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BANACH BUNDLES IN THE THEORY OF LATTICE-NORMED SPACES. III

APPROXIMATING SETS AND BOUNDED OPERATORS

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Abstract

Two questions in the general theory of lattice-normed spaces (LNSs) are considered. First, the situation is studied when a subset of an LNS is order dense in the entire LNS; the notion of order approximation is introduced and described from various points of view. Second, the situation is studied when a linear operator from one LNS to another is order bounded; several different types of boundedness are introduced and studied in detail.

Key words and phrases: vector lattice, lattice-normed space, order approximating set, order bounded linear operator.

Although the present article is the third part of the intended paper on Banach bundles in the theory of LNSs, bundles are not mentioned here at all. Actually, this part can be regarded as a separate paper, since the questions under consideration belong to the theory of LNSs in general. Moreover, the two sections of this article are not interrelated. In the first, we study the situation when a subset of an LNS is in a sense dense in the entire LNS (we introduce the notion of order approximation and describe it from various points of view). In the second section, we study the situation when a linear operator from one LNS to another is in a sense bounded (we introduce several different types of boundedness and study them in detail).

The first two parts of the intended article were published in [1] and [2] and we sometimes appeal to the material of these papers without explicit reference. The author apologizes to the reader for an erroneous announcement of the third part made in [2] and hopes to begin a study of disjointness-preserving operators in the fourth part.

Before the basic presentation, we briefly recall some general information about bands, order projections, order convergence, and Boolean homomor-

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phisms. Note that we give only the information that was not presented in [1] or [2].

Let E be a vector lattice. A subset of E having the form $F^{\perp\perp}$ for some $F \subset E$ is called a *band* of E. The band $F^{\perp\perp}$ is said to be *generated* by F. A subset $F \subset E$ is a band if and only if $F^{\perp\perp} = F$. A band $F \subset E$ is called *principal* if $F = \{e\}^{\perp\perp}$ for some $e \in E$.

The image of every order projection is a band. Every order projection is uniquely determined by its image. There are order projections onto every band (every principal band) in a K-space (in a K_σ-space). Given an element $e \in E$, the symbol $\langle e \rangle$ denotes the order projection onto the principal band $\{e\}^{\perp\perp}$ (if such a projection exists). For $e, f \in E$, we define $\langle e < f \rangle := \langle (f - e)^+ \rangle$, $\langle e \leqslant f \rangle := \langle f < e \rangle^\perp$, $\langle e > f \rangle := \langle f < e \rangle$, and $\langle e \geqslant f \rangle := \langle f \geqslant e \rangle$. (We recall that the symbols e^+ and e^- denote the positive part $e \vee 0$ and the negative part $(-e) \vee 0$ of an element $e \in E$ and $(\cdot)^\perp$ stands for the complement operation in a Boolean algebra.) It is clear that $\langle e \leqslant f \rangle = \max\{\pi \in \mathrm{Pr}(E) : \pi e \leqslant \pi f\}$.

We assume by default that all the LNSs are d-decomposable (see 0.4.1 of [1]). We also assume that the equality $\{|u| : u \in \mathcal{U}\}^{\perp\perp} = E$ holds for every LNS $\mathcal{U}$ over E that we consider. It is useful to be aware of the fact that if $\mathcal{U}$ is a Banach–Kantorovich space over a vector lattice E, then E is a K-space and $\{|u| : u \in \mathcal{U}\} = \{e \in E : e \geqslant 0\}$.

If $\mathcal{U}$ is a (d-decomposable) LNS over E, then there is a natural mapping that associates with every $\pi \in \mathrm{Pr}(E)$ a linear projection $\pi_{\mathcal{U}} : \mathcal{U} \to \mathcal{U}$ (see 0.4.1 of [1]). The set $\mathrm{Pr}(\mathcal{U}) := \{\pi_{\mathcal{U}} : \pi \in \mathrm{Pr}(E)\}$ endowed with the order

$$\pi_{\mathcal{U}} \leqslant \rho_{\mathcal{U}} \ \Leftrightarrow \ \pi_{\mathcal{U}} \circ \rho_{\mathcal{U}} = \pi_{\mathcal{U}}$$

is a Boolean algebra with Boolean operations translated from $\mathrm{Pr}(E)$ by the mapping $\pi \mapsto \pi_{\mathcal{U}}$. More precisely, for all $\pi, \rho \in \mathrm{Pr}(E)$, the following relations hold: $\pi_{\mathcal{U}} \wedge \rho_{\mathcal{U}} = (\pi \wedge \rho)_{\mathcal{U}} = \pi_{\mathcal{U}} \circ \rho_{\mathcal{U}}$, $\pi_{\mathcal{U}} \vee \rho_{\mathcal{U}} = (\pi \vee \rho)_{\mathcal{U}} = \pi_{\mathcal{U}} + \rho_{\mathcal{U}} - \pi_{\mathcal{U}} \circ \rho_{\mathcal{U}}$, and $(\pi_{\mathcal{U}})^\perp = (\pi^\perp)_{\mathcal{U}} = \mathrm{id}_{\mathcal{U}} - \pi_{\mathcal{U}}$. The mapping $\pi \mapsto \pi_{\mathcal{U}}$ is an isomorphism between the Boolean algebras $\mathrm{Pr}(E)$ and $\mathrm{Pr}(\mathcal{U})$; by means of the mapping we identify the two algebras. Given an element $u \in \mathcal{U}$, the projection $\langle |u| \rangle$ is denoted by $\langle u \rangle$.

In the sequel, we will deal with various convergences (such as o- and r-convergences) and related notions (such as o- and r-closures, o- and r-continuity, etc.). For the sake of convenience and in order to avoid duplication, we present some general definitions now.

Let X be an arbitrary set and let c be some convergence in X. The totality of the c-limits of all c-convergent in X nets constituted by elements of some subset $X_0 \subset X$ is called the *c-closure* of X_0. A set is called *c-closed* if it coincides with the c-closure of itself. The set X_0 is said to be *c-dense* in X if X is the c-closure of X_0. Suppose now that X_1 and X_2 are some sets with convergences c_1 and c_2, respectively. A mapping $f : X_1 \to X_2$

is called c_1-c_2-*continuous* if c_1-convergence $x_\alpha \to x$ implies c_2-convergence $f(x_\alpha) \to f(x)$ for every net $(x_\alpha)_{\alpha\in\mathrm{A}}$ in X_1 and every element $x \in X_1$. If the convergences c_1 and c_2 have the same notation c, then any c_1-c_2-continuous mapping is called *c-continuous.*

Considering only countable nets in the above definitions results in the notions of *countable c-closure*, *countable c-closedness*, *countable c-density*, and *countable c-continuity.* By replacing nets with sequences, we obtain the notions of *sequential c-closure*, *sequential c-closedness*, *sequential c-density*, and *sequential c-continuity.*

For the future presentation, we need one more type of closure that is not generated by o- or r-convergence. If $(u_\xi)_{\xi\in\Xi}$ is an arbitrary family in an LNS $\mathcal{U}$ and $(\pi_\xi)_{\xi\in\Xi}$ is a partition of unity in the Boolean algebra $\mathrm{Pr}(\mathcal{U})$, then the sum $o\text{-}\sum_{\xi\in\Xi}\pi_\xi u_\xi$ (if the latter exists) is called the *mixing* of the family $(u_\xi)_{\xi\in\Xi}$ with respect to $(\pi_\xi)_{\xi\in\Xi}$. Let $\mathcal{V}$ be a subset of $\mathcal{U}$. The totality of all mixings of arbitrary (finite) families in $\mathcal{V}$ is called the *cyclic hull* (the *finitely cyclic hull*) of $\mathcal{V}$ and denoted by $\mathrm{mix}\,\mathcal{V}$ (by $\mathrm{mix_{fin}}\,\mathcal{V}$). The cyclic hull of the union $\mathcal{V}\cup\{0\}$ is called the *d-closure* of $\mathcal{V}$ and denoted by $d\mathcal{V}$. Similarly, the symbol $d_{\mathrm{fin}}\mathcal{V}$ is used to denote the finitely cyclic hull of $\mathcal{V}\cup\{0\}$. The set $\mathcal{V}$ is called *cyclic* (*finitely cyclic*) if $\mathrm{mix}\,\mathcal{V} = \mathcal{V}$ ($\mathrm{mix_{fin}}\,\mathcal{V} = \mathcal{V}$). It easy to verify that the (finitely) cyclic hull of a set $\mathcal{V}$ is the smallest (finitely) cyclic set that includes $\mathcal{V}$. Obviously, for a set $\mathcal{V}$ to be finitely cyclic, it is sufficient that it contain the sums $\pi v + \pi^\perp w$ for all $v, w \in \mathcal{V}$ and $\pi \in \mathrm{Pr}(\mathcal{U})$.

Ring and Boolean homomorphisms will often arise in our further consideration. We recall the necessary definitions.

Let A and B be Boolean algebras. A mapping $h : A \to B$ is called a *ring homomorphism* if the following equalities hold for all $a_1, a_2 \in A$:

(a) $h(a_1 \vee a_2) = h(a_1) \vee h(a_2)$;
(b) $h(a_1 \wedge a_2) = h(a_1) \wedge h(a_2)$;
(c) $h(a_1 \backslash a_2) = h(a_1) \backslash h(a_2)$,

where $x\backslash y$ stands for $x \wedge y^\perp$. We observe that (a) is a consequence of (b) and (c), as well as (b) is a consequence of (a) and (c). Every ring homomorphism $h : A \to B$ preserves the order, i.e., $a_1 \leqslant a_2$ implies $h(a_1) \leqslant h(a_2)$ for all $a_1, a_2 \in A$.

A ring homomorphism $h : A \to B$ is called a *Boolean homomorphism* if $h(1) = 1$. Obviously, a mapping $h : A \to B$ is a Boolean homomorphism if and only if it satisfies one of the conditions (a) or (b) and, in addition, $h(a^\perp) = h(a)^\perp$ for all $a \in A$. Every ring homomorphism $h : A \to B$ is a Boolean homomorphism into the Boolean algebra $B_{h(1)} = \{b \in B : b \leqslant h(1)\}$. The image $h[A]$ of the homomorphism h is a Boolean subalgebra of $B_{h(1)}$.

A mapping $h : A \to B$ is called a *Boolean isomorphism* of A onto B if it possesses any of the following equivalent properties:

(1) h is a bijective Boolean homomorphism;
(2) h is a bijection and both h and h^{-1} are Boolean homomorphisms;
(3) h is a bijection and both h and h^{-1} preserve the order;
(4) h is a Boolean homomorphism, $h[A] = B$, and $h^{-1}(0) = \{0\}$;
(5) h is a Boolean homomorphism, $h[A] = B$, and $h^{-1}(1) = \{1\}$.

Algebras A and B are said to be *isomorphic* if there exists a Boolean isomorphism of A onto B.

We conclude the introduction with some description of Boolean homomorphisms. Such a description is especially convenient in studying disjointness-preserving operators.

Proposition. *Let A and B be Boolean algebras. A mapping $h : A \to B$ is a Boolean homomorphism if and only if, for every partition (a_1, a_2, a_3) of unity in A, the triple $\big(h(a_1), h(a_2), h(a_3)\big)$ is a partition of unity in B.*

Proof. Necessity is obvious; thus, we only prove sufficiency. Suppose that the mapping h preserves triple partitions. By applying this property of h to the triple $(0, 0, 1)$, we obtain the equality $h(0) = 0$. By considering the triple $(a, a^{\perp}, 0)$, we conclude that $h(a^{\perp}) = h(a)^{\perp}$ for every $a \in A$. It remains to establish the relation $h(a_1 \vee a_2) = h(a_1) \vee h(a_2)$. First, we prove this equality for disjoint a_1 and a_2. For this purpose, it is sufficient to apply the partition preservation property of h to the triples $\big(a_1, a_2, (a_1 \vee a_2)^{\perp}\big)$ and $\big(a_1 \vee a_2, (a_1 \vee a_2)^{\perp}, 0\big)$. Finally, taking arbitrary elements $a_1, a_2 \in A$ and using the above-established facts, we obtain

$$\begin{aligned} h(a_1 \vee a_2) &= h\big((a_1 \backslash a_2) \vee (a_1 \wedge a_2) \vee (a_2 \backslash a_1)\big) \\ &= h(a_1 \backslash a_2) \vee h(a_1 \wedge a_2) \vee h(a_2 \backslash a_1) \\ &= \big(h(a_1 \backslash a_2) \vee h(a_1 \wedge a_2)\big) \vee \big(h(a_1 \wedge a_2) \vee h(a_2 \backslash a_1)\big) \\ &= h(a_1) \vee h(a_2). \end{aligned}$$

1. Order approximating sets

In this section, we develop the results 0.4.5–0.4.9 of [1]. We introduce the notions of order approximating and h-approximating subsets of an LNS. We also present equivalent descriptions of the notions in terms of convergences of various types. The notion of order approximation seems to be useful in the general theory of LNSs. As for h-approximation, it will play its role in the future, in studying disjointness-preserving operators.

1.1. Lemma. *Let $\mathcal{U}$ be an LNS over a K-space E and let $\mathcal{V}$ be a finitely cyclic subset of $\mathcal{U}$. Then, for every $u \in \mathcal{U}$, there exists a net $(v_\alpha)_{\alpha \in \mathrm{A}}$ in $\mathcal{V}$ such that the net $\big(|u - v_\alpha|\big)_{\alpha \in \mathrm{A}}$ decreases and $\{|u - v_\alpha| : \alpha \in \mathrm{A}\} = \{|u - v| : v \in \mathcal{V}\}$. In particular, $|u - v_\alpha| \searrow \inf_{v \in \mathcal{V}} |u - v|$.*

Proof. Suppose that a set $\mathcal{V} \subset \mathcal{U}$ meets the hypothesis of the lemma and fix an arbitrary element $u \in \mathcal{U}$. We introduce in $\mathcal{V}$ relations of equivalence and preorder as follows:

$$v \sim w \;\Leftrightarrow\; |u-v| = |u-w|\,,$$
$$v \preccurlyeq w \;\Leftrightarrow\; |u-v| \geqslant |u-w|\,.$$

For any two elements $v, w \in \mathcal{V}$ we can find a projection $\pi \in \mathrm{Pr}(E)$ such that $\bigl|u - (\pi v + \pi^{\perp} w)\bigr| = |u-v| \wedge |u-w|$. Since $\mathcal{V}$ is finitely cyclic, the latter means that the set $(\mathcal{V}, \preccurlyeq)$ is directed. Therefore, the quotient set $\mathrm{A} := \mathcal{V}/{\sim}$ (endowed with the quotient order) is a directed ordered set. Taking an element $v_\alpha \in \alpha$ in every coset $\alpha \in \mathrm{A}$, we obtain the desired net $(v_\alpha)_{\alpha \in \mathrm{A}}$.

1.2. Let $\mathcal{V}$ be a subset of an LNS $\mathcal{U}$. We say that $\mathcal{V}$ (*orderly*) *approximates an element* $u \in \mathcal{U}$ if $\inf_{v \in \mathcal{V}} |u-v| = 0$. We say that $\mathcal{V}$ (*orderly*) *approximates a subset* $\mathcal{W} \subset \mathcal{U}$ if $\mathcal{V}$ approximates every element of $\mathcal{W}$. A subset of $\mathcal{U}$ is called (*order*) *approximating* if it approximates $\mathcal{U}$. Any order dense ideal of an LNS is an example of an approximating set.

Proposition. *Let X, Y, and Z be subsets of an LNS. If X approximates Y and Y approximates Z, then X approximates Z.*

Proof. For an arbitrary element $z \in Z$, denote $\inf_{x \in X} |x-z|$ by e and assume to the contrary that $e \neq 0$. Since $\inf_{y \in Y} |y-z| = 0$, there is an element $y \in Y$ and an order projection ρ such that $\rho\,|y-z| < \rho e/2$. Similarly, in view of the equality $\inf_{x \in X} |x-y| = 0$, there is an element $x \in X$ and an order projection π such that $\pi\,|x-y| < \pi\rho e/2$. The following contradictory relations complete the proof:

$$\pi\rho e \leqslant \pi\rho\,|x-z| \leqslant \pi\rho\,|x-y| + \pi\rho\,|y-z| < \pi\rho e/2 + \pi\rho e/2 = \pi\rho e.$$

1.3. Proposition. *Let $\mathcal{V}$ be a subset and let u be an element of an LNS. The set $\mathcal{V}$ approximates u if and only if u is the o-limit of some net in* $\mathrm{mix}_{\mathrm{fin}}\,\mathcal{V}$.

Proof. If $\mathcal{V}$ approximates u then $\inf\bigl\{|u-w| : w \in \mathrm{mix}_{\mathrm{fin}}\,\mathcal{V}\bigr\} = 0$. Therefore, in view of 1.1, there exists a net $(w_\alpha)_{\alpha \in \mathrm{A}}$ in $\mathrm{mix}_{\mathrm{fin}}\,\mathcal{V}$ such that $|u - w_\alpha| \searrow 0$.

Conversely, if u is the o-limit of a net in $\mathrm{mix}_{\mathrm{fin}}\,\mathcal{V}$ then $\mathrm{mix}_{\mathrm{fin}}\,\mathcal{V}$ approximates u. It remains to use Proposition 1.2 on observing that $\mathcal{V}$ approximates $\mathrm{mix}_{\mathrm{fin}}\,\mathcal{V}$.

Corollary. *If a subset $\mathcal{V}$ of an LNS $\mathcal{U}$ is finitely cyclic, then its o-closure consists of all elements $u \in \mathcal{U}$ approximated by $\mathcal{V}$.*

Corollary. *If a subset $\mathcal{V}$ of an LNS $\mathcal{U}$ is finitely cyclic, then its o-closure is o-closed and, hence, is the least o-closed subset of $\mathcal{U}$ that includes $\mathcal{V}$.*

Proof. The claim follows from the previous corollary and Proposition 1.2.

1.4. Proposition. *The following properties of a subset $\mathcal{V}$ of an LNS $\mathcal{U}$ are equivalent:*

(1) *$\mathcal{V}$ is an approximating subset of $\mathcal{U}$;*
(2) *for every ideal $\mathcal{U}_0 \subset \mathcal{U}$, the set $d_{\mathrm{fin}}\mathcal{V} \cap \mathcal{U}_0$ is o-dense in $\mathcal{U}_0$;*
(3) *the set $d_{\mathrm{fin}}\mathcal{V}$ is o-dense in $\mathcal{U}$;*
(4) *$d_{\mathrm{fin}}\mathcal{V}$ is an approximating subset of $\mathcal{U}$.*

Proof. The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious. It remains to prove that (1) $\Rightarrow$ (2) and (4) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2): Suppose that the set $\mathcal{V} \subset \mathcal{U}$ satisfies condition (1), fix an arbitrary ideal $\mathcal{U}_0 \subset \mathcal{U}$ and its element $u \in \mathcal{U}_0$, denote the set $d_{\mathrm{fin}}\mathcal{V} \cap \mathcal{U}_0$ by $\mathcal{W}$, and assign $e := \inf_{w\in\mathcal{W}} |u - w|$. Obviously, $e \leqslant |u|$. According to 1.1, there exists a net $(w_\alpha)_{\alpha\in\mathrm{A}}$ in $\mathcal{W}$ such that $|u - w_\alpha| \searrow e$. It remains to show that $e = 0$. If $e \neq 0$ then, in view of 1.3, there are $w \in \mathrm{mix}_{\mathrm{fin}}\,\mathcal{V}$ and $\pi \in \mathrm{Pr}(E)$ such that $\pi\,|u - w| < \pi e$. The inequalities $|\pi w| \leqslant |\pi w - \pi u| + |\pi u| \leqslant e + |u| \leqslant 2\,|u|$ ensure the containment $\pi w \in \mathcal{W}$ and, thus, we have the following contradictory relations: $\pi e \leqslant \pi\,|u - \pi w| < \pi e$.

(4) $\Rightarrow$ (1): Denote the set $d_{\mathrm{fin}}\mathcal{V}$ by $\mathcal{W}$ and suppose that it is an approximating subset of $\mathcal{U}$.

Denote $\inf_{v\in\mathcal{V}} |v|$ by e and prove that $e = 0$. If it is not so, then there is an element $u \in \mathcal{U}$ that satisfies the inequalities $0 < |u| \leqslant e/2$. Since $\inf_{w\in\mathcal{W}} |u - w| = 0$, there is an order projection $\pi \neq 0$ and an element $w = \pi_1 v_1 + \dots + \pi_n v_n \in \mathcal{W}$ ($v_i \in \mathcal{V}$) such that $\pi_0\,|u - w| < \pi_0\,|u|$ for all $0 \neq \pi_0 \leqslant \pi$. It is clear that $\pi w \neq 0$ and, hence, $\rho := \pi_i \wedge \pi \neq 0$ for some i. Now, the inequalities $\rho\,|u - v_i| < \rho\,|u| \leqslant \rho e/2$ lead to a contradiction: $\rho e \leqslant \rho\,|v_i| \leqslant \rho\,|u - v_i| + \rho\,|u| < \rho e/2 + \rho e/2 = \rho e$.

Thus, $\inf_{v\in\mathcal{V}} |v| = 0$, which implies that $\mathcal{V}$ approximates $\mathcal{V} \cup \{0\}$. However, it is obvious that the set $\mathcal{V} \cup \{0\}$ approximates $d_{\mathrm{fin}}\mathcal{V}$ and the latter approximates $\mathcal{U}$. It remains to apply Proposition 1.2.

R e m a r k . Replacing $d_{\mathrm{fin}}\mathcal{V}$ by $\mathrm{mix}_{\mathrm{fin}}\,\mathcal{V}$ in condition (2) of the last proposition can lead to a nonequivalent assertion even if $\mathcal{U} = E$. Indeed, the totality $\mathcal{V}$ of all number sequences convergent to 1 is an approximating subset of the K-space $\mathcal{U}$ of all sequences; however, the set $\mathrm{mix}_{\mathrm{fin}}\,\mathcal{V}$ coincides with $\mathcal{V}$ and has empty intersection with the order dense ideal $\mathcal{U}_0 \subset \mathcal{U}$ of all vanishing sequences.

1.5. Lemma. *If $\mathcal{V}$ is an approximating subset of a d-complete LNS $\mathcal{U}$ over E then, for every $u \in \mathcal{U}$, $e \in E$, and $n \in \mathbb{N}$, there exists an element $w \in \mathrm{mix}\,\mathcal{V}$ satisfying the inequality $\langle e\rangle\,|u - w| \leqslant e/n$.*

Proof. Suppose that $\mathcal{U}$ and $\mathcal{V}$ meet the hypotheses of the lemma and consider arbitrary elements $u \in \mathcal{U}$, $e \in E$, and $n \in \mathbb{N}$. According to 1.4, there is a net $(v_\alpha)_{\alpha\in\mathrm{A}}$ in $\mathrm{mix}_{\mathrm{fin}}\,\mathcal{V}$ o-convergent to u. We may assume that this net is order bounded. In view of 0.3.2(1) of [1], there is a partition of unity $(\pi_\alpha^n)_{\alpha\in\mathrm{A}}$ in the Boolean algebra $\mathrm{Pr}(E)$ such that $\pi_\alpha^n\langle e\rangle\,|v_\alpha - u| \leqslant e/n$ for all $\alpha \in \mathrm{A}$. It is clear that the sum $w := o\text{-}\sum_{\alpha\in\mathrm{A}} \pi_\alpha^n v_\alpha$ is the desired element of $\operatorname{mix}\mathcal{V}$.

1.6. Suppose that an order unity 1 is fixed in the norming K-space of an LNS. Then the r-convergence with regulator 1 is called the *uniform convergence* in the LNS. The notions of a *uniform dense subset* and *uniform closure* are introduced in such an LNS similarly.

Proposition. *Let $\mathcal{V}$ be a subset and let u be an element of an LNS over a K-space with order unity. The set $\mathcal{V}$ approximates u if and only if u is a uniform limit of some sequence in* $\operatorname{mix}\mathcal{V}$.

Proof. Necessity is a straightforward consequence of Lemma 1.5; sufficiency is established as in the proof of Proposition 1.3.

1.7. Proposition. *Let $\mathcal{U}$ be a d-complete LNS over a K-space with order unity. The following properties of a subset $\mathcal{V} \subset \mathcal{U}$ are equivalent:*

(1) *$\mathcal{V}$ is an approximating subset of $\mathcal{U}$;*
(2) *for every ideal $\mathcal{U}_0 \subset \mathcal{U}$, the set $d\mathcal{V} \cap \mathcal{U}_0$ is uniformly dense in $\mathcal{U}_0$;*
(3) *$d\mathcal{V}$ is uniformly dense in $\mathcal{U}$;*
(4) *$d\mathcal{V}$ is an approximating subset of $\mathcal{U}$.*

Proof. Suppose that an LNS $\mathcal{U}$ over E meets the hypotheses of the proposition and 1 is an order unity in E. The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious and the implication (4) $\Rightarrow$ (1) is established as in the proof of Proposition 1.4. It remains to show that (1) $\Rightarrow$ (2).

Suppose that a subset $\mathcal{V} \subset \mathcal{U}$ satisfies condition (1), fix an arbitrary ideal $\mathcal{U}_0 \subset \mathcal{U}$, and denote the set $d\mathcal{V} \cap \mathcal{U}_0$ by $\mathcal{W}$.

Show that $\mathcal{W}$ approximates $\mathcal{U}_0$. For this purpose, we fix an arbitrary element $u \in \mathcal{U}_0$, assign $e := \inf_{w\in\mathcal{W}} |u - w|$, and establish the equality $e = 0$. If $e \neq 0$ then, in view of 1.6, there are $w \in \operatorname{mix}\mathcal{V}$ and $\pi \in \Pr(E)$ such that $\pi\,|u - w| < \pi e$. Obviously, $e \leqslant |u|$. The inequalities $|\pi w| \leqslant |\pi w - \pi u| + |\pi u| \leqslant e + |u| \leqslant 2\,|u|$ ensure the containment $\pi w \in \mathcal{W}$ and, thus, we have the following contradictory relations: $\pi e \leqslant \pi\,|u - \pi w| < \pi e$.

Since $\mathcal{W}$ approximates $\mathcal{U}_0$, in view of 1.5 there exists a sequence $(w_n)_{n\in\mathbb{N}}$ in $\operatorname{mix}\mathcal{W}$ such that $\langle u\rangle\,|u - w_n| \leqslant (|u| \wedge 1)/n$ for all $n \in \mathbb{N}$. It is clear that the sequence $\big(\langle u\rangle w_n\big)_{n\in\mathbb{N}}$ is constituted by elements of $\mathcal{W}$ and r-converges to u with regulator 1.

Remark. Replacing $d\mathcal{V}$ by $\operatorname{mix}\mathcal{V}$ in condition (2) of the last proposition can lead to a nonequivalent assertion even if $\mathcal{U} = E$. Indeed, the totality $\mathcal{V}$ of all number sequences with every member nonzero is an approximating subset of the K-space $\mathcal{U}$ of all sequences; however, the set $\operatorname{mix}\mathcal{V}$ coincides with $\mathcal{V}$ and has empty intersection with the order dense ideal $\mathcal{U}_0 \subset \mathcal{U}$ of all finitary (= terminating) sequences.

1.8. Proposition. *Let $\mathcal{V}$ be a subset and let u be an element of an LNS. The set $\mathcal{V}$ approximates u if and only if u is the r-limit of some sequence in* $\operatorname{mix}\mathcal{V}$.

Proof. Sufficiency: Suppose that $\mathcal{V}$ approximates u. Consider an arbitrary element $v \in \mathcal{V}$ and assign $e := |u| \vee |v|$. It is sufficient to fix an $n \in \mathbb{N}$ and find an element $w \in \operatorname{mix}\mathcal{V}$ that satisfies the inequality $|u - w| \leqslant e/n$. According to Lemma 1.5, there exists an element $w_0 \in \operatorname{mix}\mathcal{V}$ that satisfies the inequality $\langle e\rangle\, |u - w_0| \leqslant e/n$. It is clear that the sum $\langle e\rangle w_0 + \langle e\rangle^{\perp} v$ belongs to $\operatorname{mix}\mathcal{V}$, coincides with $\langle e\rangle w_0$, and, thus, is the desired element w.

Necessity is established in the same way as in Proposition 1.3.

1.9. Proposition. *Let $\mathcal{U}$ be a d-complete LNS. The following properties of a subset $\mathcal{V} \subset \mathcal{U}$ are equivalent*:

(1) *$\mathcal{V}$ is an approximating subset of $\mathcal{U}$*;
(2) *for every ideal $\mathcal{U}_0 \subset \mathcal{U}$, the set $d\mathcal{V} \cap \mathcal{U}_0$ is r-dense in $\mathcal{U}_0$*;
(3) *$d\mathcal{V}$ is r-dense in $\mathcal{U}$*;
(4) *$d\mathcal{V}$ is an approximating subset of $\mathcal{U}$.*

Proof. The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious, the equivalence (4) $\Leftrightarrow$ (1) is established in Proposition 1.7, and the proof of the implication (1) $\Rightarrow$ (2) word for word repeats that of the analogous implication in Proposition 1.7, with the only difference that 1 is replaced by $|u|$.

Remark. Replacing $d\mathcal{V}$ by $\operatorname{mix}\mathcal{V}$ in condition (2) of the last proposition can lead to a nonequivalent assertion. There is an appropriate example in the previous remark (see 1.7).

1.10. A net $(e_\alpha)_{\alpha\in A}$ in a vector lattice E is said to be *asymptotically bounded* if there exists an index $\bar\alpha \in A$ such that the set $\{e_\alpha : \alpha \geqslant \bar\alpha\}$ is order bounded. Obviously, every o-convergent net is asymptotically bounded.

In the sequel, we will need some modification of Theorem 0.3.2 of [1].

Lemma. *Let $(e_\alpha)_{\alpha\in \mathrm{A}}$ be a net in a K-space E and let $e \in E$.*

(1) *The net $(e_\alpha)_{\alpha\in \mathrm{A}}$ o-converges to e if and only if it is asymptotically bounded and the relation $o\text{-}\lim_{\alpha\in\mathrm{A}}\langle d\rangle\big\langle|e_\alpha - e| > d\big\rangle = 0$ holds in the Boolean algebra $\operatorname{Pr}(E)$ for all positive $d \in E$.*

(2) *Let D be a set of positive elements in E such that the band $D^{\perp\perp}$ contains e and all the members of the net $(e_\alpha)_{\alpha\in\mathrm{A}}$. If the net $(e_\alpha)_{\alpha\in\mathrm{A}}$ is asymptotically bounded and $o\text{-}\lim_{\alpha\in\mathrm{A}}\langle d\rangle\big\langle|e_\alpha - e| > d/n\big\rangle = 0$ for all $d \in D$ and $n \in \mathbb{N}$, then $o\text{-}\lim_{\alpha\in\mathrm{A}} e_\alpha = e$.*

Proof. (1) It is easy to verify the necessity of the criterion formulated, and its sufficiency follows from (2).

(2) Let an index $\bar\alpha \in \mathrm{A}$ be such that the set $\{e_\alpha : \alpha \geqslant \bar\alpha\}$ is bounded. Assign $e_0 := \inf_{\alpha\geqslant\bar\alpha}\sup_{\beta\geqslant\alpha}|e_\beta - e|$. If the net $(e_\alpha)_{\alpha\in\mathrm{A}}$ does not converge to e

then $e_0 > 0$ and, thus, there are $\pi \in \mathrm{Pr}(E)$, $d \in D$, and $n \in \mathbb{N}$ such that $0 < \pi d/n < e_0$. Therefore, for each index $\alpha \geqslant \bar{\alpha}$, we have

$$\sup_{\beta \geqslant \alpha} \langle d \rangle \big\langle |e_\beta - e| > d/n \big\rangle = \langle d \rangle \Big\langle \sup_{\beta \geqslant \alpha} |e_\beta - e| > d/n \Big\rangle \geqslant \pi,$$

which contradicts the convergence of $\langle d \rangle \big\langle |e_\alpha - e| > d/n \big\rangle$ to zero.

Corollary. *Suppose that a K-space E has an order unity 1, $(e_\alpha)_{\alpha \in \mathrm{A}}$ is an asymptotically bounded net in E, and $e \in E$. Then $o\text{-}\lim_{\alpha \in \mathrm{A}} e_\alpha = e$ if and only if the relation $o\text{-}\lim_{\alpha \in \mathrm{A}} \big\langle |e_\alpha - e| > 1/n \big\rangle = 0$ holds in the Boolean algebra $\mathrm{Pr}(E)$ for all $n \in \mathbb{N}$.*

The condition of asymptotic boundedness presented in the above assertions is essential. Indeed, let a net $(\pi_\alpha)_{\alpha \in \mathrm{A}}$ of order projections and an element $e \in E$ be such that $o\text{-}\lim_{\alpha \in \mathrm{A}} \pi_\alpha = 0$ and $\pi_\alpha e \neq 0$ for all $\alpha \in \mathrm{A}$. Endow the Cartesian product $\mathrm{A} \times \mathbb{N}$ with the lexicographic order:

$$(\alpha, m) < (\beta, n) \quad \Leftrightarrow \quad \alpha < \beta \ \text{ or } \ (\alpha = \beta \ \text{ and } \ m < n).$$

Then $o\text{-}\lim_{(\alpha,n) \in \mathrm{A} \times \mathbb{N}} \langle d \rangle \big\langle |n\pi_\alpha e| > d \big\rangle = 0$ for all positive $d \in E$; however, the net $(n\pi_\alpha e)_{(\alpha,n) \in \mathrm{A} \times \mathbb{N}}$ is not asymptotically bounded and, hence, has no order limit.

1.11. By simplifying the proof of Lemma 1.10, we can obtain the following assertion.

Lemma. *Let $(e_\xi)_{\xi \in \Xi}$ be a family of positive elements of a K-space E.*

(1) *The equality $\inf_{\xi \in \Xi} e_\xi = 0$ is valid in the K-space E if and only if the relation $\inf_{\xi \in \Xi} \langle d \rangle \big\langle e_\xi > d \big\rangle = 0$ holds in the Boolean algebra $\mathrm{Pr}(E)$ for all positive $d \in E$.*

(2) *Let D be a set of positive elements of E such that $e_\xi \in D^{\perp\perp}$ for all $\xi \in \Xi$. If $\inf_{\xi \in \Xi} \langle d \rangle \big\langle e_\xi > d/n \big\rangle = 0$ for all $d \in D$ and $n \in \mathbb{N}$, then $\inf_{\xi \in \Xi} e_\xi = 0$.*

Corollary. *Suppose that a K-space E has an order unity 1 and $(e_\xi)_{\xi \in \Xi}$ is a family of positive elements of E. Then $\inf_{\xi \in \Xi} e_\xi = 0$ if and only if the relation $\inf_{\xi \in \Xi} \big\langle e_\xi > 1/n \big\rangle = 0$ holds in the Boolean algebra $\mathrm{Pr}(E)$ for all $n \in \mathbb{N}$.*

1.12. Throughout the section, we assume that E is a K-space, B is a complete Boolean algebra, and $h : \mathrm{Pr}(E) \to B$ is a ring homomorphism. Say that a net $(\pi_\alpha)_{\alpha \in \mathrm{A}}$ in $\mathrm{Pr}(E)$ *h-converges to zero* and write $h\text{-}\lim_{\alpha \in \mathrm{A}} \pi_\alpha = 0$ if $o\text{-}\lim_{\alpha \in \mathrm{A}} \pi_\alpha = 0$ in the Boolean algebra $\mathrm{Pr}(E)$ and $o\text{-}\lim_{\alpha \in \mathrm{A}} h(\pi_\alpha) = 0$ in the Boolean algebra B. In case $h\text{-}\lim_{\alpha \in \mathrm{A}} \pi_\alpha^\perp = 0$, i.e. if $o\text{-}\lim_{\alpha \in \mathrm{A}} \pi_\alpha = 1$ and $o\text{-}\lim_{\alpha \in \mathrm{A}} h(\pi_\alpha) = h(1)$, we say that the net $(\pi_\alpha)_{\alpha \in \mathrm{A}}$ *h-converges to unity* and write $h\text{-}\lim_{\alpha \in \mathrm{A}} \pi_\alpha = 1$. We say that a net $(e_\alpha)_{\alpha \in \mathrm{A}}$ in E *h-converges to $e \in E$*

and write $h\text{-}\lim_{\alpha\in\mathrm{A}} e_\alpha = e$ if the net $(e_\alpha)_{\alpha\in\mathrm{A}}$ is asymptotically bounded and $h\text{-}\lim_{\alpha\in\mathrm{A}}\langle d\rangle\big\langle|e_\alpha - e| > d\big\rangle = 0$ for all positive $d \in E$. In this case, we call the element e the *h-limit* of the net $(e_\alpha)_{\alpha\in\mathrm{A}}$. We say that a net $(u_\alpha)_{\alpha\in\mathrm{A}}$ in $\mathcal{U}$ *h-converges to* $u \in \mathcal{U}$ and write $h\text{-}\lim_{\alpha\in\mathrm{A}} u_\alpha = u$ if $h\text{-}\lim_{\alpha\in\mathrm{A}} |u_\alpha - u| = 0$. In this case, we call the element u the *h-limit* of the net $(u_\alpha)_{\alpha\in\mathrm{A}}$. The totality of the h-limits of all h-convergent nets in a subset $\mathcal{V} \subset \mathcal{U}$ is called the *h-closure* of $\mathcal{V}$. We call a set *h-closed* if it coincides with the h-closure of itself. We say that a set is *h-dense in* $\mathcal{U}$ if its h-closure coincides with $\mathcal{U}$.

If a family $(\pi_\xi)_{\xi\in\Xi}$ in $\mathrm{Pr}(E)$ is such that $\inf_{\xi\in\Xi}\pi_\xi = 0$ in the Boolean algebra $\mathrm{Pr}(E)$ and $\inf_{\xi\in\Xi} h(\pi_\xi) = 0$ in the Boolean algebra B, then we write $h\text{-}\inf_{\xi\in\Xi}\pi_\xi = 0$. In case $h\text{-}\inf_{\xi\in\Xi}\pi_\xi^\perp = 0$, i.e. if $\sup_{\xi\in\Xi}\pi_\xi = 1$ and $\sup_{\xi\in\Xi} h(\pi_\xi) = h(1)$, we write $h\text{-}\sup_{\xi\in\Xi}\pi_\xi = 1$. For an arbitrary family $(e_\xi)_{\xi\in\Xi}$ of positive elements of a K-space E, the notation $h\text{-}\inf_{\xi\in\Xi} e_\xi = 0$ means that $h\text{-}\inf_{\xi\in\Xi}\langle d\rangle\big\langle e_\xi > d\big\rangle = 0$ for all positive $d \in E$.

Remark. The criterion of o-convergence formulated in Corollary 1.10 has no analog for h-convergence. The same is true of Corollary 1.11. Indeed, consider as E the K-space of all number sequences. Let the Boolean homomorphism $h : \mathrm{Pr}(E) \to \{0,1\}$ be the characteristic function of some nonprincipal ultrafilter in the Boolean algebra $\mathrm{Pr}(E)$. Denote by F the set of all positive sequences convergent to 1. Obviously, the sequence $e = (m)_{m\in\mathbb{N}}$ is an order unity in E and the relation $h\text{-}\inf_{f\in F}\big\langle f > e/n\big\rangle = 0$ holds for all $n \in \mathbb{N}$. Moreover, indexing each element of F by itself and endowing the index set with the reverse pointwise order, we obtain a set $(f)_{f\in F}$ that satisfies the relation $h\text{-}\lim_{f\in F}\big\langle f > e/n\big\rangle = 0$. Nevertheless, $h\big\langle f > 1/2\big\rangle = 1$ for all $f \in F$.

The following assertion is a straightforward consequence of Lemmas 1.10 and 1.11.

Proposition. (a) *For every net* $(e_\alpha)_{\alpha\in\mathrm{A}}$ *in* E *and arbitrary element* $e \in E$, *from* $h\text{-}\lim_{\alpha\in\mathrm{A}} e_\alpha = e$ *it follows that* $o\text{-}\lim_{\alpha\in\mathrm{A}} e_\alpha = e$. *If the homomorphism* h *is o-continuous, then the relations* $h\text{-}\lim_{\alpha\in\mathrm{A}} e_\alpha = e$ *and* $o\text{-}\lim_{\alpha\in\mathrm{A}} e_\alpha = e$ *are equivalent.*

(b) *For every net* $(e_\xi)_{\xi\in\Xi}$ *of positive elements of* E, *from* $h\text{-}\inf_{\xi\in\Xi} e_\xi = 0$ *it follows that* $\inf_{\xi\in\Xi} e_\xi = 0$. *If the homomorphism* h *is o-continuous, then the relations* $h\text{-}\inf_{\xi\in\Xi} e_\xi = 0$ *and* $\inf_{\xi\in\Xi} e_\xi = 0$ *are equivalent.*

1.13. Remark. In the sequel, while establishing equalities of the form $\lim_{\alpha\in\mathrm{A}} h\big(\langle d\rangle\langle e_\alpha > d\rangle\big) = 0$ or $\inf_{\xi\in\Xi} h\big(\langle d\rangle\langle e_\xi > d\rangle\big) = 0$, we often assume that $h\langle d\rangle = 1$. This assumption does not restrict generality, since, leaving aside the trivial case $h\langle d\rangle = 0$ and replacing B by the Boolean algebra $\big\{b \in B : b \leqslant h\langle d\rangle\big\}$, we arrive at the situation $h\langle d\rangle = 1$.

1.14. Let $\mathcal{V}$ be a subset of an LNS $\mathcal{U}$. We say that $\mathcal{V}$ *h-approximates an element* $u \in \mathcal{U}$ if $h\text{-}\inf_{v\in\mathcal{V}} |u - v| = 0$. We say that $\mathcal{V}$ *h-approximates a set* $\mathcal{W} \subset \mathcal{U}$ if $\mathcal{V}$ h-approximates every element of $\mathcal{W}$. A subset of an LNS $\mathcal{U}$ is called *h-approximating* if it h-approximates $\mathcal{U}$. From Proposition 1.12 it follows that every h-approximating set is approximating and, in case the homomorphism h is o-continuous, the notions of approximating and h-approximating set coincide.

Proposition. *Let X, Y, and Z be subsets of an LNS. If X h-approximates Y and Y h-approximates Z, then X h-approximates Z.*

Proof. Consider an arbitrary element $z \in Z$, fix a positive element d of the norming lattice, and assign $b := \inf_{x\in X} h\big(\langle d\rangle\langle |x - z| > d\rangle\big)$. Due to 1.2, it is sufficient to establish the equality $b = 0$. For simplicity, we assume that $h\langle d\rangle = 1$ (see 1.13). Suppose to the contrary that $b \neq 0$. Then, in view of $\inf_{y\in Y} h\langle |y - z| > d/2\rangle = 0$, there is an element $y \in Y$ such that $b_0 := b \wedge h\langle |y - z| > d/2\rangle < b$. Similarly, in view of the equality $\inf_{x\in X} h\langle |x - y| > d/2\rangle = 0$, there is an element $x \in X$ such that $(b\backslash b_0) \wedge h\langle |x - y| > d/2\rangle < (b\backslash b_0)$. It is easy to verify that x satisfies the inequality $b \wedge h\langle |x - z| > d\rangle < b$, which contradicts the definition of b.

1.15. Proposition. *Let $\mathcal{V}$ be a subset and let u be an element of an LNS. The set $\mathcal{V}$ h-approximates u if and only if u is the h-limit of some net in* $\operatorname{mix}_{\mathrm{fin}} \mathcal{V}$.

Proof. Necessity: If $\mathcal{V}$ h-approximates u then, in view of 1.1, there exists a net $(w_\alpha)_{\alpha\in\mathrm{A}}$ in $\operatorname{mix}_{\mathrm{fin}} \mathcal{V}$ such that the net $\big(|u - w_\alpha|\big)_{\alpha\in\mathrm{A}}$ decreases and $\{|u - w_\alpha| : \alpha \in \mathrm{A}\} = \{|u - w| : w \in \operatorname{mix}_{\mathrm{fin}} \mathcal{V}\}$. It remains to observe that $h\text{-}\lim_{\alpha\in\mathrm{A}} |u - w_\alpha| = 0$.

Sufficiency: If u is the h-limit of a net in $\operatorname{mix}_{\mathrm{fin}} \mathcal{V}$, then $\operatorname{mix}_{\mathrm{fin}} \mathcal{V}$ h-approximates u. It remains to observe that $\mathcal{V}$ h-approximates $\operatorname{mix}_{\mathrm{fin}} \mathcal{V}$ and to use Proposition 1.14.

Corollary. *If a subset $\mathcal{V}$ of an LNS $\mathcal{U}$ is finitely cyclic, then its h-closure consists of all elements $u \in \mathcal{U}$ h-approximated by $\mathcal{V}$.*

Corollary. *If a subset $\mathcal{V}$ of an LNS $\mathcal{U}$ is finitely cyclic, then its h-closure is h-closed and, hence, is the least h-closed subset of $\mathcal{U}$ that includes $\mathcal{V}$.*

Proof. The claim follows from the previous corollary and Proposition 1.14.

1.16. Proposition. *Let $\mathcal{V}$ be a subset of an LNS $\mathcal{U}$ and satisfy the relation $h\text{-}\inf_{v\in\mathcal{V}} |v| = 0$. Then the following assertions are equivalent:*

(1) *$\mathcal{V}$ is an h-approximating subset of $\mathcal{U}$;*
(2) *for every ideal $\mathcal{U}_0 \subset \mathcal{U}$, the set $d_{\mathrm{fin}}\mathcal{V} \cap \mathcal{U}_0$ is h-dense in $\mathcal{U}_0$;*
(3) *the set $d_{\mathrm{fin}}\mathcal{V}$ is h-dense in $\mathcal{U}$;*
(4) *$d_{\mathrm{fin}}\mathcal{V}$ is an h-approximating subset of $\mathcal{U}$.*

Proof. The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious. It remains to prove that (1) $\Rightarrow$ (2) and (4) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2): Suppose that a subset $\mathcal{V} \subset \mathcal{U}$ satisfies condition (1). Fix an arbitrary ideal $\mathcal{U}_0 \subset \mathcal{U}$ and denote the set $d_{\mathrm{fin}}\mathcal{V} \cap \mathcal{U}_0$ by $\mathcal{W}$. Consider an arbitrary element $u \in \mathcal{U}_0$. According to 1.15, there exists a net $(w_\alpha)_{\alpha\in\mathrm{A}}$ in $\mathrm{mix}_{\mathrm{fin}}\mathcal{V}$ that h-converges to u. For each $\alpha \in \mathrm{A}$, we assign $\pi_\alpha := \langle |u - w_\alpha| \leqslant |u| \rangle$. The relations

$$|\pi_\alpha w_\alpha| \leqslant |u| + |u - \pi_\alpha w_\alpha| = |u| + (\pi_\alpha\, |u - w_\alpha| + \pi_\alpha^\perp\, |u|)$$
$$\leqslant |u| + (\pi_\alpha\, |u| + \pi_\alpha^\perp\, |u|) = 2\, |u|$$

ensure that the net $(\pi_\alpha w_\alpha)_{\alpha\in\mathrm{A}}$ is constituted by elements of $\mathcal{W}$ and the relations

$$|u - \pi_\alpha w_\alpha| = \pi_\alpha\, |u - w_\alpha| + \pi_\alpha^\perp\, |u| \leqslant \pi_\alpha\, |u - w_\alpha| + \pi_\alpha^\perp\, |u - w_\alpha| = |u - w_\alpha|$$

together with $h\text{-}\lim_{\alpha\in\mathrm{A}} |u - w_\alpha| = 0$ give $h\text{-}\lim_{\alpha\in\mathrm{A}} |u - \pi_\alpha w_\alpha| = 0$.

(4) $\Rightarrow$ (1): From the relation $h\text{-}\inf_{v\in\mathcal{V}} |v| = 0$ it follows that $\mathcal{V}$ h-approximates $\mathcal{V} \cup \{0\}$. On the other hand, the set $\mathcal{V} \cup \{0\}$ obviously h-approximates $d_{\mathrm{fin}}\mathcal{V}$, the latter in turn h-approximating $\mathcal{U}$. It remains to apply Proposition 1.14.

1.17. The difference between the statements of Propositions 1.4 and 1.16 is essential: the condition $h\text{-}\inf_{v\in\mathcal{V}} |v| = 0$ in the latter proposition cannot be omitted. Indeed, consider the K-space E of all number sequences and assign $\mathcal{U} := \{u \in E : \inf(\mathrm{Lim}|u| \backslash \{0\}) > 0\}$, where $\mathrm{Lim}|u|$ is the set of all partial limits of the sequence $|u|$. We make $\mathcal{U}$ an LNS over E by defining $|u| := |u|$ for all $u \in \mathcal{U}$. As in Remark 1.12, let the Boolean homomorphism $h : \mathrm{Pr}(E) \to \{0, 1\}$ be the characteristic function of some nonprincipal ultrafilter in the Boolean algebra $\mathrm{Pr}(E)$. Consider as $\mathcal{V}$ the set $\{u \in E : \inf \mathrm{Lim}|u| > 0\}$ and assign $d := (1/n)_{n\in\mathbb{N}}$. It is clear that $d_{\mathrm{fin}}\mathcal{V} = \mathcal{U}$; however, $h\langle |v| > d \rangle = 1$ for all $v \in \mathcal{V}$.

Proposition. *Let $\mathcal{U}$ be an LNS over E. Suppose that, for every positive $e \in E$, there is an element $u \in \mathcal{U}$ satisfying the inequalities $e \leqslant |u| \leqslant 2e$ (this is true, for instance, in case $\mathcal{U}$ is o-complete, see* 0.4.3 *of* [1]*). Then the condition $h\text{-}\inf_{v\in\mathcal{V}} |v| = 0$ in the statement of Proposition* 1.16 *can be omitted.*

Proof. Consider an arbitrary subset $\mathcal{V} \subset \mathcal{U}$, denote $d_{\mathrm{fin}}\mathcal{V}$ by $\mathcal{W}$, suppose that $\mathcal{W}$ h-approximates $\mathcal{U}$, and establish the relation $h\text{-}\inf_{v\in\mathcal{V}} |v| = 0$. Due to 1.4 (we now use the implication (4) $\Rightarrow$ (1)), it is sufficient to fix an arbitrary positive element $d \in E$ and to show that $\inf_{v\in\mathcal{V}} h\big(\langle d \rangle \langle |v| > d \rangle\big) = 0$. For the sake of simplicity, we assume that $h\langle d \rangle = 1$ (see 1.13). Denote $\inf_{v\in\mathcal{V}} h\langle |v| > d \rangle$ by b and assume to the contrary that $b \neq 0$. Consider an arbitrary element $u \in \mathcal{U}$ satisfying the inequalities $d/4 \leqslant |u| \leqslant d/2$. In view of the equality $\inf_{w\in\mathcal{W}} h\langle |u-w| > d/5\rangle = 0$, there exists an element $w = \pi_1 v_1 + \dots + \pi_n v_n \in \mathcal{W}$ ($v_i \in \mathcal{V}$) such that $b \wedge h\langle |u-w| > d/5\rangle < b$. Using the equality

$$\langle |u-w| > d/5\rangle$$
$$= \pi_1\langle |u-v_1| > d/5\rangle \vee \dots \vee \pi_n\langle |u-v_n| > d/5\rangle \vee (\pi_1 \vee \dots \vee \pi_n)^{\perp}\langle d\rangle,$$

it is easy to verify that $b \wedge h\langle |u-v_i| > d/5\rangle < b$ for at least one index $i \in \{1,\dots,n\}$. Then, applying the relations

$$\langle |v_i| > d\rangle \leqslant \langle |u-v_i| + |u| > d\rangle \leqslant \langle |u-v_i| > d/2\rangle \leqslant \langle |u-v_i| > d/5\rangle,$$

we arrive at the equality $b \wedge h\langle |v_i| > d\rangle < b$, which contradicts the definition of b.

1.18. We call a disjoint family $(\pi_\xi)_{\xi\in\Xi}$ in the Boolean algebra $\mathrm{Pr}(E)$ an *h-partition of unity* if $h\text{-}\sup_{\xi\in\Xi}\pi_\xi = 1$. If $(u_\xi)_{\xi\in\Xi}$ is an arbitrary family in an LNS $\mathcal{U}$ over E and $(\pi_\xi)_{\xi\in\Xi}$ is an h-partition of unity in $\mathrm{Pr}(E)$, then we call the sum $o\text{-}\sum_{\xi\in\Xi}\pi_\xi u_\xi$ (if it exists) the *h-mixing* of the family $(u_\xi)_{\xi\in\Xi}$ with respect to $(\pi_\xi)_{\xi\in\Xi}$. For an arbitrary subset $\mathcal{V} \subset \mathcal{U}$, the totality of various h-mixings of all (all countable) families in $\mathcal{V}$ is called the *h-cyclic hull* (the *countably h-cyclic hull*) of the set $\mathcal{V}$ and denoted by $h\text{-}\mathrm{mix}\,\mathcal{V}$ (by $h\text{-}\mathrm{mix}_\sigma\,\mathcal{V}$, respectively). A set $\mathcal{V} \subset \mathcal{U}$ is called *h-cyclic* if it coincides with the h-cyclic hull of itself. It is easy to verify that the h-cyclic hull of $\mathcal{V}$ is the least h-cyclic set that includes $\mathcal{V}$.

1.19. Remark. We confine ourselves to the criteria for h-approximation given in Propositions 1.15 and 1.16. The author did not succeed in using the notion of h-cyclic hull to obtain efficient descriptions for h-approximation analogous to those presented in 1.6, 1.7, 1.8, and 1.9.

2. Order bounded operators

In this section, we depart from the convention made in 0.4.1 of [1] and consider not only decomposable LNSs over K-spaces but also arbitrary LNSs over arbitrary vector lattices. We do it not for the sake of generality but rather to avoid duplication of formulations both for LNSs and vector lattices. Indeed, every vector lattice together with the modulus function $|\cdot|$ is an LNS over itself. Thus, a definition or an assertion formulated for LNSs can be formally extended to the case of vector lattices. Observe that a vector lattice is o-complete as an LNS (i.e., is a BKS) if and only if it is a K-space.

2.1. Let $\mathcal{U}$ be an LNS over a vector lattice E. A net $(u_\alpha)_{\alpha\in\mathrm{A}}$ in $\mathcal{U}$ is called *asymptotically bounded* if the net $(|u_\alpha|)_{\alpha\in\mathrm{A}}$ possesses this property; i.e., if there exists an index $\bar{\alpha}\in\mathrm{A}$ such that the set $\{|u_\alpha| : \alpha\geqslant\bar{\alpha}\}$ is order bounded in E.

(a) We say that a subset $\mathcal{W}\subset\mathcal{U}$ is *r-annullable* (*o-annullable*, *boundable*) if, for every net $(w_\alpha)_{\alpha\in\mathrm{A}}$ in $\mathcal{W}$ and every vanishing number net $(\varepsilon_\alpha)_{\alpha\in\mathrm{A}}$, the net $(\varepsilon_\alpha w_\alpha)_{\alpha\in\mathrm{A}}$ is r-convergent to zero (o-convergent to zero, asymptotically bounded).

(b) We say that a subset $\mathcal{W}\subset\mathcal{U}$ is *countably r-annullable* (*countably o-annullable*, *countably boundable*) if, for every countable net $(w_\alpha)_{\alpha\in\mathrm{A}}$ in $\mathcal{W}$ and every vanishing number net $(\varepsilon_\alpha)_{\alpha\in\mathrm{A}}$, the net $(\varepsilon_\alpha w_\alpha)_{\alpha\in\mathrm{A}}$ is r-convergent to zero (o-convergent to zero, asymptotically bounded).

(c) We say that a subset $\mathcal{W}\subset\mathcal{U}$ is *sequentially r-annullable* (*sequentially o-annullable*, *sequentially boundable*) if, for every sequence $(w_n)_{n\in\mathbb{N}}$ in $\mathcal{W}$ and every vanishing number sequence $(\varepsilon_n)_{n\in\mathbb{N}}$, the sequence $(\varepsilon_n w_n)_{n\in\mathbb{N}}$ is r-convergent to zero (o-convergent to zero, bounded).

(d) We say that a subset $\mathcal{W}\subset\mathcal{U}$ is *semibounded* (*countably semibounded*, *sequentially semibounded*) if, for every net (countable net, sequence) $(w_\alpha)_{\alpha\in\mathrm{A}}$ in $\mathcal{W}$ and every vanishing number net $(\varepsilon_\alpha)_{\alpha\in\mathrm{A}}$, the relation $\inf_{\alpha\in\mathrm{A}}|\varepsilon_\alpha w_\alpha| = 0$ holds in the vector lattice E.

Theorem. *Let $\mathcal{U}$ be an LNS over a vector lattice E and let $\mathcal{W}$ be a subset of $\mathcal{U}$.*

(a) *The following assertions are equivalent:*
  (1) *the set $\mathcal{W}$ is r-annullable;*
  (2) *the set $\mathcal{W}$ is o-annullable;*
  (3) *the set $\mathcal{W}$ is boundable;*
  (4) *the set $\{|w| : w\in\mathcal{W}\}$ is order bounded in E.*

(b) *The following assertions are equivalent:*
  (1) *the set $\mathcal{W}$ is countably r-annullable;*
  (2) *the set $\mathcal{W}$ is countably o-annullable;*
  (3) *the set $\mathcal{W}$ is countably boundable;*
  (4) *for every countable subset $\mathcal{W}_0\subset\mathcal{W}$, the set $\{|w| : w\in\mathcal{W}_0\}$ is order bounded in E.*

(c) *The following assertions are equivalent:*
  (1) *the set $\mathcal{W}$ is sequentially r-annullable;*
  (2) *the set $\mathcal{W}$ is sequentially o-annullable;*
  (3) *the set $\mathcal{W}$ is sequentially boundable.*

(d) *The following assertions are equivalent:*
  (1) *the set $\mathcal{W}$ is semibounded;*
  (2) *the set $\mathcal{W}$ is countably semibounded;*
  (3) *the set $\mathcal{W}$ is sequentially semibounded;*
  (4) $\inf_{n\in\mathbb{N}}|w_n|/n = 0$ *for every sequence $(w_n)_{n\in\mathbb{N}}$ in $\mathcal{W}$.*

Proof. (a) The implications (4) $\Rightarrow$ (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious. We will show that (3) $\Rightarrow$ (4). Order the Cartesian product $\mathcal{W} \times \mathbb{N}$ by comparing the second component: $(w_1, n_1) < (w_2, n_2) \ \Leftrightarrow \ n_1 < n_2$. Applying assertion (3) to the nets $(w)_{(w,n)\in\mathcal{W}\times\mathbb{N}}$ and $(1/n)_{(w,n)\in\mathcal{W}\times\mathbb{N}}$, we obtain a pair $(\bar{w}, \bar{n}) \in \mathcal{W} \times \mathbb{N}$ and an element $e \in E$ such that $\big|w/n\big| \leqslant e$ for all $(w, n) \geqslant (\bar{w}, \bar{n})$. In particular, $\big|w/(\bar{n}+1)\big| \leqslant e$ for all $w \in \mathcal{W}$, which implies that the set $\{|w| : w \in \mathcal{W}\}$ is bounded from above by $(\bar{n}+1)e$.

(b) This is established in the same way as (a).

(c) The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious. We will show that (3) $\Rightarrow$ (1). Fix an arbitrary sequence $(w_n)_{n\in\mathbb{N}}$ in $\mathcal{W}$ and a vanishing number sequence $(\varepsilon_n)_{n\in\mathbb{N}}$. According to (3), the set $\big\{\big|\, |\varepsilon_n|^{1/2} w_n \,\big| : n \in \mathbb{N}\big\}$ has some upper bound $e \in E$. In order to prove assertion (1), it remains to observe that $|\varepsilon_n w_n| \leqslant |\varepsilon_n|^{1/2} e$ for all $n \in \mathbb{N}$.

(d) The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious. We show that (4) $\Rightarrow$ (1). Fix an arbitrary net $(w_\alpha)_{\alpha\in\mathrm{A}}$ in $\mathcal{W}$ and a vanishing number net $(\varepsilon_\alpha)_{\alpha\in\mathrm{A}}$. For each natural $n \in \mathbb{N}$, choose an index $\alpha(n) \in \mathrm{A}$ so that $\varepsilon_{\alpha(n)} \leqslant 1/n$. Then, using (4), we obtain the relations $\inf_{\alpha\in\mathrm{A}} |\varepsilon_\alpha w_\alpha| \leqslant \inf_{n\in\mathbb{N}} \big|\varepsilon_{\alpha(n)} w_{\alpha(n)}\big| = 0$.

A subset $\mathcal{W} \subset \mathcal{U}$ satisfying the conditions listed in items (a), (b), and (c) of the last theorem is called *bounded*, *countably bounded*, and *sequentially bounded*, respectively.

2.2. Obviously, every bounded set is countably bounded, every countably bounded set is sequentially bounded, and every sequentially bounded set is semibounded. We observe that the four types of boundedness differ pairwise even if $\mathcal{U} = E$. Indeed, in the K-space of all functions $e : \mathbb{R} \to \mathbb{R}$ with countable supports $e^{-1}\big[\mathbb{R}\backslash\{0\}\big]$, the set $\{e_t : t \in \mathbb{R}\}$ of the characteristic functions of all singletons $\{t\} \subset \mathbb{R}$ is countably bounded but not bounded. The set $\{e_n : n \in \mathbb{N}\}$ of the characteristic functions of all singletons $\{n\} \subset \mathbb{N}$ is a sequentially bounded but not a countably bounded subset of the K-space of vanishing number sequences.

We will give an example of a semibounded but not sequentially bounded subset in the K-space $M\big([0,1]\big)$ of cosets of real-valued Lebesgue-measurable functions on the interval $[0,1]$. For this purpose, we construct a family of intervals I^n_m $\big(n \in \mathbb{N},\, m \in \{1, 2, \ldots, 2^n\}\big)$ as follows:

$$I^1_1 := \big[\tfrac{0}{2}, \tfrac{1}{2}\big], \quad I^1_2 := \big[\tfrac{1}{2}, \tfrac{2}{2}\big];$$

$$I^2_1 := \big[\tfrac{0}{4}, \tfrac{1}{4}\big], \quad I^2_2 := \big[\tfrac{1}{4}, \tfrac{2}{4}\big], \quad I^2_3 := \big[\tfrac{2}{4}, \tfrac{3}{4}\big], \quad I^2_4 := \big[\tfrac{3}{4}, \tfrac{4}{4}\big];$$

$$\cdots$$

$$I^n_1 := \big[\tfrac{0}{2^n}, \tfrac{1}{2^n}\big], \quad I^n_2 := \big[\tfrac{1}{2^n}, \tfrac{2}{2^n}\big], \quad \ldots, \quad I^n_{2^n} := \big[\tfrac{2^n-1}{2^n}, \tfrac{2^n}{2^n}\big];$$

$$\cdots$$

and denote by $\mathbf{f}_m^n$ the coset in $M\big([0,1]\big)$ containing the characteristic function of the interval I_m^n. Then the set $\big\{2^n\mathbf{f}_m^n : n \in \mathbb{N},\ m \in \{1,2,\dots,2^n\}\big\}$ is the desired one.

2.3. Theorem. *Let $\mathcal{U}$ and $\mathcal{V}$ be LNSs over respective vector lattices E and F and let T be a linear operator from $\mathcal{U}$ into $\mathcal{V}$.*

(a) *The following assertions are equivalent:*

- (1) *the operator T is r-continuous;*
- (2) *the operator T is r-o-continuous;*
- (3) *if $r\text{-}\lim_{\alpha\in\mathrm{A}} u_\alpha = 0$ in $\mathcal{U}$ then the net $(Tu_\alpha)_{\alpha\in\mathrm{A}}$ is asymptotically bounded;*
- (4) *the operator T takes bounded subsets of $\mathcal{U}$ into bounded subsets of $\mathcal{V}$;*
- (5) *for every positive element $e \in E$, the set $\{|Tu| : |u| \leqslant e\}$ is bounded in F.*

(b) *The following assertions are equivalent:*

- (1) *the operator T is countably r-continuous;*
- (2) *the operator T is countably r-o-continuous;*
- (3) *if $r\text{-}\lim_{\alpha\in\mathrm{A}} u_\alpha = 0$ in $\mathcal{U}$ and the index set A is countable, then the net $(Tu_\alpha)_{\alpha\in\mathrm{A}}$ is asymptotically bounded;*
- (4) *the operator T takes countably bounded subsets of $\mathcal{U}$ into countably bounded subsets of $\mathcal{V}$;*
- (5) *the operator T takes bounded subsets of $\mathcal{U}$ into countably bounded subsets of $\mathcal{V}$;*
- (6) *the operator T takes countable bounded subsets of $\mathcal{U}$ into bounded subsets of $\mathcal{V}$.*

(c) *The following assertions are equivalent:*

- (1) *the operator T is sequentially r-continuous;*
- (2) *the operator T is sequentially r-o-continuous;*
- (3) *if $r\text{-}\lim_{n\in\mathbb{N}} u_n = 0$ in $\mathcal{U}$ then the sequence $(Tu_n)_{n\in\mathbb{N}}$ is bounded;*
- (4) *the operator T takes sequentially bounded subsets of $\mathcal{U}$ into sequentially bounded subsets of $\mathcal{V}$;*
- (5) *the operator T takes bounded subsets of $\mathcal{U}$ into sequentially bounded subsets of $\mathcal{V}$.*

(d) *The following assertions are equivalent:*

- (1) *if $r\text{-}\lim_{\alpha\in\mathrm{A}} u_\alpha = 0$ in $\mathcal{U}$ then $\inf_{\alpha\in\mathrm{A}} |Tu_\alpha| = 0$;*
- (2) *if $r\text{-}\lim_{\alpha\in\mathrm{A}} u_\alpha = 0$ in $\mathcal{U}$ and the index set A is countable, then $\inf_{\alpha\in\mathrm{A}} |Tu_\alpha| = 0$;*
- (3) *if $r\text{-}\lim_{n\in\mathbb{N}} u_n$ in $\mathcal{U}$ then $\inf_{n\in\mathbb{N}} |Tu_n| = 0$;*
- (4) *the operator T takes semibounded subsets of $\mathcal{U}$ into semibounded subsets of $\mathcal{V}$;*
- (5) *the operator T takes bounded subsets of $\mathcal{U}$ into semibounded subsets of $\mathcal{V}$.*

Proof. (a) The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are obvious. Using boundability as a criterion for boundedness (see 2.1(a)), it is easy to deduce (4) from (3). It remains to show that (5) $\Rightarrow$ (1). Suppose that the operator T satisfies condition (5) and, for every positive element $e \in E$, denote by f_e some upper bound of the set $\{|Tu| : |u| \leqslant e\}$ in the lattice F. Let $(u_\alpha)_{\alpha \in \mathrm{A}}$ be an arbitrary net in $\mathcal{U}$ r-convergent to zero with regulator $e \in E$. Fix an arbitrary number $\varepsilon > 0$ and choose an index $\bar{\alpha} \in A$ so that $|u_\alpha| \leqslant \varepsilon e$ for all $\alpha \geqslant \bar{\alpha}$. Then, for all $\alpha \geqslant \bar{\alpha}$, we have: $|Tu_\alpha| = \varepsilon \left|Tu_\alpha/\varepsilon\right| \leqslant \varepsilon f_e$.

(b) The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) $\Rightarrow$ (6) are obvious. Using countable boundability as a criterion for countable boundedness (see 2.1(b)), it is easy to deduce (4) from (3). It remains to show that (6) $\Rightarrow$ (1). Suppose that the operator T satisfies condition (6). Let $(u_\alpha)_{\alpha \in \mathrm{A}}$ be an arbitrary countable net in $\mathcal{U}$ r-convergent to zero with regulator $e \in E$. For every natural n, denote by α_n an element of A such that $|u_\alpha| \leqslant e/n$ for all $\alpha \geqslant \alpha_n$. The set $\mathcal{U}_0 := \{nu_\alpha : n \in \mathbb{N},\ \alpha \in \mathrm{A},\ \alpha \geqslant \alpha_n\}$ is countable and bounded; hence, there is an element $f \in F$ such that $|Tu| \leqslant f$ for all $u \in \mathcal{U}_0$. Then $|Tu_\alpha| = |Tnu_\alpha|/n \leqslant f/n$ for all $\alpha \geqslant \alpha_n$.

(c) The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are obvious. Using sequential boundability as a criterion for sequential boundedness (see 2.1(c)), it is easy to deduce (4) from (3). It remains to show that (5) $\Rightarrow$ (1). Let $(u_n)_{n \in \mathbb{N}}$ be an arbitrary sequence in $\mathcal{U}$ r-convergent to zero with regulator $e \in E$. Then there exists a vanishing number sequence $\varepsilon_n > 0$ such that $|u_n| \leqslant \varepsilon_n e$ for all $n \in \mathbb{N}$. Boundedness of the set $\{u_n/\varepsilon_n : n \in \mathbb{N}\}$ and condition (5) allow us to conclude that the set $\{Tu_n/\varepsilon_n : n \in \mathbb{N}\}$ is sequentially r-annullable and, hence, the sequence $(Tu_n)_{n \in \mathbb{N}}$ r-converges to zero.

(d) The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious. We will show that (4) $\Rightarrow$ (1). Let $(u_\alpha)_{\alpha \in \mathrm{A}}$ be an arbitrary net in $\mathcal{U}$ r-convergent to zero with regulator $e \in E$. Then, for every natural $n \in \mathbb{N}$, there exists an index $\alpha(n) \in \mathrm{A}$ such that $\left|u_{\alpha(n)}\right| \leqslant e/n$. Boundedness of the set $\{nu_{\alpha(n)} : n \in \mathbb{N}\}$ and condition (4) allow us to conclude that the set $\{Tnu_{\alpha(n)} : n \in \mathbb{N}\}$ is semibounded, hence (see 2.1(d)),

$$\inf_{\alpha \in \mathrm{A}} |Tu_\alpha| \leqslant \inf_{n \in \mathbb{N}} \left|Tu_{\alpha(n)}\right| = \inf_{n \in \mathbb{N}} \left|Tnu_{\alpha(n)}\right|/n = 0.$$

An operator $T : \mathcal{U} \to \mathcal{V}$ satisfying the conditions listed in items (a), (b), (c), and (d) of the last theorem is called *bounded*, *countably bounded*, *sequentially bounded*, and *semibounded*, respectively. Obviously, every bounded operator is countably bounded, every countably bounded operator is sequentially bounded, and every sequentially bounded operator is semibounded. We will devote a large part of this section to presenting examples which will show that the four types of boundedness of operators differ pairwise. Operators arising in each of the examples below act from Banach spaces into K-spaces.

2.4. Example. There exist a Banach space X, an extended K-space F, and an operator $T : X \to F$ that is countably bounded but not bounded.

We call a sequence $(\alpha_1, \alpha_2, \dots)$ of countable ordinals α_n *finitary* if there is an index $n \in \mathbb{N}$ such that $\alpha_n \neq 0$ and $\alpha_m = 0$ for all $m > n$. In this case, the number n is called the *dimension* of the sequence α and denoted by $\dim(\alpha)$. Denote the set of all finitary sequences of countable ordinals by A and endow it with the lexicographic order by defining $\alpha < \beta$ if and only if, for some $n \in \mathbb{N}$, we have $\alpha_1 = \beta_1, \ \dots, \ \alpha_{n-1} = \beta_{n-1}$, and $\alpha_n < \beta_n$. For all $\alpha, \beta \in A$, we denote by $]\alpha, \beta[$ the open interval $\{\gamma \in A : \alpha < \gamma < \beta\}$.

For every sequence $\alpha \in A$, assign

$$\alpha + 1 := \big(\alpha_1, \dots, \alpha_{\dim(\alpha)-1}, \alpha_{\dim(\alpha)} + 1, 0, 0, \dots\big).$$

Consider $\alpha, \beta \in A$. We say that α is a *fragment* of β and write $\alpha \sqsubset \beta$ if $\alpha = (\beta_1, \beta_2, \dots, \beta_{\dim(\alpha)}, 0, 0, \dots)$.

Lemma 1. *For all $\alpha, \beta \in A$, the following relations are equivalent*:

(1) $]\alpha, \alpha+1[\, \cap \,]\beta, \beta+1[\, \neq \varnothing$;

(2) $]\alpha, \alpha+1[\, \subset \,]\beta, \beta+1[$ *or* $]\alpha, \alpha+1[\, \supset \,]\beta, \beta+1[$;

(3) $\alpha \sqsubset \beta$ *or* $\beta \sqsubset \alpha$.

Proof. If $\dim(\alpha) = \dim(\beta)$ then the claim is obvious. For definiteness, assume that $\dim(\alpha) < \dim(\beta)$. Therefore, if $\alpha < (\beta_1, \dots, \beta_{\dim(\alpha)}, 0, 0, \dots)$ then $\alpha + 1 < \beta$, and if $\alpha > (\beta_1, \dots, \beta_{\dim(\alpha)}, 0, 0, \dots)$ then $\alpha > \beta + 1$. In both cases, the intervals $]\alpha, \alpha+1[$ and $]\beta, \beta+1[$ are disjoint. The lemma is proven.

Endow the set A with the order topology, for which $\big\{]\alpha, \beta[\, : \alpha, \beta \in A\big\}$ is a base of open sets. Denote by Q the Stone compactum of the Boolean algebra $\mathrm{Rop}(A)$ of regular open subsets of A (see 0.2.3 of [1]). Let $U \mapsto \hat{U}$ be an isomorphism of $\mathrm{Rop}(A)$ onto the Boolean algebra $\mathrm{Clop}(Q)$ of clopen subsets of Q (see 0.2.2 of [1]). Observe that $\mathrm{Rop}(A)$ contains all intervals $]\alpha, \beta[$ $(\alpha, \beta \in A)$. For every sequence $\alpha \in A$, assign $Q_\alpha := \,]\alpha, \alpha+1[^\wedge \in \mathrm{Clop}(Q)$ and denote the characteristic function of the subset $Q_\alpha \subset Q$ by χ_α. Thus, $\chi_\alpha \in C(Q)$.

Lemma 2. *For every nonempty open set $U \subset A$ and every $n \in \mathbb{N}$, there is a sequence $\alpha \in A$ such that $\dim(\alpha) > n$ and $]\alpha, \alpha+1[\, \subset U$.*

Proof. By the definition of order topology, the set U includes some interval $]\alpha, \beta[$, $\alpha < \beta$. Assign $m := \min\{i \in \mathbb{N} : \alpha_i < \beta_i\}$ and $k := \max\{m, n\}$. The sequence $(\alpha_1, \dots, \alpha_k, \alpha_{k+1} + 1, 0, 0, \dots)$ is the desired one.

Lemma 3. *For every $n \in \mathbb{N}$, the relation* $\sup\big\{]\alpha, \alpha+1[\, : \, \alpha \in A$, $\dim(\alpha) \geqslant n\big\} = 1$ *holds in the Boolean algebra* $\mathrm{Rop}(A)$.

Proof. The claim follows immediately from Lemma 2.

Lemma 4. *In the K-space $C_\infty(Q)$ the sum $f_S := o\text{-}\sum_{\alpha\in S}\dim(\alpha)\chi_\alpha$ exists for every countable subset $S\subset A$.*

Proof. The formula $f(q) := \sum_{\alpha\in S}\dim(\alpha)\chi_\alpha(q)$ defines a function $f : Q\to\bar{\mathbb{R}}$. According to 0.3.3(2) of [1], in order to prove the lemma, it is sufficient to establish that $f^{-1}(\infty)$ is a meager subset of Q. Taking account of Lemma 1, we conclude the following: if a point $q\in Q$ satisfies $f(q)=\infty$, then there is a chain

$$\alpha^{(1)}\sqsubset\alpha^{(2)}\sqsubset\cdots\sqsubset\alpha^{(n)}\sqsubset\cdots$$

of pairwise different elements in S such that $q\in\bigcap_{n\in\mathbb{N}}Q_{\alpha^{(n)}}$. Thus,

$$f^{-1}(\infty)\subset\bigcap_{n\in\mathbb{N}}\bigcup_{\alpha\in S(n)}Q_\alpha,$$

where $S(n)=\{\alpha\in S:\dim(\alpha)\geqslant n\}$. Consequently, the lemma will be proven if we establish that

$$\operatorname{int}\bigcap_{n\in\mathbb{N}}\operatorname{cl}\bigcup_{\alpha\in S(n)}Q_\alpha=\varnothing,$$

i.e., $\inf_{n\in\mathbb{N}}\sup_{\alpha\in S(n)}Q_\alpha=0$ in the Boolean algebra $\operatorname{Clop}(Q)$ or, equivalently, $\inf_{n\in\mathbb{N}}\sup_{\alpha\in S(n)}\,]\alpha,\alpha+1[\,=0$ in the Boolean algebra $\operatorname{Rop}(A)$.

Assume that the last equality does not hold. Then, according to Lemma 2, there is a sequence $\beta\in A$ such that the interval $]\beta,\beta+1[$ is included in $\sup_{\alpha\in S(n)}\,]\alpha,\alpha+1[$ for every $n\in\mathbb{N}$ and, in particular, for $n=\dim(\beta)+1$. Denote the set

$$\big\{\gamma\in S\big(\dim(\beta)+1\big):\,]\beta,\beta+1[\,\cap\,]\gamma,\gamma+1[\,\neq\varnothing\big\}$$

by Γ. Obviously, $]\beta,\beta+1[\;\subset\sup_{\gamma\in\Gamma}\,]\gamma,\gamma+1[$ and, consequently, for every sequence $\alpha<\beta+1$, there exists an element $\gamma\in\Gamma$ such that $\gamma+1\geqslant\alpha$. However, Lemma 1 implies that β is a fragment of every element of Γ; therefore, for all $\gamma\in\Gamma$, we have

$$\begin{aligned}\gamma+1&=\big(\beta_1,\dots,\beta_{\dim(\beta)},\gamma_{\dim(\beta)+1},\dots,\gamma_{\dim(\gamma)}+1,0,0,\dots\big)\\&\leqslant\big(\beta_1,\dots,\beta_{\dim(\beta)},\gamma_{\dim(\beta)+1}+1,0,0,\dots\big)\\&\leqslant\Big(\beta_1,\dots,\beta_{\dim(\beta)},\sup_{\gamma'\in\Gamma}\big(\gamma'_{\dim(\beta)+1}+1\big),0,0,\dots\Big)\\&<\beta+1,\end{aligned}$$

which easily yields a contradiction.

Let $\mathcal{X}$ be the vector space of all bounded functions $x:A\to\mathbb{R}$ with countable support $\{\alpha\in A:x(\alpha)\neq0\}$. Obviously, $\mathcal{X}$ is a Banach space with respect to the uniform norm $\|\cdot\|_\infty$ and a K-space with respect to the pointwise order.

Lemma 5. *For every function $x \in \mathcal{X}$, the sum $o\text{-}\sum_{\alpha\in A}\dim(\alpha)x(\alpha)\chi_\alpha$ exists in the K-space $C_\infty(Q)$.*

Proof. Denote by S the support of the function $x \in \mathcal{X}$. Applying Lemma 4, we have

$$\sum_{\alpha\in A}\dim(\alpha)x^+(\alpha)\chi_\alpha(q) \leqslant \|x\|_\infty f_S(q)$$

at every point $q \in Q$, which implies that the sum $o\text{-}\sum_{\alpha\in A}\dim(\alpha)x^+(\alpha)\chi_\alpha$ exists in $C_\infty(Q)$. Similar arguments for the function x^- complete the proof of the lemma.

We now begin defining the spaces X and F and the operator T. The Banach space X is defined as the closure of the subspace of $\mathcal{X}$ constituted by all functions with finite supports. As the K-space F, we take $C_\infty(Q)$. Finally, the operator $T : X \to F$ is defined by the formula

$$Tx = o\text{-}\sum_{\alpha\in A}\dim(\alpha)x(\alpha)\chi_\alpha,$$

in which the existence of the o-sum is guaranteed by Lemma 5.

The operator T is countably bounded. Indeed, if the norms of all the elements of a countable subset $X_0 \subset X$ are bounded from above by a number λ and S is the union of the supports of all the functions in X_0, then, in view of Lemma 4, we have $|Tx| \leqslant \lambda f_S$ for all $x \in X_0$. Thus, the operator T satisfies condition 2.3(b)(6), i.e., it is countably bounded.

We will show that the operator T is not bounded. For every sequence $\alpha \in A$, denote the characteristic function of the singleton $\{\alpha\} \subset A$ by x_α. If the set $\{Tx : x \in X,\ \|x\|_\infty \leqslant 1\}$ had an upper bound in the K-space F, then, according to Lemma 3, for every $n \in \mathbb{N}$ we should have

$$\begin{aligned}\sup\{Tx : x \in X,\ \|x\|_\infty \leqslant 1\} &\geqslant \sup\{Tx_\alpha : \alpha \in A,\ \dim(\alpha) \geqslant n\}\\ &\geqslant \sup\{n\chi_\alpha : \alpha \in A,\ \dim(\alpha) \geqslant n\}\\ &= n\mathbb{1}_F,\end{aligned}$$

where $\mathbb{1}_F$ is the identical unity. Thus, the operator T does not satisfy condition 2.3(a)(5), i.e., it is not bounded.

2.5. Example. There exist a Banach space X, a K-space F, and an operator $T : X \to F$ that is sequentially bounded but not countably bounded.

Proof. Endowing the vector space c_0 of vanishing number sequences with the uniform norm $\|\cdot\|$, we obtain a Banach space to be denoted by X. On the other hand, endowing the space c_0 with pointwise order, we obtain a K-space which we denote by F. Consider the identity mapping $T : c_0 \to c_0$ as an operator from X into F. For every natural $n \in \mathbb{N}$, denote by e_n the characteristic function of the subset $\{n\} \subset \mathbb{N}$. The operator T is not countably bounded, since it takes a bounded countable subset $\{e_n : n \in \mathbb{N}\}$ of the Banach space X into an unbounded subset of the K-space F (see 2.3(b)(6)).

We will show that the operator T is sequentially bounded by using the criterion 2.3(c)(3). Consider an arbitrary sequence $(x_n)_{n\in\mathbb{N}}$ in X convergent in norm to zero and define a sequence x by the formula $x(m) = \sup_{n\in\mathbb{N}}|x_n(m)|$ ($m \in \mathbb{N}$). It is sufficient to show that $x(m) \to 0$ as $m \to \infty$. Fix an arbitrary number $\varepsilon > 0$. Let a number $\bar{n} \in \mathbb{N}$ be such that $\|x_n\| \leqslant \varepsilon$ for all $n > \bar{n}$ and let $\bar{m} \in \mathbb{N}$ be such that $\big(|x_1| \vee |x_2| \vee \dots \vee |x_{\bar{n}}|\big)(m) \leqslant \varepsilon$ for all $m > \bar{m}$. Then $x(m) \leqslant \varepsilon$ for all $m > \bar{m}$.

2.6. Example. There exist a Banach space X, an extended K-space F, and an operator $T : X \to F$ that is semibounded but not sequentially bounded.

Proof. Denote by Δ the set of all finite sequences of unities and zeroes: $\Delta := \big\{\big(\delta(1), \dots, \delta(n)\big) : n \in \mathbb{N},\ \delta(i) \in \{0, 1\}\big\}$. Enumerate the elements of the set Δ, listing first all the sequences of length 1, then of length 2, etc.:

$$\delta_1 := (0), \quad \delta_2 := (1);$$
$$\delta_3 := (0,0), \quad \delta_4 := (0,1), \quad \delta_5 := (1,0), \ \delta_6 := (1,1);$$
$$\dots$$
$$\delta_{2^n-1} := (0,0,\dots,0), \quad \delta_{2^n} := (0,0,\dots,1), \quad \dots, \quad \delta_{2^{n+1}-2} := (1,1,\dots,1);$$
$$\dots$$

For every element $\delta = \big(\delta(1), \dots, \delta(n)\big) \in \Delta$, denote by I_δ the following interval of the real line:

$$\Big[\frac{\delta(1)}{2^1} + \frac{\delta(2)}{2^2} + \dots + \frac{\delta(n)}{2^n},\ \frac{\delta(1)}{2^1} + \frac{\delta(2)}{2^2} + \dots + \frac{\delta(n)}{2^n} + \frac{1}{2^n}\Big].$$

By way of explication, we observe that

$$I_{\delta_1} = I^1_1, \quad I_{\delta_2} = I^1_2;$$
$$I_{\delta_3} = I^2_1, \quad I_{\delta_4} = I^2_2, \quad I_{\delta_5} = I^2_3, \quad I_{\delta_6} = I^2_4;$$
$$\dots,$$

where I^n_m are the intervals considered in 2.2.

Denote by X the Banach space $\ell^1(\Delta)$ of summable functions $x : \Delta \to \mathbb{R}$ with the norm $\|x\| = \sum_{\delta\in\Delta}|x(\delta)|$ and define F to be the K-space $M\big([0,1]\big)$ of cosets of real-valued Lebesgue-measurable functions on the interval $[0,1]$. For every element $\delta \in \Delta$, denote by f_δ the characteristic function of the interval I_δ and by $\mathbf{f}_\delta$ the coset in $M\big([0,1]\big)$ that contains the function f_δ. Define the operator $T : X \to F$ by the formula

$$Tx = o\text{-}\sum_{\delta\in\Delta} 2^{\dim\delta} x(\delta)\,\mathbf{f}_\delta\,,$$

where $\dim\delta$ is the length of a sequence δ. The last o-sum exists, since the corresponding pointwise sum $\sum_{\delta\in\Delta} 2^{\dim\delta} x(\delta) f_\delta$ is, obviously, measurable and the integral of its modulus is equal to

$$\sum_{\delta\in\Delta} 2^{\dim\delta} |x(\delta)|\,\mu(I_\delta) = \sum_{\delta\in\Delta} |x(\delta)| = \|x\|$$

and, hence, is finite. Thus, $\int|Tx| = \|x\|$, which immediately implies semi-boundedness of the operator T.

We will show that the operator T constructed is not sequentially bounded. For every element $\delta \in \Delta$, denote by e_δ the characteristic function of the singleton $\{\delta\} \subset \Delta$. Then the sequence $\big(2^{-\dim\delta_n} e_{\delta_n}\big)_{n\in\mathbb{N}}$ converges in norm to zero; however, its image $\big(\mathbf{f}_{\delta_n}\big)_{n\in\mathbb{N}}$ with respect to the operator T does not r-converge to zero.

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