)$ and Clop(Q) more explicitly by means of the canonical immersion $\tau: \Omega \to Q$ (see [3]: Chapter V, Section 3); the basic properties of the latter are also discussed in the present section.

Various approaches to defining measure and integral are treated, for instance, in [4]. In [5], measure spaces are considered from the point of view that is probably closest to ours. The monograph [3] is the main source of the information about lifting. Some properties of the Boolean algebra $B(\Omega)$ are also considered in [5] and [3].

0.6.1. In the present paper, a *measure space* is interpreted as a triple $(\Omega, \mathscr{B}, |\cdot|)$, where a set Ω, a σ-algebra $\mathscr{B}$ of its subsets, and a measure (= a positive countably additive function) $|\cdot| : \mathscr{B}(\Omega) \to \overline{\mathbf{R}}$ satisfy the following conditions:

(a) if $A \subset \Omega$ and $A \cap K \in \mathscr{B}$ for all elements $K \in \mathscr{B}$ of finite measure then $A \in \mathscr{B}$;

(b) if $A \in \mathscr{B}$ and $|A| = \infty$ then there exists an element $A_0 \in \mathscr{B}$ such that $A_0 \subset A$ and $0 < |A_0| < \infty$;

(c) if $A \in \mathscr{B}$, $|A| = 0$, and $A_0 \subset A$ then $A_0 \in \mathscr{B}$.

We write simply Ω instead of $(\Omega, \mathscr{B}, |\cdot|)$. In this case the σ-algebra $\mathscr{B}$ is denoted by $\mathscr{B}(\Omega)$, and its elements are called *measurable subsets* of Ω. The symbol $\mathscr{B}_{\mathrm{fin}}(\Omega)$ will be used to denote the totality of all the elements of $\mathscr{B}(\Omega)$ that have finite measure. As usual, we say that a particular condition holds *almost everywhere in* $A \in \mathscr{B}(\Omega)$, or *for almost all* $\omega \in A$, if it is valid for all of the elements of A aside from a negligible set (= a set of measure zero). "Almost everywhere" means "almost everywhere in Ω." We denote by $\mathscr{M}(\Omega)$ the totality of all almost everywhere defined functions that are measurable with respect to the σ-algebra $\mathscr{B}(\Omega)$. The set of essentially bounded functions in $\mathscr{M}(\Omega)$ is denoted by $\mathscr{L}^{\infty}(\Omega)$.

Any function $f \in \mathscr{M}(\Omega)$ is usually considered to be defined everywhere on Ω, assuming $f(\omega) := \infty$ at the points $\omega \in \Omega \setminus \operatorname{dom} f$. However, in this case the meaning of the symbol $\operatorname{dom} f$ remains unaltered: $f^{-1}[\mathbf{R}]$. In particular, the notations $|f(\omega)| < \infty$ and $|f(\omega)| = \infty$ read as $\omega \in \operatorname{dom} f$ and $\omega \notin \operatorname{dom} f$ respectively.

0.6.2. Let Ω be a nonzero measure space. Two sets $A, B \in \mathscr{B}(\Omega)$ are said to be *equivalent* (in writing $A \sim B$) if the symmetric difference $A \Delta B$ is negligible. The quotient set $\mathscr{B}(\Omega)/\sim$ is denoted by $B(\Omega)$. Given an arbitrary element $A \in \mathscr{B}(\Omega)$, we let the symbol $A^{\sim}$ stand for the equivalence class in $B(\Omega)$ containing A. A (partial) order can be defined on $B(\Omega)$ in the following natural way: $\mathbf{A} \le \mathbf{B}$ if and only if the difference $A \setminus B$ is negligible for some (hence, for all) representatives $A \in \mathbf{A}$ and $B \in \mathbf{B}$. Furnished with this order, the set $B(\Omega)$ is obviously a Boolean algebra with zero $\varnothing^{\sim}$ and unity $\Omega^{\sim}$. The Boolean operations in the algebra $B(\Omega)$ are defined by the formulas $A^{\sim} \vee B^{\sim} = (A \cup B)^{\sim}$, $A^{\sim} \wedge B^{\sim} = (A \cap B)^{\sim}$, and $(A^{\sim})^{\perp} = (\Omega \setminus A)^{\sim}$, where $A, B \in \mathscr{B}(\Omega)$.

Two functions $f, g \in \mathscr{M}(\Omega)$ are said to be *equivalent* (we write $f \sim g$) if they coincide almost everywhere. The quotient set $\mathscr{M}(\Omega)/\sim$ is denoted by $M(\Omega)$. Given an arbitrary element $f \in \mathscr{M}(\Omega)$, we let the symbol $f^{\sim}$ stand for the equivalence class in $M(\Omega)$ containing f. We denote $L^{\infty}(\Omega) = \{f^{\sim} : f \in \mathscr{L}^{\infty}(\Omega)\}$. The relations

$$\lambda f^{\sim} + \mu g^{\sim} = (\lambda f|_{\mathrm{dom}\, g} + \mu g|_{\mathrm{dom}\, f})^{\sim} \quad \text{and} \quad f^{\sim} g^{\sim} = (f|_{\mathrm{dom}\, g} \cdot g|_{\mathrm{dom}\, f})^{\sim},$$

where $\lambda, \mu \in \mathbf{R}$ and $f, g \in \mathscr{M}(\Omega)$, define on each of the sets $M(\Omega)$ and $L^{\infty}(\Omega)$ the structure of a vector space with zero $0^{\sim}$ and that of a commutative algebra with unity $1^{\sim}$. Moreover, a (partial) order can be defined on the set $M(\Omega)$ in the following natural way: $\mathbf{f} \le \mathbf{g}$ if and only if $f \le g$ almost everywhere for some (hence, for all) representatives $f \in \mathbf{f}$ and $g \in \mathbf{g}$. With respect to the operations just introduced, the space $M(\Omega)$ and the space $L^{\infty}(\Omega)$, which is a subspace of $M(\Omega)$, are vector lattices and ordered algebras. The bounds in these lattices are calculated by the formulas $f^{\sim} \vee g^{\sim} = (f|_{\mathrm{dom}\, g} \vee g|_{\mathrm{dom}\, f})^{\sim}$ and $f^{\sim} \wedge g^{\sim} = (f|_{\mathrm{dom}\, g} \wedge g|_{\mathrm{dom}\, f})^{\sim}$, for $f, g \in \mathscr{M}(\Omega)$.

0.6.3. A mapping $\rho : L^{\infty}(\Omega) \to \mathscr{L}^{\infty}(\Omega)$ is called a *lifting* of the quotient space $L^{\infty}(\Omega)$ if, for all $\lambda, \mu \in \mathbf{R}$ and $\mathbf{f}, \mathbf{g} \in L^{\infty}(\Omega)$, the following relations are valid:

(a) $\rho(\mathbf{f}) \in \mathbf{f}$ and $\mathrm{dom}\, \rho(\mathbf{f}) = \Omega$;

(b) if $\mathbf{f} \le \mathbf{g}$ then $\rho(\mathbf{f}) \le \rho(\mathbf{g})$ everywhere on Ω;

(c) $\rho(\lambda \mathbf{f} + \mu \mathbf{g}) = \lambda \rho(\mathbf{f}) + \mu \rho(\mathbf{g})$, $\rho(\mathbf{fg}) = \rho(\mathbf{f})\rho(\mathbf{g})$, $\rho(\mathbf{f} \vee \mathbf{g}) = \rho(\mathbf{f}) \vee \rho(\mathbf{g})$, and $\rho(\mathbf{f} \wedge \mathbf{g}) = \rho(\mathbf{f}) \wedge \rho(\mathbf{g})$;

(d) $\rho(0^{\sim}) = 0$ and $\rho(1^{\sim}) = 1$ everywhere on Ω.

(Some of the conditions listed above are consequences of the rest.) If $f \in \mathscr{L}^{\infty}(\Omega)$ then the function $\rho(f^{\sim})$ is usually denoted by $\rho(f)$. Since the lifting is a right-inverse of the operation $f \mapsto f^{\sim}$, we shall sometimes use the notation $\mathbf{f}_{\sim}$ instead of $\rho(\mathbf{f})$ if it cannot lead to confusion. Similarly, the symbol $f_{\sim}$ replaces $\rho(f)$.

Fix an arbitrary class $\mathbf{A} \in B(\Omega)$ and denote by $\chi_{\mathbf{A}}$ the class in $L^{\infty}(\Omega)$ that contains the characteristic function of some (hence, of every) element in $\mathbf{A}$. The properties of lifting obviously imply that the values of the function $\rho(\chi_{\mathbf{A}})$ are only 0 or 1. Denote by $\rho(\mathbf{A})$ the subset of Ω whose characteristic function is $\rho(\chi_{\mathbf{A}})$. The mapping $\rho : B(\Omega) \to \mathscr{B}(\Omega)$ thus obtained is a *lifting* of the quotient algebra $B(\Omega)$, i.e., for all $\mathbf{A}, \mathbf{B} \in B(\Omega)$ the following relations are valid:

(a) $\rho(\mathbf{A}) \in \mathbf{A}$;

(b) if $\mathbf{A} \le \mathbf{B}$ then $\rho(\mathbf{A}) \subset \rho(\mathbf{B})$;

(c) $\rho(\mathbf{A} \vee \mathbf{B}) = \rho(\mathbf{A}) \cup \rho(\mathbf{B})$, $\rho(\mathbf{A} \wedge \mathbf{B}) = \rho(\mathbf{A}) \cap \rho(\mathbf{B})$, and $\rho(\mathbf{A}^{\perp}) = \Omega \setminus \rho(\mathbf{A})$;

(d) $\rho(\varnothing^{\sim}) = \varnothing$ and $\rho(\Omega^{\sim}) = \Omega$.

By analogy with the lifting of $L^{\infty}(\Omega)$, we shall sometimes use the notation $\mathbf{A}_{\sim}$ for $\rho(\mathbf{A})$ and write $\rho(A)$ or $A_{\sim}$ instead of $\rho(A^{\sim})$.

Two points $\omega_1, \omega_2 \in \Omega$ are said to be *ρ-indistinguishable* if $\rho(\mathbf{f})(\omega_1) = \rho(\mathbf{f})(\omega_2)$ for every class $\mathbf{f} \in L^{\infty}(\Omega)$. Obviously, ω_1 and ω_2 are ρ-indistinguishable if and only if the relations $\omega_1 \in \rho(\mathbf{A})$ and $\omega_2 \in \rho(\mathbf{A})$ are equivalent for each $\mathbf{A} \in B(\Omega)$.

0.6.4. We say that a family $(A_\xi)_{\xi \in \Xi}$ of elements of $\mathscr{B}(\Omega)$ *approximates* a set $A \in \mathscr{B}(\Omega)$ if, for every measurable subset $A_0 \subset A$ of finite measure, there exists a sequence $(\xi_n)_{n \in \mathbf{N}}$ in Ξ such that $\bigcup_{n \in \mathbf{N}} A_{\xi_n} \cap A_0 \sim A_0$. The measure space Ω is said to *possess the direct sum property* if $\mathscr{B}(\Omega)$ contains a family of pairwise disjoint sets of finite measure that approximates Ω. A σ-finite measure space obviously possesses the direct sum property.

Theorem. *A measure space Ω possesses the direct sum property if and only if there exists a lifting of $L^{\infty}(\Omega)$.*

The existence of a lifting for an arbitrary σ-finite measure space was first established by D.Maharam [9]. A complete simple proof of the last theorem can be found in [3].

0.6.5. It was already mentioned in 0.6.2 that the totality $B(\Omega)$ of equivalence classes of measurable subsets of a measure space Ω is a Boolean algebra. Note that the Boolean algebra is σ-complete. The countable bounds in this algebra are calculated by the formulas $\sup_{n \in \mathbf{N}} A_n^{\sim} = (\bigcup_{n \in \mathbf{N}} A_n)^{\sim}$ and $\inf_{n \in \mathbf{N}} A_n^{\sim} = (\bigcap_{n \in \mathbf{N}} A_n)^{\sim}$ for $A_n \in \mathscr{B}(\Omega)$. The following theorem presents some information concerning (infinite) bounds in $B(\Omega)$. The proofs of the assertions formulated here can be found, for instance, in [3] (Chapter I).

Theorem. *Let Ω be a nonzero measure space.*

(1) *A family $(A_\xi)_{\xi \in \Xi}$ of measurable subsets of $A \in \mathscr{B}(\Omega)$ approximates the set A if and only if $\sup_{\xi \in \Xi} A_\xi^{\sim} = A^{\sim}$ in the Boolean algebra $B(\Omega)$.*

(2) *Suppose that a family $(A_\xi)_{\xi \in \Xi}$ of elements in $\mathscr{B}(\Omega)$ approximates Ω. Then, given any subset $A \subset \Omega$, the following relations are valid:*

(a) *$A \in \mathscr{B}(\Omega)$ if and only if $A \cap A_\xi \in \mathscr{B}(\Omega)$ for all $\xi \in \Xi$;*

(b) *$A \sim \varnothing$ if and only if $A \cap A_\xi \sim \varnothing$ for all $\xi \in \Xi$;*

(c) *if $A \in \mathscr{B}(\Omega)$ then $\sup_{\xi \in \Xi} (A \cap A_\xi)^{\sim} = A^{\sim}$ in the Boolean algebra $B(\Omega)$.*

(3) *If the measure space Ω possesses the direct sum property then the Boolean algebra $B(\Omega)$ is complete.*

(4) *Let ρ be a lifting of the quotient algebra $B(\Omega)$* (see 0.6.3). *Then, for every family $(\mathbf{A}_\xi)_{\xi \in \Xi}$ of elements of $\mathscr{B}(\Omega)$, the union $\Sigma := \bigcup_{\xi \in \Xi} \rho(\mathbf{A}_\xi)$ and the intersection $\Pi := \bigcap_{\xi \in \Xi} \rho(\mathbf{A}_\xi)$ are measurable and, moreover, $\Sigma^{\sim} = \sup_{\xi \in \Xi} \mathbf{A}_\xi$ and $\Pi^{\sim} = \inf_{\xi \in \Xi} \mathbf{A}_\xi$.*

0.6.6. Let ρ be a lifting of $L^{\infty}(\Omega)$. According to 0.6.4 and 0.6.5, the Boolean algebra $B(\Omega)$ is complete and, in view of 0.2.2, its Stonian compactum Q is extremally disconnected.

Lemma. *The collections of meager and nowhere-dense subsets of Q coincide.*

Proof. The direct sum property of Ω (see 0.6.4) enables us to reduce the situation to the case of finite measure considered in [10] (Theorem 22.2). □

For every point $\omega \in \Omega$, denote the ultrafilter $\{ \mathbf{A} \in B(\Omega) : \omega \in \rho(\mathbf{A}) \}$ by $\tau(\omega)$. The mapping $\tau : \Omega \to Q$ thus constructed will be called the *canonical immersion* of Ω in Q corresponding to the lifting ρ.

Theorem. *Let ρ be a lifting of $L^\infty(\Omega)$, let τ be the corresponding canonical immersion of Ω in the Stonian compactum Q of the Boolean algebra $B(\Omega)$, and let $\mathbf{A} \mapsto \widehat{\mathbf{A}}$ be the canonical isomorphism from $B(\Omega)$ onto* Clop(Q).

(1) *For each class $\mathbf{A} \in B(\Omega)$, the equality $\rho(\mathbf{A}) = \tau^{-1}[\widehat{\mathbf{A}}]$ holds. In particular, the inverse image $\tau^{-1}[U]$ of every clopen subset $U \subset Q$ is measurable.*

(2) *The mapping $U \mapsto \tau^{-1}[U]^{\sim}$ is an isomorphism from the Boolean algebra* Clop(Q) *onto $B(\Omega)$ which is the inverse of the isomorphism $\mathbf{A} \mapsto \widehat{\mathbf{A}}$.*

(3) *The image $\tau[\Omega]$ is dense in Q.*

(4) *The inverse image $\tau^{-1}[V]$ of every open subset $V \subset Q$ is measurable, and $\tau^{-1}[V] \sim \tau^{-1}[\operatorname{cl} V]$.*

(5) *The mapping $\tau : \Omega \to Q$ is Borel measurable.*

(6) *The inverse image $\tau^{-1}[N]$ of every meager (= nowhere-dense) subset $N \subset Q$ is measurable in Ω and has zero measure.*

(7) *Two points $\omega_1, \omega_2 \in \Omega$ are ρ-indistinguishable if and only if $\tau(\omega_1) = \tau(\omega_2)$.*

Proof. Assertion (1) is straightforward, (2) follows from (1), (3) follows from (2). Starting to prove assertion (4), we consider an arbitrary open subset $V \subset Q$. Let a family $(U_\xi)_{\xi \in \Xi}$ of clopen subsets of Q be such that $V = \bigcup_{\xi \in \Xi} U_\xi$. Then (1) implies measurability of the inverse images $\tau^{-1}[U_\xi]$ for all $\xi \in \Xi$ and 0.6.5(4) implies measurability of the set $\tau^{-1}[V] = \bigcup_{\xi \in \Xi} \tau^{-1}[U_\xi]$. The relation $\sup_{\xi \in \Xi} U_\xi = \operatorname{cl} V$ in the Boolean algebra Clop(Q), together with (2) and 0.6.5(4), ensures the equivalence $\tau^{-1}[V] \sim \tau^{-1}[\operatorname{cl} V]$. Assertions (5) and (6) follow from (4), and (7) immediately follows from the definition of the mapping τ. □

0.7. Real-valued measurable functions

In this section, we present the basic facts about the vector lattice $M(\Omega)$ of equivalence classes of measurable functions on a measure space Ω. Our attention is focused on the description of bounds, order convergence, and order projections in $M(\Omega)$. By the Vulikh–Ogasawara theorem, the vector lattice $C_\infty(Q)$ of extended continuous functions on an extremally disconnected compactum Q is a general form of an (extended) K-space. Therefore, in most cases, the vector lattice $M(\Omega)$ is isomorphic to $C_\infty(Q)$ for a suitable compactum Q. In this section we present an explicit description of this isomorphism by means of the canonical immersion $\tau : \Omega \to Q$ defined in 0.6.6. The basic properties of the spaces $M(\Omega)$ and $L^\infty(\Omega)$ are presented in [5] and [3].

All the vector spaces in this article are assumed to be given over the field $\mathbf{R}$ of real numbers and are nonzero by implication.

0.7.1. If Ω is a measure space then we assume that the functions $f_\xi \in \mathscr{M}(\Omega)$ in the pointwise envelopes $\mathrm{Inf}_{\xi \in \Xi} f_\xi$ and $\mathrm{Sup}_{\xi \in \Xi} f_\xi$ are defined everywhere on Ω and act into $\overline{\mathbf{R}}$ (see 0.6.1).

Proposition. *Let Ω be a nonzero measure space.*

(1) The vector lattice $M(\Omega)$ is a K_σ-space.

(2) A sequence $(f_n^{\sim})_{n \in \mathbf{N}}$ of elements of $M(\Omega)$ is order bounded from above if and only if $\sup_{n \in \mathbf{N}} f_n(\omega) < \infty$ for almost all $\omega \in \Omega$. In this case $\mathrm{Sup}_{n \in \mathbf{N}} f_n \in \sup_{n \in \mathbf{N}} f_n^{\sim}$.

(3) If a sequence $(f_n^{\sim})$ o-converges to $f^{\sim}$ in $M(\Omega)$ then $f_n(\omega) \to f(\omega)$ for almost all $\omega \in \Omega$.

0.7.2. The following theorem implies in particular that the vector lattices $M(\Omega)$ and $L^\infty(\Omega)$ are K-spaces if Ω is a finite or σ-finite measure space.

Theorem. *Suppose that the measure space Ω possesses the direct sum property and let ρ be a lifting of $L^\infty(\Omega)$.*

(1) The vector lattice $M(\Omega)$ is an extended K-space.

(2) A family $(\mathbf{f}_\xi)_{\xi \in \Xi}$ of elements of $M(\Omega)$ is order bounded from above if and only if there exists a family of representatives $f_\xi \in \mathbf{f}_\xi$ $(\xi \in \Xi)$ such that $\sup_{\xi \in \Xi} f_\xi(\omega) < \infty$ for almost all $\omega \in \Omega$.

(3) If a set $\mathbf{F} \subset M(\Omega)$ consists of positive elements and is order bounded from above then $\mathrm{Sup}_{\mathbf{f} \in \mathbf{F}^\infty} \rho(\mathbf{f}) \in \sup \mathbf{F}$, where $\mathbf{F}^\infty := \{\pi \mathbf{f} : \pi \in \mathrm{Pr}(M(\Omega)),\ \mathbf{f} \in \mathbf{F}\} \cap L^\infty(\Omega)$.

(4) If a family $(\mathbf{f}_\xi)_{\xi \in \Xi}$ of elements of $L^\infty(\Omega)$ is order bounded in $L^\infty(\Omega)$ then $\mathrm{Sup}_{\xi \in \Xi} \rho(\mathbf{f}_\xi) \in \sup_{\xi \in \Xi} \mathbf{f}_\xi$ and $\mathrm{Inf}_{\xi \in \Xi} \rho(\mathbf{f}_\xi) \in \inf_{\xi \in \Xi} \mathbf{f}_\xi$.

Proof. The proofs of assertions (1) and (2) are presented in [5] (I.6.10). Assertions (3) and (4) can be deduced from [3] (Chapter III, Theorem 3). □

0.7.3. Let Ω be a nonzero measure space. If $A \in \mathscr{B}(\Omega)$ and $f \in \mathscr{M}(\Omega)$ then the symbol $\langle A \rangle f$ denotes the pointwise product $\chi_A f$, where χ_A is the characteristic function of the set A. If $\mathbf{A} \in B(\Omega)$ and $\mathbf{f} \in M(\Omega)$ then the symbol $\langle \mathbf{A} \rangle \mathbf{f}$ stands for the class in $M(\Omega)$ that contains the function $\langle A \rangle f$ for some (hence, for all) representatives $A \in \mathbf{A}$ and $f \in \mathbf{f}$. The mapping $\mathbf{A} \mapsto \langle \mathbf{A} \rangle$ is an isomorphism between the Boolean algebra $B(\Omega)$ and $\mathrm{Pr}(M(\Omega))$.

The following proposition is a functional analog to the statement of Theorem 0.6.5(2).

Proposition. *Suppose that a family $(A_\xi)_{\xi \in \Xi}$ of elements of $\mathscr{B}(\Omega)$ approximates Ω. Then, for every function $f : \Omega \to \mathbf{R}$, the following relations are valid:*

(a) $f \in \mathscr{M}(\Omega)$ if and only if $\langle A_\xi \rangle f \in \mathscr{M}(\Omega)$ for all $\xi \in \Xi$;

(b) $f \sim 0$ if and only if $\langle A_\xi \rangle f \sim 0$ for all $\xi \in \Xi$;

(c) if $f \in \mathscr{M}(\Omega)$ and $f \geq 0$ then $\sup_{\xi \in \Xi} (\langle A_\xi \rangle f)^{\sim} = f^{\sim}$ in $M(\Omega)$.

0.7.4. A *Kantorovich–Pinsker space* is an arbitrary K-space containing an order dense ideal with a total set of order continuous functionals.

Theorem (A.G.Pinsker [6]). *(1) If Ω is a measure space possessing the direct sum property then $M(\Omega)$ is a Kantorovich–Pinsker space.*

(2) Every Kantorovich–Pinsker space is linearly and order isomorphic to an order dense ideal of $M(\Omega)$ for a suitable measure space Ω with the direct sum property.

0.7.5. Theorem. *Let Ω be a measure space, let ρ be a lifting of $L^\infty(\Omega)$, and let $\tau : \Omega \to Q$ be the corresponding canonical immersion of Ω in the Stonian compactum Q of the Boolean algebra $B(\Omega)$* (see 0.6.6).

(1) *An almost everywhere defined real-valued function e is measurable if and only if $e \sim f \circ \tau$ for some $f \in C_\infty(Q)$.*

(2) *For every class $\mathbf{e} \in M(\Omega)$, there exists a unique function $\hat{\mathbf{e}} \in C_\infty(Q)$ representing* $\mathbf{e}$ *as* $(\hat{\mathbf{e}} \circ \tau)^\sim$.

(3) *The mapping $\mathbf{e} \mapsto \hat{\mathbf{e}}$ is a linear, algebraic, and order isomorphism from $M(\Omega)$ onto $C_\infty(Q)$. The inverse isomorphism from $C_\infty(Q)$ onto $M(\Omega)$ acts by the rule $f \mapsto (f \circ \tau)^\sim$.*

(4) *The image of $L^\infty(\Omega)$ under the isomorphism $\mathbf{e} \mapsto \hat{\mathbf{e}}$ coincides with $C(Q)$. For every class $\mathbf{e} \in L^\infty(\Omega)$ the equality $\rho(\mathbf{e}) = \hat{\mathbf{e}} \circ \tau$ holds.*

Proof. Let $\mathrm{A} \mapsto \hat{\mathrm{A}}$ be the canonical isomorphism from $B(\Omega)$ onto $\mathrm{Clop}(Q)$. According to 0.7.3 and 0.3.4, there exists a (unique) isomorphism $\mathbf{e} \mapsto \hat{\mathbf{e}}$ from the extended K-space $M(\Omega)$ onto $C_\infty(Q)$ such that $\hat{\chi_{\mathrm{A}}} = \chi_{\hat{\mathrm{A}}}$ for all $\mathrm{A} \in B(\Omega)$. Let us form in $M(\Omega)$ the totality $\mathrm{St}(\Omega)$ of step-functions of the form $\sum_{i=1}^{n} \langle \mathrm{A}_i \rangle \lambda_i^\sim$, where $\mathrm{A}_i \in B(\Omega)$ and $\lambda_i \in \mathbf{R}$. In a similar manner, we define the set $\mathrm{St}(Q) \subset C(Q)$ of various sums $\sum_{i=1}^{n} \langle U_i \rangle \lambda_i$, where $U_i \in \mathrm{Clop}(Q)$ and $\lambda_i \in \mathbf{R}$. Obviously, the mapping $\mathbf{s} \mapsto \hat{\mathbf{s}}$ is an isomorphism from $\mathrm{St}(\Omega)$ onto $\mathrm{St}(Q)$. Since the set $\mathrm{St}(\Omega)$ is dense in $L^\infty(\Omega)$ under r-convergence with regulator $1^\sim$, and since $\mathrm{St}(Q)$ is uniformly dense in $C(Q)$, the image of $L^\infty(\Omega)$ under the isomorphism $\mathbf{e} \mapsto \hat{\mathbf{e}}$ coincides with $C(Q)$. From 0.6.6 it follows that $\rho(\mathbf{s}) = \hat{\mathbf{s}} \circ \tau$ for all $\mathbf{s} \in \mathrm{St}(\Omega)$. In these circumstances, we use the fact that r-convergence $\mathbf{s}_n \to \mathbf{e}$ with regulator $1^\sim$ implies uniform convergence $\rho(\mathbf{s}_n) \to \rho(\mathbf{e})$, and thus arrive at the equality $\rho(\mathbf{e}) = \hat{\mathbf{e}} \circ \tau$ for all $\mathbf{e} \in L^\infty(\Omega)$.

In order to complete the proof of the theorem, it suffices to demonstrate the inclusion $\hat{\mathbf{e}} \circ \tau \in \mathbf{e}$ for each class $\mathbf{e} \in M(\Omega)$. According to 0.3.2, every $\mathbf{e} \in M(\Omega)$ can be decomposed into a sum $o\text{-}\sum_{n \in \mathbf{N}} \langle \mathrm{A}_n \rangle \mathbf{e}$ of bounded pairwise disjoint classes $\langle \mathrm{A}_n \rangle \mathbf{e} \in L^\infty(\Omega)$. It remains to observe that $\hat{\mathbf{e}} \circ \tau = \big(o\text{-}\sum_{n \in \mathbf{N}} \widehat{\langle \mathrm{A}_n \rangle \mathbf{e}}\big) \circ \tau \sim \sum_{n \in \mathbf{N}} \rho(\langle \mathrm{A}_n \rangle \mathbf{e}) \in \mathbf{e}$. $\square$

We shall refer to the function $\hat{\mathbf{e}} \in C_\infty(Q)$ that corresponds to a class $\mathbf{e} \in M(\Omega)$ according to item (2) as the *Stonian transform* of $\mathbf{e}$.

3. MEASURABLE BANACH BUNDLES

Let $\mathcal{X}$ be a continuous Banach bundle over a locally compact space Ω with a fixed Radon measure. A section $u \in S(\Omega, \mathcal{X})$ is called measurable if, for every compact $K \subset \Omega$, there exists a sequence of continuous sections $u_n \in C(\Omega, \mathcal{X})$ converging to u almost everywhere on K. Such an approach to defining measurability of sections has been prevalent so far in the papers on Banach bundles. However, in our situation, a somewhat different approach seems to be appropriate. In this chapter, measurable sections are defined as the limits almost everywhere (on the subsets of finite measure) of sequences of elements of some set of sections given axiomatically and called a measurability structure. Such a way of introducing measurable sections is similar to Daniel's construction and is formally more general than the traditional topological approach. It is worth noting that the idea of measurability structure has been proposed by N.Dinculeanu as early as 1966, but has not been much studied since then.

In this chapter, we establish some elementary properties of measurable sections obtained by means of a measurability structure, introduce and study the notion of lifting in a quotient space of measurable sections, and state the results of applying the theory of complete Banach bundles to the study of measurable bundles.

Throughout the chapter, Ω is a nonzero measure space (see 0.6.1).

3.1. Basic notions

In the present section, we introduce the notion of measurable Banach bundle (MBB), which is a bundle endowed with so-called measurability structure that is a measurable analog to continuity structure (cf. 1.1.1). We give a definition and indicate several criteria of measurability of a section in an MBB. Studying the concept of measurability structure, we introduce and clarify the notions of equivalent and adequate structures and describe the greatest one among adequate measurability structures that coincides with the set of all measurable sections of the MBB under consideration (Theorem 3.1.12). In the same section we present general information on measurable sections and operations on them. The study of elementary properties of MBBs is concluded by considering the space of equivalence classes of measurable sections which appears to be an o-complete LNS in most cases (see 3.1.14).

3.1.1. Let $\mathscr{X}$ be a Banach bundle over Ω (see 0.5.1). We call a set of sections $\mathscr{C} \subset S_{\sim}(\Omega, \mathscr{X})$ a *measurability structure* in $\mathscr{X}$, if it satisfies the following three conditions:

(a) $\lambda_1 c_1 + \lambda_2 c_2 \in \mathscr{C}$ for all $\lambda_1, \lambda_2 \in \mathbf{R}$ and $c_1, c_2 \in \mathscr{C}$;

(b) the pointwise norm $|||c||| : \operatorname{dom} c \to \mathbf{R}$ of every element $c \in \mathscr{C}$ is measurable;

(c) the set $\mathscr{C}$ is stalkwise dense in $\mathscr{X}$.

If $\mathscr{C}$ is a measurability structure in $\mathscr{X}$ then we call the pair $(\mathscr{X}, \mathscr{C})$ a *measurable Banach bundle (MBB) over* Ω. We shall usually write simply $\mathscr{X}$ instead of $(\mathscr{X}, \mathscr{C})$ and denote the measurability structure $\mathscr{C}$ by $\mathscr{C}_{\mathscr{X}}$ (cf. 1.1.1). A set $\mathscr{C} \subset S_{\sim}(\Omega, \mathscr{X})$ satisfying condition (a) will be called *linear*.

3.1.2. Let $(\mathscr{X}, \mathscr{C})$ be an MBB over Ω. We say that $s \in S_{\sim}(\Omega, \mathscr{X})$ is a *$\mathscr{C}$-step-section* (or simply a *step-section*, if it is clear which measurability structure is spoken about), if $s = \sum_{i=1}^{n} \langle A_i \rangle c_i$ for some $n \in \mathbf{N}$, $A_1, \dots, A_n \in \mathscr{B}(\Omega)$, and $c_1, \dots, c_n \in \mathscr{C}$. A section $u \in S_{\sim}(\Omega, \mathscr{X})$ is called *$\mathscr{C}$-measurable* (or simply *measurable*) if, for every $K \in \mathscr{B}_{\mathrm{fin}}(\Omega)$, there is a sequence $(s_n)_{n \in \mathbf{N}}$ of $\mathscr{C}$-step-sections such that $s_n(\omega) \to u(\omega)$ for almost all $\omega \in K$. The set of all $\mathscr{C}$-measurable sections of $\mathscr{X}$ is denoted by $\mathscr{M}(\Omega, \mathscr{X} | \mathscr{C})$ or $\mathscr{M}(\Omega, \mathscr{X})$ for brevity.

3.1.3. Proposition. *Suppose that $\mathscr{X}$ is an MBB over Ω, $u, v \in S_{\sim}(\Omega, \mathscr{X})$, $e : \Omega \to \mathbf{R}$, $\lambda, \mu \in \mathbf{R}$.*

(1) *If the section u is measurable then the function $|||u|||$ is measurable.*

(2) *If the function e and the section u are measurable then the product eu is measurable.*

(3) *If the sections u and v are measurable then the sum $\lambda u + \mu v$ is measurable.*

3.1.4. Proposition. *If $\mathscr{X}$ is an MBB over a σ-finite measure space Ω then measurability of a section $u \in S_{\sim}(\Omega, \mathscr{X})$ is equivalent to existence of a sequence $(s_n)_{n \in \mathbb{N}}$ of step-sections such that $s_n(\omega) \to u(\omega)$ for almost all $\omega \in \Omega$.*

3.1.5. For every subset $\mathscr{V} \subset \mathscr{M}(\Omega, \mathscr{X})$ of sections of an MBB $\mathscr{X}$ over Ω, we shall denote by $d_{\mathrm{fin}}\mathscr{V}$ the totality of various mixings $\sum_{i=1}^{n} \langle A_i \rangle v_i$, where $v_i \in \mathscr{V}$ and the measurable subsets $A_i \subset \Omega$ are pairwise disjoint.

Lemma. *Let $\mathscr{X}$ be an MBB over Ω, let $\mathscr{V}$ be a countable subset of $\mathscr{M}(\Omega, \mathscr{X})$, and let $u \in S_{\sim}(\Omega, \mathscr{X})$. Suppose that, for every $v \in \mathscr{V}$, the function $|||u - v|||$ is measurable and $\inf_{v \in \mathscr{V}} \|u(\omega) - v(\omega)\| = 0$ for almost all $\omega \in \Omega$. Then there is a sequence of elements in $d_{\mathrm{fin}}\mathscr{V}$ that converges to u almost everywhere.*

Proof. Suppose that a set $\mathscr{V} = \{v_n : n \in \mathbb{N}\}$ and a section $u \in S_{\sim}(\Omega, \mathscr{X})$ meet the hypothesis of the lemma. We will construct a sequence $(w_n)_{n \in \mathbb{N}} \subset d_{\mathrm{fin}}\mathscr{V}$ by induction, defining $w_1 := v_1$ and $w_{n+1} := \langle A_n \rangle w_n + \langle \Omega \setminus A_n \rangle v_{n+1}$, where $A_n = \{\omega \in \Omega : \|w_n(\omega) - u(\omega)\| < \|v_{n+1}(\omega) - u(\omega)\|\}$. Then the sequence of functions $|||w_n - u|||$ pointwise decreases and $|||w_n - u||| \leq |||v_n - u|||$ for all $n \in \mathbb{N}$. Obviously, $(w_n)_{n \in \mathbb{N}}$ is the desired sequence. □

3.1.6. Proposition. *Let $\mathscr{X}$ be an MBB over Ω. If, for every $K \in \mathscr{B}_{\mathrm{fin}}(\Omega)$, there is a countable net of measurable sections of $\mathscr{X}$ converging to $u \in S_{\sim}(\Omega, \mathscr{X})$ almost everywhere on K, then the section u is measurable.*

Proof. Fix a $K \in \mathscr{B}_{\mathrm{fin}}(\Omega)$ and suppose that a countable net $(u_\alpha)_{\alpha \in \mathrm{A}}$ of elements of $\mathscr{U}$ converges to u almost everywhere on K. For each element $\alpha \in \mathrm{A}$, there exists a sequence $(s_n^\alpha)_{n \in \mathbb{N}}$ of step-sections converging to u_α almost everywhere on K. It is easy to verify that the functions $\langle K \rangle |||s_n^\alpha - u|||$ are measurable for all $\alpha \in \mathrm{A}$ and $n \in \mathbb{N}$ and, in addition, $\inf_{\alpha \in \mathrm{A}, n \in \mathbb{N}} \|s_n^\alpha(\omega) - u(\omega)\| = 0$ for almost all $\omega \in K$. Now, measurability of the section u follows from 3.1.5 and from arbitrariness of the set $K \in \mathscr{B}_{\mathrm{fin}}(\Omega)$. □

3.1.7. Proposition. *Let $\mathscr{X}$ be an MBB over Ω, and let $(A_\xi)_{\xi \in \Xi}$ a family approximating Ω. If the fragments $\langle A_\xi \rangle u$ of a section $u \in S_{\sim}(\Omega, \mathscr{X})$ are measurable for all $\xi \in \Xi$, then the section u is measurable.*

Proof. Suppose that a family $(A_\xi)_{\xi \in \Xi}$ and a section u meet the hypothesis of the proposition. It suffices to fix an arbitrary element $K \in \mathscr{B}_{\mathrm{fin}}(\Omega)$ and prove measurability of the fragment $\langle K \rangle u$. According to 0.6.4, there is a sequence $(\xi_n)_{n \in \mathbb{N}} \subset \Xi$ such that $\bigcup_{n \in \mathbb{N}} A_{\xi_n} \cap K \sim K$. It remains to use measurability of the fragments $\langle A_{\xi_n} \cap K \rangle u$ and Proposition 3.1.6. □

3.1.8. Let $(\mathscr{X}, \mathscr{C})$ be an MBB over Ω. Denote by $St(\Omega, \mathscr{X})$ the totality of various sections of $\mathscr{X}$ representable as $\sum_{\xi \in \Xi} \langle A_\xi \rangle c_\xi$, where $(c_\xi)_{\xi \in \Xi}$ is a family of elements of $\mathscr{C}$ and $(A_\xi)_{\xi \in \Xi}$ is a family of pairwise disjoint measurable sets approximating Ω. The totality of all the elements of $St(\Omega, \mathscr{X})$ representable as countable sums $\sum_{n \in \mathbb{N}} \langle A_n \rangle c_n$ is denoted by $St_\sigma(\Omega, \mathscr{X})$. Note that 3.1.7 implies the inclusion $St(\Omega, \mathscr{X}) \subset \mathcal{M}(\Omega, \mathscr{X})$.

Proposition. (1) *A section u is measurable if and only if, for every $K \in \mathscr{B}_{\text{fin}}(\Omega)$, there is a sequence $(s_n)_{n \in \mathbb{N}} \subset St_\sigma(\Omega, \mathscr{X})$ converging to u uniformly on some set $K_0 \sim K$.*

(2) *If the measure space Ω possesses the direct sum property, then a section $u \in S_\sim(\Omega, \mathscr{X})$ is measurable if and only if there exists a sequence $(s_n)_{n \in \mathbb{N}} \subset St(\Omega, \mathscr{X})$ converging to u uniformly on some set $\Omega_0 \sim \Omega$.*

Proof. Due to 3.1.6, only the necessity of the above conditions require proofs.

(1) Suppose that $u \in \mathcal{M}(\Omega, \mathscr{X})$ and fix an arbitrary set $K \in \mathscr{B}_{\text{fin}}(\Omega)$ and a number $n \in \mathbb{N}$. By the definition of measurability in 3.1.2, there exists a sequence $(s_m^n)_{m \in \mathbb{N}}$ of step-sections that converges to u almost everywhere on the set K. Then we have $o\text{-}\lim_{m \to \infty} (|||s_m^n - u||| \mid_K)^\sim = 0$ in the K-space $M(K)$ (see 0.7.2). According to 0.3.2, there is a partition of unity $(\pi_m^n)_{m \in \mathbb{N}}$ in the Boolean algebra $\Pr(M(K))$ of order projections such that $\pi_m^n(|||s_m^n - u||| \mid_K)^\sim \le 1/n$ for all $m \in \mathbb{N}$. We can choose a sequence $(A_m^n)_{m \in \mathbb{N}}$ of pairwise disjoint measurable subsets of K such that $\langle (A_m^n)^\sim \rangle = \pi_m^n$ for all $m \in \mathbb{N}$ (see 0.7.3). Obviously, the sequence of sections $\sum_{m \in \mathbb{N}} \langle A_m^n \rangle s_m^n \in St_\sigma(\Omega, \mathscr{X})$ $(n \in \mathbb{N})$ is the desired one.

(2) The direct sum property enables us to consider a family $(K_\xi)_{\xi \in \Xi}$ of pairwise disjoint measurable sets of finite measure approximating Ω. If $u \in \mathcal{M}(\Omega, \mathscr{X})$ then, in view of the already-proven assertion (1), for every $\xi \in \Xi$ there is a sequence $(s_n^\xi)_{n \in \mathbb{N}}$ of elements in $St_\sigma(\Omega, \mathscr{X})$ satisfying the inequality $|||u - s_n^\xi||| \le 1/n$ almost everywhere on K_ξ. Using 0.6.5, it is easy to verify that the sequence of sections $\sum_{\xi \in \Xi} \langle K_\xi \rangle s_n^\xi \in St(\Omega, \mathscr{X})$ $(n \in \mathbb{N})$ is the desired one. □

3.1.9. Lemma. *Let $\mathscr{X}$ be an MBB over Ω. Consider $\mathscr{V} \subset \mathcal{M}(\Omega, \mathscr{X})$ and $u \in S_\sim(\Omega, \mathscr{X})$ and suppose that the functions $|||u - v|||$ are measurable for all $v \in \mathscr{V}$ and $\inf_{v \in \mathscr{V}} |||u - v|||^\sim = 0$ in the K_σ-space $M(\Omega)$. Then the section u is measurable and, for every $K \in \mathscr{B}_{\text{fin}}(\Omega)$, there is a sequence of elements of $d_{\text{fin}}\mathscr{V}$ converging to u almost everywhere on K.*

Proof. Suppose that $\mathscr{X}$, $\mathscr{V}$, and u meet the hypothesis of the lemma. In order to justify measurability of the section u, it suffices to fix an arbitrary element $K \in \mathscr{B}_{\text{fin}}(\Omega)$ and show that the fragment $\langle K \rangle u$ is measurable. Since $\inf_{v \in \mathscr{V}} (|||u - v||| \mid_K)^\sim = 0$ in the K-space $M(K)$ that satisfies the countable chain condition, there is a countable subset $\{v_n : n \in \mathbb{N}\} \subset \mathscr{V}$ such that $\inf_{n \in \mathbb{N}} (|||u - v_n||| \mid_K)^\sim = 0$ (see [13]: VI.2.2). It remains to employ 3.1.5 and 3.1.6. □

3.1.10. Corollary. *Let $(\mathscr{X}, \mathscr{C})$ be an MBB over Ω. A section $u \in S_\sim(\Omega, \mathscr{X})$ is measurable if and only if the functions $|||u - c|||$ are measurable for all $c \in \mathscr{C}$ and $\inf_{c \in \mathscr{C}} |||u - c|||^\sim = 0$ in the K_σ-space $M(\Omega)$.*

R e m a r k. The formulated measurability criterion creates a conceptual possibility of developing the theory of MBBs without using any measure as such. For this, we only need a σ-complete ideal of negligible sets that can be introduced "axiomatically", without referring to the notion of measure.

3.1.11. Let $\mathscr{X}$ be an MBB over Ω. We call a subset $\mathscr{V} \subset \mathscr{M}(\Omega, \mathscr{X})$ *approximating* if, for every $u \in \mathscr{M}(\Omega, \mathscr{X})$, the relation $\inf_{v \in \mathscr{V}} |||u - v|||^{\sim} = 0$ holds in the K_σ-space $M(\Omega)$ (see 0.4.6).

Proposition. *The following relations between measurability structures $\mathscr{C}$ and $\mathscr{D}$ in a Banach bundle $\mathscr{X}$ over Ω are equivalent:*

(1) $\mathscr{M}(\Omega, \mathscr{X} | \mathscr{C}) = \mathscr{M}(\Omega, \mathscr{X} | \mathscr{D})$;

(2) $\mathscr{C} \subset \mathscr{M}(\Omega, \mathscr{X} | \mathscr{D})$ *and* $\mathscr{D} \subset \mathscr{M}(\Omega, \mathscr{X} | \mathscr{C})$;

(3) $\mathscr{C}$ *is an approximating subset of* $\mathscr{M}(\Omega, \mathscr{X} | \mathscr{D})$;

(4) $\mathscr{D}$ *is an approximating subset of* $\mathscr{M}(\Omega, \mathscr{X} | \mathscr{C})$;

(5) *the functions* $|||c - d|||$ *are measurable for all* $c \in \mathscr{C}$ *and* $d \in \mathscr{D}$ *and, in the K_σ-space $M(\Omega)$, we have* $\inf_{d \in \mathscr{D}} |||c - d|||^{\sim} = 0$ *for each* $c \in \mathscr{C}$ *and* $\inf_{c \in \mathscr{C}} |||c - d|||^{\sim} = 0$ *for each* $d \in \mathscr{D}$.

Proof. Use 3.1.6 and 3.1.10. □

Measurability structures $\mathscr{C}$ and $\mathscr{D}$ satisfying one of the equivalent conditions (1) – (5) are called *equivalent*. If $\mathscr{X}$ is an MBB then any measurability structure equivalent to its original structure $\mathscr{C}_{\mathscr{X}}$ will be called *adequate*. Thus, a subset of $\mathscr{M}(\Omega, \mathscr{X})$ is an adequate measurability structure in $\mathscr{X}$ if and only if it is linear, stalkwise dense in $\mathscr{X}$, and approximating in $\mathscr{M}(\Omega, \mathscr{X})$.

Since measurability structures are usually not encountered in explicit form, we shall conventionally identify the MBBs $(\mathscr{X}, \mathscr{C})$ and $(\mathscr{X}, \mathscr{D})$ with equivalent measurability structures. The stage for such an identification has already been set by the convention of writing $\mathscr{X}$ instead of $(\mathscr{X}, \mathscr{C})$ (cf. 1.1.9).

3.1.12. Theorem. *Suppose that a measurability structure $\mathscr{U}$ in a Banach bundle $\mathscr{X}$ over Ω contains all the fragments of its elements, i.e., $\langle A \rangle u \in \mathscr{U}$ for all $A \in \mathscr{B}(\Omega)$ and $u \in \mathscr{U}$. Then the following three properties of the set $\mathscr{U}$ are equivalent:*

(1) $\mathscr{U} = \mathscr{M}(\Omega, \mathscr{X} | \mathscr{C})$ *for some measurability structure $\mathscr{C}$ in $\mathscr{X}$.*

(2) *If a section $u \in S_{\sim}(\Omega, \mathscr{X})$ is such that, for every $K \in \mathscr{B}_{\mathrm{fin}}(\Omega)$, there is a sequence of elements of $\mathscr{U}$ converging to u almost everywhere on K, then $u \in \mathscr{U}$.*

(3) *The set $\mathscr{U}$ satisfies the following three conditions:*

(a) *if $(u_n)_{n \in \mathbb{N}}$ is a sequence of pairwise disjoint elements of $\mathscr{U}$ then $\sum_{n=1}^{\infty} u_n \in \mathscr{U}$;*

(b) *if a sequence of elements in $\mathscr{U}$ converges to a section $u \in S_{\sim}(\Omega, \mathscr{X})$ uniformly on some set $A \sim \Omega$ then $u \in \mathscr{U}$;*

(c) *if a section $u \in S_{\sim}(\Omega, \mathscr{X})$ is such that $\langle K \rangle u \in \mathscr{U}$ for each set $K \in \mathscr{B}_{\mathrm{fin}}(\Omega)$ then $u \in \mathscr{U}$.*

Proof. We will establish the chain of implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$ in which only the fragment $(3) \Rightarrow (2)$ needs a proof: the implications $(2) \Rightarrow (3)$ and $(2) \Rightarrow (1)$ are obvious and the implication $(1) \Rightarrow (2)$ is an immediate consequence of 3.1.6.

Assume conditions $(3)(a)-(c)$ to be satisfied, consider an arbitrary section $u \in S_{\sim}(\Omega, \mathscr{X})$, fix a set $K \in \mathscr{B}_{\text{fin}}(\Omega)$, and suppose that a sequence $(u_n)_{n \in \mathbf{N}}$ of elements of $\mathscr{U}$ converges to u almost everywhere on K. For arbitrary $n, m \in \mathbf{N}$, denote by A_m^n the set $\{\omega \in K : \|u_n(\omega) - u(\omega)\| < 1/m\}$. It is easy to verify that the sets A_m^n are measurable and $A_m := \bigcup_{n \in \mathbf{N}} A_m^n \sim K$. Since the set $\mathscr{B}(K) := \{A \in \mathscr{B}(\Omega) : A \subset K\}$ ordered by inclusion is a σ-complete Boolean algebra, therefore, in view of the exhaustion principle (0.2.1), for each $n \in \mathbf{N}$, there is a sequence $(B_m^n)_{n \in \mathbf{N}}$ of pairwise disjoint elements of $\mathscr{B}(K)$ such that $B_m^n \subset A_m^n$ for all $n \in \mathbf{N}$ and $\bigcup_{n \in \mathbf{N}} B_m^n = A_m$. Setting $v_m := \sum_{n \in \mathbf{N}} \langle B_m^n \rangle u_n$, we conclude by $(3)(a)$ that $v_m \in \mathscr{U}$. According to $(3)(b)$ and $(3)(c)$, for proving the inclusion $u \in \mathscr{U}$, it remains to observe that the sequence $(v_m)_{m \in \mathbf{N}}$ converges to u uniformly on the intersection $\bigcap_{m \in \mathbf{N}} A_m \sim K$. □

R e m a r k. The purpose of identifying MBBs with equivalent measurability structures and that of omitting the symbol of the measurability structure in the notation of an MBB (see 3.1.1, 3.1.2) is similar to that of the last theorem. Indeed, one can consider Theorem 3.1.12 to be suggestive of an approach to the definition of measurability excluding measurability structure as a notion, i.e. an approach consisting in some explicit presentation of the set of all measurable sections.

3.1.13. Suppose that $\mathscr{X}$ is an MBB over Ω. We consider an equivalence relation $\sim$ in the set $\mathscr{M}(\Omega, \mathscr{X})$ which is the coincidence almost everywhere: $u \sim v$ reads as $u(\omega) = v(\omega)$ for almost all $\omega \in \Omega$. The coset containing an element $u \in \mathscr{M}(\Omega, \mathscr{X})$ is denoted by $u^{\sim}$. The quotient set $\mathscr{M}(\Omega, \mathscr{X})/\sim$ is made into a vector space in the natural way: we write $\lambda u^{\sim} + \mu v^{\sim} = (\lambda u + \mu v)^{\sim}$ for $\lambda, \mu \in \mathbf{R}$ and $u, v \in \mathscr{M}(\Omega, \mathscr{X})$. In addition, for every element $u^{\sim} \in \mathscr{M}(\Omega, \mathscr{X})/\sim$, we can define its (vector) *norm* $|u^{\sim}| := |||u|||^{\sim} \in M(\Omega)$ (see 0.6.2). It is clear that the pair $(\mathscr{M}(\Omega, \mathscr{X})/\sim, |\cdot|)$ is an LNS over $M(\Omega)$; we denote it by $M(\Omega, \mathscr{X})$. Note that the space $M(\Omega, \mathscr{X})$ can be endowed with the natural structure of a module over the ring $M(\Omega)$ as follows: $e^{\sim} u^{\sim} := (eu)^{\sim}$ for all $e \in \mathscr{M}(\Omega)$ and $u \in \mathscr{M}(\Omega, \mathscr{X})$.

The *support of a class* $\mathbf{u} \in M(\Omega, \mathscr{X})$ is the class $\{\omega \in \Omega : u(\omega) \neq 0\}^{\sim} \in B(\Omega)$ defined by an arbitrary representative $u \in \mathbf{u}$. Obviously, the disjointness of elements of the LNS $M(\Omega, \mathscr{X})$ is equivalent to the disjointness of their supports in the Boolean algebra $B(\Omega)$. If $\mathbf{u} \in M(\Omega, \mathscr{X})$ and $\mathbf{A} \in B(\Omega)$ then the value $\langle \mathbf{A} \rangle \mathbf{u}$ of the order projection $\langle \mathbf{A} \rangle$ (see 0.7.3) at the class $\mathbf{u}$ is the class $(\langle A \rangle u)^{\sim}$, where $A \in \mathbf{A}$ and $u \in \mathbf{u}$.

3.1.14. Theorem. *If a measure space Ω possesses the direct sum property and if $\mathscr{X}$ is an MBB over Ω, then $M(\Omega, \mathscr{X})$ is a BKS over $M(\Omega)$.*

Proof. Decomposability of the LNS $M(\Omega, \mathscr{X})$ is obvious. In view of 0.4.3, for proving o-completeness of $M(\Omega, \mathscr{X})$, it suffices to establish its d- and r-completeness.

Let $(u_\xi^\sim)_{\xi \in \Xi}$ be a family of elements of $M(\Omega, \mathscr{X})$ with pairwise disjoint supports $\mathrm{A}_\xi \in B(\Omega)$ (see 3.1.13). Denote by $\bar{\Xi}$ the enrichment $\Xi \cup \{\infty\}$ of the set Ξ with a new element ∞ and define $u_\infty := 0$ and $\mathrm{A}_\infty := (\sup_{\xi \in \Xi} \mathrm{A}_\xi)^\perp$. Fix a lifting ρ of $L^\infty(\Omega)$ (see 0.6.4) and denote $A_\xi := \rho(\mathrm{A}_\xi)$ for each $\xi \in \bar{\Xi}$. In view of 0.6.5, the union $\bigcup_{\xi \in \bar{\Xi}} A_\xi$ is measurable and differs from Ω by a set of measure zero. It is easy to verify that the section $u := \bigcup_{\xi \in \bar{\Xi}} u_\xi|_{A_\xi}$ is measurable and the corresponding class $u^\sim \in M(\Omega, \mathscr{X})$ is the desired sum $o\text{-}\sum_{\xi \in \Xi} u_\xi^\sim$.

Now, suppose that a sequence $(u_n^\sim)_{n \in \mathbb{N}}$ of elements of $M(\Omega, \mathscr{X})$ is r-fundamental. Then, for almost all $\omega \in \Omega$, the sequence $(u_n(\omega))_{n \in \mathbb{N}}$ is fundamental. Due to completeness of the stalks of $\mathscr{X}$, there exists a section $u \in S_\sim(\Omega, \mathscr{X})$ to which the sequence $(u_n)_{n \in \mathbb{N}}$ converges almost everywhere. It is clear that the section u is measurable and the corresponding class $u^\sim$ is the desired r-limit of the sequence $(u_n^\sim)_{n \in \mathbb{N}}$. □

R e m a r k. If we do not require that the measure space Ω possesses the direct sum property, then $M(\Omega, \mathscr{X})$ becomes a countably (= sequentially) o-complete LNS over the K_σ-space $M(\Omega)$ (see 0.4.3).

3.1.15. If Ω is a measure space possessing the direct sum property and E is an ideal of $M(\Omega)$ then the set $E(\mathscr{X}) := \{\mathbf{u} \in M(\Omega, \mathscr{X}) : |\mathbf{u}| \in E\}$ endowed with the operations induced from $M(\Omega, \mathscr{X})$ is a BKS over E. We shall see below (Theorem 3.4.8) that the space $E(\mathscr{X})$ is a general form of a BKS over E in a certain sense (cf. 2.4.2). The symbol $\mathscr{L}^\infty(\Omega, \mathscr{X})$ stands for the set $\{u \in \mathcal{M}(\Omega, \mathscr{X}) : |||u||| \in \mathscr{L}^\infty(\Omega)\}$ and its elements are called *(essentially) bounded measurable sections* of $\mathscr{X}$. The equivalence classes constituted by essentially bounded sections are called *bounded classes* and the totality of all such classes is denoted by $L^\infty(\Omega, \mathscr{X})$. Obviously, the space $L^\infty(\Omega, \mathscr{X})$ coincides with $E(\mathscr{X})$, where $E = L^\infty(\Omega)$. In particular, $L^\infty(\Omega, \mathscr{X})$ is a BKS over $L^\infty(\Omega)$.

3.2. Examples

A short list of examples of MBBs presented in this section contains MBB with constant stalk, measurable subbundle and restriction of an MBB, as well as the notion of isometric MBBs.

3.2.1. Let X be a Banach space. If the totality of constant functions $c : \Omega \to X$ is taken as the measurability structure of the trivial Banach bundle $\mathscr{X} = \Omega \times \{X\}$, then the set $\mathcal{M}(\Omega, \mathscr{X})$ consists exactly of all Bochner measurable X-valued functions defined almost everywhere in Ω. In this case we use the notation $\mathcal{M}(\Omega, X)$ instead of $\mathcal{M}(\Omega, \mathscr{X})$. Note that the totality $\mathcal{M}(\Omega, X)$ of all measurable functions is another measurability structure in the trivial bundle $\Omega \times \{X\}$, equivalent to the structure of constant functions (cf. 1.2.1).

3.2.2. Consider an MBB $\mathscr{X}$ over Ω. An MBB $\mathscr{X}_0$ over Ω is called a *(measurable) subbundle* of $\mathscr{X}$, if $\mathscr{X}_0(\omega)$ is a Banach subspace of $\mathscr{X}(\omega)$ for every point $\omega \in \Omega$ and, moreover, $\mathscr{M}(\Omega, \mathscr{X}_0) = \mathscr{M}(\Omega, \mathscr{X}) \cap S_{\sim}(\Omega, \mathscr{X}_0)$ (cf. 1.2.2). If $\mathscr{X}_0$ is a subbundle of $\mathscr{X}$ then we make the convention that $M(\Omega, \mathscr{X}_0) \subset M(\Omega, \mathscr{X})$, identifying the classes $\{u \in \mathscr{M}(\Omega, \mathscr{X}_0) : u \sim u_0\} \in M(\Omega, \mathscr{X}_0)$ and $\{u \in \mathscr{M}(\Omega, \mathscr{X}) : u \sim u_0\} \in M(\Omega, \mathscr{X})$ for each section $u_0 \in \mathscr{M}(\Omega, \mathscr{X}_0)$.

Now, suppose that every stalk $\mathscr{X}_0(\omega)$ of a discrete Banach bundle $\mathscr{X}_0$ is a Banach subspace of the corresponding stalk $\mathscr{X}(\omega)$. If the intersection $\mathscr{M}(\Omega, \mathscr{X}) \cap S_{\sim}(\Omega, \mathscr{X}_0)$ is stalkwise dense in $\mathscr{X}_0$ then it is a measurability structure in $\mathscr{X}_0$, with respect to which $\mathscr{X}_0$ becomes a measurable subbundle of $\mathscr{X}$. We say that the measurability structure of $\mathscr{X}_0$ is *induced* by $\mathscr{X}$. Note that the intersection $\mathscr{M}(\Omega, \mathscr{X}) \cap S_{\sim}(\Omega, \mathscr{X}_0)$ is always stalkwise dense in $\mathscr{X}_0$, if all the one-point subsets of Ω are measurable.

Dually, any linear subset $\mathscr{U} \subset \mathscr{M}(\Omega, \mathscr{X})$ (see 3.1.1) *induces* a subbundle of $\mathscr{X}$, i.e. such a subbundle $\mathscr{X}_0$ that $\mathscr{U}$ is its measurability structure: $\mathscr{X}_0(\omega) = \mathrm{cl}\{u(\omega) : u \in \mathscr{U}\}$ $(\omega \in \Omega)$.

3.2.3. Proposition. *Let $\mathscr{X}$ be a subbundle of an MBB $\overline{\mathscr{X}}$ over Ω and let $\mathscr{C}$ and $\overline{\mathscr{C}}$ be adequate measurability structures in $\mathscr{X}$ and $\overline{\mathscr{X}}$ respectively. The following assertions are equivalent:*

(1) every section $\bar{u} \in \mathscr{M}(\Omega, \overline{\mathscr{X}})$ assumes the values $\bar{u}(\omega) \in \mathscr{X}(\omega)$ for almost all $\omega \in \Omega$;

(2) $E(\mathscr{X}) = E(\overline{\mathscr{X}})$ for every order dense ideal $E \subset M(\Omega)$ (the equality is understood in terms of the inclusion $M(\Omega, \mathscr{X}) \subset M(\Omega, \overline{\mathscr{X}})$, see 3.2.2 *);*

(3) $E(\mathscr{X}) = E(\overline{\mathscr{X}})$ for some order dense ideal $E \subset M(\Omega)$;

(4) $\mathscr{C}$ is an approximating subset of $\mathscr{M}(\Omega, \overline{\mathscr{X}})$;

(5) for all $\bar{c} \in \overline{\mathscr{C}}$, we have $\inf_{c \in \mathscr{C}} |||\bar{c} - c|||^{\sim} = 0$ in the K_σ-space $M(\Omega)$.

Proof. The implications $(1) \Rightarrow (2) \Rightarrow (3)$ and $(4) \Rightarrow (5)$ are obvious. If E is an order dense ideal of $M(\Omega)$ then the sets $\{u \in \mathscr{M}(\Omega, \mathscr{X}) : |||u|||^{\sim} \in E\}$ and $\{\bar{u} \in \mathscr{M}(\Omega, \overline{\mathscr{X}}) : |||\bar{u}|||^{\sim} \in E\}$ are adequate measurability structures in $\mathscr{X}$ and $\overline{\mathscr{X}}$ respectively. Therefore, (3) implies (4) by 3.1.11. It remains to observe that $(5) \Rightarrow (1)$. In view of 3.1.9, from (5) it follows that each section $\bar{c} \in \overline{\mathscr{C}}$ assumes the values $\bar{c}(\omega) \in \mathscr{X}(\omega)$ for almost all elements $\omega \in K$ of every set $K \in \mathscr{B}_{\mathrm{fin}}(\Omega)$. Hence, all the sections $\bar{u} \in \mathscr{M}(\Omega, \overline{\mathscr{X}})$ have this property (see 3.1.11(1) and 3.1.2). Assertion (1) now follows from 0.6.5. □

We call a subbundle $\mathscr{X}$ possessing one of the equivalent properties (1) – (5) a *dense subbundle of the MBB $\overline{\mathscr{X}}$.*

3.2.4. Let $\mathscr{X}$ be an MBB over Ω and let D be a measurable subset of Ω. If $\mathscr{C}$ is the measurability structure of $\mathscr{X}$ then the set $\{c|_D : c \in \mathscr{C}\}$ is a measurability structure in the (discrete) Banach bundle $\mathscr{X}|_D$. The MBB over D thus obtained is called the *restriction of the MBB $\mathscr{X}$ onto D* and denoted by $\mathscr{X}|_D$. Obviously, $\mathscr{M}(D, \mathscr{X}|_D) = \{u|_D : u \in \mathscr{M}(\Omega, \mathscr{X})\}$ (cf. 1.2.5).

3.2.5. Let $\mathscr{X}$ and $\mathscr{Y}$ be MBBs over the same measure space Ω. We call a mapping $H: \omega \in \Omega \mapsto H(\omega) \in B(\mathscr{X}(\omega), \mathscr{Y}(\omega))$ an *isometry* from $\mathscr{X}$ onto $\mathscr{Y}$ if, at each point $\omega \in \Omega$, the operator $H(\omega)$ is a linear isometry from $\mathscr{X}(\omega)$ onto $\mathscr{Y}(\omega)$ and, moreover, $\mathcal{M}(\Omega, \mathscr{Y}) = \{H \otimes u : u \in \mathcal{M}(\Omega, \mathscr{X})\}$. In the event that such a mapping H exists, the bundles $\mathscr{X}$ and $\mathscr{Y}$ are called *isometric*. If, for each $\mathbf{u} \in M(\Omega, \mathscr{X})$, we define the class $H \otimes \mathbf{u} \in M(\Omega, \mathscr{Y})$ by $H \otimes \mathbf{u} = (H \otimes u)^{\sim}$, where u is an arbitrary representative of $\mathbf{u}$, then the mapping $\mathbf{u} \mapsto H \otimes \mathbf{u}$ is an isometry from the LNS $M(\Omega, \mathscr{X})$ onto $M(\Omega, \mathscr{Y})$.

Proposition. *Let $\mathscr{X}$ and $\mathscr{Y}$ be MBBs over Ω and let a mapping H associate with each point $\omega \in \Omega$ a linear isometry $H(\omega)$ of the stalk $\mathscr{X}(\omega)$ onto $\mathscr{Y}(\omega)$. A necessary and sufficient condition for the mapping H to be an isometry from $\mathscr{X}$ onto $\mathscr{Y}$ is the existence of an approximating subset $\mathscr{C} \subset \mathcal{M}(\Omega, \mathscr{X})$ such that $\{H \otimes c : c \in \mathscr{C}\}$ is an approximating subset of $\mathcal{M}(\Omega, \mathscr{Y})$.*

Proof. Only necessity of the above condition needs a proof. Suppose that there is a set $\mathscr{C}$ possessing the formulated property. If $u \in \mathcal{M}(\Omega, \mathscr{X})$ then, for every element $c \in \mathscr{C}$, the function $|||H \otimes u - H \otimes c||| = |||u - c|||$ is measurable and

$$\inf_{c \in \mathscr{C}} |||H \otimes u - H \otimes c|||^{\sim} = \inf_{c \in \mathscr{C}} |||u - c|||^{\sim} = 0$$

in the K_σ-space $M(\Omega)$; hence, in view of 3.1.9, measurability of $H \otimes u$ follows. Measurability of a section $u \in S_{\sim}(\Omega, \mathscr{X})$ is deduced from that of $H \otimes u$ in an entirely similar manner. □

3.3. Lifting in spaces of sections

In this section we introduce and study the notion of lifting in a quotient space of measurable sections of an MBB. Liftable measurable Banach bundles in the class of all MBBs occupy, in a sense, the same place as complete continuous Banach bundles in the class of all CBBs. Many corroborations of it can be found in Section 3.4; in this section, the connection between liftable MBBs and complete CBBs is established explicitly. Theorem 3.3.4 proposes a method of constructing a liftable MBB given an arbitrary complete CBB over the corresponding extremally disconnected compactum. Theorem 3.3.5 asserts that such a method is universal.

Throughout the section, Ω is a nonzero measure space possessing the direct sum property.

3.3.1. Let $\mathscr{X}$ be an MBB over Ω. Consider a lifting $\rho: L^\infty(\Omega) \to \mathscr{L}^\infty(\Omega)$ (see 0.6.3). We call a mapping $\rho_{\mathscr{X}}: L^\infty(\Omega, \mathscr{X}) \to \mathscr{L}^\infty(\Omega, \mathscr{X})$ a *lifting* of $L^\infty(\Omega, \mathscr{X})$ *(associated with ρ) if, for all* $\mathbf{u}, \mathbf{v} \in L^\infty(\Omega, \mathscr{X})$ and $\mathbf{e} \in L^\infty(\Omega)$, the following relations hold:

(a) $\rho_{\mathscr{X}}(\mathbf{u}) \in \mathbf{u}$ and $\operatorname{dom} \rho_{\mathscr{X}}(\mathbf{u}) = \Omega$;

(b) $|||\rho_{\mathscr{X}}(\mathbf{u})||| = \rho(|\mathbf{u}|)$;

(c) $\rho_{\mathscr{X}}(\mathbf{u} + \mathbf{v}) = \rho_{\mathscr{X}}(\mathbf{u}) + \rho_{\mathscr{X}}(\mathbf{v})$;

(d) $\rho_{\mathscr{X}}(\mathbf{e}\mathbf{u}) = \rho(\mathbf{e})\rho_{\mathscr{X}}(\mathbf{u})$;

(e) the set $\{\rho_{\mathscr{X}}(\mathbf{u}) : \mathbf{u} \in L^\infty(\Omega, \mathscr{X})\}$ is stalkwise dense in $\mathscr{X}$.

In case there exists a lifting of $L^\infty(\Omega)$ and a lifting of $L^\infty(\Omega, \mathscr{X})$ associated with it, we say that $\mathscr{X}$ is a *liftable MBB*. If it is clear which liftings are dealt with then, given $\mathbf{e} \in L^\infty(\Omega)$ and $\mathbf{u} \in L^\infty(\Omega, \mathscr{X})$, we write $\mathbf{e}_\sim$ and $\mathbf{u}_\sim$ instead of $\rho(\mathbf{e})$ and $\rho_{\mathscr{X}}(\mathbf{u})$, respectively. For $e \in \mathscr{L}^\infty(\Omega)$ and $u \in \mathscr{L}^\infty(\Omega, \mathscr{X})$, the notation $\rho(e^\sim)$, $\rho_{\mathscr{X}}(u^\sim)$, $(e^\sim)_\sim$, and $(u^\sim)_\sim$ will be replaced by the symbols $\rho(e)$, $\rho_{\mathscr{X}}(u)$, $e_\sim$, and $u_\sim$, respectively.

3.3.2. The examples (1) and (2) below show that condition (e) in the definition of lifting in 3.3.1 does not follow from conditions (a) – (d).

(1) Let us consider a trivial bundle $\mathscr{X} = [0, 1] \times \{\mathbf{R}\}$ over the Lebesgue measure space $[0, 1]$. We take the totality of almost everywhere vanishing functions $u : [0, 1] \to \mathbf{R}$ as the measurability structure of $\mathscr{X}$. Then, choosing an arbitrary lifting of $L^\infty([0, 1])$, we see that the mapping, which associates with each class $\mathbf{u} \in L^\infty([0, 1], \mathscr{X})$ the constant zero function $\rho_{\mathscr{X}}(\mathbf{u})$, satisfies conditions 3.3.1(a) – (d), but not (e).

(2) Condition 3.3.1(e) takes account of individual properties of the lifting rather than those of the MBB $\mathscr{X}$. To be more precise, there can exist two mappings ρ_1 and ρ_2 from $L^\infty(\Omega, \mathscr{X})$ into $\mathscr{L}^\infty(\Omega, \mathscr{X})$ each satisfying conditions 3.3.1(a) – (d) and such that ρ_1 satisfies (e) as well (i.e., ρ_1 is a lifting), but ρ_2 does not. A simple example can be obtained by considering an arbitrary liftable MBB $\mathscr{X}$ over Ω, the stalk $\mathscr{X}(\varpi)$ of which over some negligible point $\varpi \in \Omega$ (i.e. such that $\{\varpi\} \sim \varnothing$) is isometric to its proper subspace $X \subset \mathscr{X}(\varpi)$. In this case, if ρ_1 is a lifting of $L^\infty(\Omega, \mathscr{X})$ and T is an isometry from $\mathscr{X}(\varpi)$ onto X, then the mapping $\rho_2 : L^\infty(\Omega, \mathscr{X}) \to \mathscr{L}^\infty(\Omega, \mathscr{X})$ defined by the rule

$$\rho_2(\mathbf{u})(\omega) \;=\; \rho_1(\mathbf{u})(\omega) \;\text{ if }\; \omega \neq \varpi; \quad T(\rho_1(\mathbf{u})(\varpi)) \;\text{ if }\; \omega = \varpi$$

satisfies conditions 3.3.1(a) – (d), but not (e).

The technique of complete continuous Banach bundles enables us to considerably strengthen example (2); see 3.4.9.

3.3.3. Let ρ be a lifting of $L^\infty(\Omega)$ and let $\mathscr{X}$ and $\mathscr{Y}$ be liftable MBBs over Ω. We call the bundles $\mathscr{X}$ and $\mathscr{Y}$ *ρ-isometric*, if their liftings $\rho_{\mathscr{X}}$ and $\rho_{\mathscr{Y}}$ are associated with ρ and there exists an isometry H from $\mathscr{X}$ onto $\mathscr{Y}$ such that $\rho_{\mathscr{Y}}(H \otimes \mathbf{u}) = H \otimes \rho_{\mathscr{X}}(\mathbf{u})$ for all $\mathbf{u} \in L^\infty(\Omega, \mathscr{X})$. We say that an MBB $\mathscr{X}$ is *stationary with respect to ρ*, or *ρ-stationary*, if $\mathscr{X}(\omega_1) = \mathscr{X}(\omega_2)$ and $\mathbf{u}_\sim(\omega_1) = \mathbf{u}_\sim(\omega_2)$ for all $\mathbf{u} \in L^\infty(\Omega, \mathscr{X})$ and for arbitrary ρ-indistinguishable points $\omega_1, \omega_2 \in \Omega$ (see 0.6.3).

The following assertion enables us to assume without loss of generality that every MBB under consideration is stationary with respect to the corresponding "scalar" lifting.

Proposition. *For every MBB over Ω having a lifting associated with ρ, there is a ρ-stationary liftable MBB ρ-isometric to it.*

Proof. Suppose that an MBB $\mathscr{X}$ meets the hypothesis of the proposition. Fix an arbitrary pair of ρ-indistinguishable points $\omega_1, \omega_2 \in \Omega$. It suffices to construct an isometry

i from the stalk $\mathscr{X}(\omega_1)$ onto $\mathscr{X}(\omega_2)$ such that $i(\mathbf{u}_\sim(\omega_1)) = \mathbf{u}_\sim(\omega_2)$ for all $\mathbf{u} \in L^\infty(\Omega, \mathscr{X})$. Indeed, in this case, the stalks over indistinguishable points can be "identified" by means of such isometries.

For each point $\omega \in \Omega$, denote by $\mathscr{X}_0(\omega)$ the subspace $\{\mathbf{u}_\sim(\omega) : \mathbf{u} \in L^\infty(\Omega, \mathscr{X})\}$ of the stalk $\mathscr{X}(\omega)$. Let classes $\mathbf{u}, \mathbf{v} \in L^\infty(\Omega, \mathscr{X})$ be such that $\mathbf{u}_\sim(\omega_1) = \mathbf{v}_\sim(\omega_1)$. Then $\mathbf{u}_\sim(\omega_2) = \mathbf{v}_\sim(\omega_2)$, since by 0.6.3 we have

$$\begin{aligned}\|\mathbf{u}_\sim(\omega_2) - \mathbf{v}_\sim(\omega_2)\| &= |\mathbf{u} - \mathbf{v}|_\sim(\omega_2) \\ = |\mathbf{u} - \mathbf{v}|_\sim(\omega_1) &= \|\mathbf{u}_\sim(\omega_1) - \mathbf{v}_\sim(\omega_1)\| = 0.\end{aligned}$$

This enables us to consider a bijection $i_0 : \mathscr{X}_0(\omega_1) \to \mathscr{X}_0(\omega_2)$ defined by the rule $i_0(\mathbf{u}_\sim(\omega_1)) = \mathbf{u}_\sim(\omega_2)$ for every class $\mathbf{u} \in L^\infty(\Omega, \mathscr{X})$. Due to the fact that the subspaces $\mathscr{X}_0(\omega)$ are dense in the corresponding stalks $\mathscr{X}(\omega)$, the isometry i_0 can be extended to the desired isometry $i : \mathscr{X}(\omega_1) \to \mathscr{X}(\omega_2)$. □

3.3.4. Suppose that Q is the Stonian compactum of the Boolean algebra $B(\Omega)$ (see 0.2.2) and $\tau : \Omega \to Q$ is the canonical immersion of Ω into Q corresponding to the lifting ρ of $L^\infty(\Omega)$ (see 0.6.6).

Theorem. *Let $\mathscr{X} = \mathscr{Y} \circ \tau$, where $\mathscr{Y}$ is a complete CBB over Q.*

(1) If $\mathscr{C}$ is a vector subspace of $C(Q, \mathscr{Y})$ that is stalkwise dense in $\mathscr{Y}$ (for instance, if $\mathscr{C} = \mathscr{C}_{\mathscr{Y}}$), then the set $\mathscr{C} \circ \tau$ is a measurability structure in $\mathscr{X}$. For arbitrary subspaces $\mathscr{C}, \mathscr{D} \subset C(Q, \mathscr{X})$, stalkwise dense in $\mathscr{Y}$, the measurability structures $\mathscr{C} \circ \tau$ and $\mathscr{D} \circ \tau$ are equivalent. (In the sequel, the bundle $\mathscr{X} = \mathscr{Y} \circ \tau$ is always regarded as an MBB with respect to the measurability structure $\mathscr{C}_{\mathscr{Y}} \circ \tau$.)

(2) An almost everywhere defined section u of the bundle $\mathscr{X}$ is measurable if and only if $u \sim v \circ \tau$ for some element $v \in C_\infty(Q, \mathscr{Y})$.

(3) For every class $\mathbf{u} \in M(\Omega, \mathscr{X})$, there exists a unique section $\hat{\mathbf{u}} \in C_\infty(Q, \mathscr{Y})$ representing $\mathbf{u}$ as $(\hat{\mathbf{u}} \circ \tau)^\sim$.

(4) The mapping $\mathbf{u} \mapsto \hat{\mathbf{u}}$ is an isometry from the BKS $M(\Omega, \mathscr{X})$ onto $C_\infty(Q, \mathscr{Y})$ associated with the isomorphism $\mathbf{e} \in M(\Omega) \mapsto \hat{\mathbf{e}} \in C_\infty(Q)$ (see 0.7.5, 0.4.2). The image of $L^\infty(\Omega, \mathscr{X})$ under this isometry is $C(Q, \mathscr{Y})$. The inverse isometry from $C_\infty(Q, \mathscr{Y})$ onto $M(\Omega, \mathscr{X})$ is defined by the rule $v \mapsto (v \circ \tau)^\sim$ and is associated with the isomorphism $e \in C_\infty(Q) \mapsto (e \circ \tau)^\sim \in M(\Omega)$.

(5) The mapping $\mathbf{u} \mapsto \hat{\mathbf{u}} \circ \tau$ is a lifting of $L^\infty(\Omega, \mathscr{X})$ associated with ρ. Endowed with this lifting, the MBB $\mathscr{X}$ is ρ-stationary.

Proof. (1) If $\mathscr{C}$ is a stalkwise dense subspace of $C(Q, \mathscr{Y})$ and $c \in \mathscr{C}$ then $|||c \circ \tau||| = |||c||| \circ \tau$ is a measurable function. (see 0.7.5). The remaining properties of the set $\mathscr{C} \circ \tau$ listed in the definition of measurability structure are obvious. Equivalence of the measurability structures $\mathscr{C} \circ \tau$ and $\mathscr{D} \circ \tau$ for arbitrary subspaces $\mathscr{C}, \mathscr{D} \subset C(Q, \mathscr{Y})$, stalkwise dense in $\mathscr{Y}$, follows from 1.5.7 and 3.1.11.

(2) First of all, we note that, in view of 0.6.6, for every $v \in C_\infty(Q, \mathscr{Y})$ the composition $v \circ \tau$ is defined almost everywhere in Ω. Denote the set $\{u \in S_\sim(\Omega, \mathscr{X}) : u \sim v \circ \tau$ for some $v \in C_\infty(Q, \mathscr{Y})\}$ by $\mathscr{U}$ and prove that $\mathscr{U} = \mathscr{M}(\Omega, \mathscr{X})$.

Let $v \in C_\infty(Q, \mathscr{Y})$. According to 2.1.4, there exists a sequence $(v_n)_{n \in \mathbb{N}}$ of pairwise disjoint elements in $C(Q, \mathscr{Y})$ such that $v = o\text{-}\sum_{n \in \mathbb{N}} v_n$. Obviously, $v \circ \tau \sim \sum_{n \in \mathbb{N}} (v_n \circ \tau)$. Thus, the inclusion $\mathscr{U} \subset \mathscr{M}(\Omega, \mathscr{X})$ holds. If $v \in C_\infty(Q, \mathscr{Y})$, $A \in \mathscr{B}(\Omega)$, and $\mathbf{A} = A^\sim$, then

$$\langle A \rangle (v \circ \tau) \sim \langle \mathbf{A}_\sim \rangle (v \circ \tau) = \langle \tau^{-1}[\hat{\mathbf{A}}] \rangle (v \circ \tau) = (\langle \hat{\mathbf{A}} \rangle v) \circ \tau,$$

which implies that the set $\mathscr{U}$ contains the fragments of all its elements. According to 3.1.12, to prove the reverse inclusion $\mathscr{U} \supset \mathscr{M}(\Omega, \mathscr{X})$, it suffices to verify conditions 3.1.12(4)(a) – (c) for the set $\mathscr{U}$.

(a) Let sections $u_n \sim v_n \circ \tau$ be pairwise disjoint. Then the sections v_n's are pairwise disjoint and, hence, we have $v := o\text{-}\sum_{n \in \mathbb{N}} v_n \in C_\infty(Q, \mathscr{Y})$. It is clear that $\sum_{n \in \mathbb{N}} u_n \sim v \circ \tau$.

(b) Suppose that a sequence of sections $u_n \sim v_n \circ \tau$ converges to a section $u \in S_\sim(\Omega, \mathscr{X})$ uniformly on a set $A \sim \Omega$. The r-convergence $|u_n^\sim - u_m^\sim| \to 0$ in $M(\Omega)$ implies, by 0.7.5, the r-convergence $|v_n - v_m| \to 0$ in $C_\infty(Q)$. Due to o-completeness of the LNS $C_\infty(Q, \mathscr{Y})$, there exists the r-limit $v \in C_\infty(Q, \mathscr{Y})$ of the sequence (v_n). Obviously, $u \sim v \circ \tau$.

(c) Let a section $u \in S_\sim(\Omega, \mathscr{X})$ be such that $\langle K \rangle u \in \mathscr{U}$ for every $K \in \mathscr{B}_{\text{fin}}(\Omega)$. The direct sum property enables us to decompose the set Ω into subsets K_ξ $(\xi \in \Xi)$ of finite measure so that $\sup_{\xi \in \Xi} K_\xi^\sim = \Omega^\sim$ (see 0.6.5). For each $\xi \in \Xi$, there is a section $v_\xi \in C_\infty(Q, \mathscr{Y})$ such that $\langle K_\xi \rangle u \sim v_\xi \circ \tau$. Since the sections v_ξ's are pairwise disjoint, there is $v := o\text{-}\sum_{\xi \in \Xi} v_\xi \in C_\infty(Q, \mathscr{Y})$. The equivalence $u \sim v \circ \tau$ follows from 0.6.5.

(3) If sections $v, w \in C_\infty(Q, \mathscr{Y})$ are connected by the relation $v \circ \tau \sim w \circ \tau$, then $|v - w| \circ \tau = 0$. Then 0.7.5 implies that $|v - w| = 0$, i.e., $v = w$.

Assertions (4) and (5) are straightforward, except that the equality $\operatorname{dom}(\hat{\mathbf{u}} \circ \tau) = \Omega$ for an arbitrary $\mathbf{u} \in L^\infty(\Omega, \mathscr{X})$ possibly needs clarification. The inequality $|\mathbf{u}| \leq \lambda$ that holds for some number λ implies, by 0.7.5, the inequality $|\hat{\mathbf{u}}| \leq \lambda$. Now, from 2.1.3 it follows that $\operatorname{dom} \hat{\mathbf{u}} = Q$ and, hence, $\operatorname{dom}(\hat{\mathbf{u}} \circ \tau) = \Omega$. □

We refer to the section $\hat{\mathbf{u}} \in C_\infty(Q, \mathscr{Y})$ associated with an element $\mathbf{u} \in M(\Omega, \mathscr{X})$ according to item (3), as the *Stonian transform* of $\mathbf{u}$ (cf. 0.7.5).

3.3.5. Theorem 3.3.4 describes a method of constructing a liftable MBB given a complete CBB over the corresponding Stonian compactum. The following result shows that every liftable MBB can be obtained exactly in such a way.

Suppose that ρ is a lifting of $L^\infty(\Omega)$ and $\tau : \Omega \to Q$ is the corresponding canonical immersion of Ω into the Stonian compactum Q of the Boolean algebra $B(\Omega)$.

Theorem. *Let $\mathscr{X}$ be a ρ-stationary MBB over Ω that has a lifting associated with ρ. Then there exists a complete CBB $\hat{\mathscr{X}}$ over Q unique to within an isometry and such that $\mathscr{X} = \hat{\mathscr{X}} \circ \tau$ and $\mathbf{u}_\sim = \hat{\mathbf{u}} \circ \tau$ for all $\mathbf{u} \in L^\infty(\Omega, \mathscr{X})$, where $\hat{\mathbf{u}}$ is the Stonian transform of $\mathbf{u}$.*

Proof. Since $\mathscr{X}$ is ρ-stationary, we can define a Banach bundle $\mathscr{Y}$ over $\tau[\Omega]$ by the formula $\mathscr{Y}(\tau(\omega)) := \mathscr{X}(\omega)$ and endow it with the continuity structure $\{ \mathbf{u}_\sim \circ \tau^{-1} : \mathbf{u} \in L^\infty(\Omega, \mathscr{X}) \}$. Then the Stone–Čech extension $\widehat{\mathscr{X}}$ of $\mathscr{Y}$ onto Q (see 1.5.10) is the desired CBB. Most of the necessary properties of the bundle $\widehat{\mathscr{X}}$ are easily verified. We only prove completeness.

According to 1.5.10, for each class $\mathbf{u} \in L^\infty(\Omega, \mathscr{X})$, the bounded section $\mathbf{u}_\sim \circ \tau \in C(\tau[\Omega], \mathscr{Y})$ can be extended to an element of $C(Q, \widehat{\mathscr{X}})$ that will be denoted by $\widehat{\mathbf{u}}$. It is easy to verify that the mapping $\mathbf{u} \mapsto \widehat{\mathbf{u}}$ is an isometry from the LNS $L^\infty(\Omega, \mathscr{X})$ onto $C(Q, \widehat{\mathscr{X}})$ associated with the isomorphism $\mathbf{e} \in L^\infty(\Omega) \mapsto \widehat{\mathbf{e}} \in C(Q)$. From 1.5.7 and 0.4.6 it follows that the image of $L^\infty(\Omega, \mathscr{X})$ under this isometry is o-dense in $C(Q, \widehat{\mathscr{X}})$. In view of the o-completeness of $L^\infty(\Omega, \mathscr{X})$, this image coincides with $C(Q, \widehat{\mathscr{X}})$ and, hence, the LNS $C(Q, \widehat{\mathscr{X}})$ is o-complete as well.

Uniqueness of the bundle $\widehat{\mathscr{X}}$ follows from 3.3.4(4) and 2.4.1. □

We call the complete CBB $\widehat{\mathscr{X}}$ presented in the statement of the last theorem the *Stonian transform of the MBB* $\mathscr{X}$. Note that $\widehat{\mathscr{X}}$ is the realization CBB for $M(\Omega, \mathscr{X})$ (see 2.4.4).

3.4. Applications of the theory of complete Banach bundles

A constructive connection between liftable MBBs and complete CBBs established by Theorems 3.3.4 and 3.3.5 enables us to transfer all the basic facts of the theory of complete CBBs to the case of MBB. The results stated in this section are obtained exactly by this transfer. Also, we present here a criterion of measurability in terms of a lifting, describe approximating subsets in a quotient space of measurable sections, indicate a way of embedding a certain class of MBBs into liftable MBBs (an analog to the completion of a CBB), list the basic properties of the measurable bundles $B(\mathscr{X}, \mathscr{Y})$ and $\mathscr{X}'$, and formulate a series of results on the representation of an LNS as a space of equivalence classes of measurable sections of a liftable MBB.

Throughout the section, Ω is a nonzero measure space possessing the direct sum property, ρ is a lifting of $L^\infty(\Omega)$, and $\tau : \Omega \to Q$ is the corresponding canonical immersion of Ω into the Stonian compactum Q of the Boolean algebra $B(\Omega)$. If $\mathscr{X}$ is a (ρ-stationary) liftable MBB over Ω and $\widehat{\mathscr{X}}$ is the Stonian transform of $\mathscr{X}$ then, as usual, the Stonian transforms of classes $\mathbf{e} \in M(\Omega)$ and $\mathbf{u} \in M(\Omega, \mathscr{X})$ are denoted by $\widehat{\mathbf{e}} \in C_\infty(Q)$ and $\widehat{\mathbf{u}} \in C_\infty(Q, \widehat{\mathscr{X}})$, and their liftings are denoted by $\mathbf{e}_\sim$ and $\mathbf{u}_\sim$, respectively.

3.4.1. Condition (e) in the definition of lifting in 3.3.1 is valid in a stronger form:

Proposition. *If* $\mathscr{X}$ *is a liftable MBB over* Ω *then, for every point* $\omega \in \Omega$, *the set* $\{ \mathbf{u}_\sim(\omega) : \mathbf{u} \in L^\infty(\Omega, \mathscr{X}) \}$ *coincides with* $\mathscr{X}(\omega)$.

Proof. The claim follows from 3.3.3, 3.3.5, and 1.3.5. □

3.4.2. Proposition. *Let* $\mathscr{X}$ *be a liftable MBB over* Ω. *A section* $u \in S(\Omega, \mathscr{X})$ *is measurable and belongs to the image of the lifting of* $L^\infty(\Omega, \mathscr{X})$ *if and only if, for every* *class* $\mathbf{v} \in L^{\infty}(\Omega, \mathscr{X})$, *the function* $|||u - \mathbf{v}_{\sim}|||$ *is measurable and belongs to the image of the lifting of* $L^{\infty}(\Omega)$.

Proof. Necessity of the suggested criterion is obvious; we will prove its sufficiency. Suppose that the lifting of $L^{\infty}(\Omega, \mathscr{X})$ is associated with ρ. According to 3.3.3, we may assume the bundle $\mathscr{X}$ to be ρ-stationary. Let $\widehat{\mathscr{X}}$ be the Stonian transform of $\mathscr{X}$. Suppose that the function $|||u - \mathbf{v}_{\sim}|||$ belongs to the image of the lifting of $L^{\infty}(\Omega)$ for every class $\mathbf{v} \in L^{\infty}(\Omega, \mathscr{X})$. Using 3.4.1 and the fact that $\mathscr{X}$ is ρ-stationary, it is easy to show that the section u assumes equal values at ρ-indistinguishable points. This enables us to consider the section $v \in S(\tau[\Omega], \widehat{\mathscr{X}})$ defined by the formula $v(\tau(\omega)) := u(\omega)$. The properties of the section u ensure continuity of the functions $|||v - \widehat{\mathbf{v}}|||$ for all $\mathbf{v} \in L^{\infty}(\Omega, \mathscr{X})$, which means continuity of the section v. Obvious boundedness of u implies boundedness of v and, since $\widehat{\mathscr{X}}$ is complete, the section v can be extended to $\bar{v} \in C(Q, \widehat{\mathscr{X}})$. It remains to observe that $u = \bar{v} \circ \tau$ and employ 3.3.4. □

3.4.3. Proposition. *Let E be an order dense ideal of the K-space $M(\Omega)$ and let $\mathscr{X}$ and $\mathscr{Y}$ be liftable MBBs over Ω (having liftings associated with ρ). The BKSs $E(\mathscr{X})$ and $E(\mathscr{Y})$ are isometric if and only if the bundles $\mathscr{X}$ and $\mathscr{Y}$ are isometric (ρ-isometric).*

Proof. The claim follows from 3.3.3, 3.3.5, and 2.4.1. □

3.4.4. Let $\mathscr{X}$ be a liftable MBB over Ω. Given an arbitrary subset $\mathscr{U} \subset L^{\infty}(\Omega, \mathscr{X})$ and a point $\omega \in \Omega$, denote the set $\{\mathbf{u}_{\sim}(\omega) : \mathbf{u} \in \mathscr{U}\}$ by $\mathscr{U}_{\sim}(\omega)$.

Theorem. *The following properties of a subset $\mathscr{U} \subset L^{\infty}(\Omega, \mathscr{X})$ are equivalent:*

(1) *every section $v \in \mathscr{M}(\Omega, \mathscr{X})$ assumes the values $v(\omega) \in \mathrm{cl}\, \mathscr{U}_{\sim}(\omega)$ for almost all $\omega \in \Omega$;*

(2) *for each class $\mathbf{v} \in L^{\infty}(\Omega, \mathscr{X})$ the inclusion $\mathbf{v}_{\sim}(\omega) \in \mathrm{cl}\, \mathscr{U}_{\sim}(\omega)$ holds for almost all $\omega \in \Omega$;*

(3) *$\mathscr{U}$ is an order approximating subset of $M(\Omega, \mathscr{X})$;*

(4) *$\{u \in \mathscr{M}(\Omega, \mathscr{X}) : u^{\sim} \in \mathscr{U}\}$ is an approximating subset of $\mathscr{M}(\Omega, \mathscr{X})$.*

Proof. The implications $(1) \Rightarrow (2)$ and $(3) \Leftrightarrow (4)$ are obvious. We will prove that $(2) \Rightarrow (3)$. Suppose that the lifting of $L^{\infty}(\Omega, \mathscr{X})$ is associated with ρ. According to 3.3.3, we may assume the bundle $\mathscr{X}$ to be ρ-stationary. Let $\widehat{\mathscr{X}}$ be the Stonian transform of $\mathscr{X}$. Denote the set $\{\widehat{\mathbf{u}} : \mathbf{u} \in \mathscr{U}\} \subset C(Q, \widehat{\mathscr{X}})$ by $\widehat{\mathscr{U}}$. In view of 3.3.4(4) and 1.5.7((2) $\Rightarrow$ (4)), to prove (3), it suffices to show that, for every class $\mathbf{v} \in L^{\infty}(\Omega, \mathscr{X})$, the section $\widehat{\mathbf{v}}$ assumes the values $\widehat{\mathbf{v}}(q) \in \mathrm{cl}\, \widehat{\mathscr{U}}(q)$ on a dense subset of Q. If the last claim is not valid then $\widehat{\mathbf{v}}(q) \notin \mathrm{cl}\, \widehat{\mathscr{U}}(q)$ for all elements q of some nonempty clopen subset $A \subset Q$. Then $\mathbf{v}_{\sim}(\omega) \notin \mathrm{cl}\, \mathscr{U}_{\sim}(\omega)$ for all elements ω of the set $\tau^{-1}[A]$ of nonzero measure, which contradicts (2). The implication $(3) \Rightarrow (1)$ follows from 3.3.5 and 1.5.7((4) $\Rightarrow$ (3)). □

R e m a r k. By analogy to 1.5.7, the last assertion can be generalized to the case of an arbitrary subset $\mathscr{U} \subset M(\Omega, \mathscr{X})$. In this case, $\mathscr{U}_{\sim}(\omega)$ must be understood to be the totality of various values $(\langle \mathbf{A}\rangle \mathbf{u})_{\sim}(\omega)$, where $\mathbf{u} \in M(\Omega, \mathscr{X})$ and $\mathbf{A} \in B(\Omega)$ are such that $\langle \mathbf{A}\rangle \mathbf{u} \in L^{\infty}(\Omega, \mathscr{X})$ (see 3.1.13).

3.4.5. Theorem. *Let $\mathscr{X}$ be an MBB over Ω. Suppose that $\mathscr{C}$ is an adequate measurability structure in $\mathscr{X}$ with all the elements $c \in \mathscr{C}$ satisfying the conditions $|||c||| \in \mathscr{L}^{\infty}(\Omega)$ and $\rho(|||c|||) = |||c|||$. Then there exists an MBB $\overline{\mathscr{X}}$ over Ω unique to within a ρ-isometry, having a lifting $\rho_{\overline{\mathscr{X}}}$ associated with ρ, and such that $\mathscr{X}$ is a dense subbundle of $\overline{\mathscr{X}}$ and $\rho_{\overline{\mathscr{X}}}(c) = c$ for all $c \in \mathscr{C}$.*

Proof. Suppose that an MBB $\mathscr{X}$ and its measurability structure $\mathscr{C}$ meet the hypothesis of the theorem. According to 2.4.3, there is a complete CBB $\mathscr{Y}$ over Q and an isometry $i: M(\Omega, \mathscr{X}) \to C_{\infty}(Q, \mathscr{Y})$ associated with the isomorphism $\mathbf{e} \in L^{\infty}(\Omega) \mapsto \hat{\mathbf{e}} \in C(Q)$. Denote the liftable MBB $\mathscr{Y} \circ \tau$ by $\overline{\mathscr{X}}$ (see 3.3.4). To prove the theorem, it suffices to construct an isometry H from $\mathscr{X}$ onto a dense subbundle of $\overline{\mathscr{X}}$ such that $\rho_{\overline{\mathscr{X}}}(H \otimes c) = H \otimes c$ for all $c \in \mathscr{C}$. (In this case, the stalks of $\overline{\mathscr{X}}$ can be "retouched" so that the isometry H become the identity embedding.)

For each point $\omega \in \Omega$, denote by $\mathscr{X}_0(\omega)$ the dense subspace $\{c(\omega) : c \in \mathscr{C}\}$ of the stalk $\mathscr{X}(\omega)$ and define a linear operator $H_0(\omega) : \mathscr{X}_0(\omega) \to \overline{\mathscr{X}}(\omega)$ as follows: $H_0(\omega)c(\omega) := i(c^{\sim})(\tau(\omega))$, $c \in \mathscr{C}$. The operator $H_0(\omega)$ is properly defined and isometric in view of the relations

$$\|i(c^{\sim})(\tau(\omega))\| = |i(c^{\sim})|(\tau(\omega)) = |c^{\sim}|^{\wedge}(\tau(\omega)) = \rho(|c^{\sim}|)(\omega) = \|c(\omega)\|$$

that are valid for all $c \in \mathscr{C}$. Let $H(\omega)$ be an extension of $H_0(\omega)$ to an isometric embedding $\mathscr{X}(\omega) \to \overline{\mathscr{X}}(\omega)$. The set $\{c^{\sim} : c \in \mathscr{C}\}$ is approximating in $M(\Omega, \mathscr{X})$. Consequently, $\{i(c^{\sim}) : c \in \mathscr{C}\}$ is an approximating subset of $C_{\infty}(Q, \mathscr{Y})$, and, since $H \otimes c = i(c^{\sim}) \circ \tau$ for all $c \in \mathscr{C}$, the set $\{(H \otimes c)^{\sim} : c \in \mathscr{C}\}$ is approximating in $M(\Omega, \overline{\mathscr{X}})$. From 3.2.5 it follows that the mapping H is an isometry from $\mathscr{X}$ onto a dense subbundle of $\overline{\mathscr{X}}$ induced by the linear subset $\{H \otimes c : c \in \mathscr{C}\} \subset \mathscr{M}(\Omega, \overline{\mathscr{X}})$ (see 3.2.2).

Uniqueness of the bundle $\overline{\mathscr{X}}$ to within a ρ-isometry follows from 3.2.3 and 3.4.3. □

3.4.6. Theorem. *Let $\mathscr{X}$ and $\mathscr{Y}$ be MBBs over Ω having liftings associated with the same lifting of $L^{\infty}(\Omega)$. Then there exists a (unique) liftable MBB $B(\mathscr{X}, \mathscr{Y})$ over Ω such that*

(a) at each point $\omega \in \Omega$, the stalk $B(\mathscr{X}, \mathscr{Y})(\omega)$ is a Banach subspace of $B(\mathscr{X}(\omega), \mathscr{Y}(\omega))$;

(b) if $u \in \mathscr{M}(\Omega, \mathscr{X})$ and $H \in \mathscr{M}(\Omega, B(\mathscr{X}, \mathscr{Y}))$ then $H \otimes u \in \mathscr{M}(\Omega, \mathscr{Y})$;

(c) $(H \otimes u)_{\sim} = H_{\sim} \otimes u_{\sim}$ for all $u \in \mathscr{L}^{\infty}(\Omega, \mathscr{X})$ and $H \in \mathscr{L}^{\infty}(\Omega, B(\mathscr{X}, \mathscr{Y}))$;

(d) if a bounded mapping $H : \omega \in \Omega \mapsto H(\omega) \in B(\mathscr{X}(\omega), \mathscr{Y}(\omega))$ is such that, for every $u \in \mathscr{L}^{\infty}(\Omega, \mathscr{X})$, the section $H \otimes u$ is measurable and $H \otimes u_{\sim} = (H \otimes u)_{\sim}$, then $H \in \mathscr{L}^{\infty}(\Omega, B(\mathscr{X}, \mathscr{Y}))$.

Proof. The claims follow from 3.3.3, 3.3.5, and 2.2.3. □

3.4.7. Denote by $\mathscr{R}$ the trivial (liftable) MBB $\Omega \times \{\mathbf{R}\}$. If $\mathscr{X}$ is a liftable MBB over Ω then the bundle $B(\mathscr{X}, \mathscr{R})$ is called the *dual MBB* of $\mathscr{X}$ and denoted by the symbol $\mathscr{X}'$. We list below the basic properties of the dual MBB that follow, due to 3.3.5, from the analogous properties of the dual CBB. (The notation $\langle u \mid u' \rangle$ is used instead of $\langle u(\cdot) \mid u'(\cdot) \rangle$, see 0.5.4.)

Theorem. *Let $\mathscr{X}$ be a liftable MBB over Ω.*

(1) *$\mathscr{X}'$ is a liftable MBB.*

(2) *If $\mathscr{Y}$ is the Stonian transform of $\mathscr{X}$ then $\mathscr{Y}'$ is the Stonian transform of $\mathscr{X}'$.*

(3) *At each point $\omega \in \Omega$, the stalk $\mathscr{X}'(\omega)$ is a Banach subspace of $\mathscr{X}(\omega)'$. The inclusion $\mathscr{X}'(\omega) \subset \mathscr{X}(\omega)'$ can be strict.*

(4) *If $u \in \mathscr{M}(\Omega, \mathscr{X})$ and $u' \in \mathscr{M}(\Omega, \mathscr{X}')$ then $\langle u \mid u' \rangle \in \mathscr{M}(\Omega)$.*

(5) *For all $u \in \mathscr{L}^{\infty}(\Omega, \mathscr{X})$ and $u' \in \mathscr{L}^{\infty}(\Omega, \mathscr{X}')$ we have $\langle u \mid u' \rangle_{\smile} = \langle u_{\smile} \mid u_{\smile}' \rangle$.*

(6) *If a bounded mapping $u : \omega \in \Omega \mapsto u'(\omega) \in \mathscr{X}(\omega)'$ is such that, for every $u \in \mathscr{L}^{\infty}(\Omega, \mathscr{X})$, the function $\langle u \mid u' \rangle$ is measurable and $\langle u_{\smile} \mid u' \rangle = \langle u \mid u' \rangle_{\smile}$ then $u' \in \mathscr{L}^{\infty}(\Omega, \mathscr{X}')$.*

(7) *For arbitrary classes $\mathbf{u} \in M(\Omega, \mathscr{X})$ and $\mathbf{u}' \in M(\Omega, \mathscr{X}')$, denote by $\langle \mathbf{u} \mid \mathbf{u}' \rangle$ the class $\langle u \mid u' \rangle^{\sim} \in M(\Omega)$, where $u \in \mathbf{u}$, $u' \in \mathbf{u}'$. Then the bilinear form $(\mathbf{u}, \mathbf{u}') \mapsto \langle \mathbf{u} \mid \mathbf{u}' \rangle$ establishes an $M(\Omega)$-valued duality between the spaces $M(\Omega, \mathscr{X})$ and $M(\Omega, \mathscr{X}')$.*

(8) *For every $\mathbf{u} \in M(\Omega, \mathscr{X})$, we have $|\mathbf{u}| = \max \{\langle \mathbf{u} \mid \mathbf{u}' \rangle : \mathbf{u}' \in L^{\infty}(\Omega, \mathscr{X}'), |u'| \leq 1\}$.*

(9) *For every $\mathbf{u}' \in M(\Omega, \mathscr{X}')$, we have $|\mathbf{u}'| = \sup \{\langle \mathbf{u} \mid \mathbf{u}' \rangle : \mathbf{u} \in L^{\infty}(\Omega, \mathscr{X}), |u| \leq 1\}$.*

(10) *If $\mathbf{u}' \in L^{\infty}(\Omega, \mathscr{X}')$ then the equality $|\mathbf{u}'|_{\smile}(\omega) = \sup \{\langle \mathbf{u} \mid \mathbf{u}' \rangle_{\smile}(\omega) : \mathbf{u} \in L^{\infty}(\Omega, \mathscr{X}), |\mathbf{u}| \leq 1\}$ holds at each point $\omega \in \Omega$.*

(11) *For every point $\omega \in \Omega$ the space $\mathscr{X}'(\omega)$ norms $\mathscr{X}(\omega)$. Moreover, $\|x\| = \max\{\langle x \mid x' \rangle : x' \in \mathscr{X}'(\omega), \|x'\| \leq 1\}$ for all $x \in \mathscr{X}(\omega)$.*

(12) *If the stalk $\mathscr{X}(\omega)$ at a point $\omega \in \Omega$ is reflexive then $\mathscr{X}'(\omega) = \mathscr{X}(\omega)'$.*

(13) *Fix an arbitrary point $\omega \in \Omega$ and consider the canonical embedding $x \mapsto x''$ of $\mathscr{X}(\omega)$ into $\mathscr{X}(\omega)''$. Then the mapping $x \mapsto x''|_{\mathscr{X}'(\omega)}$ is an isometric embedding of $\mathscr{X}(\omega)$ into $\mathscr{X}''(\omega)$, where $\mathscr{X}''$ is the dual bundle of $\mathscr{X}'$.*

(14) *Assume that $\mathscr{X}(\omega) \subset \mathscr{X}''(\omega)$ at each point $\omega \in \Omega$ by the convention that the embedding (13) of the stalks of $\mathscr{X}$ into the corresponding stalks of $\mathscr{X}''$ is identical. Then $\mathscr{X}$ is a measurable subbundle of $\mathscr{X}$.*

3.4.8. The list of facts presented below is an immediate consequence of applying Theorem 3.3.5 to the realization results of the theory of complete CBBs (see 2.4).

Theorem. *Let Ω be a measure space possessing the direct sum property.*

(1) *For every BKS $\mathcal{U}$ over an order dense ideal $F \subset M(\Omega)$, there exists a liftable MBB $\mathcal{X}$ over Ω, unique to within an isometry and such that the LNSs $\mathcal{U}$ and $F(\mathcal{X})$ are isometric.*

(2) *For every BKS $\mathcal{U}$ over E and every isomorphism i from the K-space E onto an order dense ideal $F \subset M(\Omega)$, there exists a (unique to within an isometry) liftable MBB $\mathcal{X}$ over Ω and an isometry from $\mathcal{U}$ onto $F(\mathcal{X})$ associated with i.*

(3) *For every BKS $\mathcal{U}$ over a Kantorovich – Pinsker space, there exists a measurability structure Ω possessing the direct sum property, an order dense ideal $F \subset M(\Omega)$, and a liftable MBB $\mathcal{X}$ over Ω such that the LNSs $\mathcal{U}$ and $F(\mathcal{X})$ are isometric.*

We call the space $F(\mathcal{X})$ (more precisely, the isometry from $\mathcal{U}$ onto $F(\mathcal{X})$) the (measurable) realization of the BKS $\mathcal{U}$, and call $\mathcal{X}$ the realization MBB for $\mathcal{U}$ (cf. 2.4.4). *It can be shown that the realization MBB for a given BKS is unique to within an isometry.*

(4) *If an MBB $\mathcal{X}$ over Ω is the realization MBB for a BKS $\mathcal{U}$ then the LNS $M(\Omega, \mathcal{X})$ is isometric to the maximal extension of $\mathcal{U}$.*

(5) *Let $\mathcal{X}$ be a liftable MBB over Ω. Suppose that order dense ideals E and F of an extended K-space $M(\Omega)$ form a duality pair* (see 0.4.4). *Then the LNS $E^{*}(\mathcal{X}')$ is isometric to $E(\mathcal{X})^{*}$, where an isometry is performed by associating with each class $\mathbf{u}' \in E^{*}(\mathcal{X}')$ the functional $|\,\mathbf{u}'\rangle : \mathbf{u} \in E(\mathcal{X}) \mapsto \langle \mathbf{u} |\, \mathbf{u}'\rangle \in F$. In particular, if $\mathcal{X}$ is the realization MBB for a BKS $\mathcal{U}$ then $\mathcal{X}'$ is the realization MBB for the dual BKS $\mathcal{U}^{*}$.*

3.4.9. In 3.3.2 we established that property (e) of lifting (see 3.3.1) does not follow from properties (a) – (d). Using the technique of complete CBB, we will strengthen that assertion and show that conditions (a) – (d) are consistent with the "stalkwise" negation of (e); moreover, it is so even in case the MBB under consideration is liftable.

Proposition. *For every atomless measure space Ω and every lifting ρ of $L^{\infty}(\Omega)$, there exists an MBB $\mathcal{X}$ over Ω and two mappings ρ_1 and ρ_2 from $L^{\infty}(\Omega, \mathcal{X})$ to $\mathcal{L}^{\infty}(\Omega, \mathcal{X})$, each satisfying conditions* 3.3.1(a) – (d) *and such that ρ_1 satisfies* (e) *as well (i.e., ρ_1 is a lifting), but the set $\{\rho_2(\mathbf{u})(\omega) : \mathbf{u} \in L^{\infty}(\Omega, \mathcal{X})\}$ is not dense in $\mathcal{X}(\omega)$ for every point $\omega \in \Omega$.*

Proof. Let Ω be an atomless measure space, let ρ be a lifting of $L^{\infty}(\Omega)$, and let $\tau : \Omega \to Q$ be the corresponding canonical immersion of Ω into the Stonian compactum Q of the Boolean algebra $B(\Omega)$. Fix an arbitrary (infinite-dimensional) Hilbert space Z and an isometry $T : Z \to Z_0$ onto a proper subspace $Z_0 \subset Z$. Denote by X the Hilbert space $l^2(Q, Z)$ with the inner product $\langle x,y \rangle = \sum_{q \in Q} \langle x(q), y(q) \rangle$ $(x,y \in X)$. For each point $q \in Q$, we define a linear operator $H(q) : X \to X$ by setting $(H(q)x)(q) := T(x(q))$ and $(H(q)x)(p) := x(q)$ for $p \neq q$. Obviously, the operator $H(q)$ maps isometrically the space X onto its proper subspace $\{x \in X : x(q) \in Z_0\}$. Let $\mathcal{Y}$ be the completion of the trivial CBB $Q \times \{X\}$. According to 2.1.7, all the stalks of $\mathcal{Y}$ are Hilbert spaces. Using this fact, for each point $q \in Q$, we can easily extend the isometry $H(q)$ to an isometry $\bar{H}(q)$ of the space $\mathcal{Y}(q)$ onto its proper subspace.

We will show that every section $v \in C(Q, \mathscr{Y})$ coincides with $\bar{H} \otimes v$ on a comeager subset of Ω.

(a) First, establish that, for each $x \in X$, the equality $\bar{H}(q)x = x$ holds for all elements q of a comeager subset of Q. Indeed, from the definition of the operator $H(q)$ it is clear that $H(q)x = x$ for all points $q \in Q$ at which $x(q) = 0$. It remains to observe that the set $\{q \in Q : x(q) \neq 0\}$ is countable and, in view of the fact that the algebra $B(\Omega)$ is atomless, all countable subsets of Q are meager.

(b) Denote by $\mathscr{C}$ the totality of all constant functions $c : Q \to X$ and let $d\mathscr{C}$ stand for the cyclic hull of the set $\mathscr{C}$ in the LNS $C_\infty(Q, \mathscr{Y})$ (see 0.4.1). Due to 0.1.5, from (a) it follows that, for every element $s \in d\mathscr{C}$, the section $\bar{H} \otimes s$ coincides with s on a comeager subset of Ω.

(c) Now, let v be an arbitrary element of $C(Q, \mathscr{Y})$. From 1.5.7 it follows that $\mathscr{C}$ is an approximating subset of $C_\infty(Q, \mathscr{Y})$. Then, in view of 0.4.9, the set $d\mathscr{C}$ is uniformly dense in $C_\infty(Q, \mathscr{Y})$ and, hence, there is a sequence of elements in $d\mathscr{C}$ converging to v uniformly (= with regulator 1). Now, coincidence of the sections v and $\bar{H} \otimes v$ on a comeager subset of Q follows from (b). Denote by $\mathscr{X}$ the MBB $\mathscr{Y} \circ \tau$ with the lifting $\rho_1 : \mathbf{u} \mapsto \hat{\mathbf{u}} \circ \tau$ (see 3.3.4). We define the second of the desired mappings by $\rho_2(\mathbf{u}) := (\bar{H} \otimes \hat{\mathbf{u}}) \circ \tau$, $\mathbf{u} \in L^\infty(\Omega, \mathscr{X})$. Most of the necessary properties of ρ_2 are easily verified. Only the inclusion $\rho_2(\mathbf{u}) \in \mathbf{u}$ needs clarification, to which end it suffices to establish coincidence of the sections $\rho_1(\mathbf{u})$ and $\rho_2(\mathbf{u})$ almost everywhere. The last claim follows from 0.6.6 and from the fact that the sections $\hat{\mathbf{u}}$ and $\bar{H} \otimes \hat{\mathbf{u}}$ coincide on a comeager subset of Q. □

4. SPACES OF VECTOR-FUNCTIONS

Spaces of measurable and weakly measurable vector-functions are the classical examples of Banach – Kantorovich spaces. These spaces are simple enough and the representation by means of sections in a Banach bundle does not additionally clarify their structure. However, such a representation is necessary for applying the general results about liftable MBBs to spaces of vector-functions. The present chapter is devoted to studying the structure of realization bundles for spaces of vector-functions. Also, we present here some applications of the obtained results to establishing a connection between spaces of measurable and continuous vector-functions, as well as to constructing various liftings in spaces of measurable vector-functions.

In the sequel, we assume that X and Y are Banach spaces and a bilinear form $\langle \cdot | \cdot \rangle : X \times Y \to \mathbf{R}$ establishes a duality between X and Y; moreover, Y norms X, i.e., $\|x\| = \sup \{ \langle x | y \rangle : y \in Y,\ \|y\| \leq 1 \}$ for all $x \in X$. Such a connection between the spaces X and Y enables us to assume that $X \subset Y'$, where $x \in X$ is identified with $\langle x | \in Y'$.

4.1. Measurable vector-functions

In this section, we study the question of existence for a lifting in a space of measurable functions. We also construct an embedding of an MBB with constant stalk into a liftable MBB and formulate some properties, of measurable and weakly measurable vector-functions, which result from applying the theory of MBBs.

Throughout the section, Ω is a nonzero measure space (possessing the direct sum property), ρ is a lifting of $L^\infty(\Omega)$. As before, a bilinear form $\langle \cdot | \cdot \rangle$ establishes a duality between Banach spaces X and Y such that Y norms X.

4.1.1. The symbol $\mathcal{M}(\Omega,X)$ stands for the set of all (Bochner) measurable functions defined almost everywhere in Ω with values in the space X. We denote by $M(\Omega,X)$ the totality of equivalence classes of elements in $\mathcal{M}(\Omega,X)$ with respect to the relation of coincidence almost everywhere. The symbol $\mathcal{L}^\infty(\Omega,X)$ designates the set $\{u \in \mathcal{M}(\Omega,X) : |||u||| \in \mathcal{L}^\infty(\Omega)\}$; its elements are called *(essentially) bounded* measurable functions. The equivalence classes constituted by essentially bounded functions are called *bounded classes* and the totality of all such classes is denoted by $L^\infty(\Omega,X)$.

The set $M(\Omega,X)$ is endowed with the natural structure of an LNS over $M(\Omega)$ and that of a module over $M(\Omega)$. In this case the norm $|\mathbf{u}|$ of a class $\mathbf{u} \in M(\Omega,X)$ contains the pointwise norms $|||u|||$ of all representatives $u \in \mathbf{u}$. If E is an ideal of $M(\Omega)$ then the set $E(X) := \{\mathbf{u} \in M(\Omega,X) : |\mathbf{u}| \in E\}$ endowed with the operations induced from $M(\Omega,X)$ is an LNS over E. Obviously, $L^\infty(\Omega,X)$ coincides with $E(X)$, where $E = L^\infty(\Omega)$.

4.1.2. We say that a mapping $\rho_X : L^\infty(\Omega,X) \to \mathcal{L}^\infty(\Omega,X)$ is a *lifting of* $L^\infty(\Omega,X)$ *(associated with* ρ*)* if the following relations are valid for all $\mathbf{u},\mathbf{v} \in L^\infty(\Omega,X)$ and $\mathbf{e} \in L^\infty(\Omega)$:

(a) $\rho_X(\mathbf{u}) : \Omega \to X$ and $\rho_X(\mathbf{u}) \in \mathbf{u}$;

(b) $|||\rho_X(\mathbf{u})||| = \rho(|\mathbf{u}|)$;

(c) $\rho_X(\mathbf{u}+\mathbf{v}) = \rho_X(\mathbf{u}) + \rho_X(\mathbf{v})$;

(d) $\rho_X(\mathbf{eu}) = \rho(\mathbf{e})\rho_X(\mathbf{u})$;

(e) $\rho_X(c^{\sim}) = c$ for all constant functions $c : \Omega \to X$.

As in the case of sections (cf. 3.3.1), we use the notation $\rho_X(u)$ instead of $\rho_X(u^{\sim})$ for $u \in \mathcal{L}^\infty(\Omega,X)$, $\rho(e)$ instead of $\rho(e^{\sim})$ for $e \in \mathcal{L}^\infty(\Omega)$, and $\rho(A)$ instead of $\rho(A^{\sim})$ for $A \in \mathcal{B}(\Omega)$.

Theorem. *There exists a lifting of* $L^\infty(\Omega,X)$ *if and only if the measure space* Ω *is atomic or the Banach space* X *is finite-dimensional.*

Proof. It is easy to verify that the formulated condition is sufficient, and so we prove only its necessity. Let X be infinite-dimensional and let Ω be not atomic. Without loss of generality, we may assume that Ω is an atomless finite nonzero measure space. Contrary to the claim, suppose that the space $L^\infty(\Omega,X)$ has a lifting ρ_X associated with ρ.

Since X is infinite-dimensional, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X such that $\|x_n\| = 1$ and $\|x_n - x_m\| \geq 1/2$ for all $n,m \in \mathbb{N}$, $n \neq m$. Since Ω is atomless, there exists a point $\omega_0 \in \Omega$ and a sequence $(A_n)_{n\in\mathbb{N}}$ of pairwise disjoint nonempty measurable subsets of Ω such that $\rho(A_n) = A_n$ and $\omega_0 \notin \bigcup_{n\in\mathbb{N}} A_n \sim \Omega$. Such a sequence can be constructed by induction, decomposing at step $n+1$ the remainder $\Omega \setminus \bigcup_{i=1}^{n} A_i$

into two subsets A and B of equal measures, and taking as A_{n+1} one of the sets $\rho(A)$ or $\rho(B)$ which does not contain the point ω_0.

Define a function $u \in \mathscr{L}^{\infty}(\Omega, X)$ by setting $u(\omega) = x_n$ for $\omega \in A_n$. Property (d) of the lifting ρ_X implies that the function $\rho_X(u)$ extends u. We will denote the value of $\rho_X(u)$ at ω_0 by x_0 and show that x_0 belongs to the set $\{x_n : n \in \mathbf{N}\}$. Indeed, otherwise, there is a number $\varepsilon > 0$ such that $\|x_0 - x_n\| \geq \varepsilon$ for all $n \in \mathbf{N}$, which yields the contradictory relations

$$0 = |||\rho_X(u) - x_0|||(\omega_0) = |||\rho_X(u - x_0)|||(\omega_0) = \rho(|||u - x_0|||)(\omega_0) \geq \varepsilon > 0.$$

Thus, $x_0 = x_n$ for some $n \in \mathbf{N}$. Since $|||u - x_n||| \geq 1/2$ almost everywhere on $\Omega \setminus A_n$, we have $|||\rho_X(u) - x_n||| = \rho(|||u - x_n|||) \geq 1/2$ on $\Omega \setminus A_n$. Consequently, $\omega_0 \in A_n$ and we obtain the desired contradiction. □

4.1.3. Consider the trivial measurable Banach bundle $\Omega \times \{X\}$. As was noted in 3.2.1, the totality $\mathscr{M}(\Omega, \Omega \times \{X\})$ of measurable sections of this bundle coincides with $\mathscr{M}(\Omega, X)$. From 3.3.1 it follows that existence of a lifting in $\Omega \times \{X\}$ is seldom. However, the following theorem shows that the bundle $\Omega \times \{X\}$ can be densely embedded into a liftable MBB.

Theorem. *There exists a liftable MBB $\mathscr{X}$ over Ω, unique to within a ρ-isometry and such that*

(1) *X is a Banach subspace of each stalk $\mathscr{X}(\omega)$, $\omega \in \Omega$;*

(2) *every section $u \in \mathscr{M}(\Omega, \mathscr{X})$ assumes the values $u(\omega) \in X$ for almost all $\omega \in \Omega$;*

(3) $\mathscr{M}(\Omega, X) = \{u \in \mathscr{M}(\Omega, \mathscr{X}) : \mathrm{im}\, u \subset X\} = \{u|^{X} : u \in \mathscr{M}(\Omega, \mathscr{X})\}$;

(4) *the lifting $\rho_{\mathscr{X}}$ is associated with ρ;*

(5) *$\rho_{\mathscr{X}}(c^{\sim}) = c$ for all constant functions $c : \Omega \to X$.*

Proof. The claims follow from 3.2.3 and 3.4.5. □

We denote the MBB $\mathscr{X}$ presented in the last theorem by X_{Ω}. It is clear that the trivial MBB $\Omega \times \{X\}$ is a dense subbundle of X_{Ω} in terms of 3.2.3.

4.1.4. Corollary. *The LNS $M(\Omega, X)$ is an (extended) BKS over $M(\Omega)$. The bundle X_{Ω} is the realization MBB for the space $M(\Omega, X)$. If E is an ideal of $M(\Omega)$ then the mapping $\mathbf{u} \mapsto \mathbf{u} \cap \mathscr{M}(\Omega, X)$ is an isometry from the BKS $E(X_{\Omega})$ onto $E(X)$.*

4.1.5. A function u defined almost everywhere in Ω, with values in the space X is called *Y-weakly measurable* if, for every $y \in Y$, the real-valued function $\langle u \mid y \rangle := \langle \cdot \mid y \rangle \circ u$ is measurable. We denote the totality of all such functions u by $\mathscr{M}(\Omega, X \mid Y)$. Almost everywhere defined X-valued functions u and v are called *Y-weakly equivalent* if, for every $y \in Y$, the functions $\langle u \mid y \rangle$ and $\langle v \mid y \rangle$ coincide almost everywhere. We use the symbol $\asymp$ to denote the relation of Y-weak equivalence in the set $\mathscr{M}(\Omega, X \mid Y)$, in contrast to the symbol $\sim$ designating the coincidence almost everywhere. The quotient set $\mathscr{M}(\Omega, X \mid Y)/\asymp$ is denoted by $M(\Omega, X \mid Y)$. We recall that the notation $M(\Omega, X)$ was introduced for the quotient set $\mathscr{M}(\Omega,X)/\sim$ (see 4.1.1). If $u \in \mathbf{u} \in M(\Omega,X|\,Y)$ and $y \in Y$ then the symbol $\langle \mathbf{u}\,|\,y\rangle$ stands for the class $\langle u\,|\,y\rangle^{\sim} \in M(\Omega)$.

Obviously, $\mathscr{M}(\Omega,X) \subset \mathscr{M}(\Omega,X|\,Y)$ and the inclusion may be strict. There exist examples of weakly equivalent functions that do not coincide almost everywhere (see [1]: II.1). However, it is known that X'-weakly equivalent Bochner measurable X-valued functions are equal almost everywhere (see [1]: II.2, Corollary 7). Employing the lifting ρ of $L^{\infty}(\Omega)$, we can easily strengthen this result:

Lemma. *If functions $u, v \in \mathscr{M}(\Omega,X)$ are Y-weakly equivalent then they coincide almost everywhere.*

Proof. For each step-function (= a measurable finite-valued function) $s = \sum_{i=1}^{n} \chi_{A_i} x_i$, we define the step-function $\rho(s) \sim s$ by the formula $\rho(s) = \sum_{i=1}^{n} \chi_{\rho(A_i)} x_i$. Obviously, $\langle \rho(s)\,|\,y\rangle = \rho(\langle s\,|\,y\rangle)$ for every $y \in Y$.

By a standard argument, we can reduce the assertion of the lemma to the case in which $u \in \mathscr{L}^{\infty}(\Omega,X)$, $v \equiv 0$. Consider a sequence $(s_n)_{n\in\mathbb{N}}$ of step-functions such that $|||s_n - u||| \le 1/n$ for all $n \in \mathbb{N}$. The Y-weak equivalence $u = 0$ implies that, for every element $y \in Y$ and every number $n \in \mathbb{N}$, the inequality $|\langle s_n\,|\,y\rangle| \le \|y\|/n$ holds almost everywhere and, therefore, $|\langle \rho(s_n)\,|\,y\rangle| \le \|y\|/n$ everywhere on Ω. Consequently, $\rho(s_n) \to 0$ on Ω. It remains to observe that $\rho(s_n) \to u$ almost everywhere. □

The last assertion enables us, given a function $u \in \mathscr{M}(\Omega,X)$, to identify the classes $\{v \in \mathscr{M}(\Omega,X) : v \sim u\} \in M(\Omega,X)$ and $\{v \in \mathscr{M}(\Omega,X|\,Y) : v = u\} \in M(\Omega,X|\,Y)$. In the sequel, we denote both these classes by the same symbol $u^{\sim}$.

4.1.6. The set $M(\Omega,X|\,Y)$ is made into a vector space in the natural way: we set $\lambda u^{\sim} + \mu v^{\sim} := (\lambda u|_{\mathrm{dom}\,v} + \mu v|_{\mathrm{dom}\,u})^{\sim}$ for $\lambda,\mu \in \mathbb{R}$ and $u, v \in \mathscr{M}(\Omega,X|\,Y)$. Obviously, $M(\Omega,X)$ is a vector subspace of $M(\Omega,X|\,Y)$. From 0.7.2 it follows that, for every class $\mathbf{u} \in M(\Omega,X|\,Y)$, the set $\{\langle \mathbf{u}\,|\,y\rangle : y \in Y,\ \|y\| \le 1\}$ is order bounded in $M(\Omega)$. Taking the least upper bound of the set $\{\langle \mathbf{u}\,|\,y\rangle : y \in Y,\ \|y\| \le 1\}$ in $M(\Omega)$ as the norm $|\mathbf{u}|$, we make the space $M(\Omega,X|\,Y)$ into an LNS over $M(\Omega)$. If E is an ideal of $M(\Omega)$ then the set $E(X|\,Y) := \{\mathbf{u} \in M(\Omega,X|\,Y) : |\mathbf{u}| \in E\}$ endowed with the operations induced from $M(\Omega,X|\,Y)$ is an LNS over E. If $E = L^{\infty}(\Omega)$, then the space $E(X|\,Y)$ is denoted by $L^{\infty}(\Omega,X|\,Y)$ and $\cup L^{\infty}(\Omega,X|\,Y)$ is denoted by $\mathscr{L}^{\infty}(\Omega,X|\,Y)$. Given a $u \in \mathscr{M}(\Omega,X|\,Y)$, the symbol $|u|$ stands for an arbitrary element of the class $|u^{\sim}|$ (the concrete choice of a representative $|u| \in |u^{\sim}|$ will be insignificant when the symbol $|u|$ is used).

4.1.7. If $\mathbf{u} \in M(\Omega,X|\,Y)$ then, for every class $\mathbf{K} \in B(\Omega)$ of finite measure, there exists a sequence $(y_n)_{n\in\mathbb{N}}$ of elements of the unit ball of Y such that $\langle \mathbf{K}\rangle|\mathbf{u}| = \sup_{n\in\mathbb{N}} \langle \mathbf{K}\rangle\langle \mathbf{u}\,|\,y_n\rangle$. Thus, if $(\mathbf{K}_\xi)_{\xi\in\Xi}$ is a partition of unity in the Boolean algebra $B(\Omega)$ into classes of finite measure then, for every $\mathbf{u} \in M(\Omega,X|\,Y)$, there exists a family $(y_n^{\xi})_{\xi\in\Xi, n\in\mathbb{N}}$ of elements of the unit ball of Y such that

$$|\mathbf{u}| = \sup_{\xi\in\Xi} \sup_{n\in\mathbb{N}} \langle \mathbf{K}_\xi\rangle\langle \mathbf{u}\,|\,y_n^{\xi}\rangle.$$

From the above it follows that, for every function $u \in \mathcal{M}(\Omega, X|\, Y)$, the inequality $|u| \le |||u|||$ holds almost everywhere in Ω. In general, it is not possible to say more about the connection between the functions $|u|$ and $|||u|||$. The function $|||u|||$ need not be measurable. It is possible that $|u| \sim 0$ but $|||u||| \equiv 1$. Thus, the function $|u|$ and the pointwise envelope $\sup\{\langle u|\,y\rangle : y \in Y,\ \|y\| \le 1\}$ are unrelated in general. The situation changes if we consider a lifting of $L^\infty(\Omega)$ and replace $\langle u|\,y\rangle$ by $\langle u|\,y\rangle_\sim$ (see 0.6.3): according to 0.7.2, for every function $u \in \mathcal{M}(\Omega, X|\, Y)$, the equality $|u| = \sup\{\langle u|\,y\rangle_\sim : y \in Y,\ \|y\| \le 1\}$ holds almost everywhere in Ω.

4.1.8. If $v \in \mathcal{M}(\Omega, (X_\Omega)')$ then the function $v_X : \omega \in \operatorname{dom} v \mapsto v(\omega)|_X \in X'$ obviously belongs to $\mathcal{M}(\Omega, X'|\, X)$. For each element $\mathbf{v} \in M(\Omega, (X_\Omega)')$, denote by $\mathbf{v}_X$ the class in $M(\Omega, X'|\, X)$ containing the functions v_X for $v \in \mathbf{v}$.

Lemma. *The mapping $\mathbf{v} \mapsto \mathbf{v}_X$ is an isometry form the LNS $M(\Omega, (X_\Omega)')$ onto $M(\Omega, X'|\, X)$.*

Proof. Let us show that the linear mapping $\mathbf{v} \mapsto \mathbf{v}_X$ is isometric. Fix an arbitrary class $\mathbf{v} \in M(\Omega, (X_\Omega)')$. According to Theorem 3.4.7, we have $|\mathbf{v}| = \sup\{\langle \mathbf{u}|\,\mathbf{v}\rangle : \mathbf{u} \in M(\Omega, X_\Omega),\ |\mathbf{u}| \le 1\}$. The inequality $|\mathbf{v}_X| \le |\mathbf{v}|$ is obvious. Consider an arbitrary element $\mathbf{u} \in M(\Omega, X_\Omega)$, $|\mathbf{u}| \le 1$, and observe that

$$\langle \mathbf{u}|\,\mathbf{v}\rangle = \langle \mathbf{u}^X|\,\mathbf{v}_X\rangle \le |\mathbf{u}^X| \cdot |\mathbf{v}_X| \le |\mathbf{v}_X|,$$

where $\mathbf{u}^X$ is the class in $M(\Omega, X)$ containing the functions $u|^X$ for $u \in \mathbf{u}$ (see 4.1.3). Arbitrariness of $\mathbf{u}$ provides the missing inequality $|\mathbf{v}| \le |\mathbf{v}_X|$.

It remains to show that the mapping $\mathbf{v} \mapsto \mathbf{v}_X$ is surjective. Given any element $\mathbf{w} \in M(\Omega, X'|\, X)$, there is a partition of unity $(\mathrm{K}_\xi)_{\xi \in \Xi}$ in the Boolean algebra $B(\Omega)$ such that $\langle \mathrm{K}_\xi \rangle |\mathbf{w}| \in L^\infty(\Omega)$ for all $\xi \in \Xi$ (see 0.3.2). Fix arbitrary $\xi \in \Xi$, $\omega \in (\mathrm{K}_\xi)_\sim$, and $x \in X_\Omega(\omega)$; choose $\mathbf{u} \in L^\infty(\Omega, X_\Omega)$ such that $\mathbf{u}_\sim(\omega) = x$ (see 3.4.1); and denote $v(\omega)x := \langle \mathbf{u}|\,\mathbf{w}\rangle_\sim(\omega)$. It is easily verified that the presented construction soundly defines a section $v \in \mathcal{M}(\Omega, (X_\Omega)')$ satisfying the relation $(v^\sim)_X = \mathbf{w}$. □

Corollary. *The LNS $M(\Omega, X'|\, X)$ is an (extended) BKS over $M(\Omega)$. The bundle $(X_\Omega)'$ is the realization MBB for the space $M(\Omega, X'|\, X)$. If E is an ideal of $M(\Omega)$ then the mapping $\mathbf{u} \mapsto \mathbf{u}_X$ is an isometry from the BKS $E((X_\Omega)')$ onto $E(X'|\, X)$.*

4.2. Connection between measurable and continuous vector-functions

The results formulated in this section are consequences of the relations of three types:
(a) between spaces of continuous vector-functions and CBBs (see Section 2.5);
(b) between spaces of measurable vector-functions and MBBs (see Section 4.1);
(c) between MBBs and CBBs (see Section 3.3).
The devices listed are applied here, first of all, for establishing connections between various spaces of measurable and continuous vector-functions in terms of the so-called Stonian

transform. The section ends with a series of incidentally-obtained facts concerning existence for liftings of various types in spaces of measurable vector-functions.

As before, Ω is a nonzero measure space (possessing the direct sum property), ρ is a lifting of $L^{\infty}(\Omega)$, a form $\langle\cdot|\cdot\rangle$ establishes a duality between X and Y such that Y norms X, which enables us to consider X as a Banach subspace of Y'. In addition, Q is the Stonian compactum of the Boolean algebra $B(\Omega)$ and τ is the canonical immersion of Ω into Q corresponding to the lifting ρ. It is well to bear in mind that in this context meager subsets of Q are nowhere-dense (see 0.6.6).

In the sequel, we assume that $C_\infty(Q,X)\subset C_\infty(Q,X|Y)$ by identifying the functions $v\in C_\infty(Q,X)$ and $\mathrm{ext}_{X|Y}(v)\in C_\infty(Q,X|Y)$ (see 2.5.4). We recall that $M(\Omega,X)$ was similarly embedded into $M(\Omega,X|Y)$ in 4.1.5.

4.2.1. Lemma. *The CBB X_Q is the Stonian transform of the MBB X_Ω, i.e., we may assume that $X_\Omega=X_Q\circ\tau$.*

Proof. The claim follows from 3.3.4, 3.3.5, and 4.1.3. □

4.2.2. Theorem. (1) *An X-valued function u defined almost everywhere in Ω is measurable if and only if $u\sim v\circ\tau$ for some element $v\in C_\infty(Q,X)$.*

(2) *For every class $\mathbf{u}\in M(\Omega,X)$, there exists a unique function $\hat{\mathbf{u}}\in C_\infty(Q,X)$ representing $\mathbf{u}$ as $(\hat{\mathbf{u}}\circ\tau)^\sim$.*

(3) *The mapping $\mathbf{u}\mapsto\hat{\mathbf{u}}$ is an isometry from the BKS $M(\Omega,X)$ onto $C_\infty(Q,X)$ associated with the isomorphism $\mathbf{e}\in M(\Omega)\mapsto\hat{\mathbf{e}}\in C_\infty(Q)$* (see 0.7.5). *The inverse isometry from $C_\infty(Q,X)$ onto $M(\Omega,X)$ is defined by the rule $v\mapsto(v\circ\tau)^\sim$ and is associated with the isomorphism $e\in C_\infty(Q)\mapsto(e\circ\tau)^\sim\in M(\Omega)$.*

Proof. The claims follow from 3.3.4, 4.1.3, 4.1.4, and 4.2.1. Other proofs of the isometry of $M(\Omega,X)$ and $C_\infty(Q,X)$ can be found in [11] (Theorem 2.9) and in [8] (Theorem 4.1.15). □

The function $\hat{\mathbf{u}}\in C_\infty(Q,X)$ corresponding to a class $\mathbf{u}\in M(\Omega,X)$ by item (2) is called the *Stonian transform* of $\mathbf{u}$.

4.2.3. Lemma. *Let one of the following pairs of conditions be satisfied:* (1) *$v\in C_\infty(Q,X)$ and a subset $X_0\subset X$ is closed under the norm topology or* (2) *$v\in C_\infty(Q,X|Y)$ and a set $X_0\subset X$ is closed under the Y-weak topology. Then the image of v is contained in X_0 if and only if the image of the composition $v\circ\tau$ is contained in X_0.*

Proof. If the image of v is contained in X_0 then the set $v^{-1}[X_0]$ is not dense in Q and, hence, there is a nonempty clopen in Q subset $U\subset Q\setminus v^{-1}[X_0]$. In this case, the function $v\circ\tau$ assumes values belonging to $X\setminus X_0$ almost everywhere in the set $\tau^{-1}[U]$ of nonzero measure. □

Corollary. *Let X_0 be a closed subset of X. The values $u(\omega)$ of a function $u\in\mathcal{M}(\Omega,X)$ belong to X_0 for almost all $\omega\in\Omega$ if and only if the image of the Stonian transform of the class $u^\sim$ is contained in X_0.*

4.2.4. Denote by $\mathscr{L}^c(\Omega,X)$ (by $\mathscr{L}^c_Y(\Omega,X)$) the totality of all the functions $u \in \mathcal{M}(\Omega,X)$ for which the image $u[A]$ of some set $A \sim \Omega$ is relatively compact under the norm topology (under the Y-weak topology). Observe that $\mathscr{L}^c(\Omega,X) \subset \mathscr{L}^c_Y(Q,X) \subset \mathscr{L}^\infty(\Omega,X)$ and $\mathscr{L}^c_X(\Omega,X') = \mathscr{L}^\infty(\Omega,X')$. The vector space $\{u^\sim : u \in \mathscr{L}^c_Y(\Omega,X)\} \subset L^\infty(\Omega,X)$ is denoted by $L^c_Y(\Omega,X)$; the symbol $L^c(\Omega,X)$ stands for its subspace $\{u^\sim : u \in \mathscr{L}^c(\Omega,X)\}$.

Corollary. *Let a function u with values in the space X be defined almost everywhere in* Ω.

(1) *The function u belongs to* $\mathscr{L}^c(\Omega,X)$ *if and only if* $u \sim v \circ \tau$ *for some element* $v \in C(Q,X)$. *The Stonian transform* $\mathbf{u} \mapsto \hat{\mathbf{u}}$ *is an isometry from the LNS* $L^c(\Omega,X)$ *onto* $C(Q,X)$ *associated with the isomorphism* $\mathbf{e} \in M(\Omega) \mapsto \hat{\mathbf{e}} \in C_\infty(Q)$.

(2) *The function u belongs to* $\mathscr{L}^c_Y(\Omega,X)$ *if and only if* $u \sim v \circ \tau$ *for some element* $v \in C_\infty(Q,X) \cap C(Q,X|\,Y)$. *The Stonian transform* $\mathbf{u} \mapsto \hat{\mathbf{u}}$ *is an isometry from the LNS* $L^c_Y(\Omega,X)$ *onto* $C_\infty(Q,X) \cap C(Q,X|\,Y)$ *associated with the isomorphism* $\mathbf{e} \in M(\Omega) \mapsto \hat{\mathbf{e}} \in C_\infty(Q)$.

4.2.5. Theorem (F.D.Sâentilles). *The spaces* $C_\infty(Q,X|\,X')$ *and* $C_\infty(Q,X)$ *coincide.*

Proof. From [11] (Theorems 2.5 and 2.8) the equality $C^b_\infty(U,X|\,X') = C^b_\infty(U,X)$ follows for every element $U \in \mathrm{Clop}(Q)$ such that the inverse image $\tau^{-1}[U]$ has finite measure. The direct sum property possessed by Ω enables us to extend the last equality onto the entire compactum: $C^b_\infty(Q,X|\,X') = C^b_\infty(Q,X)$. Now, the assertion of the theorem follows from the fact that, for every function $u \in C_\infty(Q,X|\,X')$, there exists a partition of unity $(U_n)_{n\in\mathbb{N}}$ in the Boolean algebra $\mathrm{Clop}(Q)$ such that $\langle U_n\rangle u \in C^b_\infty(Q,X|\,X')$ for all $n \in \mathbb{N}$ (see 0.3.2). □

4.2.6. Corollary. *An X-valued function u defined almost everywhere in* Ω *belongs to* $\mathscr{L}^c_X(\Omega,X)$ *if and only if* $u \sim v \circ \tau$ *for some element* $v \in C(Q,X|\,X')$. *The Stonian transform* $\mathbf{u} \mapsto \hat{\mathbf{u}}$ *is an isometry from the LNS* $L^c_X(\Omega,X)$ *onto* $C(Q,X|\,X')$ *associated with the isomorphism* $\mathbf{e} \in M(\Omega) \mapsto \hat{\mathbf{e}} \in C_\infty(Q)$.

Proof. The claim follows from 4.2.4(2) and 4.2.5. □

4.2.7. Theorem. (1) *An* X'*-valued function defined almost everywhere in* Ω *is X-weakly measurable if and only if it is X-weakly equivalent to the composition* $v \circ \tau$ *for some element* $v \in C_\infty(Q,X'|\,X)$.

(2) *For every class* $\mathbf{u} \in M(\Omega,X'|\,X)$, *there exists a unique function* $\hat{\mathbf{u}} \in C_\infty(Q,X'|\,X)$ *representing* $\mathbf{u}$ *as* $(\hat{\mathbf{u}} \circ \tau)^\sim$.

(3) *The mapping* $\mathbf{u} \mapsto \hat{\mathbf{u}}$ *is an isometry from the BKS* $M(\Omega,X'|\,X)$ *onto* $C_\infty(Q,X'|\,X)$ *associated with the isomorphism* $\mathbf{e} \in M(\Omega) \mapsto \hat{\mathbf{e}} \in C_\infty(Q)$. *The inverse*

isometry from $C_\infty(Q,X'|X)$ *onto* $M(\Omega,X'|X)$ *is defined by the rule* $v\mapsto(v\circ\tau)^{\sim}$ *and is associated with the isomorphism* $e\in C_\infty(Q)\mapsto(e\circ\tau)^{\sim}\in M(\Omega)$.

Proof. Due to 4.1.8, 4.2.1, 3.4.7(2), 3.3.4, and 2.5.10, we have the following chain of isometries: $M(\Omega,X'|X)=M(\Omega,(X_\Omega)')=C_\infty(Q,(X_Q)')=C_\infty(Q,X'|X)$. □

The function $\hat{\mathbf{u}}\in C_\infty(Q,X'|X)$ corresponding to a class $\mathbf{u}\in M(\Omega,X'|X)$ by item (2) is called the *Stonian transform* of $\mathbf{u}$. If the class $\mathbf{u}$ belongs to the space $M(\Omega,X')$ then its Stonian transforms in terms of 4.2.2 and in terms of 4.2.7 coincide (according to the inclusions $M(\Omega,X')\subset M(\Omega,X'|X)$ and $C_\infty(Q,X')\subset C_\infty(Q,X'|X)$), which justifies the use of the same terms and notations for them.

4.2.8. Since $M(\Omega,X|Y)\subset M(\Omega,Y'|Y)$, the Stonian transform $\hat{\mathbf{u}}\in C_\infty(Q,Y'|Y)$ can also be defined for elements $\mathbf{u}\in M(\Omega,X|Y)$. In particular, if $\mathbf{u}\in M(\Omega,X|X')$ then $\hat{\mathbf{u}}\in C_\infty(Q,X''|X')$. The value $\hat{\mathbf{u}}(q)\in Y'$ of the Stonian transform of a class $\mathbf{u}\in M(\Omega,X|Y)$ at a point $q\in\operatorname{dom}\hat{\mathbf{u}}$ can be defined as follows: $\langle y|\hat{\mathbf{u}}(q)\rangle=\langle\mathbf{u}|y\rangle^\wedge(q)$, $y\in Y$. In this form, it is presented in the papers [12] and [7] for classes $\mathbf{u}\in L^\infty(\Omega,X|X')$.

As is seen from Theorem 4.2.7, the Stonian transform is an isometric embedding of the LNS $M(\Omega,X|Y)$ into the BKS $C_\infty(Q,Y'|Y)$. Note that the image of this embedding need not coincide neither with $C_\infty(Q,Y'|Y)$ nor with $C_\infty(Q,X|Y)$.

Theorem (F.D.Sentilles [12]). *A class* $\mathbf{u}\in M(\Omega,X|X')$ *belongs to* $M(\Omega,X)$ *if and only if* $\hat{\mathbf{u}}^{-1}[X''\setminus X]$ *is a meager (= nowhere-dense) subset of* Q.

Proof. The claim follows from 4.2.2 and 4.2.5. □

4.2.9. We call a function $u\in\mathcal{M}(\Omega,X|Y)$ *compact* if its image is relatively compact in the Y-weak topology. The symbol $\mathcal{L}^c(\Omega,X|Y)$ stands for the totality of all elements in $\mathcal{M}(\Omega,X|Y)$ that are Y-weakly equivalent to compact functions. We note that the set $\mathcal{L}^c(\Omega,X'|X)$ coincides with $\mathcal{L}^\infty(\Omega,X'|X)$ and equals the totality of all almost everywhere defined X'-valued functions u satisfying the condition $\langle u|x\rangle\in\mathcal{L}^\infty(\Omega)$ for every $x\in X$. Only the inclusion $\mathcal{L}^c(\Omega,X|Y)\subset\mathcal{L}^\infty(\Omega,X|Y)$ holds in general. We denote the vector subspace $\{u^{\sim}:u\in\mathcal{L}^c(\Omega,X|Y)\}\subset L^\infty(\Omega,X|Y)$ by $L^c(\Omega,X|Y)$.

Corollary. *An X-valued function u defined almost everywhere in* Ω *belongs to* $\mathcal{L}^c(\Omega,X|Y)$ *if and only if it is Y-weakly equivalent to the composition* $v\circ\tau$ *for some element* $v\in C(Q,X|Y)$. *The Stonian transform* $\mathbf{u}\mapsto\hat{\mathbf{u}}$ *is an isometry from the LNS* $L^c(\Omega,X|Y)$ *onto* $C(Q,X|Y)$ *associated with the isomorphism* $\mathbf{e}\in M(\Omega)\mapsto\hat{\mathbf{e}}\in C_\infty(Q)$.

Proof. The claim follows from 4.2.7 and 4.2.3. □

4.2.10. Corollary (D.R.Lewis [1]: III.4). *The spaces* $L^c(\Omega,X|X')$ *and* $L^c_{X'}(\Omega,X)$ *coincide. In other words, if the image of a weakly measurable function* $u:\Omega\to X$ *is weakly* *relatively compact, then u is weakly equivalent to a Bochner measurable function $v: \Omega \to X$ having weakly relatively compact image as well.*

Proof. The claim follows from 4.2.6, 4.2.9, and 4.2.5. □

4.2.11. Given $u \in \mathbf{u} \in M(\Omega, X | Y)$ and $y \in Y$, the function $|y\rangle \circ u$ is denoted by $\langle u | y \rangle$ and the symbol $\langle \mathbf{u} | y \rangle$ stands for the corresponding class $\langle u | y \rangle^{\sim}$. Let L be a vector subspace of $L^{\infty}(\Omega, X | Y)$ and let Z be a Banach subspace of Y'. We call a mapping $\rho_L : L \to \mathcal{L}^{\infty}(\Omega, Z | Y)$ a *$Z | Y$-weak lifting* of L *(associated with ρ)* if it satisfies the following conditions:

(a) $\rho_L(\mathbf{u}) : \Omega \to Z$ for every $\mathbf{u} \in L$;

(b) $\langle \rho_L(\mathbf{u}) | y \rangle = \rho(\langle \mathbf{u} | y \rangle)$ for all $\mathbf{u} \in L$ and $y \in Y$.

If a $Z | Y$-weak lifting ρ_L additionally satisfies the conditions

(c) if $u \in \mathbf{u} \in L$ then the functions u and $\rho_L(\mathbf{u})$ coincide almost everywhere;

(d) $|||\rho_L(\mathbf{u})||| = \rho(|\mathbf{u}|)$ for each $\mathbf{u} \in L$,

then it is called a *$Z | Y$-strong lifting* of L. Note that every $Z | Y$-weak lifting ρ_L has the following properties:

(1) if $u \in \mathbf{u} \in L$ then the functions u and $\rho_L(\mathbf{u})$ are Y-weakly equivalent;

(2) if $\mathbf{u}, \mathbf{v} \in L$ then $\rho_L(\mathbf{u} + \mathbf{v}) = \rho_L(\mathbf{u}) + \rho_L(\mathbf{v})$;

(3) if $\mathbf{u} \in L$, $\mathbf{e} \in L^{\infty}(\Omega)$, and $\mathbf{eu} \in L$, then $\rho_L(\mathbf{eu}) = \rho(\mathbf{e})\rho_L(\mathbf{u})$;

(4) if a function $c : \Omega \to X$ is constant and $c^{\sim} \in L$ then $\rho_L(c^{\sim}) = c$.

4.2.12. Theorem. *For every lifting $\rho : L^{\infty}(\Omega) \to \mathcal{L}^{\infty}(\Omega)$ there exist a unique $X | X'$-strong lifting ρ_X of the space $L^c(\Omega, X)$ and a unique $X | Y$-weak lifting $\rho_{X|Y}$ of the space $L^c(\Omega, X | Y)$, both associated with ρ. Moreover, for each element $\mathbf{u} \in L^c(\Omega, X)$ (respectively, $\mathbf{u} \in L^c(\Omega, X | Y)$), the function $\rho_X(\mathbf{u})$ (respectively, $\rho_{X|Y}(\mathbf{u})$) coincides with $\hat{\mathbf{u}} \circ \tau$, where $\hat{\mathbf{u}}$ is the Stonian transform of the class $\mathbf{u}$ in terms of* 4.2.2 (respectively, in terms of 4.2.8) *and τ is the canonical immersion of Ω into Q corresponding to the lifting ρ.*

Proof. The claim follows from 4.2.4 and 4.2.9. □

4.2.13. Some consequences of the last theorem are listed below.

Corollary. *Suppose that Ω is a measure space possessing the direct sum property, X and Y are Banach spaces, and there is a duality between X and Y such that Y norms X.*

(1) *The space $L^c(\Omega, X)$ has an $X | X'$-strong lifting.*

(2) *The space $L^c_Y(\Omega, X)$ has an $X | Y$-weak lifting.*

(3) *The space $L^{\infty}(\Omega, X')$ has an $X' | X$-weak lifting.*

(4) *The space $L^c(\Omega, X | Y)$ has an $X | Y$-weak lifting.*

(5) *The space $L^{\infty}(\Omega, X | Y)$ has a $Y' | Y$-weak lifting.*

(6) *The space $L^{\infty}(\Omega, X' | X)$ has an $X' | X$-weak lifting.*

(7) *The space $L^{\infty}(\Omega, X | X')$ has an $X'' | X'$-weak lifting.*

This list is of illustrative value and the results presented in it are not new. For instance, assertions (3) and (4) can be easily deduced from [3] (Chapter VI, Sections 4–7).

References

1. DIESTEL J. and UHL J.J. (1977) *Vector Measures*. Providence, Amer. Math. Soc.
2. GUTMAN A.E. (1993) Banach bundles in the theory of lattice-normed spaces. I. Continuous Banach bundles. SIBAM, v.3, N3, 1–55.
3. IONESCU TULCEA A. and IONESCU TULCEA C. (1969) *Topics in the Theory of Lifting*. Springer, Berlin, etc.
4. JANSSEN A.J. E. M. and VAN DER STEEN P. (1984) *Integration Theory*. Springer, Berlin, etc.
5. KANTOROVICH L.V. and AKILOV G.P. (1984) *Functional Analysis*. Nauka, Moscow (Russian).
6. KANTOROVICH L.V., VULIKH B.Z., and PINSKER A.G. (1950) *Functional Analysis in Semiordered Spaces*. Gostekhizdat, Moscow–Leningrad (Russian).
7. KUO R.T. (1980) *Vector Measurable Functions Via Stonian Spaces*. Dissertation. University of Missouri, Columbia.
8. KUSRAEV A.G. (1985) *Vector Duality and Its Applications*. Nauka, Novosibirsk (Russian).
9. MAHARAM D. (1958) On a theorem of von Neumann. *Proc. Amer. Math. Soc.*, v.9, 987–994.
10. OXTOBY J.C. (1971) *Measure and Category*. Springer, New-York.
11. SENTILLES D. (1980) Stonian differentiation and representation of vector functions and measures. *Contemp. Math.*, v.2, 241–269.
12. SENTILLES D. (1983) Decomposition of weakly measurable functions. *Indiana Univ. Math. J.*, v.32, N3, 425–437.
13. VULIKH B. Z. (1961) *Introduction into the Theory of Semiordered Spaces*. Fizmatgiz, Moscow (Russian).

Translated by the author