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BANACH BUNDLES IN THE THEORY OF LATTICE-NORMED SPACES. IV

DISJOINTNESS PRESERVING OPERATORS

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Abstract

In the present article, we study disjointness preserving operators that act in K-spaces and lattice-normed spaces. In particular, we find their analytic representations and decompositions into simpler components. We study orthomorphisms, shift operators, weighted shift operators, and arbitrary disjointness preserving operators.

Key words and phrases: vector lattice, lattice-normed space, continuous Banach bundle, continuous section, orthomorphism, shift operator, weighted shift operator, disjointness preserving operator, analytic representation.

This is the fourth part of the article the author intended as a paper on Banach bundles in the theory of lattice-normed spaces (LNSs). The first three parts were published in [12–14] and we sometimes appeal to the material of these papers without explicit reference.

Disjointness preserving operators have its own theory rich in results and treating such questions as boundedness, continuity, spectral and geometric properties, multiplicativity, compactness, etc. The list of publications devoted to disjointness preserving operators is so extensive that it could serve as a reason for a separate review. Leaving aside many rather interesting directions, we will only concentrate our attention on analytic representation and decomposition of disjointness preserving operators. B. Z. Vulikh [29–31] was one of the first who considered these questions. Later, disjointness preserving operators were studied by Y. A. Abramovich, E. L. Arenson, D. R. Hart, A. K. Kitover, A.V. Koldunov, P. T. N. MacPolin, A. I. Veksler, A.W. Wickstead, A. C. Zaanen, and many others (see, for instance, [1, 2, 5, 6, 16, 22, 33, 34]). We also observe that the question of analytic representation of disjointness

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preserving operators includes such an extensive direction as descriptions of isometries of vector-valued L^p-spaces (so-called Banach–Stone theorems).

In the present article, we study disjointness preserving operators that act in K-spaces and lattice-normed spaces. In particular, we find their analytic representations and decompositions into simpler components. We begin with studying general properties of disjointness preserving operators; then we consider orthomorphisms, shift operators, weighted shift operators, and, finally, return to arbitrary operators and apply the accumulated experience.

0. Prerequisites

In this section, we present some preliminary information. Note that we give only those facts and definitions that were not presented in the previous three parts of the paper.

0.1. Proposition. *Let A and B be Boolean algebras.*

(a) *The following properties of a Boolean homomorphism $h\colon A \to B$ are equivalent:*
   (1) *h is o-continuous;*
   (2) *if a subset $C \subset A$ has a supremum then $h(\sup C) = \sup h[C]$;*
   (3) *if a subset $C \subset A$ has an infimum then $h(\inf C) = \inf h[C]$;*
   (4) *if $(a_\lambda)_{\lambda\in\Lambda}$ is a net in A and $a_\lambda \uparrow 1$ then $\sup_{\lambda\in\Lambda} h(a_\lambda) = 1$;*
   (5) *if $(a_\lambda)_{\lambda\in\Lambda}$ is a net in A and $a_\lambda \downarrow 0$ then $\inf_{\lambda\in\Lambda} h(a_\lambda) = 0$.*

(b) *The following properties of a Boolean homomorphism $h\colon A \to B$ are equivalent:*
   (1) *h is countably o-continuous;*
   (2) *if a countable subset $C \subset A$ has a supremum then $h(\sup C) = \sup h[C]$;*
   (3) *if a countable subset $C \subset A$ has an infimum then $h(\inf C) = \inf h[C]$;*
   (4) *if $(a_n)_{n\in\mathbb{N}}$ is a sequence in A and $a_n \uparrow 1$ then $\sup_{n\in\mathbb{N}} h(a_n) = 1$;*
   (5) *if $(a_n)_{n\in\mathbb{N}}$ is a sequence in A and $a_n \downarrow 0$ then $\inf_{n\in\mathbb{N}} h(a_n) = 0$.*

If the Boolean algebra A is complete (σ-complete) then each of the five conditions (a) (*respectively,* (b)) *is equivalent to the following one:* $\sup h[D] = 1$ *for every (countable) partition D of unity in A.*

In view of the equivalence of conditions (a)(1)–(a)(3), o-continuous homomorphisms are often called *full* or *complete*. Observe that the implication (b)(5)⇒(b)(1) implies equivalence of countable and sequential o-continuity of a Boolean homomorphism.

0.2. Let A and B be Boolean algebras. We say that a ring homomorphism $h\colon A \to B$ *dominates* a function $h_0\colon A \to B$ (and write $h_0 \leqslant h$), if $h_0(a) \leqslant h(a)$ for all $a \in A$.

Proposition. *Let A and B be Boolean algebras. A ring homomorphism $h\colon A \to B$ dominates a ring homomorphism $h_0\colon A \to B$ if and only if $h_0(a) = h_0(1) \wedge h(a)$ for all $a \in A$.*

$\lhd$ The equality $h_0(a) = h_0(1) \wedge h(a)$ readily ensues from the following relations:

$$h_0(a) \leqslant h_0(1) \wedge h(a),$$
$$h_0(a^{\perp}) \leqslant h_0(1) \wedge h(a^{\perp}),$$
$$h_0(a) \vee h_0(a^{\perp}) = h_0(1). \quad \rhd$$

0.3. Let f be an arbitrary positive element of a vector lattice E. We say that an $s \in E$ is an *f-step element*, if $s = \sum_{i=1}^{n} \lambda_i \pi_i f$ for some $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ and $\pi_1, \dots, \pi_n \in \mathrm{Pr}(E)$.

Proposition. *Suppose that a vector lattice E possesses the principal projection property (for instance, E is a K_σ-space). Let E_f be the ideal of E generated by a positive element $f \in E$. Then, for every element $e \in E_f$ and every number $\varepsilon > 0$, there is an f-step element $s \in E_f$ such that $|s| \leqslant |e|$ and $|e - s| \leqslant \varepsilon f$. In particular, the set of all f-step elements is r-dense in E_f.*

$\lhd$ Assume all the hypotheses of the proposition to be satisfied and consider an arbitrary element $e \in E_f$ and a number $\varepsilon > 0$. Let numbers $m, n \in \mathbb{N}$ be such that $|e| \leqslant mf$ and $1/n \leqslant \varepsilon$. Then the sum

$$\sum_{i=-mn}^{-1} \frac{i}{n} \left\langle \frac{i-1}{n} f < e \leqslant \frac{i}{n} f \right\rangle f + \sum_{i=1}^{mn} \frac{i}{n} \left\langle \frac{i}{n} f \leqslant e < \frac{i+1}{n} f \right\rangle f$$

is a desired f-step element. $\rhd$

0.4. Let $\mathcal{U}$ be an arbitrary, not necessarily d-decomposable LNS over an arbitrary vector lattice E. Suppose that a d-decomposable LNS $\overline{\mathcal{U}}$ over E contains $\mathcal{U}$ as a subspace with the induced norm. We say that the LNS $\overline{\mathcal{U}}$ is a *d-decomposable hull* of $\mathcal{U}$, if $d_{\mathrm{fin}}\mathcal{U} = \overline{\mathcal{U}}$, i.e., $\overline{\mathcal{U}}$ is a minimal d-decomposable LNS that contains $\mathcal{U}$ as a subspace with the induced norm.

Proposition. *Suppose that a vector lattice E possesses the principal projection property. Then every (not necessarily d-decomposable)* LNS *over E has a d-decomposable hull which is unique to within an isometry.*

$\triangleleft$ In order to construct a d-decomposable hull of an LNS $\mathcal{U}$ over E, we employ the schema of formal mixing which is traditionally used in similar situations (cf. [20, 27, 28]). Denote by $\widetilde{\mathcal{U}}$ the totality of all finite families $\big((\pi_i, u_i)\big)_{i\in I}$ of elements in $\mathrm{Pr}(E)\times\mathcal{U}$ such that $(\pi_i)_{i\in I}$ is a partition of unity in the Boolean algebra $\mathrm{Pr}(E)$. Introduce in $\widetilde{\mathcal{U}}$ the equivalence relation by letting $\big((\pi_i, u_i)\big)_{i\in I} \sim \big((\rho_j, v_j)\big)_{j\in J}$ if and only if $\pi_i\rho_j\big|u_i - v_j\big| = 0$ for all $i\in I$ and $j\in J$. Define $\overline{\mathcal{U}}$ to be the quotient set $\widetilde{\mathcal{U}}/{\sim}$ and agree to denote the coset of a family $\big((\pi_i, u_i)\big)_{i\in I}$ by $\sum_{i\in I}\pi_i u_i$. By identifying the elements $u\in\mathcal{U}$ with "monomials" $1u\in\overline{\mathcal{U}}$, we assume that $\mathcal{U}\subset\overline{\mathcal{U}}$. It is easy to become convinced that $\overline{\mathcal{U}}$ is an LNS over E under the operations

$$\sum_{i\in I}\pi_i u_i + \sum_{j\in J}\rho_j v_j := \sum_{i\in I, j\in J}\pi_i\rho_j(u_i+v_j),$$

$$\lambda\sum_{i\in I}\pi_i u_i := \sum_{i\in I}\pi_i\lambda u_i,$$

$$\left|\sum_{i\in I}\pi_i u_i\right| := \sum_{i\in I}\pi_i\big|u_i\big|$$

and is a d-decomposable hull of $\mathcal{U}$. Uniqueness of a d-decomposable hull is obvious. $\triangleright$

0.5. Let E be a universally complete K-space and let $(E_\xi)_{\xi\in\Xi}$ be a family of pairwise disjoint ideals of E. The symbol $\bigoplus_{\xi\in\Xi}E_\xi$ denotes the ideal of the K-space E constituted by all elements $e\in E$ that satisfy the relation $\langle E_\xi\rangle e\in E_\xi$ for each $\xi\in\Xi$. Obviously,

$$\bigoplus_{\xi\in\Xi}E_\xi = \left\{o\text{-}\sum_{\xi\in\Xi}e_\xi \ :\ (e_\xi)_{\xi\in\Xi}\in\prod_{\xi\in\Xi}E_\xi\right\}.$$

Suppose that, for every $\xi\in\Xi$, we are given an LNS $\mathcal{U}_\xi$ over E_ξ. It is not difficult to become convinced that the vector space $\prod_{\xi\in\Xi}\mathcal{U}_\xi$ is an LNS over $\bigoplus_{\xi\in\Xi}E_\xi$ with respect to the norm $\big|(u_\xi)_{\xi\in\Xi}\big| = o\text{-}\sum_{\xi\in\Xi}\big|u_\xi\big|$. This LNS is denoted by $\bigoplus_{\xi\in\Xi}\mathcal{U}_\xi$ and called the *disjoint sum of the family of* LNSs $(\mathcal{U}_\xi)_{\xi\in\Xi}$.

0.6. Let E and F be K-spaces and let $\mathcal{U}$ be an LNS over E. Suppose that a function $S\colon E \to F$ satisfies the following conditions:

(a) $S(e_1 + e_2) \leqslant Se_1 + Se_2$ for all positive $e_1, e_2 \in E$;
(b) $S(\lambda e) = \lambda Se$ for all positive $e \in E$ and $\lambda \in \mathbb{R}$;
(c) if $0 \leqslant e_1 \leqslant e_2$ then $Se_1 \leqslant Se_2$.

Consider the vector subspace $\mathcal{U}_0 := \{u \in \mathcal{U} : S|\!|u|\!| = 0\}$ and agree to denote by $S_{\mathcal{U}}u$ the coset in $\mathcal{U}/\mathcal{U}_0$ containing an $u \in \mathcal{U}$. It is easy to become convinced that the space $\mathcal{U}/\mathcal{U}_0$ is an LNS over F with respect to the norm $|\!|S_{\mathcal{U}}u|\!| := S|\!|u|\!|$. Observe that the LNS $\mathcal{U}/\mathcal{U}_0$ need not be d-decomposable $\big($for instance, in case $\mathcal{U} = E = F = \mathbb{R}^2$ and $S(x,y) = (x,x)\big)$. Slightly abusing the language, we call a d-decomposable hull of the LNS $\mathcal{U}/\mathcal{U}_0$ the *norm transformation of* $\mathcal{U}$ *by means of* S and denote it by $S\mathcal{U}$. The linear operator $S_{\mathcal{U}}\colon \mathcal{U} \to S\mathcal{U}$ is called the *operator of norm transformation of* $\mathcal{U}$ *by means of* S.

0.7. As is known, every universally complete K-space E can be endowed with multiplication so that E becomes a commutative ordered algebra. If we additionally fix an order unity in E and require it to be a multiplication unity then the way of introducing multiplication in E becomes unique. Furthermore, for every $f \in E$, there exists a unique element $g \in E$ such that $fg = \langle f \rangle 1$, where $1 \in E$ is the multiplication unity. We denote such an element g by $1/f$. The product $e(1/f)$ is denoted by e/f for brevity.

As is known, every Banach–Kantorovich space (BKS) $\mathcal{U}$ over a universally complete K-space E with a fixed order unity 1_E can be endowed with the structure of a module over E so that

$$1_E\, u = u, \quad |\!|eu|\!| = |e|\,|\!|u|\!| \quad (e \in E,\ u \in \mathcal{U}).$$

Below (see 2.8) we will see that the relation $|\!|eu|\!| = |e|\,|\!|u|\!|$ uniquely determines the structure of a module in $\mathcal{U}$.

Let $\mathcal{U}$ be an arbitrary BKS over an order-dense ideal E of a universally complete K-space $\mathcal{E}$ with a fixed order unity. Given arbitrary $e \in \mathcal{E}$ and $u \in \mathcal{U}$, we say that *the product* eu *is defined in* $\mathcal{U}$ (and write $eu \in \mathcal{U}$), if the product eu calculated in the universal completion of $\mathcal{U}$ belongs to $\mathcal{U}$. Obviously, the latter is true if and only if $|e|\,|\!|u|\!| \in E$.

0.8. The module structure of a BKS is often used for finding elements that satisfy certain conditions imposed on their norm. Here is one of typical examples.

Lemma. *Let $\mathcal{U}$ be a* BKS *over E. For all $u \in \mathcal{U}$ and $e \in E^+$, there exists an element $u_e \in \mathcal{U}$ such that $|u_e| = e$ and $|u - u_e| = ||u| - e|$.*

$\lhd$ Fix an order unity 1 in the universal completion $\overline{E}$ of the K-space E, endow $\overline{E}$ with the corresponding multiplication and introduce in the universal completion $\overline{\mathcal{U}}$ of $\mathcal{U}$ the structure of a module over $\overline{E}$. Let $\bar{u}$ be an element of $\overline{\mathcal{U}}$ such that $|\bar{u}| = 1$ and $u = |u|\bar{u}$. Obviously, $u_e := e\bar{u}$ is the desired element. $\rhd$

0.9. Let $\mathcal{X}$ and $\mathcal{Y}$ be continuous Banach bundles (CBBs) over a topological space Q. A homomorphism $i \in \mathrm{Hom}_Q(\mathcal{X}, \mathcal{Y})$ is called an *isometric embedding* of $\mathcal{X}$ into $\mathcal{Y}$ if, for each $q \in Q$, the operator $i(q)$ is an isometric embedding of $\mathcal{X}(q)$ into $\mathcal{Y}(q)$. If, in addition, all the operators $i(q)$ are surjective then the embedding i is called an *isometry* of $\mathcal{X}$ onto $\mathcal{Y}$.

The following assertion ensues immediately from definitions: *if i is an isometric embedding of $\mathcal{X}$ into $\mathcal{Y}$ then there exists a (unique) subbundle $\mathcal{Y}_0$ in $\mathcal{Y}$ such that i is an isometry of $\mathcal{X}$ onto $\mathcal{Y}_0$.*

0.10. Let Q be a nonempty extremally disconnected compact space, let $\mathcal{X}$ be a CBB over Q, and let E be an ideal of $C_\infty(Q)$. If $\mathcal{U} \subset E(\mathcal{X})$ then the set $\mathrm{cl} \bigcup_{u \in \mathcal{U}} \mathrm{supp}\, u$ is called the *support* of $\mathcal{U}$ and denoted by $\mathrm{supp}\,\mathcal{U}$. Obviously, $\langle \mathrm{supp}\,\mathcal{U} \rangle = \langle \mathcal{U} \rangle$, i.e., the operator $u \mapsto \mathrm{ext}\big(\chi_{\mathrm{supp}\,\mathcal{U}} u\big)$ is the order projection onto the band generated by $\mathcal{U}$. In particular, $\langle \mathrm{supp}\, u \rangle = \langle u \rangle$ for every section $u \in E(\mathcal{X})$.

0.11. Let $\mathcal{U}$ and $\mathcal{V}$ be LNSs over respective vector lattices E and F. A positive operator $S\colon E \to F$ is said to be a *dominant* of an operator $T\colon \mathcal{U} \to \mathcal{V}$ if $|Tu| \leqslant S|u|$ for all $u \in \mathcal{U}$. An operator possessing a dominant is called *dominated*. The totality of all dominated operators from $\mathcal{U}$ into $\mathcal{V}$ is denoted by $M(\mathcal{U}, \mathcal{V})$. Obviously, $M(\mathcal{U}, \mathcal{V})$ is a vector subspace of the space of all linear operators from $\mathcal{U}$ into $\mathcal{V}$.

Proposition. *Let E and F be vector lattices and let $\mathcal{U}$ and $\mathcal{V}$ be* LNSs.

(1) An operator $T\colon E \to F$ is regular if and only if it is dominated.

(2) If an operator $T\colon \mathcal{U} \to \mathcal{V}$ is dominated then it is bounded.

(3) If F is a K*-space and an operator $T\colon E \to F$ is bounded then it is dominated (= regular).*

$\lhd$ Assertions (1) and (2) are obvious. A proof of (3) is presented in [17: VII.1.27; 32: Theorem VIII.2.2]. $\rhd$

R e m a r k . A bounded operator need not be dominated. Indeed, by endowing the vector space ℓ^∞ of bounded numeric sequences with coordinatewise order, we obtain a K-space (hence, a BKS) which is denoted by $\mathcal{U}$. On the other hand, by endowing ℓ^∞ with the uniform norm, we obtain a Banach space (hence, a BKS) which is denoted by $\mathcal{V}$. Then the identity mapping of ℓ^∞ onto itself, as an operator from $\mathcal{U}$ into $\mathcal{V}$, is bounded but not dominated.

0.12. Theorem [20]. *Let $\mathcal{U}$ be an* LNS *over E and let $\mathcal{V}$ be an* LNS *over F.*

(1) Every dominated operator $T\colon \mathcal{U} \to \mathcal{V}$ possesses a least dominant (with respect to the order of the vector lattice $M(E,F)$ of regular operators), denoted by $|T|$ and called the exact dominant of T.

(2) If $\mathcal{V}$ is a BKS *then the mapping $|\cdot|\colon T \mapsto |T|$ is a decomposable $M(E,F)$-valued norm on $M(\mathcal{U},\mathcal{V})$ under which $M(\mathcal{U},\mathcal{V})$ is a* BKS.

0.13. Theorem [20]. *Consider a* BKS *$\mathcal{U}$ over E, an* LNS *$\mathcal{V}$ over F, and a linear operator $T\colon \mathcal{U} \to \mathcal{V}$. For each positive element $e \in E$, assign*

$$\begin{aligned}
T_{\leqslant}(e) &:= \{|Tu_1| + \dots + |Tu_n| \ : u_i \in \mathcal{U},\ |u_1| + \dots + |u_n| \leqslant e\},\\
T_{=}(e) &:= \{|Tu_1| + \dots + |Tu_n| \ : u_i \in \mathcal{U},\ |u_1| + \dots + |u_n| = e\},\\
T_{\perp}(e) &:= \{|Tu_1| + \dots + |Tu_n| \ : u_i \in \mathcal{U} \text{ are pairwise disjoint,}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad |u_1| + \dots + |u_n| \leqslant e\}.
\end{aligned}$$

The operator T is dominated if and only if, for every positive element $e \in E$, one (hence, each) of the sets $T_{\leqslant}(e)$, $T_{=}(e)$, or $T_{\perp}(e)$ is bounded. In this case, $|T|e = \sup T_{\leqslant}(e) = \sup T_{=}(e) = \sup T_{\perp}(e)$ for all $e \geqslant 0$.

1. The shadow of an operator

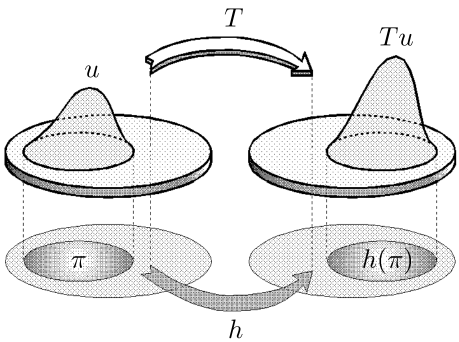
Our main tool for studying disjointness preserving operators is the so-called shadow, a ring homomorphism in Boolean algebras which is generated by the action of the operator on bands. Many properties of an operator are expressible in terms of its shadow. In particular, this is true of certain questions of continuity.

1.1. Let $\mathcal{U}$ and $\mathcal{V}$ be LNSs. An operator $T\colon \mathcal{U} \to \mathcal{V}$ is said to be *disjointness preserving* whenever $u_1 \perp u_2$ implies $Tu_1 \perp Tu_2$ for all $u_1, u_2 \in \mathcal{U}$. It is not difficult to become convinced that every disjointness preserving positive operator in K-spaces is a lattice homomorphism. The following assertion shows that all disjointness preserving operators, not only positive, are closely related to lattice homomorphisms.

Theorem. *Let E be a vector lattice, let F be a* K*-space, and let $T\colon E \to F$ be a regular disjointness preserving operator. Put $\rho := \big\langle T^+[E^+]\big\rangle$. Then the operators $\rho\circ T$ and $-\rho^\perp\circ T$ are lattice homomorphisms. In particular, $T = (\rho - \rho^\perp)|T|$.*

$\triangleleft$ The claim follows directly from [4: Theorem 3.3]. $\triangleright$

In the sequel, we repeatedly use the last theorem in order to reduce consideration of an arbitrary regular disjointness preserving operator to the case of a positive operator.



1.2. The *shadow* of a linear operator $T\colon \mathcal{U} \to \mathcal{V}$ is defined to be the mapping $h\colon \operatorname{Pr}(\mathcal{U}) \to \operatorname{Pr}(\mathcal{V})$ acting by the rule $h(\pi) = \sup_{u\in\mathcal{U}}\langle T\pi u\rangle$. In other words, $h(\pi) = \big\langle T[\pi\mathcal{U}]\big\rangle$.

Proposition. *A linear operator in* LNS*s is disjointness preserving if and only if its shadow is a ring homomorphism.*

$\triangleleft$ Only necessity requires proving. Consider a disjointness preserving linear operator $T\colon \mathcal{U} \to \mathcal{V}$ in LNSs $\mathcal{U}$ and $\mathcal{V}$. Without loss of generality, we may assume that $(\operatorname{im} T)^{\perp\perp} = \mathcal{V}$. Prove that the shadow $h\colon \operatorname{Pr}(\mathcal{U}) \to \operatorname{Pr}(\mathcal{V})$ of T is a Boolean homomorphism. To this end, use the proposition stated in [14] (see the introduction therein). Let (π_1, π_2, π_3) be a partition of unity in the algebra $\operatorname{Pr}(\mathcal{U})$. Then

$$h(\pi_1) \wedge h(\pi_2) = \sup_{u_1\in\mathcal{U}}\langle T\pi_1 u_1\rangle \wedge \sup_{u_2\in\mathcal{U}}\langle T\pi_2 u_2\rangle = \sup_{u_1,u_2\in\mathcal{U}}\langle T\pi_1 u_1\rangle \wedge \langle T\pi_2 u_2\rangle = 0,$$

i.e., $h(\pi_1) \perp h(\pi_2)$. The relations $h(\pi_1) \perp h(\pi_3)$ and $h(\pi_2) \perp h(\pi_3)$ can be established similarly. Moreover,

$$\begin{aligned} h(\pi_1) \vee h(\pi_2) \vee h(\pi_3) &= \sup_{u_1,u_2,u_3\in\mathcal{U}} \langle T\pi_1 u_1\rangle \vee \langle T\pi_2 u_2\rangle \vee \langle T\pi_3 u_3\rangle \\ &= \sup_{u_1,u_2,u_3\in\mathcal{U}} \big\langle T(\pi_1 u_1 + \pi_2 u_2 + \pi_3 u_3)\big\rangle = \sup_{u\in\mathcal{U}}\langle Tu\rangle = 1, \end{aligned}$$

whence it follows that $\big(h(\pi_1), h(\pi_2), h(\pi_3)\big)$ is a partition of unity in the algebra $\operatorname{Pr}(\mathcal{V})$. $\triangleright$

1.3. Proposition. *Consider* LNSs $\mathcal{U}$ *and* $\mathcal{V}$, *a linear operator* $T\colon \mathcal{U} \to \mathcal{V}$ *and a ring homomorphism* $h\colon \Pr(\mathcal{U}) \to \Pr(\mathcal{V})$. *The following assertions are equivalent:*

(1) h *dominates the shadow of* T (see 0.2);

(2) $\langle Tu\rangle \leqslant h\langle u\rangle$ *for all* $u \in \mathcal{U}$;

(3) $T\pi u = h(\pi)Tu$ *for all* $u \in \mathcal{U}$ *and* $\pi \in \Pr(\mathcal{U})$.

If, in addition, $h(1) = \langle \operatorname{im} T\rangle$ *then each of conditions (1)–(3) is equivalent to coincidence of the shadow of* T *with* h.

$\lhd$ The implications (3)⇒(1)⇒(2) are obvious. Assume (2) to be satisfied and prove (3). Fix arbitrary elements $u \in \mathcal{U}$ and $\pi \in \Pr(\mathcal{U})$. From (2) it follows that $T\pi u$ and $T\pi^{\perp}u$ are disjoint. Consequently, there exist a projection $\rho \in \Pr(\mathcal{V})$ such that $T\pi u = \rho Tu$ and $T\pi^{\perp}u = \rho^{\perp}Tu$. In order to ensure the equality $\rho Tu = h(\pi)Tu$, it is sufficient to show that $\rho\langle Tu\rangle = h(\pi)\langle Tu\rangle$. The relations $\rho\langle Tu\rangle = \langle T\pi u\rangle \leqslant h(\pi)$ imply the inequality $\rho\langle Tu\rangle \leqslant h(\pi)\langle Tu\rangle$. One can establish similarly that $\rho^{\perp}\langle Tu\rangle \leqslant h(\pi^{\perp})\langle Tu\rangle$. The two last inequalities directly imply the equality $\rho\langle Tu\rangle = h(\pi)\langle Tu\rangle$.

According to Proposition 0.2, condition (1) and the equality $h(1) = \langle \operatorname{im} T\rangle$ imply that the shadow of T coincides with h. $\rhd$

1.4. Proposition. *Let* T *be a dominated operator acting from a* BKS *into an* LNS. *Then the shadows of* T *and* $|T|$ *coincide.*

$\lhd$ Let an operator T acts from a BKS $\mathcal{U}$ over E into an LNS $\mathcal{V}$ over F. Denote the shadow of T by h_T and the shadow of $|T|$ by $h_{|T|}$. Of course, coincidence of the functions $h_T\colon \Pr(\mathcal{U}) \to \Pr(\mathcal{V})$ and $h_{|T|}\colon \Pr(E) \to \Pr(F)$ is understood with the identifications $\Pr(\mathcal{U}) = \Pr(E)$ and $\Pr(\mathcal{V}) = \Pr(F)$ taken into account (see the introduction in [14]). For every $\pi \in \Pr(E)$, the inequality $h_T(\pi) \leqslant h_{|T|}(\pi)$ is obvious. To prove the reverse inequality, it is sufficient to observe, that the conditions

$$e \in E, \quad \pi \in \Pr(E), \quad u_1, \dots, u_n \in \mathcal{U}, \quad |u_1| + \dots + |u_n| \leqslant \pi e$$

imply

$$\begin{aligned}\langle |Tu_1| + \dots + |Tu_n|\rangle &= \langle |T\pi u_1| + \dots + |T\pi u_n|\rangle \\ &= \langle T\pi u_1\rangle \vee \dots \vee \langle T\pi u_n\rangle \\ &\leqslant h_T(\pi),\end{aligned}$$

and to use the formula $|T|\pi e = \sup T_{\leqslant}(\pi e)$ (see 0.13). $\rhd$

Corollary. *A dominated operator* T *from a* BKS *into an* LNS *is disjointness preserving if and only if its exact dominant* $|T|$ *is disjointness preserving.*

1.5. Let $\mathcal{U}$ and $\mathcal{V}$ be LNSs and let $h\colon \mathrm{Pr}(\mathcal{U}) \to \mathrm{Pr}(\mathcal{V})$ be a ring homomorphism. Following general rules, we say that the mapping $T\colon \mathcal{U} \to \mathcal{V}$ is *h-o-continuous* whenever $h\text{-}\lim\limits_{\alpha\in\mathrm{A}} u_\alpha = u$ (see [14: 1.12]) implies $o\text{-}\lim\limits_{\alpha\in\mathrm{A}} Tu_\alpha = Tu$ for every net $(u_\alpha)_{\alpha\in\mathrm{A}}$ in $\mathcal{U}$ and every $u \in \mathcal{U}$.

Theorem. *Let E and F be* K*-spaces. Every disjointness preserving operator $T\colon E \to F$ is h-o-continuous, where h is the shadow of T.*

$\lhd$ Since the shadow of $|T|$ coincides with the shadow of T (see Proposition 1.4), we may assume that the operator T is positive. To prove h-o-continuity of T, it is sufficient to consider a net $(e_\alpha)_{\alpha\in\mathrm{A}}$ in E, which is h-convergent to zero, and to show that $o\text{-}\lim\limits_{\alpha\in\mathrm{A}} Te_\alpha = 0$. Asymptotic boundedness of the net $(Te_\alpha)_{\alpha\in\mathrm{A}}$ follows from that of $(e_\alpha)_{\alpha\in\mathrm{A}}$ and from boundedness of T. According to Lemma [14: 1.10] (2), o-convergence of Te_α to zero will be established if we prove that $o\text{-}\lim\limits_{\alpha\in\mathrm{A}} \langle Te\rangle\langle Te_\alpha > Te/n\rangle = 0$ for all $e \in E$ and $n \in \mathbb{N}$. The latter relation can be obtained as follows:

$$\begin{aligned}\langle Te\rangle\langle Te_\alpha > Te/n\rangle &= \langle Te\rangle\big\langle \big(T(e_\alpha - e/n)\big)^+\big\rangle = \langle Te\rangle\big\langle T\big((e_\alpha - e/n)^+\big)\big\rangle \\ &\leqslant h\big(\langle e\rangle\big)h\big(\big\langle (e_\alpha - e/n)^+\big\rangle\big) = h\big(\langle e\rangle\langle e_\alpha > e/n\rangle\big) \xrightarrow{o} 0. \quad \rhd\end{aligned}$$

Corollary. *Every disjointness preserving dominated operator from a* BKS *into an* LNS *is h-o-continuous, where h is its shadow.*

$\lhd$ The claim follows from Proposition 1.4 and the last theorem. $\rhd$

R e m a r k . It is sometimes useful to take the following fact into account (the fact follows directly from the last assertion): if $\mathcal{U}$ is a BKS, $\mathcal{V}$ is an LNS, and a ring homomorphism $h\colon \mathrm{Pr}(\mathcal{U}) \to \mathrm{Pr}(\mathcal{V})$ dominates the shadow of an operator $T\colon \mathcal{U} \to \mathcal{V}$, then the latter is h-o-continuous.

1.6. Corollary. *The following properties of a disjointness preserving dominated operator T from a* BKS *into an* LNS *are equivalent:*

(1) *T is (sequentially) o-continuous;*
(2) *$|\!|T|\!|$ is (sequentially) o-continuous;*
(3) *the shadow of T is (sequentially) o-continuous.*

Countable and sequential o-continuity of the operator T are equivalent.

$\lhd$ It is sufficient to combine 1.4, 0.1, [14: 1.12], and 1.5. $\rhd$

1.7. Corollary. *Consider a* BKS $\mathcal{U}$ *and an* LNS $\mathcal{V}$ *and assume that the shadows of two dominated operators* $S, T\colon \mathcal{U} \to \mathcal{V}$ *are dominated by the same ring homomorphism* $h\colon \mathrm{Pr}(\mathcal{U}) \to \mathrm{Pr}(\mathcal{V})$. *If* S *and* T *coincide on some* h*-approximating subset of* $\mathcal{U}$ (see [14: 1.14]) *then they coincide on the entire* $\mathcal{U}$.

$\lhd$ The claim follows from Remark 1.5 and Propositions [14: 1.16, 1.17]. $\rhd$

1.8. Proposition. *Let* $\mathcal{U}$ *be an* LNS *over* E, *let* $\mathcal{V}$ *be a vector subspace of* F, *let* $\mathcal{U}_0$ *let* $\mathcal{U}$, $T_0 : \mathcal{U}_0 \to \mathcal{V}$ *be a linear operator, let* $S\colon E \to F$ *be a disjointness preserving positive operator, and let* $h\colon \mathrm{Pr}(E) \to \mathrm{Pr}(F)$ *be the shadow of* S. *Denote by* $h\mathcal{U}_0$ *the* LNS *of all elements of* $\mathcal{U}$ *that are* h*-approximated by* $\mathcal{U}_0$ (see [14: 1.14, 1.15]). *Assume that* $|T_0 u_0| \leqslant S|u_0|$ (*respectively,* $|T_0 u_0| = S|u_0|$) *for all* $u_0 \in \mathcal{U}_0$. *Then there exists a unique linear extension* $T\colon h\mathcal{U}_0 \to \mathcal{V}$ *of the operator* T_0 *such that* $|Tu| \leqslant S|u|$ (*respectively,* $|Tu| = S|u|$) *for all* $u \in h\mathcal{U}_0$.

$\lhd$ First, we prove the assertion about extension preserving the inequality. If $\pi \in \mathrm{Pr}(\mathcal{U})$ and $u_0 \in \mathcal{U}_0$ are such that $\pi u_0 = 0$, then $h(\pi)T_0 u_0 = 0$, since

$$h(\pi)|T_0 u_0| \leqslant h(\pi)S|u_0| = S\pi|u_0| = 0.$$

This fact implies that the following definition of an operator $\overline{T}_0$ is sound:

$$\overline{T}_0\Big(\sum_{i=1}^{n} \pi_i u_i\Big) := \sum_{i=1}^{n} h(\pi_i)T_0 u_i \quad \big(\pi_i \in \mathrm{Pr}(\mathcal{U}) \text{ are pairwise disjoint, } u_i \in \mathcal{U}_0\big),$$

which extends T_0 onto $d_{\mathrm{fin}}\mathcal{U}_0$ and satisfies the inequality $|\overline{T}_0 u| \leqslant S|u|$ for all $u \in d_{\mathrm{fin}}\mathcal{U}_0$. In view of Proposition [14: 1.15], for every $u \in h\mathcal{U}_0$, there exists a net $(u_\alpha)_{\alpha \in \mathrm{A}}$ in $d_{\mathrm{fin}}\mathcal{U}_0$ that is h-convergent to u. From the inequality

$$\big|\overline{T}_0 u_\alpha - \overline{T}_0 u_\beta\big| \leqslant S\big|u_\alpha - u_\beta\big|$$

and h-o-continuity of S (see 1.5) it follows that the net $(\overline{T}_0 u_\alpha)_{\alpha \in \mathrm{A}}$ is o-fundamental. Since the LNS $\mathcal{V}$, is o-complete, it contains an o-limit of the net. Obviously, the limit depends only on u and, therefore, can be denoted by Tu. It is not difficult to become convinced that the operator $T\colon h\mathcal{U}_0 \to \mathcal{V}$ thus obtained is the desired one. Uniqueness of the extension constructed is ensured by its h-o-continuity inherited from S.

Assume now that $|T_0 u_0| = S|u_0|$ for all $u_0 \in \mathcal{U}_0$. In view of what was proven above, there exists an extension $T\colon h\mathcal{U}_0 \to \mathcal{V}$ of the operator T_0 such that $\|Tu\| \leqslant S\|u\|$ for all $u \in h\mathcal{U}_0$. For every $u_0 \in \mathcal{U}_0$ and $\pi \in \mathrm{Pr}(\mathcal{U})$, the relations

$$S\|u_0\| = \|Tu_0\| = \|T\pi u_0\| + \left\|T\pi^\perp u_0\right\| \leqslant S\|\pi u_0\| + S\left\|\pi^\perp u_0\right\| = S\|u_0\|$$

and the inequalities $\|T\pi u_0\| \leqslant S\|\pi u_0\|$ and $\left\|T\pi^\perp u_0\right\| \leqslant S\left\|\pi^\perp u_0\right\|$ imply $\|T\pi u_0\| = S\|\pi u_0\|$. Since $u_0 \in \mathcal{U}_0$ and $\pi \in \mathrm{Pr}(\mathcal{U})$ were chosen arbitrarily, we have $\|Tu\| = S\|u\|$ for all $u \in d_{\mathrm{fin}}\mathcal{U}_0$. The equality $\|Tu\| = S\|u\|$ for all $u \in h\mathcal{U}_0$ is now deduced from what was proven with the help of Proposition [14: 1.16]. $\triangleright$

Corollary. *Let $\mathcal{U}$ be an* LNS *over E, let $\mathcal{V}$ be a* BKS *over F, let $\mathcal{U}_0$ be an approximating vector subspace of $\mathcal{U}$, let $T_0 \colon \mathcal{U}_0 \to \mathcal{V}$ be a linear operator, and let $S\colon E \to F$ be a disjointness preserving o-continuous positive operator. Assume that $\|T_0u_0\| \leqslant S\|u_0\|$ (respectively, $\|T_0u_0\| = S\|u_0\|$) for all $u_0 \in \mathcal{U}_0$. Then there exists a unique linear extension $T\colon \mathcal{U} \to \mathcal{V}$ of T_0 such that $\|Tu\| \leqslant S\|u\|$ (respectively, $\|Tu\| = S\|u\|$) for all $u \in \mathcal{U}$.*

1.9. If D is a subset of a K-space E then $|D|$ denotes the set $\{|d| : d \in D\}$, and $\mathrm{lin}|D|$ stands for the linear span of $|D|$. The smallest ideal of E that contains D is conventionally denoted by E_D.

Lemma. *Let E be a K-space, let D be a subset of E, let $\mathcal{V}$ and $\mathcal{W}$ be arbitrary* LNSs *over the same K-space F, and let $S\colon E \to \mathcal{V}$ and $T\colon E \to \mathcal{W}$ be dominated operators. Assume that the shadows of S and T are dominated by the same ring homomorphism $h\colon \mathrm{Pr}(E) \to \mathrm{Pr}(F)$ and denote the h-closure of the ideal E_D by hE_D.*

(1) If $\mathcal{V} = \mathcal{W}$ and the operators S and T coincide on D, then they coincide on hE_D.

(2) If $\|Se\| = \|Te\|$ for all $e \in \mathrm{lin}|D|$ then $\|Se\| = \|Te\|$ for all $e \in hE_D$.

$\triangleleft$ We only prove assertion (1), since (2) can be proven similarly and even easier. Assume that the operators S and T meet all the hypotheses of the lemma and coincide on D. We will prove that S and T agree on hE_D in several steps.

(a) Suppose that $e \in |D|$, i.e., $e = |d|$ for some $d \in D$. Then

$$\begin{aligned} Se &= S\langle d^+\rangle d + S\langle d^-\rangle d = h\big(\langle d^+\rangle\big)Sd + h\big(\langle d^-\rangle\big)Sd \\ &= h\big(\langle d^+\rangle\big)Td + h\big(\langle d^-\rangle\big)Td = T\langle d^+\rangle d + T\langle d^-\rangle d = Te. \end{aligned}$$

(b) From (a) it follows that S and T agree on the set $\mathrm{lin}|D|$.

(c) Let e be a d-step element of E with $d \in \operatorname{lin}|D|$, i.e., $e = \sum_{i=1}^{n} \pi_i \lambda_i d$ for some numbers λ_i and pairwise disjoint projections $\pi_i \in \Pr(E)$. Then, in view of (b), we have

$$Se = \sum_{i=1}^{n} S(\pi_i \lambda_i d) = \sum_{i=1}^{n} \lambda_i h(\pi_i) Sd = \sum_{i=1}^{n} \lambda_i h(\pi_i) Td = \sum_{i=1}^{n} T(\pi_i \lambda_i d) = Te.$$

(d) Suppose now that $e \in E_D$. Then $|e| \leqslant d$ for some $d \in \operatorname{lin}|D|$. In view of 0.3, there exists a sequence $(e_n)_{n \in \mathbb{N}}$ of d-step elements of E that is r-convergent to e. According to (c), the operators S and T coincide on the elements e_n. Therefore, using r-continuity of S and T, we arrive at the equality $Se = Te$.

(e) Finally, if e is an arbitrary element of hE_D then the equality $Se = Te$ follows from (d) and h-o-continuity of S and T. $\triangleright$

Corollary. *Let $\mathcal{U}$ be a* BKS *over E, let D be a set of positive elements in E, let $\mathcal{V}$ and $\mathcal{W}$ be arbitrary* LNSs *over the same* K*-space F, and let $S\colon \mathcal{U} \to \mathcal{V}$ and $T\colon \mathcal{U} \to \mathcal{W}$ be dominated operators. Assume that the shadows of S and T are dominated by the same ring homomorphism $h\colon \Pr(E) \to \Pr(F)$ and denote by hE_D the h-closure of the ideal E_D.*

(1) If $\mathcal{V} = \mathcal{W}$ and the operators S and T coincide on the set $\{u \in \mathcal{U} : |u| \in D\}$ then they coincide on the set $\{u \in \mathcal{U} : |u| \in hE_D\}$.

(2) If $|Su| = |Tu|$ for all $u \in \mathcal{U}$ with norm $|u| \in \operatorname{lin} D$ then $|Su| = |Tu|$ for all $u \in \mathcal{U}$ with norm $|u| \in hE_D$.

$\triangleleft$ Prove assertion (1) (assertion (2) can be proven similarly). Assume that the operators S and T meet all the hypotheses of the corollary and coincide on the set $\{u \in \mathcal{U} : |u| \in D\}$. Consider an arbitrary element $u \in \mathcal{U}$ with norm $|u| \in hE_D$ and establish the equality $Su = Tu$.

Fix an order unit 1 in a universal completion $\overline{E}$ of the K-space E, introduce the corresponding multiplication in $\overline{E}$ and endow a universal completion $\overline{\mathcal{U}}$ of $\mathcal{U}$ with the structure of a module over $\overline{E}$ (see 0.7). Let $\bar{u}$ be an element of $\overline{\mathcal{U}}$ such that $|\bar{u}| = 1$ and $u = |u|\bar{u}$. Consider operators $S_u, T_u\colon E \to \mathcal{V}$ acting by the rules $S_u e = S(e\bar{u})$ and $T_u e = T(e\bar{u})$. It is clear that the shadows of S_u and T_u are dominated by the homomorphism h and the operators themselves coincide on D. Therefore, according to assertion (1) of the last lemma, the operators S_u and T_u coincide on hE_D. In particular, $Su = S_u|u| = T_u|u| = Tu$. $\triangleright$

1.10. As is seen from the following theorem, all the four types of boundedness introduced in [14: 2.3] coincide for each disjointness preserving operator defined on a vector lattice.

Theorem. *Let E be a vector lattice and let $\mathcal{V}$ be an* LNS. *The following properties of a disjointness preserving operator $T\colon E \to \mathcal{V}$ are equivalent:*

(1) *T is bounded;*
(2) *T is countably bounded;*
(3) *T is sequentially bounded;*
(4) *T is semibounded;*
(5) *if $e_1, e_2 \in E$ and $|e_1| \leqslant |e_2|$ then $\lvert Te_1 \rvert \leqslant \lvert Te_2 \rvert$.*

$\lhd$ The implications (5)⇒(1)⇒(2)⇒(3)⇒(4) are obvious. The proof of Theorem 2.1 in [22] that establishes the implication (4)⇒(5) is presented for the case $\mathcal{V} = E$; however, it remains valid for an operator with values in an arbitrary LNS. $\rhd$

The proof of the implication (4)⇒(5) becomes particularly simple and clear in the case when E possesses the principal projection property (for instance, when E is a K_σ-space). Indeed, assume that an operator T meets condition (4), fix arbitrary elements $e_1, e_2 \in E$ satisfying the inequality $|e_1| \leqslant |e_2|$, and denote by S the set

$$\left\{ \sum_{i=1}^{n} \pi_i \lambda_i |e_2| : \ \pi_i \in \Pr(E),\ |\lambda_i| \leqslant 1 \right\}.$$

It is not difficult to become convinced that $\lvert Ts \rvert \leqslant \lvert Te_2 \rvert$ for all $s \in S$. Moreover, in view of 0.3, there exists a sequence $(s_n)_{n\in\mathbb{N}}$ of elements in S r-convergent to e_1 with regulator $|e_2|$. Condition (4) together with the relations $\lvert Te_1 \rvert \leqslant \lvert Te_1 - Ts_n \rvert + \lvert Te_2 \rvert$ $(n \in \mathbb{N})$ now yields the desired inequality $\lvert Te_1 \rvert \leqslant \lvert Te_2 \rvert$.

1.11. The analog of Theorem 1.10 for operators defined on LNSs is not true. Moreover, all the four types of boundedness are pairwise different for this class of operators. Indeed, every normed space is an LNS over $\mathbb{R}$ and every linear operator from a normed space into an arbitrary LNS is disjointness preserving. Consequently, operators considered in the Examples [14: 2.4–2.6] act from BKSs into BKSs and are disjointness preserving.

1.12. Lemma. *Let $\mathcal{U}$ be a* BKS *over E, let $\mathcal{V}$ be an* LNS, *let $T\colon \mathcal{U} \to V$ be a disjointness preserving semibounded operator, and let e be a positive element of E. For each $u \in \mathcal{U}$ satisfying the inequality $|u| \leqslant e$, there is an element $\bar{u} \in \mathcal{U}$ such that $|\bar{u}| = e$ and $|Tu| \leqslant |T\bar{u}|$.*

$\lhd$ Suppose that $|u| \leqslant e$. In view of the equality $\{|u| : u \in \mathcal{U}\} = \{e \in E : e \geqslant 0\}$, we do not restrict generality by assuming that $\langle u\rangle = \langle e\rangle$. Obviously, the product $(e'/|u|)u$ is defined in $\mathcal{U}$ for all $e' \in E$ (see 0.7). Define an operator $S\colon E \to \mathcal{V}$ by the formula $S(e') = T\big((e'/|u|)u\big)$ and assign $\bar{u} := (e/|u|)u$. It is easy to see that the operator S is disjointness preserving and semibounded. According to Theorem 1.10, the operator S meets condition 1.10 (5). This allows us to conclude that $|Tu| = S|u| \leqslant Se = |T\bar{u}|$. It remains to observe that $|\bar{u}| = e$. $\rhd$

Proposition. *Let $\mathcal{U}$ be a* BKS *over E and let $\mathcal{V}$ be an* LNS. *A disjointness preserving operator $T\colon \mathcal{U} \to V$ is dominated if and only if it is bounded. Furthermore, for all positive $e \in E$,*

$$
\begin{aligned}
|T|e &= \sup\{|Tu| :\ u \in \mathcal{U},\ |u| \leqslant e\}\\
&= \sup\{|Tu| :\ u \in \mathcal{U},\ |u| = e\}
\end{aligned}
$$

$\lhd$ For an arbitrary positive element $e \in E$, the equality

$$
|T|e = \sup\{|Tu| :\ u \in \mathcal{U},\ |u| \leqslant e\}
$$

is easily deduced from the criterion 0.13 involving the set $T_\perp(e)$. It remains to employ the lemma proven above. $\rhd$

The last result does not provide any new information about operators in vector lattices, since dominatedness and boundedness are always equivalent for operators with values in a K-space $\big($see Proposition 0.11 (3)$\big)$. However, an analog of the last proposition is true in the case of vector lattices:

Theorem [23]. *Let E and F be arbitrary vector lattices. A disjointness preserving operator $T\colon E \to F$ is regular (= dominated) if and only if it is bounded.*

1.13. As was noted in 1.11, countable boundedness is not sufficient for boundedness of a disjointness preserving operator. It is interesting to clarify which (easily verified) additional assumptions yield boundedness of operators bounded in a weaker sense. Leaving this question open, we only formulate one corollary to Lemma 1.12 which is a small step in the indicated direction.

Proposition. *Let $\mathcal{U}$ be a* BKS *over E and let $\mathcal{V}$ be an* LNS *over F. A disjointness preserving operator $T\colon \mathcal{U} \to \mathcal{V}$ is bounded if and only if it is semibounded and, for every positive element $e \in E$, the set $\{|Tu| : u \in \mathcal{U}, |u| = e\}$ is order-bounded in F.*

Note that any semibounded disjointness preserving operator defined on a vector lattice obviously meets the hypotheses of the last proposition. This allows us to consider Proposition 1.13 as a generalization of Theorem 1.10.

1.14. One of the main results concerning disjointness preserving operators provides their representation as sums of certain special operators taking pairwise disjoint values (see Section 4). Here we pay attention to such sums.

Lemma. *Let $\mathcal{U}$ and $\mathcal{V}$ be* LNSs *and let $S, T\colon \mathcal{U} \to \mathcal{V}$ be linear operators. The following assertions are equivalent:*

(1) *$Su \perp Tu$ for all $u \in \mathcal{U}$;*
(2) *$Su_1 \perp Tu_2$ for all $u_1, u_2 \in \mathcal{U}$, i.e.,* $\operatorname{im} S \perp \operatorname{im} T$.

$\lhd$ Only the implication (1)$\Rightarrow$(2) requires proving. Let u_1 and u_2 be arbitrary elements of $\mathcal{U}$. The relations $Su_1 \perp Tu_1$ and $Su_2 \perp Tu_2$ imply:

$$|Su_1| \wedge |Tu_2| = |Su_1| \wedge |Tu_1 + Tu_2| \leqslant \big|T(u_1 + u_2)\big|,$$
$$|Su_1| \wedge |Tu_2| = |Su_1 + Su_2| \wedge |Tu_2| \leqslant \big|S(u_1 + u_2)\big|.$$

It remains to observe that $S(u_1 + u_2) \perp T(u_1 + u_2)$. $\rhd$

Operators S and T meeting each of the equivalent conditions (1) or (2) are called *strongly disjoint.* Let $\mathcal{U}$ and $\mathcal{V}$ be LNSs and let $(T_\xi)_{\xi\in\Xi}$ be a family of linear operators from $\mathcal{U}$ into $\mathcal{V}$. We say that an operator $T\colon \mathcal{U} \to \mathcal{V}$ is decomposable into the *strongly disjoint sum* of operators T_ξ $\big($and write $T = \bigoplus_{\xi\in\Xi} T_\xi\big)$, whenever the operators T_ξ are strongly disjoint and, for every $u \in \mathcal{U}$, the relation $Tu = o\text{-}\sum_{\xi\in\Xi} T_\xi u$ holds.

Assume that $T = \bigoplus_{\xi\in\Xi} T_\xi$ and assign $\rho_\xi := \langle \operatorname{im} T_\xi \rangle$ for each $\xi \in \Xi$. According to the lemma, the projections ρ_ξ are pairwise disjoint; therefore, for all $\xi \in \Xi$ the equality $T_\xi = \rho_\xi \circ T$ holds. In particular, this implies that the strongly disjoint sum $\bigoplus_{\xi\in\Xi} T_\xi$ is disjointness preserving if and only if so is each summand T_ξ.

2. Orthomorphisms

This section is devoted to one of the simplest classes of disjointness preserving operators, the class of band preserving operators.

Throughout the section, G is a universally complete K-space with a fixed order unity 1_G, Q is the Stone compact space of the Boolean algebra $\mathrm{Pr}(G)$ (recall that this algebra is the base of G), E and F are order-dense ideals of G, and $\mathcal{U}$ and $\mathcal{V}$ are LNSs over E and F, respectively. We introduce a multiplication in the K-space G which makes it a commutative ordered algebra with unity 1_G (see 0.7). Recall also that we identify the Boolean algebras $\mathrm{Pr}(G)$, $\mathrm{Pr}(E)$, $\mathrm{Pr}(F)$, $\mathrm{Pr}(\mathcal{U})$ and $\mathrm{Pr}(\mathcal{V})$.

2.1. A linear operator $T\colon \mathcal{U} \to \mathcal{V}$ is said to be *band preserving* if it satisfies one of the following equivalent conditions:

(1) $\langle Tu\rangle \leqslant \langle u\rangle$ for all $u \in \mathcal{U}$;
(2) $T\pi u = \pi T u$ for all $u \in \mathcal{U}$ and $\pi \in \mathrm{Pr}(G)$;
(3) $\pi u = 0$ implies $\pi T u = 0$ for all $u \in \mathcal{U}$ and $\pi \in \mathrm{Pr}(G)$;
(4) $|u| \perp g$ implies $|Tu| \perp g$ for all $u \in \mathcal{U}$ and $g \in G$;
(5) $|u| \perp g$ implies $|Tu| \perp g$ for all $u \in \mathcal{U}$ and all elements g of some order-dense ideal of the K-space G.

Obviously, the last definition generalizes the familiar notion of band preserving operator acting in vector lattices (see [5, 6, 22, 33, 34]).

2.2. Bounded band preserving operators are called *orthomorphisms.* The totality of all orthomorphisms from $\mathcal{U}$ into $\mathcal{V}$ is denoted by $\mathrm{Orth}(\mathcal{U},\mathcal{V})$. We write $\mathrm{Orth}(\mathcal{U})$ instead of $\mathrm{Orth}(\mathcal{U},\mathcal{U})$.

It seems interesting to clarify the additional requirements that, being imposed on band preserving operators, yield their boundedness. Of course, band preserving operators are disjointness preserving and, therefore, they are subject for such boundedness criteria as 1.10 and 1.13. It is known (see [14: 2.4–2.6]), that semiboundedness, sequential boundedness, and even countable boundedness of a disjointness preserving operator do not imply its boundedness. In the case of band preserving operators, the situation is different:

Theorem. *The following properties of a band preserving operator T from a* BKS *into an* LNS *are equivalent:*

(1) *T is bounded;*
(2) *T is countably bounded;*
(3) *T is sequentially bounded;*
(4) *T is semibounded.*

$\lhd$ The implications (1)$\Rightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(4) are obvious. It remains to show that (4)$\Rightarrow$(1). Assume that an LNS $\mathcal{U}$ is order-complete and an operator $T\colon \mathcal{U} \to \mathcal{V}$ is band preserving and semibounded. Fix an arbitrary positive element $e \in G$ and prove that the set $\{|Tu| : |u| \leqslant e\}$ is order-bounded in F. We divide the proof into two steps.

(a) Show first that the set $\{|Tu| : |u| \leqslant e\}$ is order-bounded in the universally complete K-space G. Without loss of generality, we may assume that $G = C_\infty(Q)$, where Q is an extremally disconnected compact space (see Theorem [12: 0.3.4]). Denote by D the totality of those points $q \in Q$, for which

$$\sup\{|Tu|(q) : |u| \leqslant e\} = \infty.$$

Assume that the set $\{|Tu| : |u| \leqslant e\}$ is not bounded in $C_\infty(Q)$. Then, according to [17: Chapter XIII, Theorem 2.32], the clopen set $U := \operatorname{int}\operatorname{cl} D$ is nonempty. For each natural n and each point $q \in U \cap D$, consider an element $u_n^q \in \mathcal{U}$ satisfying the conditions $\left|u_n^q\right| \leqslant e$ and $\left|Tu_n^q\right|(q) > n$. Denote by U_n^q a clopen subset of Q such that $q \in U_n^q \subset U$ and $\left|Tu_n^q\right|(p) \geqslant n$ for all $p \in U_n^q$. It is clear that, for each $n \in \mathbb{N}$ the relation $\sup_{q\in U\cap D} U_n^q = U$ holds in the Boolean algebra $\operatorname{Clop}(Q)$. In view of the exhaustion principle, there exists a family $\left(V_n^q\right)_{q\in U\cap D}$ of pairwise disjoint elements of $\operatorname{Clop}(Q)$ such that $V_n^q \subset U_n^q$ for all $q \in U \cap D$, and $\sup_{q\in U\cap D} V_n^q = U$. According to [12: 0.4.3], the sum $o\text{-}\sum_{q\in U\cap D}\langle V_n^q\rangle u_n^q$ exists in the BKS $\mathcal{U}$. Denote the sum by u_n. For all $n \in \mathbb{N}$ and $q \in U \cap D$, we have

$$\langle V_n^q\rangle|Tu_n| = \left|T\langle V_n^q\rangle u_n\right| = \left|T\langle V_n^q\rangle u_n^q\right| = \langle V_n^q\rangle|Tu_n^q| \geqslant n\chi_{V_n^q}.$$

After passing to the supremum over $q \in U \cap D$, we obtain $|Tu_n| \geqslant n\chi_U$ for all $n \in \mathbb{N}$; which, together with the inequalities $|u_n| \leqslant e$, yields a contradiction with semiboundedness of T.

(b) Denote by f the upper envelope of the set $\{|Tu| : |u| \leqslant e\}$ in the K-space G and show that $f \in F$. Without loss of generality, we may assume that $f > 0$ on some comeager subset of Q. Then, according to [17: Chapter XIII, Theorem 2.32], the set of all points $q \in Q$, for which

$$0 < \sup\{|Tu|(q) : |u| \leqslant e\} = f(q) < \infty,$$

is comeager in Q. For any such point q, consider an element $u_q \in \mathcal{U}$ satisfying the conditions $\left|u_q\right| \leqslant e$ and $\left|Tu_q\right|(q) > f(q)/2$. By repeating the idea of step (a) and “mixing up” the elements u_q in an appropriate way, we can construct an element $u \in \mathcal{U}$ such that $|Tu| \geqslant f/2$; whence the containment $f \in F$ follows directly. $\rhd$

Additional requirements, yielding boundedness of band preserving operators, can be imposed on the spaces rather than on operators acting in them. In the present article, we are not going to develop this idea. We only observe that many results in the indicated direction are presented in [5: Theorem 2; 6: Theorem 3.2 and 3.3; 22: Corollaries 2.3 and 2.4].

2.3. It is easy to become convinced that $\mathrm{Orth}(E,F)$ is an ideal of the K-space $M(E,F)$ and, therefore, is also a K-space.

If an element $g \in G$ is such that $g \cdot e \in F$ for all $e \in E$ then the operator of multiplication by g is obviously an orthomorphism from E into F. Many papers about disjointness preserving operators contain results in this direction (see, for instance, [2, 5, 6, 8, 9, 33, 34]). The following statement generalizes, in a sense, the experience from finding multiplication representation of orthomorphisms acting in K-spaces.

Theorem. *For every orthomorphism $T\colon E \to F$, there exists a unique element $g_T \in G$ such that $Te = g_T \cdot e$ for all $e \in E$. The mapping $T \mapsto g_T$ performs a linear and order isomorphism of the* K-*space* $\mathrm{Orth}(E,F)$ *onto the ideal* $\{g \in G : \ g \cdot e \in F \text{ for all } e \in E\}$ *of the* K-*space* G.

Identifying an orthomorphism T with the element $g_T \in G$, we assume in the sequel that $\mathrm{Orth}(E,F) \subset G$. Obviously, $\mathrm{Orth}(E)$ contains 1_G and is a subalgebra of G. In particular, $\mathrm{Orth}(E)$ is an f-algebra (see [16, 34]). The last theorem justifies the term *weight operator* which is sometimes used instead of "orthomorphism."

2.4. Proposition. *Let an* LNS $\mathcal{U}$ *be order-complete. A linear operator $T\colon \mathcal{U} \to \mathcal{V}$ is an orthomorphism if and only if it is dominated and its exact dominant $|T|\colon E \to F$ is an orthomorphism. In particular, the space* $\mathrm{Orth}(\mathcal{U},\mathcal{V})$ *endowed with the dominant-norm is a* BKS *over the* K-*space* $\mathrm{Orth}(E,F)$.

$\lhd$ The claim follows directly from Propositions 1.12 and 1.4. $\rhd$

2.5. Corollary. *Every orthomorphism from a* BKS *into an* LNS *is o-continuous.*

2.6. Corollary. *If two orthomorphisms from a* BKS $\mathcal{U}$ *into an* LNS $\mathcal{V}$ *coincide on some order-approximating subset of* $\mathcal{U}$ (see [14: 1.2]), *then they coincide on the entire* $\mathcal{U}$.

$\lhd$ The claim follows from 2.5 and Proposition [14: 1.4]. $\rhd$

2.7. Corollary. *If two orthomorphisms $S, T \in \mathrm{Orth}(E,\mathcal{V})$ coincide on a subset $E_0 \subset E$ then they coincide on $E_0^{\perp\perp}$. In particular, if the* K-*space E has an order unity* 1 *and $S(1) = T(1)$ then $S = T$.*

2.8. Proposition. *For every* BKS $\mathcal{U}$ *over* E *there exists a unique operation* $\mathrm{Orth}(E) \times \mathcal{U} \to \mathcal{U}$ *making* $\mathcal{U}$ *a module over* $\mathrm{Orth}(E)$ *such that* $|gu| = |g||u|$ *for all* $g \in \mathrm{Orth}(E)$ *and* $u \in \mathcal{U}$. *Furthermore,* $\mathcal{U}$ *is a unital module, i.e.,* $1_G u = u$ *for all* $u \in \mathcal{U}$. *For every* $g \in \mathrm{Orth}(E)$ *and* $u \in \mathcal{U}$, *the element* gu *coincides with the product of* g *and* u *calculated in the universal completion of* $\mathcal{U}$ (see 0.7).

$\lhd$ Let a BKS $m\mathcal{U}$ over G be a universal completion of $\mathcal{U}$. Then $\mathcal{U} = \{u \in m\mathcal{U} : |u| \in E\}$. In view of 0.7, the space $m\mathcal{U}$ can be endowed with the structure of a module over the ring G so that $1_G u = u$ and $|gu| = |g||u|$ for all $g \in G$ and $u \in m\mathcal{U}$. In order to prove existence of a desired module structure in the BKS $\mathcal{U}$, it is sufficient to observe that, for all $g \in \mathrm{Orth}(E)$ and $u \in \mathcal{U}$, we have $|g||u| \in E$ and, consequently, $gu \in \mathcal{U}$.

We now prove uniqueness. Assume that, together with the operation $(g, u) \mapsto gu$ introduced above, there is another one, $(g, u) \mapsto g * u$, also making $\mathcal{U}$ a module over $\mathrm{Orth}(E)$ and satisfying the condition $|g * u| = |g||u|$ for all $g \in \mathrm{Orth}(E)$ and $u \in \mathcal{U}$. Fix an element $u \in \mathcal{U}$ and define the mappings $S, T\colon \mathrm{Orth}(E) \to \mathcal{V}$ by the formulas $S(g) = gu$ and $T(g) = g * u$. Obviously, S and T are orthomorphisms. Observe that $T(1_G) = S(1_G)$, i.e., $1_G * u = u$. Indeed,

$$\begin{aligned} |1_G * u - u| = 1_G \cdot |1_G * u - u| &= \big|1_G * (1_G * u - u)\big| \\ &= \big|(1_G \cdot 1_G) * u - 1_G * u\big| = 0. \end{aligned}$$

For proving the equality $S = T$, it remains to employ 2.7. $\rhd$

The fact that any BKS over G can be endowed with the structure of a module over G allows us to define a simple class of orthomorphisms. If a BKS $\mathcal{U}$ over E and a BKS $\mathcal{V}$ over F are order-dense ideals of the same BKS over G and $g \in \mathrm{Orth}(E, F)$, then the operator $u \mapsto gu$ is an orthomorphism from $\mathcal{U}$ into $\mathcal{V}$. We call such operators *scalar orthomorphisms.*

2.9. Proposition. *Let* $\mathcal{U}$ *be an order-complete* LNS, $T \in \mathrm{Orth}(\mathcal{U}, \mathcal{V})$, $g \in G$, *and* $u \in \mathcal{U}$. *If the product* gu *is defined in* $\mathcal{U}$ (see 0.7) *then the product* $gT(u)$ *is defined in* $\mathcal{V}$ *and the equality* $T(gu) = gT(u)$ *holds. In particular,* $T \circ g = g \circ T$ *for every orthomorphism* $g \in \mathrm{Orth}(E)$.

$\lhd$ Fix an arbitrary element $u \in \mathcal{U}$ and denote by G_u the order-dense ideal $\{g \in G : gu \in \mathcal{U}\}$ of the K-space G. Let $m\mathcal{V}$ be the universal completion of $\mathcal{V}$. Consider the mappings $L, R\colon G_u \to m\mathcal{V}$ defined by the formulas $L(g) = T(gu)$ and $R(g) = gT(u)$. Obviously, L and R are orthomorphisms and $L(1_G) = R(1_G)$. From 2.7 it follows that $L = R$. $\rhd$

2.10. We conclude this section by a useful fact, which will be repeatedly employed in the sequel.

Theorem [21]. *Let E be a vector lattice and let F be a* K*-space. A positive operator $T\colon E \to F$ is disjointness preserving if and only if, for every operator $S\colon E \to F$ satisfying the inequalities $0 \leqslant S \leqslant T$, there is an orthomorphism $g \in \mathrm{Orth}(F)$ such that $0 \leqslant g \leqslant \mathrm{id}_F$ and $S = g \circ T$, where $\mathrm{id}_F\colon F \to F$ is the identity operator.*

Combining the last theorem with Theorem 1.1, we obtain the following result.

Corollary. *Let E be a vector lattice and let F be a* K*-space. A regular operator $T\colon E \to F$ is disjointness preserving if and only if, for every regular operator $S\colon E \to F$ satisfying the inequality $|S| \leqslant |T|$, there is an orthomorphism $g \in \mathrm{Orth}(F)$ such that $|g| \leqslant \mathrm{id}_F$ and $S = g \circ T$, where $\mathrm{id}_F\colon F \to F$ is the identity operator.*

3. Shift operators

Another class of disjointness preserving operators is considered in this section. Here, we introduce and study so-called shift operators, which are abstract analogs of the composition mappings $f \mapsto f \circ s$. This class of operators is closely related to another notion discussed here, the notion of operator "wide on a set." While studying shift operators, we suggest their equivalent characterizations, describe the maximal domain of definition on which they can be extended, and show that the notions of shift operator and that of a multiplicative operator coincide. We also introduce here the notion of the shift of a disjointness preserving operator, which in a sense concentrates multiplicative properties of the operator.

Throughout the section, $\mathcal{E}$ and $\mathcal{F}$ are universally complete K-spaces. In case order unities $1_{\mathcal{E}}$ and $1_{\mathcal{F}}$ are fixed in $\mathcal{E}$ and $\mathcal{F}$, we regard the K-spaces as ordered algebras with unities $1_{\mathcal{E}}$ and $1_{\mathcal{F}}$ (see 0.7). The ideal of the K-space $\mathcal{E}$ generated by $d \in \mathcal{E}$ is denoted by $\mathcal{E}_d$. In particular, $\mathcal{E}_1$ stands for the ideal of $\mathcal{E}$ generated by $1_{\mathcal{E}}$. We point out that some notions introduced in this section depend on a concrete choice of $1_{\mathcal{E}}$ and $1_{\mathcal{F}}$.

3.1. Let E be a K-space, let D be a subset of E, and let $\mathcal{V}$ be an LNS. We say that an operator $T\colon E \to \mathcal{V}$ is *wide on the set* D whenever the equality $T[D]^{\perp\perp} = T[E]^{\perp\perp}$ holds.

Proposition. *Suppose that E is a* K*-space, D is a subset of E, $\mathcal{V}$ is an* LNS*, $T\colon E \to \mathcal{V}$ is a disjointness preserving operator, and $h\colon \mathrm{Pr}(E) \to \mathrm{Pr}(\mathcal{V})$ is its shadow. The following assertions are equivalent:*

(1) *T is wide on the set D;*
(2) *T is wide on the ideal E_D;*
(3) *the shadow of the restriction of T onto E_D coincides with the shadow of T;*
(4) *the set $T[E_D]$ is o-dense in $T[E]$;*
(5) *the ideal E_D h-approximates the space E.*

$\lhd$ The implications (1)$\Rightarrow$(2)$\Leftarrow$(4) are obvious. Since the shadow of T dominates that of the restriction of T onto E_D, the equivalence (2)$\Leftrightarrow$(3) readily follows from Proposition 0.2. We show that (1)$\Leftarrow$(2)$\Rightarrow$(5)$\Rightarrow$(4).

(2)$\Rightarrow$(5): Assume condition (2) to be satisfied, consider an arbitrary element $e \in E$, and show that $h\text{-}\inf_{\pi\in\Pi} \pi e = e$, where $\Pi = \{\pi \in \mathrm{Pr}(E) : \pi e \in E_D\}$. For every $n \in \mathbb{N}$ and $d \in E_D$, assign $\pi_n^d := \big\langle |e| \leqslant n|d| \big\rangle$. Obviously, $\pi_n^d \in \Pi$. Since

$$\Big| d - \pi_n^d d \Big| = \big(\pi_n^d\big)^\perp |d| \leqslant \big(\pi_n^d\big)^\perp |e|/n \leqslant |e|/n$$

for all $n \in \mathbb{N}$, we have $r\text{-}\lim\limits_{n\to\infty} \pi_n^d d = d$. Using r-continuity of the operator T and taking account of the equality $T\big(\pi_n^d d\big) = h\big(\pi_n^d\big) T d$, we arrive at the relation $\sup_{n\in\mathbb{N}} h\big(\pi_n^d\big) \geqslant \langle Td \rangle$. Since the element $d \in E_D$ was chosen arbitrarily, we conclude by (2) that $\sup_{\pi\in\Pi} h(\pi) = h(1)$ and, consequently, $h\text{-}\inf_{\pi\in\Pi} \pi e = e$.

(5)$\Rightarrow$(4): Consider an arbitrary element $e \in E$. From (5) and Proposition [14: 1.3] it follows that e is the h-limit of some net $(e_\alpha)_{\alpha\in\mathrm{A}}$ of elements in E_D. In view of Corollary 1.5, we have $o\text{-}\lim\limits_{\alpha\in\mathrm{A}} T(e_\alpha) = Te$.

(2)$\Rightarrow$(1): For every element $e \in E_D$, there exist $d_1, \dots, d_n \in D$ such that $|e| \leqslant |d_1| + \dots + |d_n|$. In view of Theorem 1.10, we conclude that $\langle Te \rangle \leqslant \langle Td_1 \rangle \vee \dots \vee \langle Td_n \rangle$. It remains to employ condition (2). $\rhd$

R e m a r k . As is seen from the last proposition, the fact that an operator T is wide on a set D reflects the connection of D with the domain of definition and with the shadow of T rather then with the operator T itself.

3.2. Let $\mathcal{U}$ and $\mathcal{V}$ be LNSs and let D be a subset of the norming lattice of $\mathcal{U}$. We say that an operator $T\colon \mathcal{U} \to \mathcal{V}$ is *wide on the set* D, whenever $\{Tu : |u| \in D\}^{\perp\perp} = (\mathrm{im}\, T)^{\perp\perp}$. If $\mathcal{U}$ and $\mathcal{V}$ are K-spaces then the last definition is equivalent to that given in 3.1, which justifies preservation of terminology.

Lemma. *Let $\mathcal{U}$ be a* BKS *over a* K*-space E, let $\mathcal{V}$ be an arbitrary* LNS, *and let D be a subset of positive elements in E. A disjointness preserving operator $T\colon \mathcal{U} \to \mathcal{V}$ is wide on D if and only if its exact dominant $|T|$ is wide on D.*

$\lhd$ A proof can be easily obtained with the help of Proposition 1.12. Indeed, the relations

$$\big\langle |T|e \big\rangle = \sup_{|u|=e} \langle Tu \rangle \leqslant \sup_{|u|\in D} \langle Tu \rangle = \sup_{d\in D} \sup_{|u|=d} \langle Tu \rangle = \sup_{d\in D} \big\langle |T|d \big\rangle,$$

which hold for every positive element $e \in E$, prove necessity; whereas the relations

$$\langle Tu \rangle \leqslant \big\langle |T||u| \big\rangle \leqslant \sup_{d\in D} \big\langle |T|d \big\rangle = \sup_{d\in D} \sup_{|u|=d} \langle Tu \rangle = \sup_{|u|\in D} \langle Tu \rangle,$$

that are valid for each element $u \in \mathcal{U}$, establish sufficiency. $\rhd$

Proposition. *Suppose that $\mathcal{U}$ is a* BKS *over a* K*-space E, D is a subset of positive elements in E, $\mathcal{V}$ is an arbitrary* LNS, *$T\colon \mathcal{U} \to \mathcal{V}$ is a disjointness preserving bounded operator, and $h\colon \mathrm{Pr}(\mathcal{U}) \to \mathrm{Pr}(\mathcal{V})$ is its shadow. The following assertions are equivalent:*

(1) *T is wide on the set D;*
(2) *T is wide on the ideal E_D;*
(3) *the shadow of the restriction of T onto the set $\{u \in \mathcal{U} : |u| \in E_D\}$ coincides with the shadow of T;*
(4) *the set $\{Tu : |u| \in E_D\}$ is o-dense in* $\operatorname{im} T$;
(5) *the ideal E_D h-approximates the space E.*

$\lhd$ The equivalence (2)$\Leftrightarrow$(3) is established in the same way as in 3.1. Equivalence of assertions (1), (2), and (5) ensues from Propositions 1.4 and 3.1 and the last lemma. The implication (4)$\Rightarrow$(2) is obvious. It remains to show that (5)$\Rightarrow$(4).

Let u be an arbitrary element of $\mathcal{U}$. From (5) and Proposition [14: 1.3] it follows that $|u|$ is the h-limit of some net $(e_\alpha)_{\alpha\in\mathrm{A}}$ of positive elements in E_D. In view of Lemma 0.8, there exists a net $(u_\alpha)_{\alpha\in\mathrm{A}}$ in $\mathcal{U}$ such that $|u_\alpha| = e_\alpha$ and $|u - u_\alpha| = \big||u| - e_\alpha\big|$. Then $h\text{-}\lim\limits_{\alpha\in\mathrm{A}} u_\alpha = u$ and, according to Corollary 1.5, we have $o\text{-}\lim_{\alpha\in\mathrm{A}} Tu_\alpha = Tu$. $\rhd$

3.3. Proposition. *Let E be an ideal of $\mathcal{E}$ generated by a positive element $d \in \mathcal{E}$. For every ring homomorphism $h\colon \mathrm{Pr}(\mathcal{E}) \to \mathrm{Pr}(\mathcal{F})$, the following sets coincide:*

(1) *the h-closure of E;*
(2) *the h-cyclic hull of E;*
(3) *the countably h-cyclic hull of E;*
(4) *the set of such $e \in \mathcal{E}$ that $\inf_{n\in\mathbb{N}} h\langle |e| > nd\rangle = 0$.*

$\lhd$ The relations (4)⊂(3)⊂(2)⊂(4) are obvious. The inclusion (4)⊂(1) can be easily established with the help of the first corollary in [14: 1.18]. It remains to show that (1)⊂(4). Suppose that a net $(e_\alpha)_{\alpha\in\mathrm{A}}$ of elements in E h-converges to $e \in \mathcal{E}$. For each $\alpha \in \mathrm{A}$, denote by n_α the natural number satisfying the inequality $|e_\alpha| \leqslant n_\alpha d$. By using the relations $h\text{-}\inf_{\alpha\in\mathrm{A}} |e - e_\alpha| = 0$ and

$$h\langle |e| > 2n_\alpha d\rangle \leqslant h\langle |e| > 2|e_\alpha|\rangle$$
$$= h\big(\langle e\rangle\big\langle |e| - |e_\alpha| > |e|/2\big\rangle\big) \leqslant h\big(\langle e\rangle\big\langle |e - e_\alpha| > |e|/2\big\rangle\big),$$

we obtain the desired equality $\inf_{n\in\mathbb{N}} h\langle |e| > nd\rangle = 0$. $\rhd$

The coincident sets (1)–(4) described in the last proposition are denoted by hE.

3.4. Proposition. *Fix an order unity $1_{\mathcal{E}}$ in the* K*-space $\mathcal{E}$. Then the set $h\mathcal{E}_1$ is a subalgebra of $\mathcal{E}$.*

$\lhd$ This fact ensues from 3.3 (we mean the equality $h\mathcal{E}_1 = (4)$ for $d = 1_{\mathcal{E}}$) and from the following relations:

$$\begin{aligned}
\inf_{n\in\mathbb{N}} h\langle |ef| > n1_{\mathcal{E}}\rangle &= \inf_{m,n\in\mathbb{N}} h\langle |ef| > mn1_{\mathcal{E}}\rangle \\
&\leqslant \inf_{m,n\in\mathbb{N}} h\big(\langle |e| > m1_{\mathcal{E}}\rangle \vee \langle |f| > n1_{\mathcal{E}}\rangle\big) \\
&= \inf_{m,n\in\mathbb{N}} \big(h\langle |e| > m1_{\mathcal{E}}\rangle \vee h\langle |f| > n1_{\mathcal{E}}\rangle\big) \\
&= \inf_{m\in\mathbb{N}} h\langle |e| > m1_{\mathcal{E}}\rangle \vee \inf_{n\in\mathbb{N}} h\langle |f| > n1_{\mathcal{E}}\rangle. \quad \rhd
\end{aligned}$$

3.5. Lemma. *Let d be an arbitrary order unity in $\mathcal{E}$. For every sequence $(\pi_n)_{n\in\mathbb{N}}$ of projections in $\mathrm{Pr}(\mathcal{E})$ that decreases to zero, there is an element $e \in \mathcal{E}$ such that $\pi_n = \langle |e| > nd\rangle$ for all $n \in \mathbb{N}$.*

$\lhd$ Since the K-space $\mathcal{E}$ is universally complete, the series $\sum_{n=1}^{\infty} \pi_n d$ has an o-sum in it. Denote the sum by s. It is clear that $\langle s > nd\rangle = \pi_{n+1}$ for all $n \in \mathbb{N}$ and, consequently, we can take $s + d$ as the desired element e. $\rhd$

Corollary. *Let $h\colon \mathrm{Pr}(\mathcal{E}) \to \mathrm{Pr}(\mathcal{F})$ be a ring homomorphism and let d be an arbitrary order unity in $\mathcal{E}$. The equality $h\mathcal{E}_d = \mathcal{E}$ holds if and only if the homomorphism $h\colon \mathrm{Pr}(\mathcal{E}) \to \mathrm{Pr}(\mathcal{F})$ is sequentially o-continuous.*

3.6. Let $\mathcal{U}$ be an LNS over an order-dense ideal E of the universally complete K-space $\mathcal{E}$, let d be a positive element of $\mathcal{E}$, and let $\mathcal{V}$ be an arbitrary LNS. We say that an operator $T\colon \mathcal{U} \to \mathcal{V}$ is *wide at the element* d whenever it is wide on the set $\{e \in E : e \text{ is a fragment of } d\}$.

Lemma. *Suppose that E is an order-dense ideal of $\mathcal{E}$, d is a positive element of $\mathcal{E}$, $\mathcal{V}$ is an LNS, $T\colon E \to \mathcal{V}$ is a disjointness preserving bounded operator, and h is its shadow. Assign $\Pi := \{\pi \in \mathrm{Pr}(\mathcal{E}) : \pi d \in E\}$. The following assertions are equivalent:*

(1) *the operator T is wide at the element d;*
(2) $\sup_{\pi\in\Pi} h(\pi) = h(1)$ *and, for all $\pi \in \Pi$ the equality $\langle T\pi d\rangle = h(\pi)$ holds;*
(3) $E \subset h\mathcal{E}_d$.

$\lhd$ The equivalence of (1) and (3) is contained in Proposition 3.1, the implication (2)$\Rightarrow$(1) is obvious. It remains to show that (1)$\Rightarrow$(2). If (1) is valid then, for every projection $\pi_0 \in \Pi$, we have

$$h(\pi_0) = h(\pi_0)\sup_{e\in E}\langle Te\rangle = h(\pi_0)\sup_{\pi\in\Pi}\langle T\pi d\rangle = \sup_{\pi\in\Pi}\langle T\pi_0\pi d\rangle = \langle T\pi_0 d\rangle. \quad \rhd$$

3.7. Proposition. *Fix arbitrary order unities $1_{\mathcal{E}}$ and $1_{\mathcal{F}}$ in the K-spaces $\mathcal{E}$ and $\mathcal{F}$. For every ring homomorphism $h\colon \mathrm{Pr}(\mathcal{E}) \to \mathrm{Pr}(\mathcal{F})$, there exists a unique regular operator $S\colon h\mathcal{E}_1 \to \mathcal{F}$ such that the shadow of S is equal to h and $S(1_{\mathcal{E}}) = h(1)1_{\mathcal{F}}$. Furthermore, the operator S is positive.*

$\lhd$ For the sake of convenience, assume that $h(1) = 1$. We construct the operator S in three steps.

1. Define the operator S on the set of step-elements of $\mathcal{E}$ by letting

$$S\left(\sum_{i=1}^{n}\lambda_i\pi_i 1_{\mathcal{E}}\right) := \sum_{i=1}^{n}\lambda_i h(\pi_i)1_{\mathcal{F}}$$

for arbitrary $\lambda_1,\dots,\lambda_n \in \mathbb{R}$ and $\pi_1,\dots,\pi_n \in \mathrm{Pr}(\mathcal{E})$.

2. Extend the operator S onto $\mathcal{E}_1$. To this end, fix an arbitrary element $e \in \mathcal{E}_1$ and choose a sequence $(e_n)_{n\in\mathbb{N}}$ of step-elements in $\mathcal{E}$ so that it r-converges to e with regulator $1_{\mathcal{E}}$. It is easy to verify that the sequence $(Se_n)_{n\in\mathbb{N}}$ is r-fundamental (with regulator $1_{\mathcal{F}}$). Assign $Se := r\text{-}\lim_{n\to\infty} Se_n$.

3. Finally, extend S onto the entire set $h\mathcal{E}_1$. Every element $e \in h\mathcal{E}_1$ can be represented as the mixing $o\text{-}\sum_{n\in\mathbb{N}} \pi_n e_n$ of elements $e_n \in \mathcal{E}_1$ by means of an h-partition $(\pi_n)_{n\in\mathbb{N}}$. Assign $Se := o\text{-}\sum_{n\in\mathbb{N}} h(\pi_n) Se_n$.

It is easy to verify that the definition of S is sound at each of the steps. Obvious positiveness of S ensures its regularity. In order to prove uniqueness of S, it is sufficient to observe that, at step 3, the sequence $\big(\sum_{n=1}^m \pi_n e_n\big)_{m\in\mathbb{N}}$ is r-convergent to e with regulator $o\text{-}\sum_{n\in\mathbb{N}} n\pi_n|e_n| \in h\mathcal{E}_1$. $\rhd$

The operator S, whose existence is asserted in the last proposition, is called the *shift by h* and denoted by S_h. Let E be an order-dense ideal of $\mathcal{E}$ and F be an order-dense ideal of $\mathcal{F}$. We say that an operator $S\colon E \to F$ is a *shift operator*, if there exists a ring homomorphism $h\colon \mathrm{Pr}(\mathcal{E}) \to \mathrm{Pr}(\mathcal{F})$ such that $E \subset h\mathcal{E}_1$ and $S = S_h$ on E. It is clear that, in this case, the homomorphism h is the shadow of S. Observe that the notion of the shift and that of shift operator depend on the choice of unities $1_{\mathcal{E}}$ and $1_{\mathcal{F}}$ in the K-spaces $\mathcal{E}$ and $\mathcal{F}$.

3.8. Proposition. *Fix order unities in the universally complete* K*-spaces $\mathcal{E}$ and $\mathcal{F}$. Let E be an order-dense ideal of $\mathcal{E}$, let F be an order-dense ideal of $\mathcal{F}$, and let $S, \overline{S}\colon E \to F$ be shift operators. If $S \leqslant \overline{S}$ then $S = \rho \circ \overline{S}$ for some projection $\rho \in \mathrm{Pr}(F)$.*

$\lhd$ The claim ensues from Propositions 0.2 and 3.7. $\rhd$

Let $\rho \in \mathrm{Pr}(\mathcal{F})$, let $h\colon \mathrm{Pr}(\mathcal{E}) \to \mathrm{Pr}(\mathcal{F})$ be a ring homomorphism, and let S be the shift by h. Then the shift by the homomorphism $\rho \circ h$ is denoted by ρS. Observe that, in general, $\operatorname{dom} \rho S$ is wider than $\operatorname{dom} S$; therefore, ρS differs from the composition $\rho \circ S$. However, in view of the last proposition, the operators ρS and $\rho \circ S$ coincide on $\operatorname{dom} S$ and, thus, ρS extends $\rho \circ S$.

3.9. Theorem. *Fix order unities $1_{\mathcal{E}}$ and $1_{\mathcal{F}}$ in the* K*-spaces $\mathcal{E}$ and $\mathcal{F}$. Let E be an order-dense ideal of $\mathcal{E}$ and let F be an order-dense ideal of $\mathcal{F}$. A linear operator $S\colon E \to F$ is a shift operator if and only if it satisfies the following conditions:*

(a) *S is disjointness preserving;*
(b) *S is regular;*
(c) *S takes fragments of $1_{\mathcal{E}}$ into fragments of $1_{\mathcal{F}}$;*
(d) *S is wide at $1_{\mathcal{E}}$.*

$\lhd$ Necessity of conditions (a)–(c) is obvious and necessity of (d) follows from 3.6. Let us show sufficiency. Suppose that the operator S satisfies conditions (a)–(d), denote the shadow of S by h and assign $\Pi := \{\pi \in \mathrm{Pr}(\mathcal{E}) : \pi 1_{\mathcal{E}} \in E\}$. Lemma 3.6 implies the equality $\langle S(\pi 1_{\mathcal{E}})\rangle = h(\pi)$ for each $\pi \in \Pi$, which, together with condition (c), yields $S(\pi 1_{\mathcal{E}}) = S_h(\pi 1_{\mathcal{E}})$. The same lemma ensures the inclusion $E \subset h\mathcal{E}_1$. In view of Lemma 1.9, we now conclude that $S = S_h$ on E. $\triangleright$

Corollary. *Fix order unities $1_{\mathcal{E}}$ and $1_{\mathcal{F}}$ in the* K*-spaces $\mathcal{E}$ and $\mathcal{F}$. Let E be an order-dense ideal of $\mathcal{E}$ containing $1_{\mathcal{E}}$ and let F be an arbitrary order-dense ideal of $\mathcal{F}$. A linear operator $S\colon E \to F$ is a shift operator if and only if it satisfies the following conditions:*

(a) *S is disjointness preserving;*
(b) *S is regular;*
(c) *$S(1_{\mathcal{E}})$ is a fragment of $1_{\mathcal{F}}$;*
(d) $\big\{S(1_{\mathcal{E}})\big\}^{\perp\perp} = (\operatorname{im} S)^{\perp\perp}$.

Remark. Conditions (d) in the statements of the theorem and the corollary may not be omitted. Indeed, let $\mathcal{F} = \mathbb{R}$, let $\mathcal{E}$ be the space of all sequences, and let E be the ideal of $\mathcal{E}$ generated by the sequence $e_0(n) = n$ ($n \in \mathbb{N}$). Denote by Q the Stone-Čech compactification of the discrete topological space $\mathbb{N}$ and fix an arbitrary point $q \in Q\backslash\mathbb{N}$. Naturally identifying the spaces $\mathcal{E}$ and $C_\infty(Q)$, define an operator $S\colon E \to \mathcal{F}$ by the formula $Se = \big(e/e_0\big)(q)$. Letting $1_{\mathcal{E}}(n) = 1$ ($n \in \mathbb{N}$) and $1_{\mathcal{F}} = 1$, we see that the operator S satisfies conditions (a)–(c) of the last lemma, but $S(1_{\mathcal{E}}) = 0$.

Remark. From the last corollary it is clear that the domain of definition $h\mathcal{E}_1$ of the shift by h is maximally wide. More precisely, $h\mathcal{E}_1$ contains the domain of definition of every regular operator S acting from an order-dense ideal of $\mathcal{E}$ into $\mathcal{F}$, having shadow h, and satisfying the equality $S(1_{\mathcal{E}}) = h(1)1_{\mathcal{F}}$.

3.10. Fix order unities $1_{\mathcal{E}}$ and $1_{\mathcal{F}}$ in the K-spaces $\mathcal{E}$ and $\mathcal{F}$. A linear operator $S\colon E \to \mathcal{F}$ defined on an order-dense ideal $E \subset \mathcal{E}$ is called *multiplicative* if $Se_1 Se_2 = S(e_1 e_2)$ for any two elements $e_1, e_2 \in E$, whose product belongs to E. Observe that the notion of multiplicative operator depends on the choice of unities $1_{\mathcal{E}}$ and $1_{\mathcal{F}}$.

Theorem. *Let E be an order-dense ideal of $\mathcal{E}$. A linear operator $S\colon E \to \mathcal{F}$ is a shift operator if and only if it is multiplicative.*

$\triangleleft$ The fact that every shift operator is multiplicative is easily established by checking all the steps of its construction in 3.7. We will show that any multiplicative operator $S\colon E \to \mathcal{F}$ is a shift operator by verifying conditions (a)–(d) of Theorem 3.9.

(a) Disjointness of elements $e_1, e_2 \in E$ is equivalent to the equality $e_1 e_2 = 0$. The same is true for elements of $\mathcal{F}$. Consequently, S is disjointness preserving.

(b) We prove, that the operator S is positive, in three steps.

(b$_1$) If $e \in E$ and $0 \leqslant e \leqslant 1_{\mathcal{E}}$ then $Se \geqslant 0$. Indeed, in this case, e^3 and $e\sqrt{e}$ belong to E in view of the inequalities $e^3 \leqslant e$ and $e\sqrt{e} \leqslant e$; consequently, $(Se)^3 = S(e^3) = S\big((e\sqrt{e})^2\big) = S\big(e\sqrt{e}\big)^2 \geqslant 0$.

(b$_2$) If $e \in E$ and $e \geqslant 1_{\mathcal{E}}$ then $Se \geqslant 0$. Indeed, in this case, $\sqrt{e} \in E$ in view of the inequality $\sqrt{e} \leqslant e$; consequently, $Se = S\big((\sqrt{e})^2\big) = S\big(\sqrt{e}\big)^2 \geqslant 0$.

(b$_3$) If $e \in E$ and $e \geqslant 0$ then $Se \geqslant 0$. Indeed, $Se = S\langle e \leqslant 1_{\mathcal{E}}\rangle e + S\langle e > 1_{\mathcal{E}}\rangle e \geqslant 0$ in view of (b$_1$) and (b$_2$).

(c) The fact that an element $e \in E$ is a fragment of $1_{\mathcal{E}}$ is equivalent to the equality $e^2 = e$. The same is true for fragments of $1_{\mathcal{F}}$. Consequently, S takes fragments of $1_{\mathcal{E}}$ into fragments of $1_{\mathcal{F}}$.

(d) Show that $\{Se : |e| \leqslant 1_{\mathcal{E}}\}^{\perp\perp} = (\operatorname{im} S)^{\perp\perp}$. Consider the projection $\rho \in \operatorname{Pr}(\mathcal{F})$ onto the band $\{Se : |e| \leqslant 1_{\mathcal{E}}\}^{\perp}$ and define an operator $T\colon E \to \mathcal{F}$ by the formula $Te = \rho Se$. The proof will be completed if we establish that $T = 0$. Obviously, the operator T is multiplicative and $Te = 0$ whenever $|e| \leqslant 1_{\mathcal{E}}$. We also observe that, in view of (b), the operator T is positive. Let e be an arbitrary positive element of E. For each $n \in \mathbb{N}$, the equality $T(e/n) = Te_n$ holds, where $e_n = \langle e/n > 1_{\mathcal{E}}\rangle e/n$. Since $\sqrt{e_n} \leqslant e_n \leqslant e/n$, we have the inclusions $\sqrt{e_n}, e_n \in E$ and the inequality $T\sqrt{e_n} \leqslant Te_n$. Consequently,

$$Te = nTe_n = nT\big(\sqrt{e_n}^{\,2}\big) = n\big(T\sqrt{e_n}\big)^2 \leqslant n(Te_n)^2 = n(Te/n)^2 = (Te)^2/n$$

for all $n \in \mathbb{N}$, which is possible only in case $Te = 0$. $\triangleright$

3.11. R e m a r k . There are a number of results describing multiplicative operators (= shift operators) as extreme points of certain sets of operators (see [10, 11, 24]).

3.12. R e m a r k . It is known (see [32: Theorem VIII.10.1]) that every regular operator $T\colon \mathcal{E}_1 \to \mathcal{F}$ admits an integral representation

$$Te = \int_{-\infty}^{\infty} \lambda \, d\varphi\big(\langle e \leqslant \lambda 1_{\mathcal{E}}\rangle\big) \quad (e \in \mathcal{E}_1),$$

where φ is an arbitrary order-bounded additive function from $\operatorname{Pr}(\mathcal{E})$ into $\mathcal{F}$. It is not difficult to become convinced that T is a shift operator if and only if the values of the function φ are fragments of $1_{\mathcal{F}}$. Furthermore, the shadow h of T is defined by the formula $h(\pi) = \langle \varphi(\pi)\rangle$. Some classes of multiplicative operators (= shift operators) are described form the viewpoint of the integral representation in the papers by B. Z. Vulikh [29, 31].

3.13. Fix order unities $1_{\mathcal{E}}$ and $1_{\mathcal{F}}$ in the K-spaces $\mathcal{E}$ and $\mathcal{F}$. Let $\mathcal{U}$ and $\mathcal{V}$ be LNSs over order-dense ideals $E \subset \mathcal{E}$ and $F \subset \mathcal{F}$, let $T\colon \mathcal{U} \to \mathcal{V}$ be a disjointness preserving operator, and let $h\colon \mathrm{Pr}(E) \to \mathrm{Pr}(F)$ be its shadow. Then the shift $S_h\colon h\mathcal{E}_1 \to \mathcal{F}$ by h is called the *shift of the operator* T.

Proposition. *Let $\mathcal{U}$ and $\mathcal{V}$ be* LNSs *over order-dense ideals $E \subset \mathcal{E}$ and $F \subset \mathcal{F}$ and suppose that the* LNS *$\mathcal{U}$ is order-complete. Assume that $T\colon \mathcal{U} \to \mathcal{V}$ is a disjointness preserving bounded operator with shift S. If elements $e \in \operatorname{dom} S$ and $u \in \mathcal{U}$ are such that the product eu is defined in $\mathcal{U}$, then the product $S(e)T(u)$ is defined in $\mathcal{V}$ and the equality $T(eu) = S(e)T(u)$ holds. In particular, $T \circ g = S(g) \circ T$ for every orthomorphism $g \in \mathrm{Orth}(E) \cap \operatorname{dom} S$.*

$\lhd$ Fix an arbitrary element $u \in \mathcal{U}$ and denote by $\mathcal{E}_u$ the order-dense ideal $\{e \in \operatorname{dom} S : eu \in \mathcal{U}\}$ of the K-space $\mathcal{E}$. Let $m\mathcal{V}$ be the universal completion of $\mathcal{V}$. Consider the mappings $L, R\colon \mathcal{E}_u \to m\mathcal{V}$ defined by the formulas $L(e) = T(eu)$ and $R(e) = S(e)T(u)$. Obviously, the operators L and R are bounded (= dominated) and disjointness preserving; moreover, their shadows are dominated by the shadow of T. Since $L(1_{\mathcal{E}}) = R(1_{\mathcal{E}})$ and $\mathcal{E}_u \subset \operatorname{dom} S$, Lemma 1.9 implies the equality $L = R$. $\rhd$

3.14. Fix order unities $1_{\mathcal{E}}$ and $1_{\mathcal{F}}$ in the K-spaces $\mathcal{E}$ and $\mathcal{F}$. Let $\mathcal{U}$ be an LNS over an order-dense ideal $E \subset \mathcal{E}$ and let $\mathcal{V}$ be an LNS over an order-dense ideal $F \subset \mathcal{F}$. An operator $S\colon \mathcal{U} \to \mathcal{V}$ is called a *shift operator* if there exists a shift operator $s\colon E \to F$ such that $|Su| = s|u|$ for all $u \in \mathcal{U}$. Obviously, $s = |S|$, i.e., the operator s is the exact dominant of S (see 0.12).

R e m a r k . Thus, if $S\colon \mathcal{U} \to \mathcal{V}$ is a shift operator then it is dominated and its exact dominant $|S|\colon E \to F$ if a shift operator. The converse is false in general. Indeed, if $\mathcal{U}$ and $\mathcal{V}$ are Banach spaces and the norm of an operator $S\colon \mathcal{U} \to \mathcal{V}$ is equal to unity then its exact dominant $|S|\colon \mathbb{R} \to \mathbb{R}$ is the identity operator (and, hence, a shift operator), while the operator S itself is a shift operator only if it is an isometric embedding.

Proposition. *Let $\mathcal{U}$ be an* LNS *over an order-dense ideal $E \subset \mathcal{E}$ and let $\mathcal{V}$ be an* LNS *over an order-dense ideal $F \subset \mathcal{F}$. An operator $S\colon \mathcal{U} \to \mathcal{V}$ is a shift operator if and only if there exist a shift operator $s\colon E \to F$ and an F-isometric embedding $\iota\colon s\mathcal{U} \to \mathcal{V}$ such that $S = \iota \circ s_{\mathcal{U}}$, where $s_{\mathcal{U}}\colon \mathcal{U} \to s\mathcal{U}$ is the norm transformation of $\mathcal{U}$ by means of s* (see 0.6).

$$
\begin{tikzcd}
E \arrow[r, "s"] & F \\
\mathcal{U} \arrow[u, "|\cdot|"] \arrow[r, "S"] \arrow[dr, "s_{\mathcal{U}}"'] & \mathcal{V} \arrow[u, "|\cdot|"'] \\
& s\mathcal{U} \arrow[u, "\iota"']
\end{tikzcd}
$$

$\lhd$ Only necessity requires proving. An elementary verification shows that the formula

$$\iota\left(\sum_{i=1}^{n} \rho_i s_{\mathcal{U}} u_i\right) = \sum_{i=1}^{n} \rho_i S u_i \quad (u_i \in \mathcal{U},\ \rho_i \in \mathrm{Pr}(\mathcal{V}))$$

soundly defines a function $\iota\colon s\mathcal{U} \to \mathcal{V}$ that is the desired isometry. $\rhd$

3.15. The following description of shift operators generalizes criterion 3.9 to the case of LNSs.

Theorem. *Fix order unities $1_{\mathcal{E}}$ and $1_{\mathcal{F}}$ in the* K*-spaces $\mathcal{E}$ and $\mathcal{F}$. Let $\mathcal{U}$ be a* BKS *over an order-dense ideal $E \subset \mathcal{E}$ and let $\mathcal{V}$ be an* LNS *over an order-dense ideal $F \subset \mathcal{F}$. An operator $S\colon \mathcal{U} \to \mathcal{V}$ is a shift operator if and only if it satisfies the following conditions:*

(a) *S is disjointness preserving;*
(b) *S is bounded;*
(c) *if $u \in \mathcal{U}$ and $|u|$ is a fragment of $1_{\mathcal{E}}$ then $|Su|$ is a fragment of $1_{\mathcal{F}}$;*
(d) *S is wide at $1_{\mathcal{E}}$.*

$\lhd$ Necessity of conditions (a)–(d) follows immediately from Theorem 3.9. Assume that an operator S satisfies conditions (a)–(d). Denote by $|S|$ the exact dominant of S and show first that $|S|\colon E \to F$ is a shift operator by verifying conditions (a)–(d) of Theorem 3.9. Condition (a) ensues from Corollary 1.4, condition (b) is ensured by the fact that $|S|$ is positive, condition (c) follows from Proposition 1.12, and condition (d) from Lemma 3.2. Thus, $|S|$ is a shift operator. Since the shadows of S and $|S|$ coincide (see Proposition 1.4), the operator $|S|$ is the restriction of the shift of S onto E.

Assign $\mathcal{U}_1 := \{u \in \mathcal{U} : |u| \text{ is a fragment of } 1_{\mathcal{E}}\}$, consider an arbitrary element $u \in \mathcal{U}_1$, and show that $|Su| = |S||u|$. For the sake of convenience, we assume that $|u| = 1_{\mathcal{E}}$ and $|S|1_{\mathcal{E}} = 1_{\mathcal{F}}$. This assumption does not restrict generality, since $S[\langle u\rangle\mathcal{U}] \subset \langle |S||u|\rangle\mathcal{V}$, and, therefore, we may regard S as an operator from $\langle u\rangle\mathcal{U}$ into $\langle |S||u|\rangle\mathcal{V}$. Denote the projection $\langle Su\rangle^{\perp}$ by ρ. Since $|Su|$ is a fragment of $1_{\mathcal{F}}$, it is sufficient to show that $\rho = 0$. Assume to the contrary that $\rho \neq 0$. Then, by Proposition 1.12, there is an element $u_1 \in \mathcal{U}$ such that $|u_1| = 1_{\mathcal{E}}$ and $\rho S u_1 \neq 0$. Assign $e := |u_1 + 3u|$. The equalities $|u| = |u_1| = 1_{\mathcal{E}}$ readily imply $2 \cdot 1_{\mathcal{E}} \leqslant e \leqslant 4 \cdot 1_{\mathcal{E}}$; hence, $\frac{1}{4}1_{\mathcal{E}} \leqslant 1/e \leqslant \frac{1}{2}1_{\mathcal{E}}$. The last inequality proves that the product $\bar{u} := (1/e)(u_1 + 3u)$ is defined in $\mathcal{U}$.

By using Proposition 3.13 and the equality $\rho Su = 0$, we obtain:

$$\begin{aligned}\rho|S\bar{u}| &= \rho\big|S\big((1/e)(u_1+3u)\big)\big| \\ &= \rho|S|(1/e)\big|S(u_1+3u)\big| \\ &= |S|(1/e)|\rho Su_1 + 3\rho Su| \\ &= \rho|S|(1/e)|Su_1| \\ &= \langle \rho Su_1\rangle |S|(1/e).\end{aligned}$$

Observe that $|\bar{u}| = 1_{\mathcal{E}}$ and, consequently, $|S\bar{u}|$ is a fragment of $1_{\mathcal{F}}$. Therefore, the relations

$$\rho|S\bar{u}| = \langle \rho Su_1\rangle |S|(1/e) \geqslant \langle \rho Su_1\rangle |S|\Big(\frac{1}{4}1_{\mathcal{E}}\Big) = \frac{1}{4}\langle \rho Su_1\rangle 1_{\mathcal{F}},$$

yield the inequality $\rho|S\bar{u}| \geqslant \langle \rho Su_1\rangle 1_{\mathcal{F}}$ which contradicts the following relations:

$$\langle \rho Su_1\rangle 1_{\mathcal{F}} \leqslant \rho|S\bar{u}| = \langle \rho Su_1\rangle |S|(1/e) \leqslant \langle \rho Su_1\rangle |S|\Big(\frac{1}{2}1_{\mathcal{E}}\Big) = \frac{1}{2}\langle \rho Su_1\rangle 1_{\mathcal{F}}.$$

Thus, we established that $|Su| = |S||u|$ for all $u \in \mathcal{U}_1$. Denote by h the shadow of S. As is known, h coincides with the shadow of $|S|$. Then, applying Corollary 1.9 (2) to the operators $S\colon \mathcal{U} \to \mathcal{V}$ and $|S|_{\mathcal{U}}\colon \mathcal{U} \to |S|\mathcal{U}$, we obtain the equality $|Su| = |S||u|$ for all $u \in \mathcal{U}$ with norm in $h\mathcal{E}_1$. It remains to observe that $\{u \in \mathcal{U} : |u| \in h\mathcal{E}_1\} = \mathcal{U}$, since $E = \operatorname{dom}|S| \subset h\mathcal{E}_1$. $\rhd$

4. Weighted shift operators

Weighted shift operators considered in this section are the compositions $W \circ S \circ w$ of two orthomorphisms w and W and a shift operator S. Representability of a disjointness preserving operator as such a composition is related to existence of a bounded set on which the operator is wide. In addition to this criterion, we also suggest some sufficient conditions for representability of an operator in the form $W \circ S \circ w$. The main result of the present section is representation of an arbitrary disjointness preserving operator as the strongly disjoint sum of weighted shift operators. Thus, operators of the form $W \circ S \circ w$ play the role of simple elements, from which wider classes of operators are constituted. In the sequel, this fact will allow us to construct one of analytic representations of disjointness preserving operators.

Throughout the section, E and F and order-dense ideals of the K-spaces $\mathcal{E}$ and $\mathcal{F}$. In the spaces $\mathcal{E}$ and $\mathcal{F}$, we fix order unities $1_{\mathcal{E}}$ and $1_{\mathcal{F}}$ and consider the multiplication that makes the spaces commutative ordered algebras with unities $1_{\mathcal{E}}$ and $1_{\mathcal{F}}$, respectively (see 0.7). We recall that orthomorphisms in the K-spaces under consideration are multiplication operators and we identify them with the corresponding multipliers (see 2.3). The ideal of the K-space $\mathcal{E}$ generated by the element $1_{\mathcal{E}}$ is denoted by $\mathcal{E}_1$. Observe that some notions introduced in this section depend on a concrete choice of unities $1_{\mathcal{E}}$ and $1_{\mathcal{F}}$.

4.1. We say that a linear operator $T\colon E \to F$ is a *weighted shift operator* if there exist order-dense ideals $E' \subset \mathcal{E}$ and $F' \subset \mathcal{F}$, orthomorphisms $w\colon E \to E'$ and $W\colon F' \to F$, and a shift operator $S\colon E' \to F'$ such that $T = W \circ S \circ w$. The composition $W \circ S \circ w$ is called a WSW-*representation* of T, and the operators W, S, and w are called the *outer weight*, the *shift*, and the *inner weight* of the representation $W \circ S \circ w$.

$$\begin{array}{ccc} E & \xrightarrow{T} & F \\ {\scriptstyle w}\downarrow & & \uparrow{\scriptstyle W} \\ E' & \xrightarrow{S} & F' \end{array}$$

Observe that, in view of Theorem 1.1, a regular operator $T\colon E \to F$ is a weighted shift operator if and only if so is its modulus $|T|$. Moreover, if one of the operators T or $|T|$ admits a WSW-representation then the other one admits a WSW-representation with the same shift and inner weight. Thus, while discussing the question of whether an operator is a weighted shift operator, we may always assume the operator positive.

From the viewpoint of the above definition, the property of a mapping to be a weighted shift operator depends on the choice of $1_{\mathcal{E}}$ and $1_{\mathcal{F}}$. Actually, there is no such a dependence. Indeed, let an operator T admit a WSW-representation

$$Te = W * S(w * e) \quad (e \in E),$$

where "$*$" is the multiplication corresponding to the unities $1_{\mathcal{E}}$ and $1_{\mathcal{F}}$. Then, after replacing $1_{\mathcal{E}}$ and $1_{\mathcal{F}}$ by $1'_{\mathcal{E}}$ and $1'_{\mathcal{F}}$ and introducing the new multiplication "$\cdot$" in the K-spaces under consideration, the operator T remains a weighted shift operator and admits the WSW-representation

$$Te = W \cdot S'(w' \cdot e) \quad (e \in E),$$

where

$$S'x = (1_{\mathcal{F}}'/1_{\mathcal{F}}) \cdot S(1_{\mathcal{E}} \cdot x) \quad \bigl(x \in (\operatorname{dom} S)/1_{\mathcal{E}}\bigr)$$

and $w' = w/1_{\mathcal{E}}^2$ (here, the division and the power operation also correspond to the new unities). Thus, the notion of a weighted shift operator $T\colon E \to F$ makes sense for "pure" K-spaces E and F, without any dependence on their embedding into universally complete K-spaces and introducing the multiplicative structure. In particular, this implies that a positive operator $T\colon E \to F$ is a weighted shift operator if and only if it can be made a shift operator by an appropriate choice of unities $1_{\mathcal{E}}$ and $1_{\mathcal{F}}$.

Simple examples show that a single weighted shift operator can have different WSW-representations. However, variety of the components of a WSW-representation for a given operator T is naturally restricted by their connection with T and with each other. Two main aspects of this connection are reflected in the following proposition:

Proposition. *Let $T\colon E \to F$ be a weighted shift operator and let $W \circ S \circ w$ be a* WSW*-representation of it. Assign $\rho := \langle \operatorname{im} T \rangle$.*

(1) Denote the shift of T by S_T. Then S_T extends $\rho \circ S$ and the equality $W \circ S \circ w = W \circ S_T \circ w$ holds.

(2) Identify w and W with the corresponding elements of $\mathcal{E}$ and $\mathcal{F}$ and assign $W_T := o\text{-}\lim_{\pi\in\Pi} T\pi(1_{\mathcal{E}}/w) \in \mathcal{F}$, where $\Pi = \{\pi \in \operatorname{Pr}(\mathcal{E}) : \pi(1_{\mathcal{E}}/w) \in E\}$. Then $\rho W = W_T$ and $W \circ S \circ w = W_T \circ S \circ w$.

$\lhd$ Assertion (1) readily follows from 3.7 and 3.8. Let us prove (2). Due to the obvious equality $T \circ \langle w \rangle^{\perp} = 0$, we do not restrict generality by assuming that $\langle w \rangle = \langle 1 \rangle$. Then

$$o\text{-}\lim_{\pi\in\Pi} T\pi(1_{\mathcal{E}}/w) = o\text{-}\lim_{\pi\in\Pi} WS_T w\pi(1_{\mathcal{E}}/w) = o\text{-}\lim_{\pi\in\Pi} WS_T\pi 1_{\mathcal{E}} = \Big(\sup_{\pi\in\Pi} h(\pi)\Big)W,$$

where h is the shadow of T. Since $\rho = h(1)$, it is sufficient to show the relation $\sup_{\pi\in\Pi} h(\pi) = h(1)$. From $E \subset \operatorname{dom}(S_T \circ w)$ it follows that $w[E] \subset \operatorname{dom} S_T = h\mathcal{E}_1$ and, hence, $E \subset h\mathcal{E}_{1/w}$. It remains to employ Lemma 3.6. $\rhd$

Thus, a WSW-representation of a concrete operator determines to a great extent by the choice of the inner weight. Observe that every weighted shift operator admits a WSW-representation with positive inner weight. Indeed, consider an arbitrary WSW-representation $W \circ S \circ w$. Identifying the orthomorphism w with an element of $\mathcal{E}$ (see 2.3), denote the projection $\langle w^+ \rangle \in \operatorname{Pr}(E)$ by π and assign $\rho := \big\langle S(\pi 1_{\mathcal{E}}) \big\rangle$. Then

$$\begin{gathered} W \circ S \circ w = W \circ S \circ \big(\pi|w| - \pi^{\perp}|w|\big) \\ = W \circ \big(\rho \circ S \circ |w| - \rho^{\perp} \circ S \circ |w|\big) = (\rho W - \rho^{\perp} W) \circ S \circ |w|. \end{gathered}$$

Remark. If $W \circ S \circ w$ is a WSW-representation of an operator T with positive inner weight w, then the operators T^+, T^-, and $|T|$ admit the following WSW-representations: $T^+ = W^+ \circ S \circ w$, $T^- = W^- \circ S \circ w$, and $|T| = |W| \circ S \circ w$.

4.2. Theorem. *Let w be an arbitrary positive element of $\mathcal{E}$. A linear operator $T\colon E \to F$ admits a* WSW*-representation with inner weight w if and only if it is disjointness preserving, regular, and wide at the element $1_{\mathcal{E}}/w$.*

$\lhd$ Necessity ensues from Proposition 4.1 (2). Let us prove sufficiency. Suppose that a disjointness preserving operator $T\colon E \to F$ is wide at $1_{\mathcal{E}}/w$. Without loss of generality, we may assume that the operator T is positive. Assign $\Pi := \{\pi \in \mathrm{Pr}(\mathcal{E}) : \pi(1_{\mathcal{E}}/w) \in E\}$ and denote by W the orthomorphism of multiplication by $\sup_{\pi\in\Pi} T\pi(1_{\mathcal{E}}/w) \in \mathcal{F}$. Consider the composition $(1_{\mathcal{F}}/W) \circ T \circ (1_{\mathcal{E}}/w)$ as an operator from $w[E]$ into $\mathcal{F}$ and denote it by S. By proving that S is a shift operator, we will obtain the desired WSW-representation $W \circ S \circ w$ for T. In accordance with Theorem 3.9, it is sufficient to show that the operator S satisfies conditions (a)–(d) presented in the statement of that theorem. Verification of the conditions causes no difficulties. $\rhd$

We say that a subset of the K-space E is *$\mathcal{E}$-bounded* if it is bounded in $\mathcal{E}$. A subset $\mathcal{U}_0$ of an LNS over E is called *$\mathcal{E}$-bounded* if the set $\{\lvert u_0\rvert : u_0 \in \mathcal{U}_0\}$ is $\mathcal{E}$-bounded.

Corollary. *A linear operator $T\colon E \to F$ is a weighted shift operator if and only if it is disjointness preserving, regular, and wide on some $\mathcal{E}$-bounded subset of E.*

$\lhd$ If the operator T is wide on a set $D \subset E$ and an element $e \in \mathcal{E}$ is such that $|d| \leqslant e$ for all $d \in D$, then the operator T is wide at e and, in view of the last theorem, it admits a WSW-representation with inner weight $1_{\mathcal{E}}/e$. $\rhd$

4.3. Proposition. *Assume that regular operators $T, \overline{T}\colon E \to F$ are disjointness preserving and satisfy the inequality $|T| \leqslant |\overline{T}|$. Then T is a weighted shift operator if and only if so is $\overline{T}$. Moreover, the following assertions are true:*

(1) If $\overline{W} \circ \overline{S} \circ \overline{w}$ is a WSW*-representation of $\overline{T}$ then the operator T admits a* WSW*-representation of the form $W \circ \overline{S} \circ \overline{w}$, where $|W| \leqslant |\overline{W}|$.*

(2) If $W \circ S \circ w$ is a WSW*-representation of T then the operator $\overline{T}$ admits a* WSW*-representation of the form $\overline{W} \circ \overline{S} \circ w$, where $\langle \operatorname{im} T\rangle |W| \leqslant |\overline{W}|$.*

$\lhd$ Without loss of generality, we may assume that the operators T and $\overline{T}$ are positive.

(1) The claim is ensured by Corollary 2.10.

(2) Assume that T admits a WSW-representation $W \circ S \circ w$ and assign $\rho := \langle \operatorname{im} T\rangle$. According to Theorem 4.2, the operator T is wide at the element $1_{\mathcal{E}}/w$. Then the operator $\overline{T}$ also has this property and, by the same Theorem 4.2, it admits a WSW-representation $\overline{W} \circ \overline{S} \circ w$. The desired interrelation between W and $\overline{W}$ ensues from Proposition 4.1. $\rhd$

4.4. In accordance with Theorem 4.2, it seems interesting to study situations in which an operator $T\colon E \to F$ turns out to be wide on some $\mathcal{E}$-bounded subset of E. Without touching the general problem, we will only discuss several particular cases.

First of all, we point out a trivial corollary to Theorem 4.2: *if* $\{Te\}^{\perp\perp} = (\operatorname{im} T)^{\perp\perp}$ *for some element* $e \in E$ *then* T *is a weighted shift operator (and it admits a* WSW*-representation with inner weight* $1_{\mathcal{E}}/e$*).* In particular, the following assertion holds:

Proposition. *If there exists a strong order unity* e *in the* K*-space* E *then every disjointness preserving regular operator* $T\colon E \to F$ *is a weighted shift operator and admits a* WSW*-representation with inner weight* $1_{\mathcal{E}}/e$*.*

Of course, the indicated cases admit generalizations. For instance, since every set of pairwise disjoint elements in E is $\mathcal{E}$-bounded, we have the following:

Proposition. *Let* $T\colon E \to F$ *be a disjointness preserving regular operator. If* $\{Te_\xi : \xi \in \Xi\}^{\perp\perp} = (\operatorname{im} T)^{\perp\perp}$ *for some family* $(e_\xi)_{\xi\in\Xi}$ *of pairwise disjoint elements in* E*, then* T *is a weighted shift operator.*

The hypothesis of this proposition is not necessary. Indeed, let $\mathcal{E} = C_\infty(P)$, where P is an extremally disconnected compact space containing a nonisolated point $p \in P$. Denote by E the order-dense ideal $\{e \in \mathcal{E} : e(p) = 0\}$ of the K-space $\mathcal{E}$. Consider the set $Q := P\backslash\{p\}$ and let $\mathcal{F}$ be the K-space of all real-valued functions defined on Q. Define an operator $T\colon E \to \mathcal{F}$ as follows: $Te = e\big|_Q$. Obviously, the operator T is wide on the $\mathcal{E}$-bounded set $\{e \in E : |e| \leqslant 1\}$ (and, therefore, it is a weighted shift operator), but the family $(e_\xi)_{\xi\in\Xi}$ mentioned in the statement of the last proposition does not exist.

Another class of weighted shift operators resulted by combining Lemma 3.6 and Corollaries 1.6 and 3.5.

Theorem. *Every disjointness preserving sequentially o-continuous regular operator* $T\colon E \to F$ *is a weighted shift operator. Moreover, for every order unity* $w \in \mathcal{E}$*, such an operator* T *admits a* WSW*-representation with inner weight* w*.*

4.5. It is known that not every disjointness preserving regular operator is a weighted shift operator. For the sake of completeness, we present here the corresponding example from [1], which is moreover typical in a sense (see below).

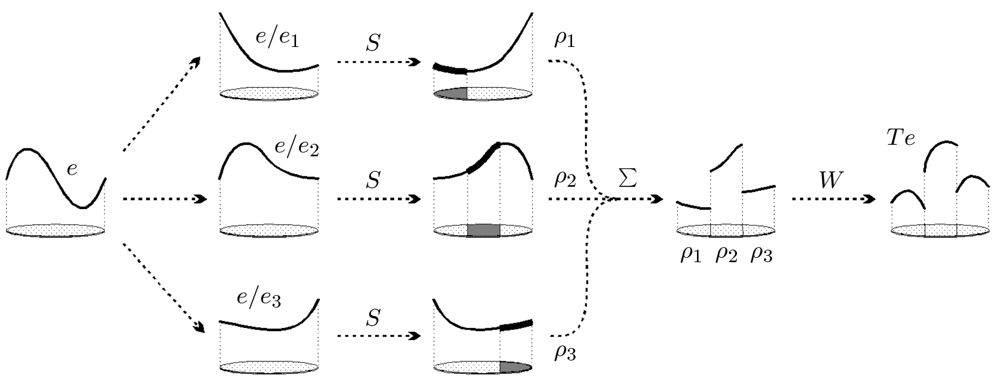
Let Q be an extremally disconnected compact space without isolated points. In this case, we can find an order-dense ideal $E \subset C_\infty(Q)$, a family $(e_\xi)_{\xi\in\Xi}$ in E, and a family $(q_\xi)_{\xi\in\Xi}$ in Q so that the following conditions be satisfied: the set $\{q_\xi : \xi \in \Xi\}$ is dense in Q, $e_\xi(q_\xi) = \infty$ for all $\xi \in \Xi$, and, for each $e \in E$, the number set $\{(e/e_\xi)(q_\xi) : \xi \in \Xi\}$ is bounded. Then the operator $T\colon E \to \ell^\infty(\Xi)$ acting by the rule $(Te)(\xi) = (e/e_\xi)(q_\xi)$ is disjointness preserving and regular (even positive), but is not a weighted shift operator.

The above construction of an operator T possesses the following property: if we denote by ρ_ξ the operator of multiplication by the characteristic function $\chi_{\{\xi\}}$, then we obtain a partition of unity $(\rho_\xi)_{\xi\in\Xi}$ in the algebra $\mathrm{Pr}\big(\ell^\infty(\Xi)\big)$ such that all fragments of the form $\rho_\xi \circ T$ are weighted shift operators. It turns out that of all disjointness preserving regular operators are structured in the same way.

Theorem. *Let $T\colon E \to F$ be a disjointness preserving regular operator. Then there exists a partition of unity $(\rho_\xi)_{\xi\in\Xi}$ in the algebra $\mathrm{Pr}(F)$ such that, for each $\xi \in \Xi$, the composition $\rho_\xi \circ T$ is a weighted shift operator. Moreover, the projections ρ_ξ can be taken so that each composition $\rho_\xi \circ T$ admit a WSW-representation with inner weight $1_{\mathcal{E}}/e_\xi$, where e_ξ is a positive element of E. In this case, the operator T is decomposed into the strongly disjoint sum*

$$T = \bigoplus_{\xi\in\Xi} W \circ \rho_\xi S \circ (1_{\mathcal{E}}/e_\xi),$$

where S is the shift of T and $W\colon \mathcal{F} \to \mathcal{F}$ is the orthomorphism of multiplication by $o\text{-}\sum_{\xi\in\Xi} \rho_\xi T e_\xi$.



◁ By applying the exhaustion principle to the relation $\sup_{e\in E^+}\langle Te\rangle = \langle \mathrm{im}\, T\rangle$, we obtain a disjoint family $(\rho_\xi)_{\xi\in\Xi}$ in the algebra $\mathrm{Pr}(F)$ and a family $(e_\xi)_{\xi\in\Xi}$ of positive elements in E such that $\sup_{\xi\in\Xi} \rho_\xi\langle Te_\xi\rangle = \langle \mathrm{im}\, T\rangle$. After adding the projection $\langle \mathrm{im}\, T\rangle^\perp$ to the family $(\rho_\xi)_{\xi\in\Xi}$ and the zero element to the family $(e_\xi)_{\xi\in\Xi}$, we make $(\rho_\xi)_{\xi\in\Xi}$ a partition of unity and preserve the relation $\sup_{\xi\in\Xi} \rho_\xi\langle Te_\xi\rangle = \langle \mathrm{im}\, T\rangle$. By Theorem 4.2, for each $\xi \in \Xi$, the composition $\rho_\xi \circ T$ is a weighted shift operator and admits a WSW-representation with inner weight $1_{\mathcal{E}}/e_\xi$. If S is the shift of T then the shift of $\rho_\xi \circ T$ is equal to $\rho_\xi S$ (see 3.8); thus, using Proposition 4.1, we conclude that $\rho_\xi \circ T = \rho_\xi T e_\xi \circ \rho_\xi S \circ (1_{\mathcal{E}}/e_\xi)$. ▷

4.6. Let $\mathcal{U}$ be a BKS over an order-dense ideal $E \subset \mathcal{E}$ and let $\mathcal{V}$ be a BKS over an order-dense ideal $F \subset \mathcal{F}$. We say that a linear operator $T\colon \mathcal{U} \to \mathcal{V}$ is a *weighted shift operator* if there exist a BKS $\mathcal{U}'$ over an order-dense ideal $E' \subset \mathcal{E}$, a BKS $\mathcal{V}'$ over an order-dense ideal $F' \subset \mathcal{F}$, orthomorphisms $w\colon \mathcal{U} \to \mathcal{U}'$ and $W\colon \mathcal{V}' \to \mathcal{V}$, and a shift operator $S\colon \mathcal{U}' \to \mathcal{V}'$ such that $T = W \circ S \circ w$. As in the case of an operator in K-spaces, the composition $W \circ S \circ w$ is called a WSW-*representation* of T and the operators W, S, and w are respectively called the *outer weight*, the *shift*, and the *inner weight* of the representation $W \circ S \circ w$.

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{T} & \mathcal{V} \\ {\scriptstyle w}\downarrow & & \uparrow{\scriptstyle W} \\ \mathcal{U}' & \xrightarrow{S} & \mathcal{V}' \end{array}$$

Of course, use of the terminology of 4.1 in the case of operators in LNSs is not quite correct, since a K-space is a particular case of an LNS. Therefore, in order to avoid confusion, we sometimes call a weighted shift operator *scalar* or *vector*, referring to Definition 4.1 or 4.6, respectively. By analogous reasons, we speak about scalar or vector WSW-representations. A vector WSW-representation $W \circ S \circ w$ of an operator $T\colon \mathcal{U} \to \mathcal{V}$ will be called *semivector* if w is a scalar orthomorphism (see 2.8), i.e., $\mathcal{U}$ and $\mathcal{U}'$ are order-dense ideals of the same BKS over $\mathcal{E}$ and the orthomorphism w acts by the rule $u \mapsto eu$ for some fixed orthomorphisms $e \in \operatorname{Orth}(E, E')$.

Theorem. *Let $\mathcal{U}$ be a* BKS *over an order-dense ideal $E \subset \mathcal{E}$ and let $\mathcal{V}$ be a* BKS *over an order-dense ideal $F \subset \mathcal{F}$. A linear operator $T\colon \mathcal{U} \to \mathcal{V}$ is a vector weighted shift operator if and only if it is dominated and its exact dominant $|T|\colon E \to F$ is a scalar weighted shift operator. Moreover, the following assertions hold:*

(1) If $\overline{W} \circ \overline{S} \circ \overline{w}$ is a vector WSW*-representation of T then $|T|$ admits a scalar* WSW*-representation $W \circ |\overline{S}| \circ |\overline{w}|$ such that $0 \leqslant W \leqslant |\overline{W}|$.*

(2) Let $W \circ S \circ w$ be a scalar WSW*-representation of $|T|$ with positive weights W and w. Then T admits a semivector* WSW*-representation $\overline{W} \circ \overline{S} \circ \overline{w}$ such that $|\overline{W}| = W$, $|\overline{S}| = S$, and $\overline{w}$ is the orthomorphism of multiplication by w.*

$\lhd$ (1) The claim readily follows from 4.3 (1).

(2) Suppose that $W \circ S \circ w$ is a scalar WSW-representation of $|T|$, where $w\colon E \to E'$, $S\colon E' \to F'$ and $W\colon F' \to F$. Let $m\mathcal{U}$ be the universal completion of $\mathcal{U}$, let $\mathcal{U}'$ be the ideal $\{u \in m\mathcal{U} : |u| \in E'\}$ of the BKS $m\mathcal{U}$, and let $\overline{w}\colon \mathcal{U} \to \mathcal{U}'$ be the orthomorphism of multiplication by w. Denote by $\mathcal{V}'$ the o-completion of the norm transformation of $\mathcal{U}'$ by means of S (see 0.6) and consider the corresponding operator of norm transformation $\overline{S}\colon \mathcal{U}' \to \mathcal{V}'$. Now, we are to construct an orthomorphism $\overline{W}\colon \mathcal{V}' \to \mathcal{V}$.

Assign $\mathcal{V}_0' := (\overline{S} \circ \overline{w})[\mathcal{U}]$ and define a linear operator $\overline{W}_0 \colon \mathcal{V}_0' \to \mathcal{V}$ as follows: $\overline{W}_0(\overline{S}\overline{w}u) := Tu$. Such a definition is sound, since the equality $\overline{S}\overline{w}u_1 = \overline{S}\overline{w}u_2$ implies

$$\begin{aligned} |Tu_1 - Tu_2| &\leqslant |T||u_1 - u_1| = WSw|u_1 - u_1| \\ &= WS|\overline{w}u_1 - \overline{w}u_2| = W\big|\overline{S}\overline{w}u_1 - \overline{S}\overline{w}u_2\big| = 0. \end{aligned}$$

Assign $\rho := \langle \operatorname{im} T\rangle$. Since $\rho \leqslant \big\langle (\overline{S}\circ\overline{w})[\mathcal{U}]\big\rangle$ and $\overline{w}[\mathcal{U}] = \big\{v' \in \mathcal{V}' : \big|v'\big| \in w[E]\big\}$, the operator $\rho \circ \overline{S}$ is wide on the ideal $w[E] \subset E'$. Consequently, by Proposition 3.2$\big((2)\Rightarrow(3)\big)$, the set $\mathcal{V}_0' = (\rho \circ \overline{S})\big[\overline{w}[\mathcal{U}]\big]$ approximates $(\rho \circ \overline{S})[\mathcal{U}']$. The latter set, by the definition of the norm transformation $S\mathcal{U}'$, approximates the set $\rho[S\mathcal{U}']$, which in turn approximates $\rho[\mathcal{V}']$. Therefore, in view of [14: 1.2], the set $\mathcal{V}_0'$ approximates $\rho[\mathcal{V}']$. Obviously, $\big|\overline{W}_0 v_0'\big| \leqslant W\big|v_0'\big|$ for all $v_0' \in \mathcal{V}_0'$. According to Corollary 1.8, the operator $\overline{W}_0$ admits a (unique) linear extension $\overline{W}_1 \colon \rho[\mathcal{V}'] \to \mathcal{V}$ such that $\big|\overline{W}_1 v'\big| \leqslant W\big|v'\big|$ for all $v' \in \mathcal{V}'$. Then the composition $\overline{W}_1 \circ \rho \colon \mathcal{V}' \to \mathcal{V}$ satisfies the inequality $\big|\overline{W}_1 \circ \rho\big| \leqslant W$ and, consequently, it is an orthomorphism. Thus, we have already constructed a WSW-representation $(\overline{W}_1 \circ \rho) \circ \overline{S} \circ \overline{w}$ of the operator T. However, we cannot assign $\overline{W} := \overline{W}_1 \circ \rho$ at this moment, since the equality $\big|\overline{W}\big| = W$ will not be guaranteed.

For all positive $e \in E$, we have

$$\begin{aligned} \big|\overline{W}_1 \circ \rho\big| Swe &= \sup\big\{\big|\overline{W}_1 \rho v'\big| : \ v' \in \mathcal{V}', \ \big|v'\big| = Swe\big\} \\ &\geqslant \sup\big\{\rho\big|\overline{W}_0 v_0'\big| : \ v_0' \in \mathcal{V}_0', \ \big|v_0'\big| = Swe\big\} \\ &= \sup\big\{\rho\big|\overline{W}_0 \overline{S}\overline{w}u\big| : u \in \mathcal{U}, \ \big|\overline{S}\overline{w}u\big| = Swe\big\} \\ &= \sup\big\{|Tu| : \ Sw|u| = Swe\big\} \\ &\geqslant \sup\big\{|Tu| : \ |u| = e\big\} \\ &= |T|e \\ &= WSwe, \end{aligned}$$

whence $\big|\overline{W}_1 \circ \rho\big| Swe = WSwe$ by the inequality $\big|\overline{W}_1 \circ \rho\big| \leqslant W$. Thus, $W \circ S \circ w$ and $\big|\overline{W}_1 \circ \rho\big| \circ S \circ w$ are two WSW-representations of the operator $|T|$. Hence, according to Proposition 4.1 (2), the equality $\big|\overline{W}_1 \circ \rho\big| = \rho W$ holds. To ensure the equality $\big|\overline{W}\big| = W$, it is sufficient to define $\overline{W}$ as the sum of the orthomorphism $\overline{W}_1 \circ \rho$ and some "inactive" supplement with norm $\rho^\perp W$. Proposition 2.4 implies existence of an orthomorphism $\overline{W}_2 \in \operatorname{Orth}(\mathcal{U}, \mathcal{V})$ such that $\big|\overline{W}_2\big| = W$. We assign $\overline{W} := \overline{W}_1 \circ \rho + \overline{W}_2 \circ \rho^\perp$. $\rhd$

Remark. (1) The inequality $W \leqslant |\overline{W}|$ presented in assertion (1) of the last theorem can be strict. In other words, the equality $|T| = |\overline{W}|\circ|\overline{S}|\circ|\overline{w}|$ cannot be guaranteed for every WSW-representation $T = \overline{W}\circ\overline{S}\circ\overline{w}$. (A simple counterexample can be given in the case when $\mathcal{U}$ and $\mathcal{V}$ are Banach spaces.) However, (2) implies that every weighted shift operator $T\colon \mathcal{U} \to \mathcal{V}$ admits a WSW-representation $\overline{W}\circ\overline{S}\circ\overline{w}$ such that $|T| = |\overline{W}|\circ|\overline{S}|\circ|\overline{w}|$.

(2) From the last theorem it follows that each vector weighted shift operator admits a semivector WSW-representation. Moreover, if an operator admits a vector WSW-representation with inner weight w then it admits a semivector WSW-representation with inner weight the operator of multiplication by $|w|$.

(3) If we consider each of the K-spaces E and F as a BKS (over itself) then the exact dominant of every regular operator $T\colon E \to F$ coincides with its modulus $|T|$. This observation and the last theorem allow us to conclude the following: a mapping $T\colon E \to F$ is a vector weighted shift operator if and only if it is a scalar weighted shift operator. This fact justifies correctness of using the common term "weighted shift operator" for operators in BKSs as well as for operators in K-spaces.

4.7. Each of the assertions stated in the following theorem readily follows from a similar "scalar" assertion (see 4.1–4.4) and Theorem 4.6.

Theorem. *Let $\mathcal{U}$ be a* BKS *over an order-dense ideal $E \subset \mathcal{E}$ and let $\mathcal{V}$ be a* BKS *over an order-dense ideal $F \subset \mathcal{F}$.*

(1) The property of a mapping $T\colon \mathcal{U} \to \mathcal{V}$ to be a weighted shift operator does not depend on choosing unities $1_{\mathcal{E}}$ and $1_{\mathcal{F}}$.

(2) A linear operator $T\colon \mathcal{U} \to \mathcal{V}$ is a weighted shift operator if and only if it is bounded, preserves disjointness, and satisfies the relation $T[\mathcal{U}_0]^{\perp\perp} = T[\mathcal{U}]^{\perp\perp}$ for some $\mathcal{E}$-bounded subset $\mathcal{U}_0 \subset \mathcal{U}$.

(3) Let w be an arbitrary positive element of $\mathcal{E}$. A linear operator $T\colon \mathcal{U} \to \mathcal{V}$ admits a WSW*-representation with inner weight of norm w if and only if it is disjointness preserving, bounded, and wide at the element $1_{\mathcal{E}}/w$.*

(4) Suppose that $T\colon \mathcal{U} \to \mathcal{V}$ is a disjointness preserving bounded operator. If $\{Tu\}^{\perp\perp} = (\operatorname{im} T)^{\perp\perp}$ for some element $u \in \mathcal{U}$ then T is a weighted shift operator and admits a WSW*-representation with inner weight of norm $1_{\mathcal{E}}/|u|$.*

(5) If there exists a strong order unity e in the K*-space E then every disjointness preserving bounded operator $T\colon \mathcal{U} \to \mathcal{V}$ is a weighted shift operator and admits a* WSW*-representation with inner weight of norm $1_{\mathcal{E}}/e$.*

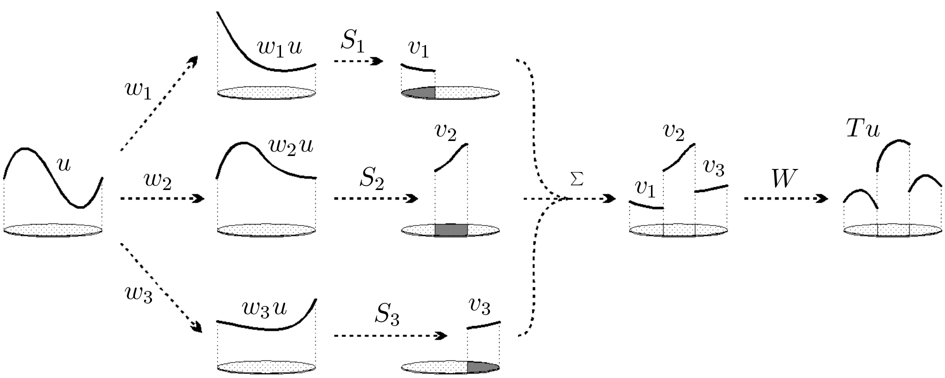
(6) Every disjointness preserving sequentially o-continuous bounded operator $T\colon \mathcal{U} \to \mathcal{V}$ is a weighted shift operator. Moreover, for every order unity $w \in \mathcal{E}$, such an operator T admits a WSW*-representation with inner weight of norm w.*

4.8. Theorem. *Suppose that $\mathcal{U}$ is a* BKS *over an order-dense ideal $E \subset \mathcal{E}$, $\mathcal{V}$ is a* BKS *over an order-dense ideal $F \subset \mathcal{F}$, $m\mathcal{U}$ and $m\mathcal{V}$ are universal completions of $\mathcal{U}$ and $\mathcal{V}$, and $T\colon \mathcal{U} \to \mathcal{V}$ is a disjointness preserving bounded operator. Then there exists a partition of unity $(\rho_\xi)_{\xi\in\Xi}$ in the algebra* $\mathrm{Pr}(\mathcal{V})$ *such that, for each $\xi \in \Xi$, the composition $\rho_\xi \circ T$ is a weighted shift operator.*

The projections ρ_ξ can be chosen so that each composition $\rho_\xi \circ T$ admit WSW*-representation with inner weight of norm $1_{\mathcal{E}}/e_\xi$, where e_ξ is a positive element of E.*

For each $\xi \in \Xi$, assign $E_\xi := \{e/e_\xi : e \in E\}$ and $\mathcal{U}_\xi := \{u \in m\mathcal{U} : |u| \in E_\xi\}$, where $m\mathcal{U}$ is the universal completion of $\mathcal{U}$, and denote by $w_\xi\colon \mathcal{U} \to \mathcal{U}_\xi$ the scalar orthomorphism of multiplication by $1_{\mathcal{E}}/e_\xi$. Then there exist a BKS *$\mathcal{V}'$ over $\mathcal{F}$, strongly disjoint shift operators $S_\xi\colon \mathcal{U}_\xi \to \mathcal{V}'$ ($\xi \in \Xi$), and an orthomorphism $W\colon \mathcal{V}' \to m\mathcal{V}$ such that the operators T and $|T|$ decompose into the following strongly disjoint sums:*

$$T = \bigoplus_{\xi\in\Xi} W \circ S_\xi \circ w_\xi, \qquad |T| = \bigoplus_{\xi\in\Xi} |W| \circ |S_\xi| \circ |w_\xi|.$$



$\lhd$ Let $T\colon \mathcal{U} \to \mathcal{V}$ be an arbitrary disjointness preserving bounded operator. By Theorem 4.5, there exists a partition of unity $(\rho_\xi)_{\xi\in\Xi}$ in the algebra $\mathrm{Pr}(F)$ such that, for each $\xi \in \Xi$, the composition $\rho_\xi \circ |T|$ is a weighted shift operator and, moreover, admits a WSW-representation with inner weight $1_{\mathcal{E}}/e_\xi$, where e_ξ is a positive element of E. Define BKSs $\mathcal{U}_\xi$ and orthomorphisms $w_\xi\colon \mathcal{U} \to \mathcal{U}_\xi$ in the same way as in the statement of the theorem being proved. By Theorem 4.6, for each $\xi \in \Xi$, there exist a BKS $\mathcal{V}_\xi$ over an order-dense ideal $F_\xi \subset \rho_\xi[\mathcal{F}]$, a shift operator $S_\xi\colon \mathcal{U}_\xi \to \mathcal{V}_\xi$, and an orthomorphism $W_\xi\colon \mathcal{V}_\xi \to \rho_\xi[\mathcal{V}]$ such that

$$\rho_\xi \circ T = W_\xi \circ S_\xi \circ w_\xi,$$
$$\rho_\xi \circ |T| = |W_\xi| \circ |S_\xi| \circ |w_\xi|.$$

In order to complete the proof, it remains to construct the desired BKS $\mathcal{V}'$ and "glue" the orthomorphisms W_ξ together to obtain a single orthomorphism W.

Assign $\mathcal{V}_0' := \bigoplus_{\xi\in\Xi} \mathcal{V}_\xi$ (see 0.5) and denote by $\mathcal{V}'$ a universal completion of the BKS $\mathcal{V}_0'$. Naturally identifying $\mathcal{V}_\xi$ and $\rho_\xi[\mathcal{V}_0']$, we regard S_ξ as an operator from $\mathcal{U}_\xi$ into $\mathcal{V}'$. For each element

$$v_0' = (v_\xi)_{\xi\in\Xi} \in \mathcal{V}_0',$$

assign

$$W_0(v') := o\text{-}\sum_{\xi\in\Xi} W_\xi(v_\xi) \in m\mathcal{V}.$$

Due to Corollary 1.8, the orthomorphism $W_0\colon \mathcal{V}_0' \to m\mathcal{V}$ admits a unique extension to an orthomorphism $W\colon \mathcal{V}' \to m\mathcal{V}$. $\triangleright$

5. Representation of disjointness preserving operators

Constructing analytic representations of disjointness preserving operators is an old tradition. This question was studied by everyone who was interested in these operators from an abstract point of view. Representation of various classes of operators as composition and multiplication mappings is presented, for instance, in [1, 3–6, 19, 30, 31, 33, 34]. According to the Vulikh–Ogasawara theorem [12: 0.3.4], an order-dense ideal of the K-space $C_\infty(Q)$, with Q an extremally disconnected compact space, is a general form of a K-space. Furthermore, according to the Corollary [12: 2.4.4], order-dense ideals of the LNS $C_\infty(Q,\mathcal{X})$, where $\mathcal{X}$ is a continuous Banach bundle over Q, exhaust all BKSs. These two facts provide a base for representation methods of studying operators in K-spaces and BKSs. Analytic representations of operators are constructed in this section with the help of such operations as continuous change of variable, pointwise multiplication by a real-valued function, and pointwise evaluation of an operator-valued function.

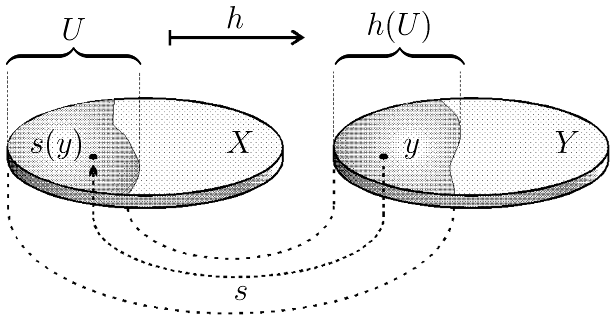
Throughout the section, X and Y are totally disconnected, and P and Q extremally disconnected compact spaces. The symbol 1_M denotes the function on a set M which is identically equal to unity.

5.1. Assume that some "abstract" objects A and B (for instance, Boolean algebras, K-spaces, or BKSs) are represented via isomorphisms $i\colon A \to \widehat{A}$ and $j\colon B \to \widehat{B}$ in the form of some "concrete" objects $\widehat{A}$ and $\widehat{B}$ (for instance, algebras of sets or spaces of functions). Then the *interpretation* of a mapping $f\colon A \to B$ (with respect to the representations i and j) is defined to be the composition $j \circ f \circ i^{-1}\colon \widehat{A} \to \widehat{B}$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d, "j"] \\ \widehat{A} \arrow[r, "\text{interpre-}", "\text{tation of } f"'] & \widehat{B} \end{tikzcd}$$

5.2. Denote by $C_0(Y,X)$ the totality of all continuous functions $s\colon Y_0 \to X$ defined on various clopen subsets $Y_0 \subset Y$.

Proposition. *A mapping $h\colon \mathrm{Clop}(X) \to \mathrm{Clop}(Y)$ is a ring homomorphism if and only if there exists a function $s \in C_0(Y,X)$ such that $h(U) = s^{-1}[U]$ for all $U \in \mathrm{Clop}(X)$. For every ring homomorphism h, such a function s is unique.*

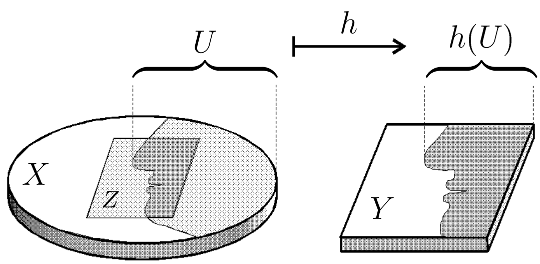


◁ The claim follows directly from the well-known theorem of R. Sikorski (see [26: Section 11; 25]). ▷

The relation $h(U) = s^{-1}[U]$ is called the *representation* of the ring homomorphism h by means of the function s. Observe that, due to the Stone theorem, the above proposition describes the structure of ring homomorphisms acting in arbitrary Boolean algebras.

5.3. The following proposition shows that every ring homomorphism (to within an isomorphism) is the mapping of intersection with a fixed set.

Proposition. *Let $h\colon \mathrm{Clop}(X) \to \mathrm{Clop}(Y)$ be a ring homomorphism. Then there exist a closed subset $Z \subset X$ and an order isomorphism i of the Boolean algebra $\mathrm{Clop}(Z)$ onto $\mathrm{im}\, h$ such that $h\big(U\big) = i(U \cap Z)$ for all $U \in \mathrm{Clop}(X)$.*



◁ Let $h(U) = s^{-1}[U]$ be the representation of h by means of a function $s \in C_0(Y,X)$. Assign $Z := \mathrm{im}\, s$ and, for each element $W \in \mathrm{Clop}(Z)$, define the set $i(W) \in \mathrm{Clop}(Y)$ by the formula $i(W) := s^{-1}[W]$. Verification of the assertions of the theorem causes no difficulties. ▷

5.4. Proposition. *Let E and F be order-dense ideals of $C_\infty(Q)$. A mapping $W\colon E \to F$ is an orthomorphism if and only if there exists a function $w \in C_\infty(Q)$ such that*

$$W(e) = we \quad (e \in E).$$

For every orthomorphism W, such a function w is unique.

$\lhd$ The assertions stated are a reformulation of Theorem 2.3 with account taken of Theorem [12: 0.3.4]. $\rhd$

The relation $W(e) = we$ is called the *representation* of the orthomorphism W by means of the function w. Observe that, due to Theorem [12: 0.3.4], the last proposition describes the structure of orthomorphisms acting in arbitrary K-spaces.

5.5. Given arbitrary functions $s \in C_0(Q, P)$ and $e \in C_\infty(P)$, the function $e \bullet s\colon Q \to \overline{\mathbb{R}}$ is defined as follows:

$$(e \bullet s)(q) := \begin{cases} e\big(s(q)\big) & \text{if } q \in \operatorname{dom} s, \\ 0 & \text{if } q \in Q \setminus \operatorname{dom} s. \end{cases}$$

Of course, to ensure correctness, while using the notation $e \bullet s$, we must always have in mind a fixed set Q containing $\operatorname{dom} s$. Obviously, the function $e \bullet s$ is continuous but, in general, does not belong to $C_\infty(Q)$, since it can assume infinite values on a set with nonempty interior. The totality of all functions $e \in C_\infty(P)$ for which $e \bullet s \in C_\infty(Q)$ is denoted by $C_s(P)$.

Proposition. *Let $h\colon \operatorname{Pr}\big(C_\infty(P)\big) \to \operatorname{Pr}\big(C_\infty(Q)\big)$ be a ring homomorphism and let $hC(P)$ be the order-dense ideal of $C_\infty(P)$ defined in 3.3. Then*

$$hC(P) = C_s(P),$$

where $h(U) = s^{-1}[U]$ is the representation of h by means of an $s \in C_0(Q, P)$ (with respect to the natural representations of $\operatorname{Pr}\big(C_\infty(P)\big)$ and $\operatorname{Pr}\big(C_\infty(Q)\big)$).

$\lhd$ The claim follows from Propositions 3.3 and 5.2. $\rhd$

A continuous function $s\colon Q \to P$ is called *σ-exact*, if $s^{-1}[\operatorname{cl} G] = \operatorname{cl} s^{-1}[G]$ for every open σ-closed subset $G \subset P$. Below (see. 6.1), this property of a function is considered in more detail.

Lemma. *Denote the image of a function $s \in C_0(Q,P)$ by R.*

(1) For every function $e \in C_s(P)$, the intersection $R \cap \operatorname{dom} e$ is dense in R, i.e., $C_s(P) \subset \{e \in C_\infty(P) : e|_R \in \overline{C}_\infty(R)\}$.

(2) If the restriction $s|^R$ is a σ-exact function then $C_s(P) = \{e \in C_\infty(P) : e|_R \in \overline{C}_\infty(R)\}$ and $\overline{C}_\infty(R) = \{e|_R : e \in C_s(P)\}$.

$\lhd$ (1) Consider a function $e \in C_s(P)$. If there were a nonempty open set $W \subset R$ disjoint from $\operatorname{dom} e$ then the function $e \bullet s$ would assume infinite values on the nonempty open set $s^{-1}[W]$, which would contradict the inclusion $e \bullet s \in C_\infty(Q)$. Consequently, the intersection $R \cap \operatorname{dom} e$ is dense in R.

(2) Let a function $e \in C_s(P)$ be such that the intersection $R \cap \operatorname{dom} e$ is dense in R. Then, using the fact that the function $s|^R$ is σ-exact and the intersection $R \cap \operatorname{dom} e$ is a σ-closed open subset of R, we obtain

$$\begin{aligned} \operatorname{cl}(e \circ s)^{-1}[\mathbb{R}] &= \operatorname{cl} s^{-1}[\operatorname{dom} e] = \operatorname{cl} s^{-1}[R \cap \operatorname{dom} e] \\ &= s^{-1}\big[\operatorname{cl}(R \cap \operatorname{dom} e)\big] = s^{-1}[R] = \operatorname{dom} s, \end{aligned}$$

and the first equality is established. The second equality follows from the first one due to the Tietze–Urysohn theorem. $\rhd$

Remark. The requirement in condition (2) of the lemma, that the function $s|^R$ be σ-exact, is essential since the set $C_s(P)$ is not in general determined by the image of s. Indeed, suppose that $p \in P$ is not a P-point, i.e., the intersection of some sequence of neighborhoods of p is not a neighborhood of p. Let $\overline{P} := P \cup \{\infty\}$ be the enrichment of P by a new isolated point ∞. Then the identity function $s \colon P \to P$ and the function $\bar{s} := s \cup \{(\infty, p)\} \colon \overline{P} \to P$ have the same image, while the sets $C_s(P)$ and $C_{\bar{s}}(P)$ does not coincide.

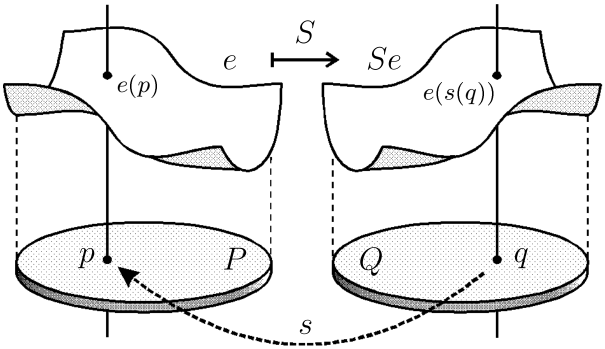
5.6. If $E \subset C_\infty(P)$ and $R \subset P$ then the set $\{e|_R : e \in E\}$ is denoted by $E|_R$.

Lemma. *Denote the image of a function $s \in C_0(Q,P)$ by R and assume that the function $s|^R$ is σ-exact. Then*

(1) *$\overline{C}_\infty(R)$ is a vector sublattice of $C_\infty(R)$;*
(2) *if E is an ideal of the* K-*space $C_s(P)$ then $E|_R$ is an ideal of the vector lattice $\overline{C}_\infty(R)$.*

$\lhd$ Assertion (1) readily follows from Lemma 5.5 (2). Let us prove (2). Assume that a function $g \in \overline{C}_\infty(R)$ satisfies the inequalities $0 \leqslant g \leqslant e|_R$ for some positive element $e \in E$. In view of Lemma 5.5 (2), there is a positive function $\bar{e} \in C_s(P)$ such that $g = \bar{e}|_R$. Then $\bar{e} \wedge e \in E$ and $g = (\bar{e} \wedge e)|_R$. $\rhd$

5.7. Proposition. *Let E be an order-dense ideal of $C_\infty(P)$ and let F be an order-dense ideal of $C_\infty(Q)$. A mapping $S\colon E \to F$ is a shift operator if and only if there exists a function $s \in C_0(Q,P)$ such that $Se = e \bullet s$ for all $e \in E$.*



$\triangleleft$ Sufficiency can be easily established with the help of Theorem 3.9. Let us show necessity. Suppose that $S\colon E \to F$ is a shift operator and $h\colon \mathrm{Pr}(E) \to \mathrm{Pr}(F)$ is its shadow. Represent the algebras $\mathrm{Pr}(E)$ and $\mathrm{Pr}(F)$ as $\mathrm{Clop}(P)$ and $\mathrm{Clop}(Q)$ and consider the representation $\hat{h}(U) = s^{-1}[U]$ of the corresponding interpretation $\hat{h}\colon \mathrm{Clop}(P) \to \mathrm{Clop}(Q)$ of the homomorphism h by means of an $s \in C_0(Q,P)$. According to Proposition 5.5, the equality $hC(P) = C_s(P)$ holds. Since the operators $(e \mapsto e \bullet s)\colon C_s(P) \to C_\infty(Q)$ and $S_h\colon hC(P) \to C_\infty(Q)$ have the same shadow h and satisfy the equalities $1_P \bullet s = S_h(1_P) = h(1)1_Q$, they coincide in view of Proposition 3.7. Therefore, $Se = S_h e = e \bullet s$ for all $e \in E$. $\triangleright$

5.8. The function s connected with the shift operator S in the way described in the last proposition is not unique in general. Indeed, assume that the compact space P contains two distinct nonisolated points p_1 and p_2, assign $E := \{e \in C_\infty(P) : e(p_1) = e(p_2) = 0\}$ and consider the functions $s_1, s_2\colon Q \to P$ identically equal to p_1 and p_2, respectively. Then $e \bullet s_1 = e \bullet s_2 = 0$ for all $e \in E$.

The following proposition clarifies the question about unique representation of a shift operator.

Proposition. *Let E be an order-dense ideal of $C_\infty(P)$, let F be an order-dense ideal of $C_\infty(Q)$, and let $S\colon E \to F$ be a shift operator. Assign $Q_0 := \operatorname{supp} \operatorname{im} S = \operatorname{cl} \bigcup_{e\in E} \operatorname{supp} Se$.*

(1) If functions $s_1, s_2 \in C_0(Q,P)$ satisfy the equalities $Se = e \bullet s_1 = e \bullet s_2$ for all $e \in E$ then $Q_0 \subset \operatorname{dom} s_1 \cap \operatorname{dom} s_2$ and $s_1 = s_2$ on Q_0.

(2) There exists a unique function $s \in C(Q_0,P)$ such that $Se = e \bullet s$ for all $e \in E$. Furthermore, if s is such a function then $h(U) = s^{-1}[U]$ is a representation of the shadow h of the operator S.

◁ (1) Denote by D the totality of all points in P, at which some functions in E are nonzero. Obviously, the set $s_1^{-1}[D]$ is dense in Q_0; therefore, it is sufficient to establish the equality $s_1 = s_2$ on this set. Take an arbitrary point $q \in s_1^{-1}[D]$ and assume to the contrary that $s_1(q) \neq s_2(q)$. Since $s_1(q) \in D$, there exists a function $e \in E$ that satisfies the relations $e\big(s_1(q)\big) \neq 0$ and $e\big(s_2(q)\big) = 0$, which contradicts the equality $e \bullet s_1 = e \bullet s_2$.

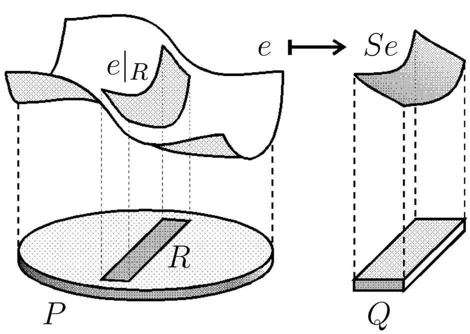
(2) Existence of the function s follows from Proposition 5.7, and its uniqueness from assertion (1). The fact that s represents the shadow of S ensues from the proof of the Proposition 5.7. ▷

If a function s satisfies the conditions of assertion (2) then the relation $Se = e \bullet s$ is called the *representation* of the shift operator S by means of the function s. Observe that, due to Theorem [12: 0.3.4], Propositions 5.7 and 5.8 describe the structure of shift operators acting in arbitrary K-spaces.

5.9. The following proposition shows that every shift operator (to within an isomorphism) is the operator of restriction onto a fixed set.

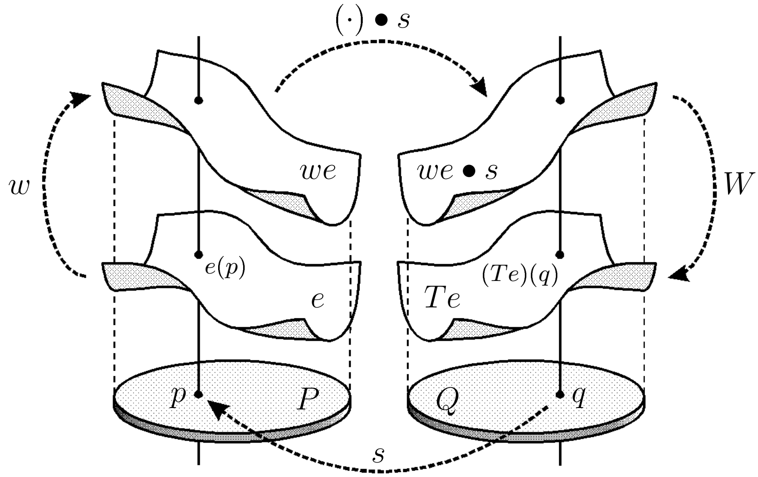
Proposition. *Let E be an order-dense ideal of $C_\infty(P)$, let F be an order-dense ideal of $C_\infty(Q)$, and let $S\colon E \to F$ be a shift operator. Then there exist a closed subset $R \subset P$ and a mapping $i\colon E|_R \to F$ such that*

(1) *$E|_R$ is a vector sublattice of the K-space $C_\infty(R)$;*
(2) *i is a linear and order isomorphism of $E|_R$ onto $\operatorname{im} S$;*
(3) *$Se = i\big(e|_R\big)$ for all $e \in E$.*



◁ Let $Se = e \bullet s$ be the representation of S by means of a function $s \in C_0(Q, P)$. Assign $R := \operatorname{im} s$ and, for each element $g \in E|_R$, define the function $i(g) \in C(Q, \overline{\mathbb{R}})$ by the formula $i(g) := g \bullet s$. Verification of assertions (1)–(3) causes no difficulties. ▷

5.10. Theorem. *Let E be an order-dense ideal of $C_\infty(P)$ and let F be an order-dense ideal of $C_\infty(Q)$. A mapping $T\colon E \to F$ is a weighted shift operator if and only if there exist functions $s \in C_0(Q, P)$, $w \in C_\infty(P)$, and $W \in C_\infty(Q)$ such that $we \bullet s \in C_\infty(Q)$ and $Te = W(we \bullet s)$ for all $e \in E$.*



$\lhd$ The claim readily follows from Propositions 5.4 and 5.7. $\rhd$

5.11. Simple examples show that the components of a representation $Te = W(we \bullet s)$ of a weighted shift operator T are not unique. However, omitting certain details, we may say that the function s is unique and W is uniquely determined by the choice of w. This observation can be precisely stated as follows:

Proposition. *Let E be an order-dense ideal of $C_\infty(P)$, let F be an order-dense ideal of $C_\infty(Q)$, and let $T\colon E \to F$ be a disjointness preserving regular operator. Assign $Q_0 := \operatorname{supp} \operatorname{im} T$.*

(1) Let functions $s_1, s_2 \in C_0(Q, P)$, $w_1, w_2 \in C_\infty(P)$, and $W_1, W_2 \in C_\infty(Q)$ be such that $Te = W_1(w_1 e \bullet s_1) = W_2(w_2 e \bullet s_2)$ for all $e \in E$. Then $Q_0 \subset \operatorname{dom} s_1 \cap \operatorname{dom} s_2$ and $s_1 = s_2$ on Q_0. If, in addition, $w_1 = w_2$ then $W_1 = W_2$ on Q_0.

(2) Let a positive function $w \in C_\infty(P)$ be such that T is wide at $1/w$ (see 3.6). Then there exist unique functions $s \in C(Q_0, P)$ and $W \in C_\infty(Q)$ such that $W = 0$ outside Q_0 and $Te = W(we \bullet s)$ for all $e \in E$. Furthermore, $\operatorname{supp} W = s^{-1}[\operatorname{supp} w] = Q_0$, $Se = e \bullet s$ is a representation of the shift S of the operator T, and $h(U) = s^{-1}[U]$ is a representation of its shadow h.

$\lhd$ Assertion (1) follows immediately from Proposition 4.1 (due to 5.4 and 5.8). Let us show (2). Existence of functions s and W ensues from Theorems 4.2 and 5.10, and their uniqueness from assertion (1). Connection of the function s with the shift and shadow of the operator T follows from Propositions 4.1 (1) and 5.8 (2). $\rhd$

If s, w, and W satisfy the conditions stated in assertion (2), then the relation $Te = W(we \bullet s)$ is called the *representation* of the weighted shift operator T by means of the functions s, w, and W. Observe that, due to the Theorem [12: 0.3.4], assertions 5.10 and 5.11 describe the structure of weighted shift operators acting in arbitrary K-spaces.

Remark. If $Te = W(we \bullet s)$ is a representation of a weighted shift operator T then the operators T^+, T^-, and $|T|$ admit the following representations: $T^+e = W^+(we \bullet s)$, $T^-e = W^-(we \bullet s)$, $|T|e = |W|(we \bullet s)$.

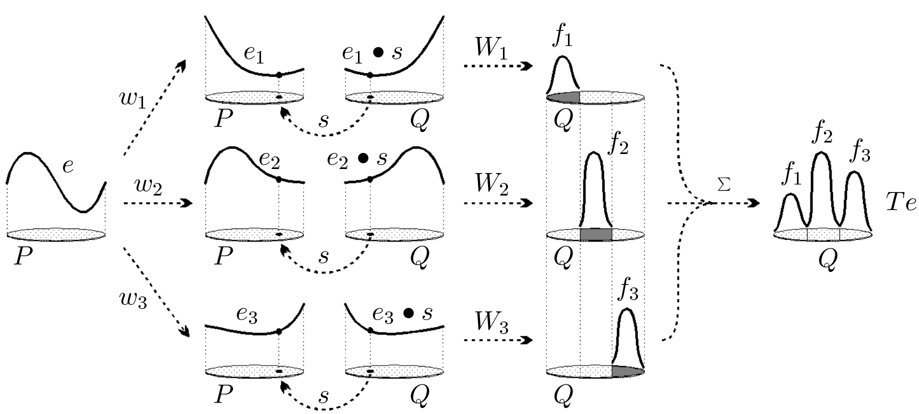
5.12. Given arbitrary functions $f, g \in C(Q, \overline{\mathbb{R}})$, the product $fg \in C(Q, \overline{\mathbb{R}})$ is defined by the rule

$$(fg)(q) := \begin{cases} f(q)g(q) & \text{if the product } f(q)g(q) \text{ makes sense,} \\ & \text{i.e., does not have the form } 0 \cdot \pm\infty \text{ or } \pm\infty \cdot 0, \\ 0 & \text{if } f \equiv 0 \text{ or } g \equiv 0 \text{ in a neighborhood of } q \end{cases}$$

on a dense subset of Q and then extends onto the entire space Q by continuity.

Theorem. *Let E be an order-dense ideal of $C_\infty(P)$, let F be an order-dense ideal of $C_\infty(Q)$, and let $T\colon E \to F$ be a disjointness preserving regular operator. Consider the representation $h(U) = s^{-1}[U]$ of the shadow h of the operator T by means of a function $s \in C_0(Q, P)$. Then there exist a family $(w_\xi)_{\xi\in\Xi}$ of positive functions in $C_\infty(P)$ and a family $(W_\xi)_{\xi\in\Xi}$ of pairwise disjoint functions in $C_\infty(Q)$ such that $1/w_\xi \in E$ for all $\xi \in \Xi$ and*

$$Te = o\text{-}\sum_{\xi\in\Xi} W_\xi(w_\xi e \bullet s) \quad (e \in E). \qquad (*)$$



◁ The assertion stated is a reformulation of Theorem 4.5 with account taken of Proposition 5.11 (2). ▷

Observe that the functions $w_\xi e \bullet s$ in the representation $(*)$, being continuous functions from Q into $\overline{\mathbb{R}}$, need not belong to $C_\infty(Q)$ while the products $W_\xi(w_\xi e \bullet s)$ do belong to $C_\infty(Q)$.

We call the relation

$$Te = o\text{-}\sum_{\xi\in\Xi} W_\xi(w_\xi e \bullet s)$$

the *representation* of the operator T by means of the functions s, w_ξ, and W_ξ. Observe that, due to Theorem [12: 0.3.4], the last theorem describes the structure of disjointness preserving regular operators acting in arbitrary K-spaces.

R e m a r k . If $Te = o\text{-}\sum_{\xi\in\Xi} W_\xi(w_\xi e \bullet s)$ is a representation of the operator T then the operators T^+, T^-, and $|T|$ admit the following representations:

$$T^+e = o\text{-}\sum_{\xi\in\Xi} W_\xi^+(w_\xi e \bullet s),$$

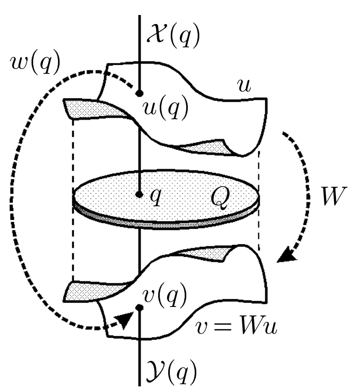
$$T^-e = o\text{-}\sum_{\xi\in\Xi} W_\xi^-(w_\xi e \bullet s),$$

$$|T|e = o\text{-}\sum_{\xi\in\Xi} \big|W_\xi\big|(w_\xi e \bullet s).$$

The remainder of the current section is devoted to representation of operators in Banach–Kantorovich spaces.

5.13. If $\mathcal{X}$ and $\mathcal{Y}$ are ample CBBs over Q (see [12: Chapter 2], wherein ample CBBs are called “complete”), $u \in C_\infty(Q,\mathcal{X})$ and $w \in C_\infty\big(Q, B(\mathcal{X},\mathcal{Y})\big)$, then the section $\mathrm{ext}(w \otimes u) \in C_\infty(Q,\mathcal{Y})$ is denoted by $w \mathbin{\overline{\otimes}} u$.

Proposition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be ample* CBBs *over Q and let E and F be order-dense ideals of $C_\infty(Q)$. A mapping $W\colon E(\mathcal{X}) \to F(\mathcal{Y})$ is an orthomorphism if and only if there exists a section $w \in C_\infty\big(Q, B(\mathcal{X},\mathcal{Y})\big)$ such that $Wu = w \mathbin{\overline{\otimes}} u$ for all $u \in E(\mathcal{X})$. For every orthomorphism W, such a section w is unique. Furthermore, $\boldsymbol{|}W\boldsymbol{|}(e) = \boldsymbol{|}w\boldsymbol{|}e$ for all $e \in E$.*



$\lhd$ Let $W\colon E(\mathcal{X}) \to F(\mathcal{Y})$ be an orthomorphism. Consider the representation $\boldsymbol{|}W\boldsymbol{|}(e) = ge$ of the orthomorphism $\boldsymbol{|}W\boldsymbol{|}\colon E \to F$ by means of a function $g \in C_\infty(Q)$ (see 2.4 and 5.4). Denote by D the (open, dense) set of all points of Q at which the function g is finite and some functions in E are nonzero. In addition, assign $E_1 := E \cap C(Q)$. Define the mapping $w_0 \colon q \in D \mapsto w(q) \in B\big(\mathcal{X}(q), \mathcal{Y}(q)\big)$ as follows: for every $q \in D$ and $x \in \mathcal{X}(q)$, take a section $u \in E_1(\mathcal{X})$ satisfying $u(q) = x$ (such a section exists in view of [12: 1.3.11]) and assign $w_0(q)x := (Wu)(q)$. Correctness of this definition and boundedness of the operator $w_0(q)$ are ensured by the relations

$$\|(Wu)(q)\| = |Wu|(q) \leqslant \big(|W||u|\big)(q) = \big(g|u|\big)(q) = g(q)\|u(q)\|$$

that are valid for all $q \in D$ and $u \in E_1(\mathcal{X})$. By Theorem [12: 2.2.13], we have $w_0 \in C\big(D, B(\mathcal{X}, \mathcal{Y})\big)$. Assign $w := \operatorname{ext}(w_0) \in C_\infty\big(Q, B(\mathcal{X}, \mathcal{Y})\big)$. The construction of the section w directly implies that $Wu = w \mathbin{\overline{\otimes}} u$ for all $u \in E_1(\mathcal{X})$. The set $E_1(\mathcal{X})$ is an order-dense ideal of the BKS $E(\mathcal{X})$ and thus approximates the latter. Therefore, in view of 2.6, the orthomorphisms W and $u \mapsto w \mathbin{\overline{\otimes}} u$ coincide on the entire space $E(\mathcal{X})$.

Show uniqueness of w. Suppose that sections $w_1, w_2 \in C_\infty\big(Q, B(\mathcal{X}, \mathcal{Y})\big)$ satisfy the equality $w_1 \mathbin{\overline{\otimes}} u = w_2 \mathbin{\overline{\otimes}} u$ for all $u \in E(\mathcal{X})$. Denote by D_0 the set of all points of Q, at which some functions in E are nonzero, and assign $D := D_0 \cap \operatorname{dom} w_1 \cap \operatorname{dom} w_2$. Consider a $q \in D$ and an $x \in \mathcal{X}(q)$. In view of [12: 1.3.11], there is a section $u \in E(\mathcal{X})$ such that $u(q) = x$. Therefore,

$$w_1(q)x = (w_1 \mathbin{\overline{\otimes}} u)(q) = (w_2 \mathbin{\overline{\otimes}} u)(q) = w_2(q)x.$$

For proving the equality $w_1 = w_2$, it remains to observe that the set D is dense in Q.

Let us establish the equality $|W|(e) = |w|e$. From [12: 1.3.11] it follows that

$$|w| = \sup\{\big|w \mathbin{\overline{\otimes}} u\big| :\ u \in C(Q, \mathcal{X}),\ |u| \leqslant 1\}.$$

Therefore, for all positive $e \in E$, we have

$$\begin{aligned} |W|e &= \sup_{|u| \leqslant e} |Wu| = \sup_{|u| \leqslant 1} \big|W(eu)\big| \\ &= \sup_{|u| \leqslant 1} \big|w \mathbin{\overline{\otimes}} (eu)\big| = \sup_{|u| \leqslant 1} \big|w \mathbin{\overline{\otimes}} u\big|e = |w|e. \quad \triangleright \end{aligned}$$

The relation $Wu = w \mathbin{\overline{\otimes}} u$ is called the *representation* of the orthomorphism W by means of the section w. Observe that, due to the Corollary [12: 2.4.4], the last proposition describes the structure of orthomorphisms acting in arbitrary BKSs.

5.14. Proposition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be arbitrary* CBBs *over Q and let $\mathcal{Y}$ be ample. Suppose that E is an order-dense ideal of $C_\infty(Q)$, $\mathcal{U}$ and $\mathcal{V}$ are lattice-normed subspaces of $E(\mathcal{X})$ and $E(\mathcal{Y})$, and $\mathcal{U}$ approximates $E(\mathcal{X})$. A mapping $I\colon \mathcal{U} \to \mathcal{V}$ is an isometric embedding if and only if there exists an isometric embedding i of $\mathcal{X}$ into $\mathcal{Y}$ such that $I(u) = i \mathbin{\overline{\otimes}} u$ for all $u \in \mathcal{U}$.*

$\lhd$ Only necessity requires verification. Let $I\colon \mathcal{U} \to \mathcal{V}$ be an isometric embedding. In view of Corollary 1.8, there exists an isometric embedding $\bar{I}\colon C_\infty(Q,\mathcal{X}) \to C_\infty(Q,\mathcal{Y})$ that extends I. Denote by $\overline{\mathcal{X}}$ the ample hull of $\mathcal{X}$, represent $C_\infty(Q,\mathcal{X})$ as $C_\infty(Q,\overline{\mathcal{X}})$ (see [12: 2.4.8]) and consider the representation $\hat{I}(u) = \bar{i} \mathbin{\overline{\otimes}} u$ of the corresponding interpretation $\hat{I}\colon C_\infty(Q,\overline{\mathcal{X}}) \to C_\infty(Q,\mathcal{Y})$ of the orthomorphism $\bar{I}$ by means of a section $\bar{i} \in C_\infty\big(Q, B(\overline{\mathcal{X}},\mathcal{Y})\big)$. It is not difficult to become convinced that $\bar{i}$ is an isometric embedding of $\overline{\mathcal{X}}$ into $\mathcal{Y}$. For each point $q \in Q$, assign $i(q) := \bar{i}(q)\big|_{\mathcal{X}(q)}$. By the definition of a homomorphism (see [12: 1.4.2]), we have $Q \otimes \bar{i} \in C(Q \otimes \overline{\mathcal{X}}, Q \otimes \mathcal{Y})$. Therefore, $Q \otimes i = (Q \otimes \bar{i})\big|_{Q\otimes\mathcal{X}} \in C(Q\otimes\mathcal{X}, Q\otimes\mathcal{Y})$, i.e., $i \in \mathrm{Hom}_Q(\mathcal{X},\mathcal{Y})$; consequently, i is an isometric embedding of $\mathcal{X}$ into $\mathcal{Y}$. It remains to observe that $I(u) = \bar{I}(u) = \bar{i} \mathbin{\overline{\otimes}} u = i \mathbin{\overline{\otimes}} u$ for all $u \in \mathcal{U}$. $\rhd$

The following result supplements the interpretation [12: 2.4.1] of isometric LNSs.

Corollary. *Let $\mathcal{X}$ and $\mathcal{Y}$ be ample* CBBs *over Q and let E be an order-dense ideal of $C_\infty(Q)$. A mapping $I\colon E(\mathcal{X}) \to E(\mathcal{Y})$ is an isometric embedding (an isometry) if and only if there exists an isometric embedding (an isometry) i of $\mathcal{X}$ into (onto) $\mathcal{Y}$ such that $I(u) = i \mathbin{\overline{\otimes}} u$ for all $u \in E(\mathcal{X})$.*

Due to the Corollary [12: 2.4.4], the last assertion describes the structure of isometric embeddings of arbitrary BKSs.

5.15. Lemma. *Suppose that $\mathcal{X}$ is an ample* CBB *over P, $s \in C_0(Q,P)$, and $u \in C_\infty(P,\mathcal{X})$. If $|u| \in C_s(P)$ then $u \bullet s \in C_\infty(Q, \mathcal{X} \bullet s)$.*

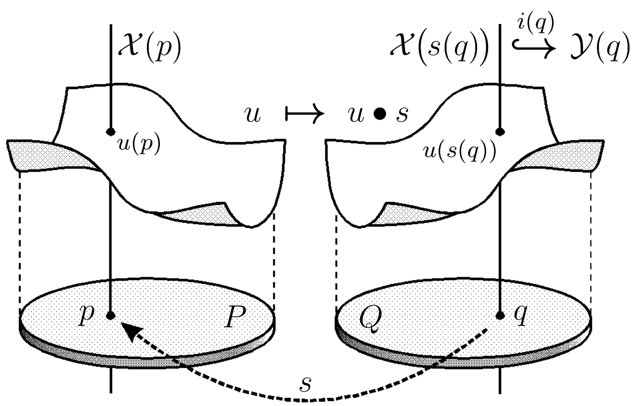
$\lhd$ First, the domain of definition of the section $u \bullet s$ coincides with $\mathrm{dom}(|u| \bullet s)$ and, therefore, it is dense in Q due to the containment $|u| \in C_s(P)$. Second, if the section $u \bullet s$ has a limit at the point $q \in Q$ then $q \in \mathrm{dom}\,|u \bullet s| = \mathrm{dom}(|u| \bullet s) = \mathrm{dom}(u \bullet s)$. $\rhd$

Observe that ampleness of $\mathcal{X}$ does not ensure that of $\mathcal{X} \bullet s$. Indeed, if the stalk $\mathcal{X}(p)$ is infinite-dimensional, the space Q is infinite, and the function $s\colon Q \to P$ is constant and equal to p, then, by Theorem [12: 2.5.3], the bundle $\mathcal{X} \bullet s$ is not ample.

5.16. Lemma. *Suppose that $\mathcal{X}$ is an ample* CBB *over P, E is an order-dense ideal of $C_\infty(P)$, F is an order-dense ideal of $C_\infty(Q)$, and $S\colon E \to F$ is a shift operator. Denote the* BKS *$E(\mathcal{X})$ by $\mathcal{U}$ and consider the operator of norm transformation $S_{\mathcal{U}}\colon \mathcal{U} \to S\mathcal{U}$* (see 0.6). *Let $Se = e \bullet s$ be the representation of the operator S by means of a function $s \in C_0(Q,P)$. Then there exists an F-isometric embedding $i\colon S\mathcal{U} \to F(\mathcal{X} \bullet s)$ such that $iS_{\mathcal{U}}u = u \bullet s$ for all $u \in E(\mathcal{X})$.*

◁ Define an operator $i_0\colon S_{\mathcal{U}}[\mathcal{U}] \to F(\mathcal{X} \bullet s)$ by letting $i_0(S_{\mathcal{U}}u) := u \bullet s$ for all $u \in \mathcal{U}$. Correctness of this definition is justified by the equalities $|u \bullet s| = |u| \bullet s = S|u| = |S_{\mathcal{U}}u|$ $(u \in \mathcal{U})$, which, in particular, imply that $\bigl|i_0(v)\bigr| = |v|$ for all $v \in S_{\mathcal{U}}[\mathcal{U}]$. By Corollary 1.8, the operator i_0 extends to the desired isometric embedding $i\colon S\mathcal{U} \to F(\mathcal{X} \bullet s)$. ▷

5.17. Proposition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be ample* CBBs *over P and Q and let E and F be order-dense ideals of $C_\infty(P)$ and $C_\infty(Q)$, respectively. A mapping $S\colon E(\mathcal{X}) \to F(\mathcal{Y})$ is a shift operator if and only if there exist a function $s \in C_0(Q,P)$ and an isometric embedding i of $\mathcal{X} \bullet s$ into $\mathcal{Y}$ such that $Su = i \otimes (u \bullet s)$ for all $u \in E(\mathcal{X})$. In this case, $|S|e = e \bullet s$ for all $e \in E$.*



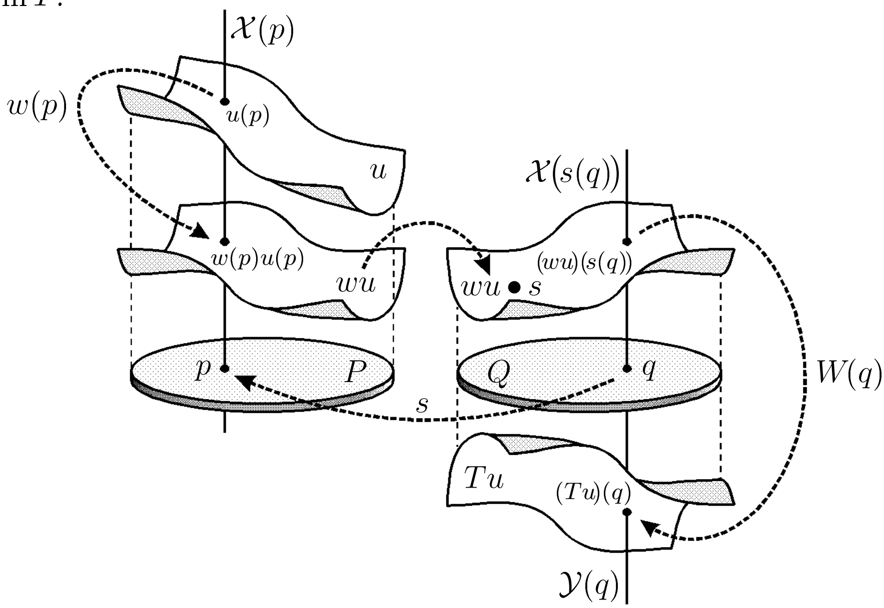
◁ Sufficiency is easily verified with the help of Theorem 3.15, and necessity can be established by consequent application of Proposition 3.14, Lemma 5.16, and Proposition 5.14. ▷

If we additionally require in the statement of the last proposition that the function s be defined on $\operatorname{supp}\operatorname{im} S$ (see 0.10), then the choice of s and i that provide the representation $Su = i \otimes (u \bullet s)$ becomes unique (this can be easily deduced from Proposition 5.8). In this case, the relation $Su = i \otimes (u \bullet s)$ is called the *representation* of the shift operator S by means of the function s and the embedding i. Observe that, due to the Corollary [12: 2.4.4], the last proposition describes the structure of shift operators acting in arbitrary BKSs.

5.18. Theorem. *Let $\mathcal{X}$ and $\mathcal{Y}$ be ample* CBBs *over P and Q and let E and F be order-dense ideals of $C_\infty(P)$ and $C_\infty(Q)$, respectively. A mapping $T\colon E(\mathcal{X}) \to F(\mathcal{Y})$ is a weighted shift operator if and only if there exist a positive function $w \in C_\infty(P)$, a mapping $s \in C_0(Q, P)$, and a section $W \in C_\infty\big(Q, B(\overline{\mathcal{X} \bullet s}, \mathcal{Y})\big)$ (where $\overline{\mathcal{X} \bullet s}$ is the ample hull of $\mathcal{X} \bullet s$) such that*

$$Tu = W \mathbin{\overline{\otimes}} (wu \bullet s) \quad \text{for all } \ u \in E(\mathcal{X}),$$
$$|T|e = |W|(we \bullet s) \quad \text{for all } \ e \in E.$$

Furthermore, we may assume that $\operatorname{dom} s = \operatorname{supp}\operatorname{im} T$ *and* $W = 0$ *outside* $\operatorname{supp}\operatorname{im} T$.



◁ Sufficiency is easily verified with the help of Propositions 5.13 and 5.17. Let us show necessity. If $T\colon E(\mathcal{X}) \to F(\mathcal{Y})$ is a weighted shift operator then, according to Theorem 4.6, there exist a BKS $\mathcal{V}'$ over an order-dense ideal $F' \subset C_\infty(Q)$, a scalar orthomorphism $\overline{w}\colon E(\mathcal{X}) \to C_\infty(P, \mathcal{X})$ generated by a positive orthomorphism $w\colon E \to C_\infty(P)$, a shift operator $\overline{S}\colon (wE)(\mathcal{X}) \to \mathcal{V}'$, and an orthomorphism $\overline{W}\colon \mathcal{V}' \to F(\mathcal{Y})$ such that

$$\begin{tikzcd}
E(\mathcal{X}) \arrow[rr, "T"] \arrow[d, "\overline{w}"'] & & F(\mathcal{Y}) \\
(wE)(\mathcal{X}) \arrow[rr, "\overline{S}"] \arrow[dr, "(\cdot)\bullet s"'] & & F'(\mathcal{Z}) \arrow[u, "\overline{W}"'] \\
& F'(\mathcal{X} \bullet s) \arrow[ur, "i \otimes (\cdot)"] \arrow[dr, "\mathrm{ext}"'] & \\
& & F'\big(\overline{\mathcal{X} \bullet s}\big) \arrow[u, "\bar{i} \otimes (\cdot)"'] \arrow[uuu, bend right=60, "W \mathbin{\overline{\otimes}} (\cdot)"']
\end{tikzcd}$$

$$T = \overline{W} \circ \overline{S} \circ \overline{w}, \quad |T| = \big|\overline{W}\big| \circ \big|\overline{S}\big| \circ |\overline{w}|.$$

Due to the Corollary [12: 2.4.4], we may assume that $\mathcal{V}' = F'(\mathcal{Z})$, where

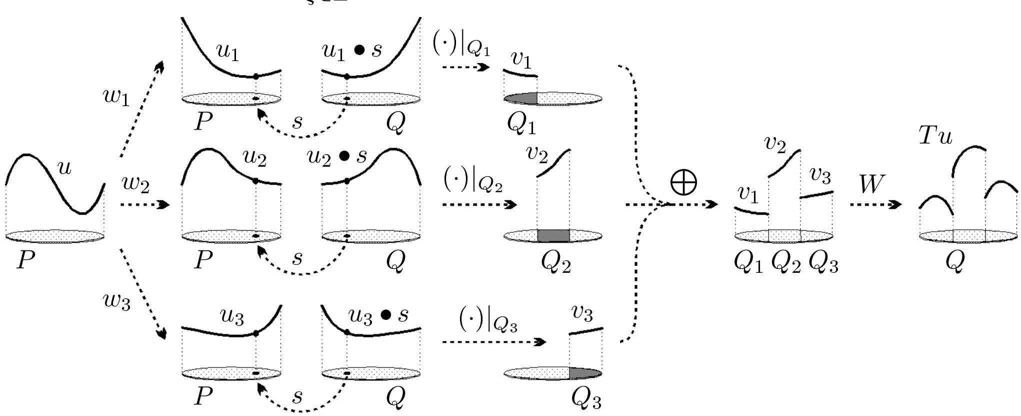
$\mathcal{Z}$ is an ample CBB over Q. According to Proposition 5.17, there is a function $s \in C_0(Q, P)$ and an isometric embedding i of $\mathcal{X} \bullet s$ into $\mathcal{Z}$ such that $\overline{S}u = i \otimes (u \bullet s)$ for all $u \in (wE)(\mathcal{X})$. In view of the Corollary [12: 2.1.10], the homomorphism i extends to an isometric embedding $\bar{i}$ of $\overline{\mathcal{X} \bullet s}$ into $\mathcal{Z}$. Due to Proposition 5.13, the orthomorphism $\big(v \mapsto \overline{W}(\bar{i} \otimes v)\big)\colon F'(\overline{\mathcal{X} \bullet s}) \to F(\mathcal{Y})$ can be represented as $v \mapsto W \mathbin{\overline{\otimes}} v$, where $W \in C_\infty\big(Q, B(\overline{\mathcal{X} \bullet s}, \mathcal{Y})\big)$. It is clear that the constructed functions w, s, and W are those desired. $\triangleright$

If functions w, s, and W satisfy the conditions stated in Theorem 5.18 then the relation $Tu = W \mathbin{\overline{\otimes}} (wu \bullet s)$ is called the *representation* of the operator T by means of s, w, and W. Due to the Corollary [12: 2.4.4], the last theorem describes the structure of weighted shift operators acting in arbitrary BKSs.

5.19. Let $\mathcal{Y}$ be a CBB over Q and let $(Q_\xi)_{\xi\in\Xi}$ be a family of pairwise disjoint elements of $\operatorname{Clop}(Q)$. Suppose that, for each $\xi \in \Xi$, we are given a section $v_\xi \in C(D_\xi, \mathcal{Y})$ over a dense subset $D_\xi \subset Q_\xi$. Assign $D := Q \setminus \operatorname{cl} \bigcup_{\xi\in\Xi} Q_\xi$. It is clear that the union $\bigcup_{\xi\in\Xi} D_\xi \cup D$ is a dense subset of Q and the function $\bigcup_{\xi\in\Xi} v_\xi \cup 0|_D$ defined on it is a continuous section of $\mathcal{Y}$. In the sequel, the maximal extension of this continuous section is denoted by $\bigoplus_{\xi\in\Xi} v_\xi$.

Theorem. *Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are ample* CBB*s over P and Q, E and F are order-dense ideals of $C_\infty(P)$ and $C_\infty(Q)$, $T\colon E(\mathcal{X}) \to F(\mathcal{Y})$ is a disjointness preserving bounded operator, and $h(U) = s^{-1}[U]$ is the representation of the shadow h of T by means of a function $s \in C_0(Q, P)$. Then there exist a family $(w_\xi)_{\xi\in\Xi}$ of positive functions in $C_\infty(P)$, a disjoint family $(Q_\xi)_{\xi\in\Xi}$ of elements in* $\operatorname{Clop}(Q)$*, and a section $W \in C_\infty\big(Q, B(\overline{\mathcal{X} \bullet s}, \mathcal{Y})\big)$ such that* $\operatorname{supp} W = \operatorname{cl} \bigcup_{\xi\in\Xi} Q_\xi = \operatorname{dom} s = \operatorname{supp} \operatorname{im} T$, $1/w_\xi \in E$ *for all $\xi \in \Xi$, and*

$$Tu = \bigoplus_{\xi\in\Xi} W \otimes (w_\xi u \bullet s)|_{Q_\xi} \quad \big(u \in E(\mathcal{X})\big). \qquad (**)$$



$\triangleleft$ According to 5.18, this assertion reformulates Theorem 4.8. $\triangleright$

Observe that the functions $w_\xi u \bullet s$ in the representation $(**)$ being continuous sections of the bundle $\mathcal{X} \bullet s$, need not belong to $C_\infty\big(Q, \mathcal{X} \bullet s\big)$, while the restrictions $(w_\xi u \bullet s)|_{Q_\xi}$ do belong to $C_\infty\big(Q_\xi, \mathcal{X} \bullet s\big)$.

The relation $Tu = \bigoplus_{\xi\in\Xi} W \otimes (w_\xi u \bullet s)|_{Q_\xi}$ is called the *representation* of the operator T (by means of s, w_ξ, Q_ξ, and W). Observe that, due to the Corollary [12: 2.4.4], the last theorem describes the structure of disjointness preserving bounded operators acting in arbitrary BKSs.

6. Interpretation for the properties of operators

The representation theorems of Section 5 allow us to interpret various properties of orthomorphisms, shift operators, weighted shift operators, and arbitrary disjointness preserving operators in terms of the properties of certain components of their representation. As an illustration, we consider order continuous operators, injective operators, and operators with ideal image.

Throughout the section, P and Q are extremally disconnected compact spaces.

6.1. Lemma. *Let X and Y be totally disconnected compact spaces and let $s\colon X \to Y$ be a continuous function.*

(a) *The following assertions are equivalent:*
  (1) $s^{-1}[\operatorname{int} F] = \operatorname{int} s^{-1}[F]$ *for every closed subset $F \subset Y$;*
  (2) $s^{-1}[\operatorname{cl} G] = \operatorname{cl} s^{-1}[G]$ *for every open subset $G \subset Y$;*
  (3) *if F is a closed subset of Y and* $\operatorname{int} F = \varnothing$ *then* $\operatorname{int} s^{-1}[F] = \varnothing$*;*
  (4) *if G is an open subset of Y and* $\operatorname{cl} G = Y$ *then* $\operatorname{cl} s^{-1}[G] = X$*;*
  (5) *the inverse image $s^{-1}[D]$ of every meager subset $D \subset Y$ is a meager subset of X;*
  (6) *the inverse image $s^{-1}[D]$ of every comeager subset $D \subset Y$ is a comeager subset of X.*

(b) *The following assertions are equivalent:*
  (1) $s^{-1}[\operatorname{int} F] = \operatorname{int} s^{-1}[F]$ *for every closed σ-open subset $F \subset Y$;*
  (2) $s^{-1}[\operatorname{cl} G] = \operatorname{cl} s^{-1}[G]$ *for every open σ-closed subset $G \subset Y$;*
  (3) *if F is a closed σ-open subset of Y and* $\operatorname{int} F = \varnothing$ *then* $\operatorname{int} s^{-1}[F] = \varnothing$*;*
  (4) *if G is an open σ-closed subset of Y and* $\operatorname{cl} G = Y$ *then* $\operatorname{cl} s^{-1}[G] = X$.

A function s satisfying any of the conditions in (a) $\big($in (b)$\big)$ is called *exact* (*σ-exact*).

R e m a r k . In the case when Y is an extremally disconnected compact space, the list (a) can be supplemented by the following equivalent assertions:

(7) *if U is a clopen subset of X then $s[U]$ is a clopen subset of Y;*
(8) *if U is an open subset of X then $s[U]$ is an open subset of Y.*

As is known, a function s satisfying condition (8) is called *open*. Thus, if Y is extremally disconnected then the classes of exact and open functions $s \in C(X, Y)$ coincide. The author does not know analogs of assertions (7) and (8) equivalent to the fact that the function s is σ-exact.

6.2. Proposition. *Let X and Y be totally disconnected compact spaces and let $h\colon \mathrm{Clop}(X) \to \mathrm{Clop}(Y)$ be a ring homomorphism. Consider the representation $h(U) = s^{-1}[U]$ of h by means of a function $s \in C_0(Y, X)$. The homomorphism h is o-continuous (sequentially o-continuous) if and only if the function s is exact (σ-exact).*

$\triangleleft$ There is a proof in [26: Section 22]. $\triangleright$

6.3. Let $\mathcal{U}$ and $\mathcal{V}$ be LNSs over order-dense ideals of the K-spaces $C_\infty(P)$ and $C_\infty(Q)$, respectively. If $T\colon \mathcal{U} \to \mathcal{V}$ is a disjointness preserving operator and $h(U) = s^{-1}[U]$ is the representation of the shadow h of the operator T by means of a function $s \in C_0(Q, P)$, then we say that s is the *shift function* of the operator T.

Theorem. *Suppose that E and F are order-dense ideals of $C_\infty(P)$ and $C_\infty(Q)$ (respectively), $\mathcal{U}$ is a* BKS *over E, $\mathcal{V}$ is an* LNS *over F, $T\colon \mathcal{U} \to \mathcal{V}$ is a disjointness preserving bounded operator, and $s \in C_0(Q, P)$ is its shift function. The operator T is o-continuous (sequentially o-continuous) if and only if the function s is exact (σ-exact).*

$\triangleleft$ Since the function s represents the shadow of T, the claim follows from 6.2 and 1.6. $\triangleright$

6.4. Proposition. *Let X and Y be totally disconnected compact spaces and let $h\colon \mathrm{Clop}(X) \to \mathrm{Clop}(Y)$ be a ring homomorphism. Consider the representation $h(U) = s^{-1}[U]$ of h by means of a function $s \in C_0(Y, X)$. The homomorphism h is injective if and only if the function s is surjective.*

6.5. Theorem. *Suppose that E and F are order-dense ideals of $C_\infty(P)$ and $C_\infty(Q)$ (respectively), $T\colon E \to F$ is a disjointness preserving regular operator, and $s \in C_0(Q, P)$ is its shift function. The operator T is injective if and only if the function s is surjective.*

$\lhd$ *Necessity*: In view of Proposition 6.4, it is sufficient to assume injectivity of the operator T and establish injectivity of its shadow $h\colon \mathrm{Pr}(E) \to \mathrm{Pr}(F)$. Consider an arbitrary projection $\pi \in \mathrm{Pr}(E)$ and suppose that $h(\pi) = 0$. Then $T\pi e = 0$ for all $e \in E$. Due to injectivity of T, the latter means that $\pi e = 0$ for all $e \in E$, i.e., $\pi = 0$.

Sufficiency: Let

$$Te = \bigoplus_{\xi \in \Xi} W(w_\xi e \bullet s)|_{Q_\xi}$$

be the representation of the operator T by means of $s \in C_0(Q, P)$, $w_\xi \in C_\infty(P)$, $Q_\xi \in \mathrm{Clop}(Q)$, and $W \in C_\infty(Q)$. Assume that the function s is surjective. For each $\xi \in \Xi$, assign $P_\xi := \operatorname{supp} w_\xi$. Consider an arbitrary functions $e \in E$ and suppose that $Te = 0$. Then $W(w_\xi e \bullet s)|_{Q_\xi} = 0$ for all $\xi \in \Xi$. The latter means that, for each $\xi \in \Xi$, the equality $w_\xi e \bullet s = 0$ holds on Q_ξ, which implies the equality $w_\xi e = 0$ on $s[Q_\xi]$ and, hence, the equality $e = 0$ on $s[Q_\xi] \cap P_\xi$. Thus, the function e is equal to zero on the union

$$D := \bigcup_{\xi \in \Xi} s[Q_\xi] \cap P_\xi.$$

It remains to show that the set D is dense in P.

Let a clopen set U be contained in the difference $P \backslash D$. Then, for all $e \in E$ and $\xi \in \Xi$, the equality $w_\xi \langle U \rangle e = 0$ holds on $U^\perp \cup P_\xi^\perp$. From the inclusion $s[Q_\xi] \cap P_\xi \subset U^\perp$ it follows that $w_\xi \langle U \rangle e = 0$ on $s[Q_\xi]$. Therefore, $(w_\xi \langle U \rangle e) \bullet s = 0$ on Q_ξ and, hence, $W\big((w_\xi \langle U \rangle e) \bullet s\big)|_{Q_\xi} = 0$. Arbitrariness of $\xi \in \Xi$ allows us to conclude that $T\langle U \rangle e = 0$, and arbitrariness of $e \in E$ yields the equality $h\langle U \rangle = 0$. According to injectivity of h (see Proposition 6.4), the latter means that $U = \varnothing$. $\rhd$

6.6. R e m a r k. The author did not succeed in obtaining an adequate criterion for injectivity of an operator in BKSs. Simple examples show that direct generalization of the last theorem to the case of an operator in BKSs fails. Interpretation for injectivity of such operator must involve the outer weight of the representation.

6.7. Proposition. *Let X and Y be totally disconnected compact spaces and let $h\colon \mathrm{Clop}(X) \to \mathrm{Clop}(Y)$ be a ring homomorphism. Consider the representation $h(U) = s^{-1}[U]$ of the homomorphism h by means of a function $s \in C_0(Y, X)$. The equality $\operatorname{im} h = [0, h(1)]$ holds if and only if the function s is injective.*

6.8. Lemma. *A continuous function $s\colon Q \to P$ is injective if and only if the operator $(e \mapsto e \circ s)\colon C(P) \to C(Q)$ is surjective.*

$\lhd$ If the function s is injective then it is a homeomorphism of Q onto $\operatorname{im} s$. In this case, every function $f \in C(Q)$ can be represented as $g \circ s$, where $g \in C(\operatorname{im} s)$. By the Tietze–Urysohn theorem, the function g extends to an $e \in C(P)$.

If points $q_1, q_2 \in Q$ are different then there is a clopen set $V \subset Q$ that contains only one of them. If the operator $e \mapsto e \circ s$ is surjective then the characteristic function of V can be represented as $e \circ s$, whence $s(q_1) \neq s(q_2)$. $\rhd$

6.9. In the sequel, we discuss interpretation of the fact that an operator has ideal image. In order to clarify this property, we present a result of [16: Lemma 2.7].

Lemma. *Let E and F be vector lattices and let $T\colon E \to F$ be a disjointness preserving regular operator. The following assertions are equivalent:*

(1) $\operatorname{im} T$ *is an ideal of F;*
(2) $\operatorname{im} |T|$ *is an ideal of F;*
(3) $|T|[0, e] = [0, |T|e]$ *for all positive $e \in E$.*

The list of equivalent properties (1)–(3) of the operator T can be supplemented by the following one: the operator T takes ideals of E into ideals of F, i.e., for every ideal $E_0 \subset E$, the set $T[E_0]$ is an ideal of F.

6.10. Proposition. *Suppose that E and F are order-dense ideals of $C_\infty(P)$ and $C_\infty(Q)$ (respectively), $T\colon E \to F$ is a disjointness preserving regular operator, and $s \in C_0(Q, P)$ is its shift function. Assume that $\langle T\bar{e} \rangle = \langle \operatorname{im} T \rangle$ for some element $\bar{e} \in E$. The image of T is an ideal of F if and only if the function s is injective.*

$\lhd$ Due to Theorem 1.1, we may assume that the operator T is positive and $\bar{e} \geqslant 0$. Moreover, for the sake of convenience, we assume that $\langle \operatorname{im} T \rangle = 1$, i.e., $\operatorname{dom} s = Q$.

Let the image of T be an ideal. In view of Lemma 6.8, to prove injectivity of s, it is sufficient to fix an arbitrary function $\beta \in C(Q)$, $0 \leqslant \beta \leqslant 1$, and represent it as $\alpha \circ s$, where $\alpha \in C(P)$. According to Lemma 6.9, the inequalities $0 \leqslant \beta T\bar{e} \leqslant T\bar{e}$ imply existence of an element $e \in E$ such that $0 \leqslant e \leqslant \bar{e}$ and $Te = \beta T\bar{e}$. Let a function $\alpha \in C(P)$ be such that $e = \alpha\bar{e}$. Then, according to 3.13, we have

$$(\alpha \circ s)T\bar{e} = T(\alpha\bar{e}) = Te = \beta T\bar{e},$$

whence $\alpha \circ s = \beta$ due to the equality $\langle T\bar{e} \rangle = 1$.

Suppose now that the function s in injective. Fix arbitrary elements $e \in E$ and $f \in F$ satisfying the inequalities $0 \leqslant f \leqslant Te$ and show that $f \in \operatorname{im} T$. Let a function $\beta \in C(Q)$ be such that $f = \beta Te$. By injectivity of the operator $(e \mapsto e \circ s)\colon C(P) \to C(Q)$ (see 6.8), there exists a function $\alpha \in C(P)$ such that $\alpha \circ s = \beta$. Then $\alpha e \in E$ and, in view of 3.13, we have $T(\alpha e) = (\alpha \circ s)Te = \beta Te = f$. $\rhd$

6.11. Existence of an element $\bar{e} \in E$ satisfying the equality $\langle T\bar{e}\rangle = \langle \operatorname{im} T\rangle$ is an essential condition in the statement of Proposition 6.10. Without this requirement, the function s need not be injective even when T is a surjective shift operator. We will give a corresponding example in this subsection.

Lemma. *Consider functions $s \in C_0(Q,P)$ and $f \in C_\infty(Q)$. Suppose that there is an open set $D \subset P$ such that s is injective on $s^{-1}[D]$ and f is identically zero outside $s^{-1}[D]$. Then $f = e \bullet s$ for some function $e \in C_\infty(P)$. For a positive and/or bounded function f, the corresponding function e can be chosen with the same property.*

$\lhd$ Denote the image of s by R and define a function $g\colon R \to \overline{\mathbb{R}}$ as follows:

$$g(p) := \begin{cases} f\big(s^{-1}(p)\big) & \text{if } p \in R \cap D, \\ 0 & \text{if } p \in R \backslash D. \end{cases}$$

Fix an arbitrary point $p \in R$ and show that the function g is continuous at p.

(1) Suppose that $p \in R \cap D$. Since the set D is open, we thus have a clopen set $U \subset P$ such that $p \in U \subset D$. From injectivity of s on $s^{-1}[D]$ it follows that the restriction $s|^U$ is a homeomorphism of $s^{-1}[U]$ onto $R \cap U$. Therefore, the function $g|_U = f \circ \big(s|^U\big)^{-1}$ is continuous.

(2) Suppose now that $p \in R\backslash D$. Fix an arbitrary number $\varepsilon > 0$ and show that $|g| < \varepsilon$ in a neighborhood of p. Assign $Q_\varepsilon := \big\{q \in Q : |f(q)| \geqslant \varepsilon\big\}$. Taking account of the fact that $f = 0$ outside $s^{-1}[D]$, we have the inclusion $Q_\varepsilon \subset s^{-1}[D]$; hence, $s[Q_\varepsilon] \subset D$. Since $|f| < \varepsilon$ outside Q_ε, we conclude that $|g| < \varepsilon$ outside $s[Q_\varepsilon]$. It remains to observe that $R \backslash s[Q_\varepsilon]$ is a neighborhood of p in the space R.

Thus, the function g is continuous. Obviously, $g \bullet s = f$. This implies that $g \in \overline{C}_\infty(R)$ $\big($if $|g| = \infty$ on a nonempty open set $W \subset R$ then $|f| = |g \bullet s| = \infty$ on the nonempty open set $s^{-1}[W]$, which contradicts the containment $f \in C_\infty(Q)\big)$. According to the Tietze–Urysohn theorem, there exists a function $e \in C_\infty(P)$ such that $e = g$ on R. Obviously, e is the desired function. Observe that positiveness and/or boundedness of the function f imply the same property of g, which in turn allows us to choose a function e with the appropriate property. $\rhd$

E x a m p l e. As is known, the remainder $\beta\mathbb{N}\backslash\mathbb{N}$ contains a discrete set D of cardinality continuum (see [7: Chapter IV, Problem 52]). Let $s\colon \beta D \to \beta\mathbb{N}$ be the continuous extension of the identity mapping of D. Introduce the notation

$$\overline{D} := \mathrm{cl}_{\beta\mathbb{N}}\, D,$$

$$E := \{e \in C(\beta\mathbb{N}) : \ e = 0 \text{ on } \overline{D}\backslash D\},$$

$$F := \{f \in C(\beta D) : \ f = 0 \text{ on } \beta D\backslash D\},$$

and assign $Se := e \circ s$ for all $e \in E$. Then $S\colon E \to F$ is a surjective shift operator, while its shift function s is not injective.

$\lhd$ First of all, show that s is actually the shift function of the operator S. To this end, we should establish the equality $\operatorname{supp}\operatorname{im} S = \beta D$ (see 5.8). Since the subset $D \subset \beta\mathbb{N}$ is discrete, each point $q \in D$ has a neighborhood $U \subset \beta\mathbb{N}$ such that $U \cap D = \{q\}$. Then $\chi_U \in E$ and

$$(S\chi_U)(q) = \chi_U\big(s(q)\big) = \chi_U(q) = 1.$$

Thus, $D \subset \operatorname{supp}\operatorname{im} S$, whence $\operatorname{supp}\operatorname{im} S = \beta D$.

Now, show that the operator S is surjective. Fix an arbitrary element $f \in F$ and assign $\mathcal{D} := \beta\mathbb{N}\backslash(\overline{D}\backslash D)$. Then $\mathcal{D}$ is an open subset of $\beta\mathbb{N}$, $s^{-1}[\mathcal{D}] = s^{-1}[D] = D$, s is injective on D, and f is the identical zero outside D. Therefore, in view of the last lemma, there exists a function $e \in C(\beta\mathbb{N})$ such that $f = e \circ s$. It is clear that $e \in E$ and, therefore, $f \in \operatorname{im} S$.

It remains to observe that the function $s\colon \beta D \to \beta\mathbb{N}$ is not injective, since (see [7: Chapter VI, Problem 180])

$$|\beta D| \ = \ 2^{2^{|D|}} \ > \ 2^{2^{|\mathbb{N}|}} \ = \ |\beta\mathbb{N}|,$$

where $|X|$ stands for the cardinality of a set X. $\rhd$

6.12. Theorem. *Suppose that E and F are order-dense ideals of $C_\infty(P)$ and $C_\infty(Q)$ (respectively), $T\colon E \to F$ is a disjointness preserving regular operator and $s \in C_0(Q,P)$ is its shift function. The image of T is an ideal of F if and only if, for every element $e \in E$, the function s is injective on the set* $\operatorname{supp} Te$. *The last property of the function s is equivalent to its injectivity on the union* $\cup\{\operatorname{supp} Te : \ e \in E\}$ *(which is an open dense subset of* $\operatorname{dom} s$*).*

$\lhd$ *Necessity*: Suppose that the image of T is an ideal and consider an arbitrary element $e \in E$. It is clear that the image of the composition $\langle Te\rangle \circ T$ is an ideal too and, in view of Proposition 6.10, its shift function is injective. It remains to observe that the shift function of the operator $\langle Te\rangle \circ T$ coincides with the restriction of s onto $\operatorname{supp} Te$.

Sufficiency: Theorem 1.1 allows us to assume that the operator T is positive. Fix arbitrary positive elements $e \in E$ and $f \in F$ satisfying the inequality $f \leqslant Te$ and show that $f \in \operatorname{im} T$. Since the function s is injective on the set $\operatorname{supp} Te$, in view of Proposition 6.10, the image of the composition $\langle Te\rangle \circ T$ is an ideal of F. According to Lemma 6.9, the inequalities $0 \leqslant f \leqslant \langle Te\rangle Te$ imply existence of an element $e_0 \in E$ such that $0 \leqslant e_0 \leqslant e$ and $\langle Te\rangle Te_0 = f$; whence $Te_0 = f$.

Injectivity of the function s on each set of the form $\operatorname{supp} Te$ $(e \in E)$ implies injectivity of s on the union $\cup\{\operatorname{supp} Te : e \in E\}$, since the containments $q_1 \in \operatorname{supp} Te_1$ and $q_2 \in \operatorname{supp} Te_2$ yield $q_1, q_2 \in \operatorname{supp} T\big(|e_1| \vee |e_2|\big)$. $\rhd$

Remark. Under the hypotheses of the last theorem, injectivity of the function s on the union $\cup\{\operatorname{Supp} Te : e \in E\}$ is not sufficient for the image of T to be an ideal $\big($here $\operatorname{Supp} f = \{q \in Q : f(q) \neq 0\}\big)$. Indeed, assign $P = Q = \beta\mathbb{N}$, fix a point $p \in P\backslash\mathbb{N}$, and, naturally identifying $C(Q)$ and ℓ^∞, consider the operator $T\colon C(P) \to C(Q)$ acting by the rule

$$(Te)(n) = \begin{cases} e(p) & \text{if } n = 1, \\ e(n)/n & \text{if } n > 1 \end{cases} \qquad (n \in \mathbb{N})$$

for all $e \in C(P)$. The image of T is not an ideal, since, for instance, $(1, \frac{1}{2}, \dots, \frac{1}{n}, \dots)$ belongs to $\operatorname{im} T$, but $(1, 0, 0, \dots)$ does not. However, the shift function s of the operator T is injective on the set $\cup\{\operatorname{Supp} Te : e \in E\} = \mathbb{N}$, since $s(1) = p$ and $s(n) = n$ whenever $n \in \mathbb{N}\backslash\{1\}$.

6.13. As is known (see 2.8), every BKS over an order-dense ideal of $C_\infty(P)$ is a module over $C(P)$. A subset $\mathcal{U}_0$ of such BKS is called a $C(P)$-*submodule* of it, if $\alpha u \in \mathcal{U}_0$ for all $u \in \mathcal{U}_0$ and $\alpha \in C(P)$.

Lemma. *Suppose that E and F are order-dense ideals of $C_\infty(P)$ and $C_\infty(Q)$ (respectively), $\mathcal{U}$ is a* BKS *over E, and $\mathcal{V}$ is a* BKS *over F. The following properties of an operator $T\colon \mathcal{U} \to \mathcal{V}$ are equivalent:*

(1) *T takes $C(P)$-submodules of $\mathcal{U}$ into $C(Q)$-submodules of $\mathcal{V}$;*
(2) *for every $u \in \mathcal{U}$ and every $\beta \in C(Q)$, there exists a function $\alpha \in C(P)$ such that $T(\alpha u) = \beta Tu$.*

$\lhd$ It is sufficient to observe that the set $\big\{\alpha u : \alpha \in C(P)\big\}$ is a $C(P)$-submodule of $\mathcal{U}$. $\rhd$

6.14. Proposition. *Suppose that E and F are order-dense ideals of $C_\infty(P)$ and $C_\infty(Q)$ (respectively), $\mathcal{U}$ is a* BKS *over E, $\mathcal{V}$ is a* BKS *over F, $T\colon \mathcal{U} \to \mathcal{V}$ is a disjointness preserving bounded operator, and $s \in C_0(Q,P)$ is its shift function. Assume that $\langle T\bar{u}\rangle = \langle \operatorname{im} T\rangle$ for some element $\bar{u} \in \mathcal{U}$. The operator T takes $C(P)$-submodules of $\mathcal{U}$ into $C(Q)$-submodules of $\mathcal{V}$ if and only if the function s is injective.*

$\lhd$ For convenience, we assume that $\langle \operatorname{im} T\rangle = 1$, i.e., $\operatorname{dom} s = Q$. Suppose that T takes $C(P)$-submodules of $\mathcal{U}$ into $C(Q)$-submodules of $\mathcal{V}$. In view of 6.8, to prove injectivity of s, it is sufficient to fix an arbitrary function $\beta \in C(Q)$ and represent it as $\alpha \circ s$, where $\alpha \in C(P)$. According to Lemma 6.13, there exists a function $\alpha \in C(P)$ such that $T(\alpha\bar{u}) = \beta T\bar{u}$. Then, due to 3.13, we have

$$|\alpha \circ s - \beta| \mathopen{|\!|} T\bar{u} \mathclose{|\!|} = \big|\!\big|(\alpha \circ s)T\bar{u} - \beta T\bar{u}\big|\!\big| = \big|\!\big|T(\alpha\bar{u}) - \beta T\bar{u}\big|\!\big| = 0;$$

whence $\alpha \circ s = \beta$ in view of the equality $\langle T\bar{u}\rangle = 1$.

Now, suppose that s is injective. Fix arbitrary elements $u \in \mathcal{U}$ and $\beta \in C(Q)$. According to surjectivity of the operator $(e \mapsto e \circ s)\colon C(P) \to C(Q)$ (see 6.8), there exists a function $\alpha \in C(P)$ such that $\alpha \circ s = \beta$. Then, due to 3.13, we have

$$T(\alpha u) = (\alpha \circ s)Tu = \beta Tu.$$

It remains to employ Lemma 6.13. $\rhd$

6.15. Lemma. *Let $\mathcal{U}$ be a* BKS *over an order-dense ideal of $C_\infty(Q)$. For any $u, v \in \mathcal{U}$, there is a function $f \in C(Q)$ such that*

$$\langle u + fv\rangle = \langle u\rangle \vee \langle v\rangle.$$

$\lhd$ As f we can take any function that is different from $|\!|u|\!|/|\!|v|\!|$ everywhere. For instance, we may let

$$f := \big\langle |\!|u|\!|/|\!|v|\!| \leqslant 2\big\rangle 3 + \big\langle |\!|u|\!|/|\!|v|\!| > 2\big\rangle 1.$$

Then the equality $\langle u + fv\rangle = \langle u\rangle \vee \langle v\rangle$ ensues from the following relations:

$$\langle u\rangle \vee \langle v\rangle \leqslant \big\langle |\!|u|\!| \neq f|\!|v|\!|\big\rangle \leqslant \langle u + fv\rangle \leqslant \langle u\rangle \vee \langle v\rangle. \quad \rhd$$

Theorem. *Suppose that E and F are order-dense ideals of $C_\infty(P)$ and $C_\infty(Q)$ (respectively), $\mathcal{U}$ is a* BKS *over E, $\mathcal{V}$ is a* BKS *over F, $T\colon \mathcal{U} \to \mathcal{V}$ is a disjointness preserving bounded operator, and $s \in C_0(Q,P)$ is its shift function. The operator T takes $C(P)$-submodules of $\mathcal{U}$ into $C(Q)$-submodules of $\mathcal{V}$ if and only if, for every element $u \in \mathcal{U}$, the function s is injective on the set* $\operatorname{supp}|Tu|$. *This property of the function s is equivalent to its injectivity on $\cup\{\operatorname{supp}|Tu| : u \in \mathcal{U}\}$ (which is an open dense subset of* $\operatorname{dom} s$*).*

$\lhd$ *Necessity*: Suppose that the operator T takes $C(P)$-submodules of $\mathcal{U}$ into $C(Q)$-submodules of $\mathcal{V}$ and consider an arbitrary element $u \in \mathcal{U}$. It is clear that the composition $\langle Tu\rangle \circ T$ preserves submodules too and, in view of Proposition 6.14, its shift function is injective. It remains to observe that the shift function of the operator $\langle Tu\rangle \circ T$ coincides with the restriction of s onto $\operatorname{supp}|Tu|$.

Sufficiency: Fix arbitrary elements $u \in \mathcal{U}$ and $\beta \in C(Q)$. Since the function s is injective on the set $\operatorname{supp}|Tu|$, in view of Proposition 6.14, the composition $\langle Tu\rangle \circ T$ takes $C(P)$-submodules of $\mathcal{U}$ into $C(Q)$-submodules of $\mathcal{V}$. According to Lemma 6.13, there exists a function $\alpha \in C(P)$ such that $\langle Tu\rangle T(\alpha u) = \beta Tu$; whence, due to the relations

$$\langle T(\alpha u)\rangle = \langle (\alpha \bullet s)Tu\rangle \leqslant \langle Tu\rangle,$$

we have $T(\alpha u) = \beta Tu$.

Show that injectivity of the function s on each set of the form $\operatorname{supp}|Tu|$ ($u \in \mathcal{U}$) implies injectivity of s on the union $\cup\{\operatorname{supp}|Tu| : u \in \mathcal{U}\}$. To this end, it is sufficient to fix arbitrary elements $u_1, u_2 \in \mathcal{U}$ and find a $u \in \mathcal{U}$ such that

$$\operatorname{supp}|Tu| = \operatorname{supp}|Tu_1| \cup \operatorname{supp}|Tu_2|.$$

According to the last lemma, there is a function $\beta \in C(Q)$ that satisfies the relation

$$\operatorname{supp}|Tu_1 + \beta Tu_2| = \operatorname{supp}|Tu_1| \cup \operatorname{supp}|Tu_2|.$$

Injectivity of s on the set $\operatorname{supp}|Tu_2|$, in view of Lemma 6.8, implies existence of a function $\alpha \in C(P)$ such that $\alpha \circ s = \beta$ on $\operatorname{supp}|Tu_2|$. It remains to observe that

$$T(u_1 + \alpha u_2) = Tu_1 + (\alpha \bullet s)Tu_2 = Tu_1 + \beta Tu_2. \quad \rhd$$

7. Comments

It is worth noting that as a rule we confine ourselves to considering K-spaces and Banach–Kantorovich spaces. Generalizations of the obtained results to the case of arbitrary vector lattices and lattice-normed spaces will appear elsewhere.

7.0. Section 0 only contains the information that was not exposed in the previous parts of the paper. For the basic definitions and facts about the objects under consideration, we refer the reader to [12–15].

The schema of a formal mixing employed in the proof of Proposition 0.4 stems from [20, 27, 28]. The notion of the disjoint sum of a family of LNSs (see 0.5) is introduced to be employed in the main result 4.8 on decomposition of a disjointness preserving operator into weighted shift operators. The new notion of the norm transformation of an LNS (see 0.6) is used for describing vector shift operators in Section 3.

7.1. The shadow of an operator as a Boolean homomorphism (without introducing the corresponding term) was first considered in [19] for lattice homomorphisms and disjointness preserving operators in lattice-normed spaces.

In Section 1, we develop this notion and show that many properties of disjointness preserving operators can be expressed in terms of their shadows. In particular, this is true of certain questions of continuity. Most results stated in Section 1 are published for the first time.

The problem of finding sufficient conditions for an operator to be bounded or dominated is traditionally studied for disjointness preserving operators (see [19: 6.5]). Y. A. Abramovich's condition (R) [1: Theorem A] was the first equivalent for boundedness of disjointness preserving operators weaker than sequential r-o-continuity. Later, this condition was also weakened. P. T. N. MacPolin and A. W. Wickstead showed [22: Theorem 2.1] that, for a disjointness preserving operator in vector lattices to be bounded, it is sufficient that the operator under test be semibounded (the latter term is introduced in [14: 2.3] and the result is presented in 1.10).

Attempts at generalizing the Abramovich–MacPolin–Wickstead criterion to the case of operators in lattice-normed spaces cannot lead to a success, since all the four types of boundedness considered in [14: 2.3] are pairwise different for that class of operators (the corresponding examples are presented in [14: 2.4–2.6]). Thus the main problem about sufficient conditions for boundedness remains open for disjointness preserving operators in LNSs. A small step in this direction is made in 1.13.

7.2. An orthomorphism is a band preserving operator that is order-bounded. The problem of finding sufficient conditions for boundedness of disjointness preserving operators is actually solved for operators in vector lattices (see [1: Theorem A; 22: Theorem 2.1] and Theorem 1.10). However, the problem remains actual for operators in lattice-normed spaces (see 7.1). Our Theorem 2.2 asserts that, for band preserving operators in LNSs, all the types of boundedness coincide.

Subsections 2.3–2.10 are devoted to a study of the module structure in a Banach–Kantorovich space and its relation to the notion of orthomorphism. The results presented here are essentially known (see, for instance, [18]).

7.3. The study of multiplicative operators in vector lattices was initiated by B. Z. Vulikh [29, 31] who proved that o-continuous shift operators in K-spaces with unity are multiplicative. Theorem 3.10 generalizes this result to the case of arbitrary shift operators in arbitrary K-spaces. The idea of considering the shift of a disjointness preserving operator is not new. Analogs of this notion occur, for instance, in [16] and in many papers about isometries of L^p-spaces.

7.4. The main criterion for WSW-representability stated in 4.2 is close to [4: 3.12]. Some of the criteria presented in 4.4 and 4.7 are also known (see [2–4]). Note that one of the sufficient conditions for WSW-representability (the second proposition in 4.4) is not necessary. The corresponding counterexample is given in 4.4. Existence of a similar example due to A. V. Koldunov is mentioned in [4: 3.14].

It is worth observing that our notion of weighted shift operator differs slightly from the analogous construction in the literature. The classical construction does not contain an inner weight (see [5: Theorem 6; 6: Theorem 4.1; 19: Theorems 2.8 and 2.9; 2: Theorem 6; 4: 3.8–3.18]). We regard this circumstance as a small demerit of the theory which, in particular, restricts the class of representations of vector lattices providing the WSW-representability and makes the problem of a global WSW-representation more difficult.

None of the known results ensured representation of an arbitrary bounded disjointness preserving operator on the entire domain of definition. Each representation theorem either restricted the class of operators under consideration (for instance by requiring order continuity), or restricted the class of spaces (for instance, by considering only Banach lattices), or did not guarantee a representation on the entire domain of definition (but only, for instance, on its principal ideals). In our opinion, the failure in searching for a global representation of disjointness preserving operators is mainly determined by the absence of an inner weight in the definition of a weighted shift operator. Involving an inner weight allows us to decompose an arbitrary bounded disjointness preserving operator in lattice-normed spaces into the strongly disjoint sum of weighted shift operators (Theorem 4.8). This result is new even for the case of operators in K-spaces (Theorem 4.5).

7.5. Many facts presented in Section 5 are essentially known. Some of them just repeat Y. A. Abramovich's results and treat the corresponding representations in more detail. Items 5.12–5.19 contain new material. The main Theorems 5.12 and 5.19 interpret the decompositions 4.5 and 4.8 of disjointness preserving operators into sums of weighted shift operators in terms of their functional representations.

7.6. The global representations 5.12 and 5.19 for a disjointness preserving operator, as well as the notions of the shift of an operator and the corresponding shift function, allow us to interpret the abstract properties of the operator in terms of its concrete function representation or in terms of the properties of its shift function. Some examples of similar interpretations can be found, for instance, in [2–4].

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