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LOCALLY ONE-DIMENSIONAL K-SPACES AND σ-DISTRIBUTIVE BOOLEAN ALGEBRAS

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Abstract

It is known that all band preserving operators acting in a universally complete K-space are regular if and only if the K-space is locally one-dimensional. In addition, a K-space with base B is locally one-dimensional if and only if $\mathbb{R}^\wedge = \mathcal{R}$ in $\mathbb{V}^{(B)}$. It seems to have been unknown so far whether there exist nondiscrete locally one-dimensional K-spaces. In the present note we give a positive answer to the question. As an auxiliary result, we establish that a K-space is locally one-dimensional if and only if its base is σ-distributive.

Key words and phrases: locally one-dimensional K-space, discrete K-space, σ-distributive Boolean algebra, σ-inductive Boolean algebra, atomic Boolean algebra, regular operator, real numbers in a Boolean-valued universe.

Throughout the text, E is an arbitrary universally complete K-space, E^+ is the totality of positive elements in E, Q is the Stone compact space of the base of E, and $\mathrm{Clop}(Q)$ is the Boolean algebra of all clopen (= simultaneously closed and open) subsets of Q. Given an arbitrary Boolean algebra B, the symbols 0 and 1 denote the greatest (unity) and the least (zero) elements of B. A subset of B with supremum unity is called a cover of B. A cover of B constituted by pairwise disjoint elements is called a partition of unity in B and is referred to as a partition of B for brevity. An element $e \in E^+$ is called *locally constant* with respect to an $f \in E^+$ if $e = \bigvee_{\xi\in\Xi} \lambda_\xi \pi_\xi f$ for some number family $(\lambda_\xi)_{\xi\in\Xi}$ and a family $(\pi_\xi)_{\xi\in\Xi}$ of pairwise disjoint order projections. A universally complete K-space E is called *locally one-dimensional* if it satisfies one of the following equivalent conditions (see [1]: Theorem 3.1):

(1) all elements of E^+ are locally constant with respect to some order unity of E;

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(2) all elements of E^+ are locally constant with respect to every order unity of E;

(3) for every function $e \in C_\infty(Q)$, there exists a partition $(U_\xi)_{\xi\in\Xi}$ of the algebra $\mathrm{Clop}(Q)$ such that the function e is constant on each of the sets U_ξ.

A linear operator $T : E \to E$ is said to be *band preserving* if, for all $e, f \in E$, from $|e| \wedge |f| = 0$ it follows that $|Te| \wedge |f| = 0$.

The following statement combines a result of Y. A. Abramovich, A. I. Veksler, and A. V. Koldunov ([2]: Theorem 2.1) and that of P. T. N. McPolin and A. W. Wickstead ([1]: Theorem 3.2).

Theorem 1. *Let E be a universally complete K-space. Every band preserving operator $T : E \to E$ is regular if and only if E is locally one-dimensional.*

In order to avoid misunderstanding, while reading the articles [2] and [1], one should be aware of the following two circumstances. First, despite of the fact that an arbitrary nondiscrete K-space is mentioned in the statement of Theorem 2.1 of [2], the proof of the theorem is given only for locally one-dimensional K-spaces. Second, the example of a nondiscrete locally one-dimensional K-space presented in [1] contains an error, which was recently reported by A. W. Wickstead in the article [3]. Thus, the question whether every locally one-dimensional K-space must be discrete (i.e. have an atomic base) seems to have been open so far.

The notion of a locally one-dimensional K-space admits the following Boolean-valued interpretation. (For an explanation of the main notions of Boolean-valued analysis, we refer the reader to the second part of the monograph [4].) Let B be a complete Boolean algebra, let $\mathcal{R}$ be the field of reals inside $\mathbb{V}^{(B)}$, and let $\mathbb{R}^\wedge$ be the canonical embedding of $\mathbb{R}$ into $\mathbb{V}^{(B)}$.

Theorem 2. *The equality $\mathbb{R}^\wedge = \mathcal{R}$ holds if and only if the descent of $\mathcal{R}$ is a locally one-dimensional K-space.*

$\lhd$ Knowing the general structure of the descents of objects that have the form $X^\wedge$, it is easy to deduce the stated assertion from the well-known E. I. Gordon's theorem (see [4]: 3.1.1(1), 5.2.1, and 5.2.2). $\rhd$

From private conversations with colleagues, the author of the present note is aware that, among the specialists in the domain of Boolean-valued analysis, the hypothesis is rather popular about atomicity of all Boolean algebras B that provide the equality $\mathbb{R}^\wedge = \mathcal{R}$ in $\mathbb{V}^{(B)}$. Thus, the question about the connection between discrete and locally one-dimensional K-spaces has a rather wide domain of applications, at least including vector lattices, positive operators, and Boolean-valued analysis.

After a certain preliminary discussion of the main notions, we will give an example of a purely nonatomic locally one-dimensional K-space. Due to Theorem 1, we shall thus obtain a purely nonatomic universally complete K-space E, for which all band preserving operators $T : E \to E$ are regular. Due to Theorem 2, we shall have a purely nonatomic complete Boolean algebra B, for which $\mathbb{R}^\wedge = \mathcal{R}$ in $\mathbb{V}^{(B)}$.

A σ-complete Boolean algebra B is called *σ-distributive* if it satisfies one of the following equivalent conditions (see [5]: 19.1):

(1) $\bigwedge_{n\in\mathbb{N}} \bigvee_{m\in\mathbb{N}} b^n_m = \bigvee_{m\in\mathbb{N}^\mathbb{N}} \bigwedge_{n\in\mathbb{N}} b^n_{m(n)}$ for all $b^n_m \in B$ $(n, m \in \mathbb{N})$;

(2) $\bigvee_{n\in\mathbb{N}} \bigwedge_{m\in\mathbb{N}} b^n_m = \bigwedge_{m\in\mathbb{N}^\mathbb{N}} \bigvee_{n\in\mathbb{N}} b^n_{m(n)}$ for all $b^n_m \in B$ $(n, m \in \mathbb{N})$;

(3) $\bigvee_{\varepsilon\in\{1,-1\}^\mathbb{N}} \bigwedge_{n\in\mathbb{N}} \varepsilon(n)b_n = 1$ for all $b_n \in B$ $(n \in \mathbb{N})$, where $1b_n = b_n$ and $(-1)b_n$ is the complement of b_n.

Let B be an arbitrary Boolean algebra and let C be a cover of B. A subset C_0 of the algebra B is said to be *refined* from C if, for each $c_0 \in C_0$, there exists a $c \in C$ such that $c_0 \leqslant c$. An element $b \in B$ is called refined from C if the set $\{b\}$ is refined from C, i.e., $b \leqslant c$ for some element $c \in C$. If $(C_n)_{n\in\mathbb{N}}$ is a sequence of covers of the algebra B and an element $b \in B$ is refined from each of the covers C_n $(n \in \mathbb{N})$, then we say that b is refined from the sequence $(C_n)_{n\in\mathbb{N}}$. We also refer to a cover, all elements of which are refined from the sequence $(C_n)_{n\in\mathbb{N}}$, as refined from the sequence.

Proposition 3. *Let B be a σ-complete Boolean algebra. The following assertions are equivalent:*

(1) *the algebra B is σ-distributive;*

(2) *from every sequence of countable covers of B, one can refine a (possibly, uncountable) cover;*

(3) *from every sequence of finite covers of B, one can refine a (possibly, infinite) cover;*

(4) *from every sequence of two-element partitions of B, one can refine a cover.*

$\lhd$ A proof of the equivalence (1)$\Leftrightarrow$(2) can be found in [5] (see 19.3). Assertion (4) is a reformulation of condition (3) in the definition of σ-distributivity. The implications (2)$\Rightarrow$(3)$\Rightarrow$(4) are obvious. $\rhd$

A proof of the following result is presented in [5] (see 20.2).

Theorem 4 (the exhaustion principle). *From every cover of a complete Boolean algebra, one can refine a partition.*

Corollary 5. *Let B be a complete Boolean algebra. The following assertions are equivalent:*

1. *the algebra B is σ-distributive;*
2. *from every sequence of countable partitions of B, one can refine a (possibly, uncountable) partition;*
3. *from every sequence of finite partitions of B, one can refine a (possibly, infinite) partition;*
4. *from every sequence of two-element partitions of B, one can refine a partition.*

We say that a function $e \in C_\infty(Q)$ is *refined* from a cover C of the Boolean algebra $\mathrm{Clop}(Q)$ if, for every two points $q', q'' \in Q$ satisfying the equality $e(q') = e(q'')$, there exists an element $U \in C$ such that $q', q'' \in U$. If $(C_n)_{n\in\mathbb{N}}$ is a sequence of covers of the algebra $\mathrm{Clop}(Q)$ and a function e is refined from each of the covers C_n ($n \in \mathbb{N}$), then we say that the function e is refined from the sequence $(C_n)_{n\in\mathbb{N}}$.

Lemma 6. *From every sequence of finite covers of the algebra* $\mathrm{Clop}(Q)$, *one can refine a function of $C(Q)$.*

$\lhd$ Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of finite covers of the algebra $\mathrm{Clop}(Q)$. With the help of induction, it is not difficult to construct a sequence of partitions $P_m = \{U_1^m, U_2^m, \dots, U_{2^m}^m\}$ of the algebra $\mathrm{Clop}(Q)$ possessing the following properties:

1. for every $n \in \mathbb{N}$ there is a number $m \in \mathbb{N}$ such that the partition P_m is refined from the cover C_n;
2. $U_j^m = U_{2j-1}^{m+1} \vee U_{2j}^{m+1}$ for all $m \in \mathbb{N}$ and $j \in \{1, 2, \dots, 2^m\}$.

For each number $m \in \mathbb{N}$, define a two-valued function $\chi_m \in C(Q)$ as follows:

$$\chi_m := \sum_{i=1}^{2^{m-1}} \chi(U_{2i}^m),$$

where $\chi(U)$ is the characteristic function of a subset $U \subset Q$. Since the series $\sum_{m=1}^{\infty} \frac{1}{3^m}\chi_m$ is uniformly convergent, its sum e belongs to $C(Q)$. We will show that the function e is refined from $(C_n)_{n\in\mathbb{N}}$. Due to property (1) of the sequence $(P_m)_{m\in\mathbb{N}}$, it is sufficient for this to establish that the function e is refined from $(P_m)_{m\in\mathbb{N}}$.

Assume the contrary and consider the smallest number $m \in \mathbb{N}$, for which the function e is not refined from the partition P_m. In this case, there are two points $q', q'' \in Q$ that satisfy the equality $e(q') = e(q'')$ and belong to distinct elements of P_m. Since the function e is refined from the partition P_{m-1} (for $m > 1$), from property (2) of the sequence $(P_m)_{m\in\mathbb{N}}$ it follows that the points q' and q'' belong to adjacent elements of P_m, i.e. elements of the form U_j^m and U_{j+1}^m, where $j \in \{1,\dots,2^m - 1\}$. For definiteness, suppose that q' belongs to an element with even subscript and q'' with odd one, i.e., $\chi_m(q') = 1$ and $\chi_m(q'') = 0$. Therefore, taking into account the fact that $\chi_i(q') = \chi_i(q'')$ for all $i \in \{1,\dots,m-1\}$, we have:

$$e(q') - e(q'') = \frac{1}{3^m} + \sum_{i=m+1}^{\infty} \frac{1}{3^i}\bigl(\chi_i(q') - \chi_i(q'')\bigr) \geqslant \frac{1}{3^m} - \sum_{i=m+1}^{\infty} \frac{1}{3^i} = \frac{1}{2\cdot 3^m} > 0,$$

which contradicts the equality $e(q') = e(q'')$. $\rhd$

Theorem 7. *A universally complete K-space is locally one-dimensional if and only if its base is σ-distributive.*

$\lhd$ Let E be a universally complete K-space and let Q be the Stone compact space of its base. Suppose that E is locally one-dimensional and consider an arbitrary sequence $(P_n)_{n\in\mathbb{N}}$ of finite partitions of the Boolean algebra $\mathrm{Clop}(Q)$. According to Corollary 5, in order to prove σ-distributivity of the base of E, it is sufficient to refine a cover of $\mathrm{Clop}(Q)$ from $(P_n)_{n\in\mathbb{N}}$. In view of Lemma 6, one can refine a function $e \in C_\infty(Q)$ from the sequence $(P_n)_{n\in\mathbb{N}}$. Since E is locally one-dimensional, there exists a partition $(U_\xi)_{\xi\in\Xi}$ of the algebra $\mathrm{Clop}(Q)$ such that the function e is constant on each of the sets U_ξ. Show that the partition $(U_\xi)_{\xi\in\Xi}$ is refined from the sequence $(P_n)_{n\in\mathbb{N}}$. To this end, we fix arbitrary indices $\xi \in \Xi$ and $n \in \mathbb{N}$ and establish that the set U_ξ is refined from the partition P_n. We may assume that $U_\xi \neq \varnothing$. Let q_0 be an element of U_ξ. Finiteness of the partition P_n allows us to find an element U of it such that $q_0 \in U$. It remains to observe that $U_\xi \subset U$. Indeed, if $q \in U_\xi$ then $e(q) = e(q_0)$ and, since the function e is refined from P_n, the points q and q_0 belong to the same element of the partition P_n, i.e., $q \in U$.

Now, assume that the base of E is σ-distributive and consider an arbitrary function $e \in C_\infty(Q)$. According to condition (3) of the definition of a locally one-dimensional K-space, it is sufficient to construct a partition $(U_\xi)_{\xi\in\Xi}$ of the algebra $\mathrm{Clop}(Q)$ such that the function e is constant on each of the sets U_ξ. For every natural n and every integer m, denote by U_m^n the interior of the closure of the set of all points $q \in Q$ for which $\frac{m}{n} \leqslant e(q) < \frac{m+1}{n}$ and define $P_n := \bigl\{U_m^n : m \in \mathbb{Z}\bigr\}$. Due to Corollary 5, from the sequence $(P_n)_{n\in\mathbb{N}}$ of countable partitions of the algebra $\mathrm{Clop}(Q)$, one can refine some partition $(U_\xi)_{\xi\in\Xi}$. It is not difficult to become convinced that the partition constructed is the desired one. $\rhd$

Thus, the question about existence of a purely nonatomic locally one-dimensional K-space is reduced to existence of a purely nonatomic σ-distributive complete Boolean algebra. The remainder of the note is devoted to constructing such an algebra.

A Boolean algebra B is called *σ-inductive* if every decreasing sequence of nonzero elements of B admits a nonzero lower bound. A subalgebra B_0 of a Boolean algebra B is said to be *dense* if, for every nonzero element $b \in B$, there exists a nonzero element $b_0 \in B_0$ such that $b_0 \leqslant b$.

Lemma 8. *If a σ-complete Boolean algebra contains a σ-inductive dense subalgebra then it is σ-distributive.*

$\lhd$ Let B be a σ-complete Boolean algebra and let B_0 be a σ-inductive dense subalgebra of B. Consider an arbitrary sequence $(C_n)_{n\in\mathbb{N}}$ of countable covers of B, denote by C the set of all elements in B that are refined from $(C_n)_{n\in\mathbb{N}}$, and assume by way of contradiction that C is not a cover of B. Then there exists a nonzero element $b \in B$ that is disjoint with all elements of C.

By induction, we construct sequences $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ as follows. Let c_1 be an element of C_1 such that $b \wedge c_1 \neq 0$. Since B_0 is dense, there is an element $b_1 \in B_0$ such that $0 < b_1 \leqslant b \wedge c_1$. Suppose that the elements b_n and c_n are already constructed. Let c_{n+1} be an element of C_{n+1} such that $b_n \wedge c_{n+1} \neq 0$. As b_{n+1} we take an arbitrary element of B_0 that satisfies the inequalities $0 < b_{n+1} \leqslant b_n \wedge c_{n+1}$.

Thus, we have constructed sequences $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ such that $b_n \in B_0$, $b_n \leqslant c_n \in C_n$ and $0 < b_{n+1} \leqslant b_n \leqslant b$ for all $n \in \mathbb{N}$. Due to the fact that B_0 is σ-inductive, it contains an element b_0 which satisfies $b_0 \leqslant b_n$ for all $n \in \mathbb{N}$. In view of the inequalities $b_0 \leqslant c_n$, the element b_0 is refined from $(C_n)_{n\in\mathbb{N}}$, i.e., belongs to C. On the other hand, $b_0 \leqslant b$, which contradicts disjointness of b with all elements of C. $\rhd$

As is known, for every Boolean algebra B, there exists a complete Boolean algebra $\overline{B}$ that contains B as a dense subalgebra (see [5]: Section 35). Such an algebra $\overline{B}$ is unique to within an isomorphism and called a *completion* of B. Obviously, a completion of a purely nonatomic Boolean algebra is purely nonatomic. In addition, due to Lemma 8, a completion of a σ-inductive algebra is σ-distributive. Therefore, in order to prove existence of a purely nonatomic σ-distributive complete Boolean algebra, it is sufficient to present an arbitrary purely nonatomic σ-inductive Boolean algebra. Examples of such algebras are readily available. For the sake of completeness, we present here one of the simplest constructions.

Example 9. Let B be the Boolean algebra of all subsets of $\mathbb{N}$ and let I be the ideal of B consisting of all finite subsets of $\mathbb{N}$. Then the quotient algebra B/I (see [5]: Section 10) is purely nonatomic and σ-inductive.

$\lhd$ Pure nonatomicity of the algebra B/I is obvious. In order to prove that the algebra is σ-inductive, it is sufficient to consider an arbitrary decreasing sequence $(b_n)_{n\in\mathbb{N}}$ of infinite subsets of $\mathbb{N}$ and construct an infinite subset $b \subset \mathbb{N}$ such that the difference $b\backslash b_n$ is finite for each $n \in \mathbb{N}$. We can easily obtain the desired set $b = \{m_n : n \in \mathbb{N}\}$ with the help of induction by letting $m_1 := \min b_1$ and $m_{n+1} := \min\{m \in b_{n+1} : m > m_n\}$. $\rhd$

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