Siberian Advances in Mathematics
1998, v. 8, N1, 99–120

FUNCTION REPRESENTATION OF THE BOOLEAN-VALUED UNIVERSE

A. E. Gutman * *and G. A. Losenkov* **

Abstract

For an abstract Boolean-valued system, a function analog is proposed that is a model whose elements are functions and the basic logical operations are calculated "pointwise."

The new notion of continuous polyverse is introduced and studied which is a continuous bundle of models of set theory. It is shown that the class of continuous sections of a continuous polyverse is a Boolean-valued system satisfying all basic principles of Boolean-valued analysis and, conversely, every Boolean-valued algebraic system can be represented as the class of sections of a suitable continuous polyverse.

Key words and phrases: Boolean-valued analysis, function representation, Stone space, continuous bundle, continuous section.

The methods of Boolean-valued analysis base on nonstandard models of set theory with multivalued truth. More exactly, the truth value of an assertion in such a model acts into some complete Boolean algebra.

At present, Boolean-valued analysis is a rather powerful theory rich of deep results and various applications, mainly, to set theory. As regards functional analysis, the methods of Boolean-valued analysis found successful applications in such domains as the theory of vector lattices and lattice-normed spaces, the theory of positive and dominated operators, the theory of von Neumann algebras, convex analysis, and the theory of vector measures.

Contemporary methods of Boolean-valued analysis, due to their nature, involve rather bulky logical technique. We can say that, from a pragmatic viewpoint, this technique might distract the user-analyst from a concrete aim: to apply the results of Boolean-valued analysis for solving analytical problems.

Received November, 1997.

Supported by the Russian Foundation for Basic Research (grant 97-01-00001).

* Sobolev Institute of Mathematics, Novosibirsk, RUSSIA.

E-mail address: gutman@math.nsc.ru.

** Novosibirsk State University, Novosibirsk, RUSSIA.

Various function spaces are common in functional analysis, and so the intention is natural of replacing an abstract Boolean-valued system by some function analog, a model whose elements are functions and in which the basic logical operations are calculated "pointwise." An example of such a model is presented by the class $\mathbb{V}^Q$ of all functions defined on a fixed nonempty set Q and acting into the class $\mathbb{V}$ of all sets. Truth values in the model $\mathbb{V}^Q$ are various subsets of Q and, in addition, the truth value $\|\varphi(u_1,\dots,u_n)\|$ of an assertion $\varphi(t_1,\dots,t_n)$ at functions $u_1,\dots,u_n \in \mathbb{V}^Q$ is calculated as follows:

$$\|\varphi(u_1,\dots,u_n)\| = \bigl\{q \in Q \bigm| \varphi\bigl(u_1(q),\dots,u_n(q)\bigr)\bigr\}.$$

In the present article, a solution is proposed to the above problem. To this end, we introduce and study the new notion of continuous polyverse, the latter being a continuous bundle of models of set theory. It is shown that the class of continuous sections of a continuous polyverse is a Boolean-valued system satisfying all basic principles of Boolean-valued analysis and, conversely, every Boolean-valued algebraic system can be represented as the class of sections of a suitable continuous polyverse.

1. Prerequisites

1.1. Let X and Y be topological spaces. A mapping $f\colon X \to Y$ is called *open* if it satisfies one (and hence all) of the following equivalent conditions:

(1) for every open subset $A \subset X$, the image $f(A)$ is open in Y;
(2) for every point $x \in X$ and every neighborhood $A \subset X$ about x, the image $f(A)$ is a neighborhood about $f(x)$ in Y;
(3) $f^{-1}(\operatorname{cl} B) \subset \operatorname{cl} f^{-1}(B)$ for every subset $B \subset Y$.

Observe that the equality $f^{-1}(\operatorname{cl} B) = \operatorname{cl} f^{-1}(B)$ holds for all subsets $B \subset Y$ if and only if the mapping f is continuous and open.

A mapping $f\colon X \to Y$ is called *closed* if it satisfies one (and hence all) of the following equivalent conditions:

(1) for every closed subset $A \subset X$, the image $f(A)$ is closed in Y;
(2) $\operatorname{cl} f(A) \subset f(\operatorname{cl} A)$ for every subset $A \subset X$.

The equality $\operatorname{cl} f(A) = f(\operatorname{cl} A)$ holds for every subset $A \subset X$ if and only if the mapping $f\colon X \to Y$ is continuous and closed.

1.2. Given a class X, the symbol $\mathcal{P}(X)$ denotes the class of all subsets of X.

Let X be a class. A subclass $\tau \subset \mathcal{P}(X)$ is called a *topology* on X whenever

(1) $\cup\tau = X$;
(2) $U \cap V \in \tau$ for all $U, V \in \tau$;
(3) $\cup\mathcal{U} \in \tau$ for every subset $\mathcal{U} \subset \tau$.

As usual, a class X endowed with a topology is called a *topological space.*

All basic topological concepts (such as neighborhood about a point, closed set, interior, closure, continuous function, Hausdorff space, etc.) can be introduced by analogy to the case of a topology on a set. However, observe that not all classical approaches to the definition of these concepts remain formally valid in the case of a class-topology. For instance, considering the two definitions of a closed set

(a) as a subset of X whose complement belongs to τ,
(b) as a subset of X whose complement, together with each point of it, contains an element of τ,

we should choose the second.

Defining the closure of a set A as the smallest closed subset of X that contains A, we take a risk: some sets may turn out to have no closure. However, the problem disappears if the topology τ is Hausdorff. (Indeed, in the case of a Hausdorff topology, every convergent filter has a unique limit and, hence, the totality of all limits of convergent filters over a given set makes a set rather than a proper class.)

The symbol $\mathrm{Clop}(X)$ denotes the class of all clopen subsets of X (i.e., subsets that are closed and open simultaneously). Henceforth the notation $U \sqsubset X$ means that $U \in \mathrm{Clop}(X)$. The class $\{A \sqsubset X \mid x \in A\}$ is denoted by $\mathrm{Clop}(x)$.

A topology is called *extremally disconnected* if the closure of every open set is again open.

Most of the necessary information about topological spaces can be found, for instance, in [1, 2].

1.3. Let B be a complete Boolean algebra. A triple $(\mathfrak{U},\ \|\cdot=\cdot\|,\ \|\cdot\in\cdot\|)$ is called a *Boolean-valued algebraic system* over B (or a B*-valued algebraic system*) if the classes $\|\cdot=\cdot\|$ and $\|\cdot\in\cdot\|$ are class-functions from $\mathfrak{U}\times\mathfrak{U}$ into B that satisfy the following conditions:

(1) $\|u = u\| = \mathbf{1}$;
(2) $\|u = v\| = \|v = u\|$;
(3) $\|u = v\| \wedge \|v = w\| \leqslant \|u = w\|$;
(4) $\|u = v\| \wedge \|v \in w\| \leqslant \|u \in w\|$;
(5) $\|u = v\| \wedge \|w \in v\| \leqslant \|w \in u\|$

for all $u, v, w \in \mathfrak{U}$.

The class-functions $\|\cdot=\cdot\|$ and $\|\cdot\in\cdot\|$ are called the Boolean-valued (B-valued) *truth values* of equality and membership.

Instead of $(\mathfrak{U},\ \|\cdot=\cdot\|,\ \|\cdot\in\cdot\|)$, we usually write simply $\mathfrak{U}$ and, if necessary, furnish the symbols of truth values with the index: $\|\cdot=\cdot\|_{\mathfrak{U}}$ and $\|\cdot\in\cdot\|_{\mathfrak{U}}$.

A Boolean-valued system $\mathfrak{U}$ is called *separated* whenever, for all $u, v \in \mathfrak{U}$, the equality $\|u = v\| = \mathbf{1}$ implies $u = v$.

1.4. Consider Boolean-valued algebraic systems $\mathfrak{U}$ and $\mathfrak{V}$ over complete Boolean algebras B and C and assume that there is a Boolean isomorphism $j\colon B \to C$. By an *isomorphism between the Boolean-valued algebraic systems* $\mathfrak{U}$ *and* $\mathfrak{V}$ (*associated with the isomorphism* j) we mean a bijective class-function $i\colon \mathfrak{U} \to \mathfrak{V}$ that satisfies the following relations:

$$j(\|u_1 = u_2\|_{\mathfrak{U}}) = \|i(u_1) = i(u_2)\|_{\mathfrak{V}},$$
$$j(\|u_1 \in u_2\|_{\mathfrak{U}}) = \|i(u_1) \in i(u_2)\|_{\mathfrak{V}}$$

for all $u_1, u_2 \in \mathfrak{U}$. Boolean-valued systems are said to be *isomorphic* if there is an isomorphism between them. In case $\mathfrak{U}$ and $\mathfrak{V}$ are Boolean-valued algebraic systems over the same algebra B, each isomorphism $i\colon \mathfrak{U} \to \mathfrak{V}$ is assumed by default to be associated with the identity isomorphism: $\|u_1 = u_2\|_{\mathfrak{U}} = \|i(u_1) = i(u_2)\|_{\mathfrak{V}}$, $\|u_1 \in u_2\|_{\mathfrak{U}} = \|i(u_1) \in i(u_2)\|_{\mathfrak{V}}$. For emphasizing this convention, whenever necessary, we call such an isomorphism B*-isomorphism* and refer to the corresponding systems as B*-isomorphic.*

1.5. In what follows, using an expression like $\varphi(t_1, \dots, t_n)$, we assume that φ is a formula of set-theoretic signature with all free variables included in the list $(t_1, \dots, t_n)$.

An arbitrary tuple $(u_1, \dots, u_n)$ of elements in a system $\mathfrak{U}$ is called a *valuation* of the list of variables $(t_1, \dots, t_n)$. By recursion on the complexity of a formula, the (Boolean) *truth value* $\|\varphi(u_1, \dots, u_n)\|$ of a formula $\varphi(t_1, \dots, t_n)$ can be defined with respect to a given valuation $(u_1, \dots, u_n)$ of the variables $(t_1, \dots, t_n)$. If a formula φ is atomic, i.e., has the form $t_1 = t_2$ or $t_1 \in t_2$; then its truth value with respect to a valuation (u_1, u_2) is defined to be $\|u_1 = u_2\|$ or $\|u_1 \in u_2\|$. Considering compound formulas, we define their truth values as follows:

$$\|\varphi(u_1, \dots, u_n) \mathbin{\&} \psi(u_1, \dots, u_n)\| := \|\varphi(u_1, \dots, u_n)\| \wedge \|\psi(u_1, \dots, u_n)\|,$$
$$\|\varphi(u_1, \dots, u_n) \vee \psi(u_1, \dots, u_n)\| := \|\varphi(u_1, \dots, u_n)\| \vee \|\psi(u_1, \dots, u_n)\|,$$
$$\|\varphi(u_1, \dots, u_n) \to \psi(u_1, \dots, u_n)\| := \|\varphi(u_1, \dots, u_n)\| \Rightarrow \|\psi(u_1, \dots, u_n)\|,$$
$$\|\neg\varphi(u_1, \dots, u_n)\| := \|\varphi(u_1, \dots, u_n)\|^{\perp},$$
$$\|(\forall t)\, \varphi(t, u_1, \dots, u_n)\| := \bigwedge_{u \in \mathfrak{U}} \|\varphi(u, u_1, \dots, u_n)\|,$$
$$\|(\exists t)\, \varphi(t, u_1, \dots, u_n)\| := \bigvee_{u \in \mathfrak{U}} \|\varphi(u, u_1, \dots, u_n)\|,$$

where the symbol $b^{\perp}$ denotes the complement of b in the Boolean algebra B. A formula $\varphi(t_1, \dots, t_n)$ is said to be *true* in an algebraic system $\mathfrak{U}$ with respect to a valuation $(u_1, \dots, u_n)$ if the equality $\|\varphi(u_1, \dots, u_n)\| = \mathbf{1}$ holds. In this case, we write $\mathfrak{U} \models \varphi(u_1, \dots, u_n)$.

1.6. Proposition. *If a formula* $\varphi(t_1,\dots,t_n)$ *is provable in the first-order predicate calculus then* $\|\varphi(u_1,\dots,u_n)\| = \mathbf{1}$ *for all* $u_1,\dots,u_n \in \mathfrak{U}$.

◁ It is easy to verify that all the axioms of the first-order predicate calculus are true in $\mathfrak{U}$ and the rules of inference preserve the truth value. The latter means that derivability (in the first-order predicate calculus) of a formula φ from formulas $\varphi_1,\dots,\varphi_n$ ensures the inequality $\|\varphi_1 \wedge \dots \wedge \varphi_n\| \leqslant \|\varphi\|$. ▷

In particular, the last proposition implies that, for an arbitrary formula $\varphi(t, t_1,\dots,t_n)$ and arbitrary elements $u, v, w_1,\dots,w_n \in \mathfrak{U}$, we have the inequality $\|u = v\| \wedge \|\varphi(u, w_1,\dots,w_n)\| \leqslant \|\varphi(v, w_1,\dots,w_n)\|$.

1.7. Let $u \in \mathfrak{U}$ be such that $\mathfrak{U} \models u \neq \varnothing$. The *descent* of the element u is the class $\{v \in \mathfrak{U} \mid \mathfrak{U} \models v \in u\}$ denoted by $u{\downarrow}$.

1.8. Let $(u_\xi)_{\xi\in\Xi}$ be a family of elements in $\mathfrak{U}$ and let $(b_\xi)_{\xi\in\Xi}$ be a family of elements in the Boolean algebra B. An element $u \in \mathfrak{U}$ is called an *ascent of the family* $(u_\xi)_{\xi\in\Xi}$ *with weights* $(b_\xi)_{\xi\in\Xi}$, if $\|v \in u\| = \bigvee_{\xi\in\Xi} b_\xi \wedge \|v = u_\xi\|$ for all $v \in \mathfrak{U}$.

Let $\mathcal{U}$ be a subset of $\mathfrak{U}$. An element $\overline{u} \in \mathfrak{U}$ is called an *ascent of the set* $\mathcal{U}$, if $\|v \in \overline{u}\| = \bigvee_{u\in\mathcal{U}} \|v = u\|$ for all $v \in \mathfrak{U}$, i.e., $\overline{u}$ is an ascent of the family $(u)_{u\in\mathcal{U}}$ with unit weights.

Assume that $(b_\xi)_{\xi\in\Xi}$ is an antichain in the algebra B. An element $u \in \mathfrak{U}$ is called a *mixing* of the family $(u_\xi)_{\xi\in\Xi}$ with weights $(b_\xi)_{\xi\in\Xi}$, if $\|u = u_\xi\| \geqslant b_\xi$ for all $\xi \in \Xi$, and $\|u = \varnothing\| \geqslant (\bigvee_{\xi\in\Xi} b_\xi)^\perp$.

If the system $\mathfrak{U}$ is separated and the extensionality axiom is true in $\mathfrak{U}$, then an ascent (mixing) of a family $(u_\xi)_{\xi\in\Xi}$ with weights $(b_\xi)_{\xi\in\Xi}$ is uniquely determined. In this case, whenever the ascent (mixing) exists, we denote it by $\operatorname{asc}_{\xi\in\Xi} b_\xi u_\xi$ ($\operatorname{mix}_{\xi\in\Xi} b_\xi u_\xi$). For the ascent of a set $\mathcal{U} \subset \mathfrak{U}$, we use the notation $\mathcal{U}{\uparrow}$.

1.9. In Boolean-valued analysis, three basic principles play a particular role, namely, the maximum principle, the mixing principle, and the ascent principle. This is explained by the fact that, in algebraic systems satisfying the principles, there is a possibility of constructing new elements from available elements.

In the current section, we state the above-mentioned principles and study interrelations between them, leaving aside the verification of the principles for concrete algebraic systems.

Let B be a complete Boolean algebra and let $\mathfrak{U}$ be a B-valued algebraic system.

The maximum principle. *For every formula $\varphi(t, t_1, \dots, t_n)$ and arbitrary elements $u_1, \dots, u_n \in \mathfrak{U}$, there exists an element $u \in \mathfrak{U}$ such that $\|(\exists t)\, \varphi(t, u_1, \dots, u_n)\| = \|\varphi(u, u_1, \dots, u_n)\|$.*

The mixing principle. *For every family $(u_\xi)_{\xi\in\Xi}$ of elements in $\mathfrak{U}$ and every antichain $(b_\xi)_{\xi\in\Xi}$ in the algebra B, there exists a mixing $(u_\xi)_{\xi\in\Xi}$ with weights $(b_\xi)_{\xi\in\Xi}$.*

The ascent principle. (1) *For every family $(u_\xi)_{\xi\in\Xi}$ of elements in $\mathfrak{U}$ and every family $(b_\xi)_{\xi\in\Xi}$ of elements in the algebra B, there exists an ascent $(u_\xi)_{\xi\in\Xi}$ with weights $(b_\xi)_{\xi\in\Xi}$.*

(2) *For every element $u \in \mathfrak{U}$, there exist a family $(u_\xi)_{\xi\in\Xi}$ of elements in $\mathfrak{U}$ and a family $(b_\xi)_{\xi\in\Xi}$ of elements in the algebra B such that u is an ascent of $(u_\xi)_{\xi\in\Xi}$ with weights $(b_\xi)_{\xi\in\Xi}$.*

1.10. Theorem. *If a B-valued system $\mathfrak{U}$ satisfies the mixing principle then $\mathfrak{U}$ satisfies the maximum principle.*

◁ Consider a formula $\varphi(t, t_1, \dots, t_n)$, denote by $\vec{u}$ a tuple of arbitrary elements $u_1, \dots, u_n \in \mathfrak{U}$, and put $b = \|(\exists t)\, \varphi(t, \vec{u})\|$. By the definition of truth value, $b = \bigvee_{v\in\mathfrak{U}} \|\varphi(v, \vec{u})\|$. According to the exhaustion principle, there exist an antichain $(b_\xi)_{\xi\in\Xi}$ in the algebra B and a family $(v_\xi)_{\xi\in\Xi}$ of elements in $\mathfrak{U}$ such that $\bigvee_{\xi\in\Xi} b_\xi = b$ and $b_\xi \leqslant \|\varphi(v_\xi, \vec{u})\|$. By the hypothesis of the theorem, there exists a mixing $v \in \mathfrak{U}$ of the family $(v_\xi)_{\xi\in\Xi}$ with weights $(b_\xi)_{\xi\in\Xi}$. In particular, $\|v = v_\xi\| \geqslant b_\xi$. In view of Proposition 1.6, the following inequalities hold: $\|\varphi(v, \vec{u})\| \geqslant \|v = v_\xi\| \wedge \|\varphi(v_\xi, \vec{u})\| \geqslant b_\xi$. Consequently, $\|\varphi(v, \vec{u})\| \geqslant \bigvee_{\xi\in\Xi} b_\xi = b$. The inequality $\|\varphi(v, \vec{u})\| \leqslant b$ is obvious. ▷

1.11. Theorem. *Let a B-valued algebraic system $\mathfrak{U}$ satisfy the ascent principle and let the extensionality axiom be true in $\mathfrak{U}$. Then the mixing principle is valid for $\mathfrak{U}$.*

◁ Let $(u_\xi)_{\xi\in\Xi}$ be a family of elements in $\mathfrak{U}$ and let $(b_\xi)_{\xi\in\Xi}$ be an antichain in the algebra B. By the hypothesis of the theorem, for every $\xi \in \Xi$, there exist a family $(u_\xi^\alpha)_{\alpha\in A(\xi)}$ of elements in $\mathfrak{U}$ and a family $(b_\xi^\alpha)_{\alpha\in A(\xi)}$ of elements in the algebra B such that

$$\|v \in u_\xi\| = \bigvee_{\alpha\in A(\xi)} b_\xi^\alpha \wedge \|v = u_\xi^\alpha\| \quad \text{for all } v \in \mathfrak{U}.$$

Consider the set $\Gamma = \{(\xi, \alpha) \mid \xi \in \Xi,\ \alpha \in A(\xi)\}$ and, for each pair $\gamma = (\xi, \alpha) \in \Gamma$, put $c_\gamma = b_\xi \wedge b_\xi^\alpha$ and $v_\gamma = u_\xi^\alpha$. Let $u \in \mathfrak{U}$ be an ascent of the family $(v_\gamma)_{\gamma\in\Gamma}$ with weights $(c_\gamma)_{\gamma\in\Gamma}$. Using straightforward calculation and employing definitions, we obtain:

$$\begin{aligned}\|v \in u\| &= \bigvee_{\gamma\in\Gamma} c_\gamma \wedge \|v = v_\gamma\| \\ &= \bigvee_{\xi\in\Xi} \bigvee_{\alpha\in A(\xi)} b_\xi \wedge b_\xi^\alpha \wedge \|v = u_\xi^\alpha\| \\ &= \bigvee_{\xi\in\Xi} b_\xi \wedge \|v \in u_\xi\|.\end{aligned}$$

Show that u is a mixing of the family $(u_\xi)_{\xi\in\Xi}$ with weights $(b_\xi)_{\xi\in\Xi}$. We begin with establishing the inequality $\|u = u_\xi\| \geqslant b_\xi$. Since the extensionality axiom is true, it is sufficient to show that $\big(\|v \in u\| \Leftrightarrow \|v \in u_\xi\|\big) \geqslant b_\xi$ or, which is equivalent, $b_\xi \wedge \|v \in u\| = b_\xi \wedge \|v \in u_\xi\|$. Employing the fact that $b_\xi \wedge b_\eta = \mathbf{0}$ for $\xi \neq \eta$, we have:

$$b_\xi \wedge \|v \in u\| = \bigvee_{\eta\in\Xi} b_\xi \wedge b_\eta \wedge \|v \in u_\eta\| = b_\xi \wedge \|v \in u_\xi\|.$$

We now show that $\|u \neq \varnothing\| \leqslant \bigvee_{\xi\in\Xi} b_\xi$. Indeed,

$$\|u \neq \varnothing\| = \|(\exists t)\, t \in u\| = \bigvee_{v\in\mathfrak{A}} \|v \in u\| = \bigvee_{v\in\mathfrak{A}} \bigvee_{\xi\in\Xi} b_\xi \wedge \|v \in u_\xi\| \leqslant \bigvee_{\xi\in\Xi} b_\xi. \quad \triangleright$$

1.12. Theorem. *If a B-valued algebraic system $\mathfrak{A}$ satisfies the maximum and ascent principles then $\mathfrak{A}$ satisfies the mixing principle.*

$\triangleleft$ Let $\varnothing^\wedge \in \mathfrak{A}$ be an ascent of the empty subset of $\mathfrak{A}$. It is easy to verify that $\|\varnothing^\wedge = \varnothing\| = \mathbf{1}$. (Here and in the sequel, the notation $u = \varnothing$ means $(\forall t)\, t \notin u$.)

Consider a family $(u_\xi)_{\xi\in\Xi}$ of elements in $\mathfrak{A}$ and an antichain $(b_\xi)_{\xi\in\Xi}$ in the algebra B. Put $b = (\bigvee_{\xi\in\Xi} b_\xi)^\perp$. Define a family $(v_\xi)_{\xi\in\Xi'}$ and a partition of unity $(c_\xi)_{\xi\in\Xi'}$ as follows: $\Xi' = \Xi \cup \{\Xi\}$, $v_\xi = u_\xi$, $c_\xi = b_\xi$ for $\xi \in \Xi$, and $v_\Xi = \varnothing^\wedge$, $c_\Xi = b$. Let $u \in \mathfrak{A}$ be an ascent of the family $(v_\xi)_{\xi\in\Xi'}$ with weights $(c_\xi)_{\xi\in\Xi'}$. It is easily seen that $\|u \neq \varnothing\| = \mathbf{1}$. Indeed, $\|v_\xi \in u\| \geqslant c_\xi$ for $\xi \in \Xi'$, which implies

$$\|u \neq \varnothing\| = \bigvee_{v\in\mathfrak{A}} \|v \in u\| \geqslant \bigvee_{\xi\in\Xi'} c_\xi = \mathbf{1}.$$

Thus, $\|(\exists t)\, t \in u\| = \mathbf{1}$. According to the maximum principle, there exists an element $v \in \mathfrak{A}$ such that $\|v \in u\| = \mathbf{1}$. Then, by the definition of ascent,

$$c_\xi = \mathbf{1} \wedge c_\xi = \bigvee_{\eta\in\Xi'} c_\eta \wedge \|v = v_\eta\| \wedge c_\xi = \|v = v_\xi\| \wedge c_\xi$$

and, hence, $\|v = v_\xi\| \geqslant c_\xi$ for all $\xi \in \Xi'$. In particular, for $\xi \in \Xi$, we have $\|v = u_\xi\| \geqslant b_\xi$. In addition, by Proposition 1.6, the following relations hold:

$$\Big(\bigvee_{\xi\in\Xi} b_\xi\Big)^{\perp} \leqslant \|v = \varnothing^\wedge\| = \|v = \varnothing^\wedge\| \wedge \|\varnothing^\wedge = \varnothing\| \leqslant \|v = \varnothing\|.$$

Consequently, v is a mixing of the family $(u_\xi)_{\xi\in\Xi}$ with weights $(b_\xi)_{\xi\in\Xi}$. ▷

1.13. Let B be a complete Boolean algebra and let $\mathfrak{U}$ be a B-valued algebraic system. The system $\mathfrak{U}$ is called a *Boolean-valued universe over* B (a B*-valued universe*) if it satisfies the following three conditions:

(1) $\mathfrak{U}$ is separated;
(2) $\mathfrak{U}$ satisfies the ascent principle;
(3) the extensionality and regularity axioms are true in $\mathfrak{U}$.

Theorem ([3]). *For every complete Boolean algebra* B*, there exists a* B*-valued universe which is unique up to isomorphism.*

A detailed presentation of the theories of Boolean algebras and Boolean-valued algebraic systems can be found in [4–7].

2. The notion of continuous bundle

2.1. Let Q be an arbitrary nonempty set and let $V^Q \subset Q \times \mathbb{V}$ be a class-correspondence. (Here and in the sequel, $\mathbb{V}$ denotes the class of all sets.) For each point $q \in Q$, denote the class

$$\{q\} \times V^Q(q) = \big\{(q,x) \mid (q,x) \in V^Q\big\}$$

by V^q. Obviously, $V^p \cap V^q = \varnothing$ for $p \neq q$. The correspondence V^Q is called a *bundle* on Q and the class V^q is called the *stalk* of the bundle V^Q at a point q.

Let $D \subset Q$. A function $u\colon D \to V^Q$ is called a *section* of the bundle V^Q on D if $u(q) \in V^q$ for all $q \in D$. The class of all sections of V^Q on D is denoted by $S(D, V^Q)$. The sections defined on Q are called *global*. If X is a subset of V^Q then the symbol $S(D, X)$ stands for the set of all sections of X on D.

A point $q \in Q$ is called the *projection of an element* $x \in V^Q$ and denoted by $\mathrm{pr}(x)$ if $x \in V^q$. The *projection of a set* $X \subset V^Q$ is defined to be $\{\mathrm{pr}(x) \mid x \in X\}$ and denoted by $\mathrm{pr}(X)$.

2.2. Assume now Q to be a topological space and suppose that some topology is given on a class $V^Q \subset Q \times \mathbb{V}$. In this case, we call V^Q a *continuous bundle* on Q.

By a *continuous section* of the bundle V^Q we mean a section that is a continuous function. Given a subset $D \subset Q$, the symbol $C(D, V^Q)$ stands for the class of all continuous sections of V^Q on D. Analogously, if X is a subset of V^Q then $C(D, X)$ stands for the totality of all continuous sections of X on D. Obviously, $C(D, X) = C(D, V^Q) \cap S(D, X)$.

Henceforth we suppose that Q is an extremally disconnected Hausdorff compact space and assume satisfied the following conditions:

(1) $\forall q \in Q \quad \forall x \in V^q \quad \exists u \in C(Q, V^Q) \quad u(q) = x$;
(2) $\forall u \in C(Q, V^Q) \quad \forall A \sqsubset Q \quad u(A) \sqsubset V^Q$.

2.3. Proposition. *The continuous bundle V^Q possesses the following properties*:

(1) *the topology of V^Q is Hausdorff*;
(2) *for every $u \in C(Q, V^Q)$ and $q \in Q$, the family $\{u(A) \mid A \in \mathrm{Clop}(q)\}$ is a neighborhood base of the point $u(q)$*;
(3) *all elements of $C(Q, V^Q)$ are open and closed mappings* (see 1.1).

◁ Let x and y be different elements of V^Q. Put $p = \mathrm{pr}(x)$ and $q = \mathrm{pr}(y)$. In view of 2.2 (1), there are sections $u, v \in C(Q, V^Q)$ such that $u(p) = x$ and $v(q) = y$.

Suppose first that $p = q$. The set

$$A = \{q \in Q \mid u(q) \neq v(q)\} = Q \backslash u^{-1}\big(v(Q)\big)$$

is clopen in view of 2.2 (2). Then $u(A)$ and $v(A)$ are disjoint neighborhoods about the points x and y.

Suppose now that $p \neq q$. In this case, there exist $A, B \sqsubset Q$ such that $A \cap B = \varnothing$, $p \in A$, and $q \in B$. Then $u(A)$ and $v(B)$ are disjoint neighborhoods about the points x and y.

Assertion (2) follows readily from 2.2 (2).

Assertion (3) is equivalent to 2.2 (2) due to the fact that $\mathrm{Clop}(Q)$ is a base both for the open and close topologies of Q. ▷

2.4. Lemma. *A subset $X \subset V^Q$ is clopen if and only if $u^{-1}(X) \sqsubset Q$ for all $u \in C(Q, V^Q)$.*

◁ Only sufficiency requires some comments. Consider an arbitrary element $x \in V^Q$. Let a section $u \in C(Q, V^Q)$ and a point $q \in Q$ be such that $u(q) = x$.

Suppose first that $x \in X$. The set $A = u^{-1}(X)$ is clopen in Q and, therefore, $u(A)$ is a neighborhood about x contained in X. Since x is arbitrary, we conclude that X is open.

If $x \notin X$ then the set $A = Q \backslash u^{-1}(X)$ is clopen in Q and, hence, $u(A)$ is a neighborhood about x disjoint from X. Since x is arbitrary, we conclude that X is closed. ▷

2.5. Proposition. *The topology of V^Q is extremally disconnected.*

◁ Let X be an open subset of V^Q. Since the topology of V^Q is Hausdorff, the closure $\operatorname{cl} X$ is a set (see 1.2). Furthermore, for every section $u \in C(Q, V^Q)$, the set $u^{-1}(\operatorname{cl} X) = \operatorname{cl} u^{-1}(X)$ is clopen. In view of Lemma 2.4, the set $\operatorname{cl} X$ is open. ▷

2.6. Lemma. *For every subset $X \subset V^Q$ the following equalities hold:*

$$X = \bigcup_{u \in C(Q,V^Q)} u\big(u^{-1}(X)\big),$$

$$\operatorname{int} X = \bigcup_{u \in C(Q,V^Q)} u\big(\operatorname{int} u^{-1}(X)\big),$$

$$\operatorname{cl} X = \bigcup_{u \in C(Q,V^Q)} u\big(\operatorname{cl} u^{-1}(X)\big).$$

◁ The claim is an obvious consequence of 2.2 (1) and the fact that all continuous sections are open. ▷

2.7. Lemma. *Let X and Y be subclasses of V^Q. The equality $X = Y$ holds if and only if $u^{-1}(X) = u^{-1}(Y)$ for all $u \in C(Q, V^Q)$.*

◁ Take arbitrary $q \in Q$ and $x \in V^q$ and consider a section $u \in C(Q, V^Q)$ such that $u(q) = x$. If $x \in X$ then $q \in u^{-1}(X) = u^{-1}(Y)$ and, consequently, $x = u(q) \in Y$. The reverse inclusion can be established similarly. ▷

2.8. Proposition. *A section $u \in S(D, V^Q)$ defined on an open subset $D \subset Q$ is continuous if and only if* $\operatorname{im} u$ *is an open subset of V^Q.*

◁ Suppose that a section u is continuous. For every $q \in D$, choose a section $u_q \in C(Q, V^Q)$ such that $u_q(q) = u(q)$. The set $D_q = \{p \in D \mid u(p) = u_q(p)\} = u^{-1}(\operatorname{im} u_q)$ is open in D and, hence, it is also open in Q. Therefore, the image $u(D_q) = u_q(D_q)$ is open in view of the fact that global continuous sections are open. Obviously, $D = \bigcup_{q \in D} D_q$, since $q \in D_q$. Thus, $\operatorname{im} u = u(D) = u\big(\bigcup_{q \in D} D_q\big) = \bigcup_{q \in D} u(D_q)$ is an open set.

Suppose now that $\operatorname{im} u$ is an open set. Consider an arbitrary point $q \in D$ and choose a section $u_q \in C(Q, V^Q)$ such that $u(q) = u_q(q)$. The open set

$\{p \in D \mid u(p) = u(p)\} = u^{-1}(\operatorname{im} u)$ is a neighborhood about q, whence it follows that u is continuous at q. ▷

2.9. Lemma. *For every subset* $X \subset V^Q$, *the following relations hold*:

(1) $\operatorname{pr}(\operatorname{cl} X) \subset \operatorname{cl} \operatorname{pr}(X)$;

(2) $\operatorname{pr}(\operatorname{int} X) \subset \operatorname{int} \operatorname{pr}(X)$.

◁ Consider an arbitrary section $u \in C(Q, V^Q)$. In view of the properties of the closure, we have $u^{-1}(\operatorname{cl} X) = \operatorname{cl} u^{-1}(X) \subset \operatorname{cl} \operatorname{pr}(X)$, whence, due to the equality $\operatorname{pr}(X) = \bigcup_{u \in C(Q,V^Q)} u^{-1}(X)$, it follows that $\operatorname{pr}(\operatorname{cl} X) \subset \operatorname{cl} \operatorname{pr}(X)$.

Relation (2) can be established similarly. ▷

3. A continuous polyverse

3.1. Consider a nonempty set Q and a bundle $V^Q \subset Q \times \mathbb{V}$. Suppose that, for each point $q \in Q$, the class V^q is an algebraic system of signature $\{\in\}$.

Given an arbitrary formula $\varphi(t_1, \dots, t_n)$ and sections $u_1, \dots, u_n$ of the bundle V^Q, we denote by $\{\varphi(u_1, \dots, u_n)\}$ the set

$$\big\{q \in \operatorname{dom} u_1 \cap \dots \cap \operatorname{dom} u_n \;\big|\; V^q \models \varphi\big(u_1(q), \dots, u_n(q)\big)\big\}.$$

For every element $x \in V^q$, put $x{\downarrow} = \{y \in V^q \mid V^q \models y \in x\}$. Obviously, if the extensionality axiom is true in the system V^q, then $x{\downarrow} = y{\downarrow} \;\leftrightarrow\; x = y$ for all $x, y \in V^q$. If X is a subset of V^Q then the symbol $\sqcup X$ denotes the union $\bigcup_{x \in X} x{\downarrow}$.

Henceforth we assume that Q is an extremally disconnected Hausdorff compact space and V^Q is a continuous bundle on Q.

For an arbitrary section $u \in C(Q, V^Q)$, the class $\bigcup_{q \in Q} u(q){\downarrow}$ is called the *unpack* of the section u and denoted by $\llcorner u \lrcorner$.

3.2. A continuous bundle V^Q is called a *continuous polyverse* on Q, if the extensionality and regularity axioms are true in each stalk V^q $(q \in Q)$ and, in addition, the following conditions hold:

(1) $\forall\, q \in Q \quad \forall\, x \in V^q \quad \exists\, u \in C(Q, V^Q) \quad u(q) = x$;

(2) $\forall\, u \in C(Q, V^Q) \quad \forall\, A \in \operatorname{Clop}(Q) \quad u(A) \in \operatorname{Clop}(V^Q)$;

(3) $\forall\, u \in C(Q, V^Q) \quad \llcorner u \lrcorner \in \operatorname{Clop}(V^Q)$;

(4) $\forall\, X \in \operatorname{Clop}(V^Q) \quad \exists\, u \in C(Q, V^Q) \quad \llcorner u \lrcorner = X$.

3.3. For arbitrary sections $u, v \in C(Q, V^Q)$, the equalities $\{u = v\} = u^{-1}(\operatorname{im} v)$ and $\{u \in v\} = u^{-1}(\llcorner v \lrcorner)$ imply that the sets $\{u = v\}$ and $\{u \in v\}$ are clopen, which allows us to introduce two class-functions

$$\| \cdot = \cdot \|, \| \cdot \in \cdot \| \colon C(Q, V^Q) \times C(Q, V^Q) \to \operatorname{Clop}(Q)$$

by letting $\|u = v\| = \{u = v\}$ and $\|u \in v\| = \{u \in v\}$.

It is easy to verify that the triple $\big(C(Q, V^Q),\ \|\cdot = \cdot\|,\ \|\cdot \in \cdot\|\big)$ is a separated $\mathrm{Clop}(Q)$-valued algebraic system (see 1.3).

The definition 3.2 (4) of continuous polyverse implies that there exists a continuous section $\varnothing^\wedge$ satisfying the condition $\llcorner\varnothing^\wedge\lrcorner = \varnothing$. Obviously, this section is unique. It is easy that $V^q \models \varnothing^\wedge(q) = \varnothing$, $\|\varnothing^\wedge = \varnothing\| = Q$, and, in addition, $\|u = \varnothing^\wedge\| = \|u = \varnothing\|$ for all $u \in C(Q, V^Q)$.

3.4. Lemma. *For every subset $X \subset V^Q$, the following relations hold*:

(1) *if $X \sqsubset V^Q$ then* $\mathrm{pr}(X) \sqsubset Q$;
(2) *if X is open then* $\mathrm{pr}(\mathrm{cl}\, X) = \mathrm{cl}\,\mathrm{pr}(X)$.

◁ (1) If $X \sqsubset V^Q$ then there is a section $u \in C(Q, V^Q)$ such that $\sqcup \mathrm{im}\, u = \llcorner u \lrcorner = X$. Obviously, $\mathrm{pr}\big(\sqcup \mathrm{im}\, u\big) = \|u \neq \varnothing\|$, whence $\mathrm{pr}(X)$ is clopen.

(2) Let X be an open subset of V^Q. Then the closure $\mathrm{cl}\, X$ is clopen, the same is true of its projection $\mathrm{pr}(\mathrm{cl}\, X)$. The obvious inclusion $\mathrm{pr}(X) \subset \mathrm{pr}(\mathrm{cl}\, X)$ implies $\mathrm{cl}\,\mathrm{pr}(X) \subset \mathrm{pr}(\mathrm{cl}\, X)$. The reverse inclusion is established in 2.9. ▷

3.5. The *support* $\mathrm{supp}\, u$ of a section $u \in S(D, V^Q)$ on $D \subset Q$ is defined to be the set $\{q \in D \mid V^q \models u(q) \neq \varnothing\}$. Obviously, $\mathrm{supp}\, u = \{u \neq \varnothing\} = \{u \neq \varnothing^\wedge\}$. So, if $u \in C(Q, V^Q)$ then $\mathrm{supp}\, u$ is a clopen set.

Let u be a continuous section of V^Q and let D be a subset of $\mathrm{supp}\, u$. The symbol $C(D, u)$ denotes the class

$$\big\{v \in C(D, V^Q) \;\big|\; (\forall q \in D)\ V^q \models v(q) \in u(q)\big\}.$$

Obviously, $C(D, u) = C(D, \llcorner u \lrcorner)$.

By the *descent* of a section u we mean the class $C(\mathrm{supp}\, u, u)$ and denote it by $u{\downarrow}$. It is easily seen that $u{\downarrow} = C(\mathrm{supp}\, u, \llcorner u \lrcorner)$. Obviously, in case $\|u \neq \varnothing\| = Q$, the descent of u is the descent of the section u regarded as an element of a Boolean-valued algebraic system (see 1.7).

3.6. Proposition. *For arbitrary $X \sqsubset V^Q$ and $u \in C(Q, V^Q)$, the following assertions are equivalent*:

(1) $\llcorner u \lrcorner = X$;
(2) $u(q){\downarrow} = X \cap V^q$ *for all* $q \in Q$;
(3) $\mathrm{supp}\, u = \mathrm{pr}(X)$ *and* $u{\downarrow} = C\big(\mathrm{pr}(X), X\big)$;
(4) $\|v \in u\| = v^{-1}(X)$ *for all* $v \in C(Q, V^Q)$.

◁ (1)→(3): It suffices to observe that $\operatorname{supp} u = \|u \neq \varnothing\| = \operatorname{pr}(\llcorner u \lrcorner)$ and employ the equality $u{\downarrow} = C(\operatorname{supp} u, \llcorner u \lrcorner)$.

(3)→(2): Put $A = \operatorname{supp} u$. It is clear that $X \cap V^q = \varnothing = u(q){\downarrow}$ for all $q \in Q \backslash A$.

Given an arbitrary point $q \in A$, there are $x \in u(q){\downarrow}$ and $v_q \in C(Q, V^Q)$ such that $v_q(q) = x$. Put $B_q = \|v_q \in u\|$. The family $(B_q)_{q \in A}$ is an open covering of the compact set A; therefore, we can refine a subcovering $(B_q)_{q \in F}$, where $F \subset A$ is finite. By the exhaustion principle, there is an antichain $(C_q)_{q \in F}$ such that $C_q \subset B_q$ for $q \in F$ and $\bigcup_{q \in F} C_q = \bigvee_{q \in F} C_q = \bigvee_{q \in F} B_q = A$. Construct a section $v \in S(A, V^Q)$ by putting $v(p) = v_q(p)$ for each point $p \in A$, where q is a (unique) element of F such that $p \in C_q$. The section v is continuous, since $v = v_q$ on C_q $(q \in F)$. It is easily seen that $v \in u{\downarrow} = C(A, X)$.

Let q be an arbitrary element of A.

Consider an $x \in u(q){\downarrow}$, choose a section $w \in C(Q, V^Q)$ such that $w(q) = x$, and construct a section $\overline{w} \in S(A, V^Q)$ as follows:

$$\overline{w}(p) = \begin{cases} w(p) & \text{if } p \in \|w \in u\|, \\ v(p) & \text{if } p \in A \backslash \|w \in u\|. \end{cases}$$

Obviously, the section $\overline{w}$ is continuous and $\overline{w} \in u{\downarrow} = C(A, X)$, whence $x = \overline{w}(q) \in X$ in view of the containment $q \in \|w \in u\|$.

Now let $x \in X \cap V^q$. As before, choose a section $w \in C(Q, V^Q)$ such that $w(q) = x$. Consider the section $\overline{w} \in S(A, V^Q)$ defined as follows:

$$\overline{w}(p) = \begin{cases} w(p) & \text{if } p \in w^{-1}(X), \\ v(p) & \text{if } p \in A \backslash w^{-1}(X). \end{cases}$$

The obvious relations $\overline{w} \in C(A, X) = u{\downarrow}$ and $q \in w^{-1}(X)$ imply that $x = w(q) = \overline{w}(q) \in u(q){\downarrow}$.

(2)→(4): Consider an arbitrary section $v \in C(Q, V^Q)$. If $q \in \|v \in u\| = v^{-1}(\llcorner u \lrcorner)$ then $v(q) \in \llcorner u \lrcorner$; consequently, $v(q) \in u(q){\downarrow} = X \cap V^q$, i.e., $q \in v^{-1}(X)$.

If $q \in v^{-1}(X)$ then $v(q) \in X \cap V^q = u(q){\downarrow}$ and, hence, $V^q \models v(q) \in u(q)$ and $q \in \|v \in u\|$.

(4)→(1): Observe that $v^{-1}(\llcorner u \lrcorner) = \|v \in u\| = v^{-1}(X)$ for all $v \in C(Q, V^Q)$. Therefore, in view of Lemma 2.7, the equality $X = \llcorner u \lrcorner$ holds. ▷

Obviously, for every $X \sqsubset V^Q$, a section u satisfying conditions (1)–(4) is unique. We call this section the *pack* of the set X and denote it by $\ulcorner X \urcorner$.

It is easy to verify validity of the following assertion:

Proposition. *Let X be an open subset of V^Q. A section $\overline{u} \in C(Q, V^Q)$ coincides with $\ulcorner \operatorname{cl} X \urcorner$ if and only if $\overline{u}$ is pointwise the least section among $u \in C(Q, V^Q)$ satisfying the inclusion $X \cap V^q \subset u(q){\downarrow}$ for all $q \in Q$.*

3.7. Lemma. *If* $u \in C(Q, V^Q)$ *and* $A \in \mathrm{Clop}(Q)$ *then* $\sqcup u(A) \in \mathrm{Clop}(V^Q)$.

◁ For every section $v \in C(Q, V^Q)$, the set $v^{-1}\big(\sqcup u(A)\big) = A \cap \|v \in u\|$ is clopen; whence, in view of 2.4, the set $\sqcup u(A)$ is clopen too. ▷

3.8. Proposition. *Every continuous section of* V^Q *defined on an open or closed subset of* Q *can be extended to a global continuous section.*

◁ Consider $A \subset Q$ and $u \in C(A, V^Q)$. For every point $q \in A$, there exist a section $u_q \in C(Q, V^Q)$ and a set $B_q \sqsubset Q$ such that $q \in B_q$ and $u_q = u$ on $B_q \cap A$.

Suppose that the set A is open. Without loss of generality, we may assume that $B_q \subset A$. Consider the open set $X = \bigcup_{q \in Q} u(q){\downarrow} = \bigcup_{q \in A} \sqcup u_q(B_q)$ and show that $(\mathrm{cl}\, X) \cap V^q = u(q){\downarrow}$ for all $q \in A$. We only establish the inclusion $(\mathrm{cl}\, X) \cap V^q \subset u(q){\downarrow}$ (the reverse inclusion follows from the obvious properties of closure). Take an $x \in \mathrm{cl}\, X \cap V^q$. There is a section $v \in C(Q, V^Q)$ such that $v(q) = x$. Evidently, for each neighborhood $B \sqsubset Q$ about q, the intersection $v(B) \cap X$ is nonempty and, thus, there exists a point $p \in B \cap B_q$ such that $v(p) \in u(p){\downarrow}$. On the other hand, $u(p) = u_q(p)$; consequently, $v(B) \cap \sqcup u_q(B_q) \neq \varnothing$. The set $\sqcup u_q(B_q)$ is closed and, therefore, $x \in \sqcup u_q(B_q)$, whence $x \in u_q(q){\downarrow} = u(q){\downarrow}$. Put $\overline{u} = \ulcorner \mathrm{cl}\, X \urcorner$. From what was established above it follows that $\overline{u}(q){\downarrow} = u(q){\downarrow}$ for all $q \in A$. Thus, $\overline{u}$ is a sought global extension of the section u.

Suppose now that the set A is closed. The family $(B_q)_{q \in A}$ forms an open covering of the compact set A and, therefore, we can refine a subcovering $(B_q)_{q \in F}$, where F is a finite subset of A. Without loss of generality, we may assume that $\bigcup_{q \in F} B_q = Q$. By the exhaustion principle, there is an antichain $(C_q)_{q \in F}$ such that $C_q \subset B_q$ for all $q \in F$ and $\bigcup_{q \in F} C_q = Q$. Construct a section $\overline{u} \in S(Q, V^Q)$ by putting $\overline{u}(p) = u_q(p)$ for each point $p \in Q$, where q is a (unique) element of F such that $p \in C_q$. The section $\overline{u}$ is continuous, since $\overline{u} = u_q$ on C_q $(q \in F)$. Obviously, $\overline{u} = u$ on A. ▷

Corollary. *If* A *is an open or closed subset of* Q *then* $C(A, V^Q) = \{u|_A : u \in C(Q, V^Q)\}$.

The extension principle. *For every section* $u \in C(A, V^Q)$ *defined on an open subset* $A \subset Q$, *there exists a unique section* $\overline{u} \in C(\mathrm{cl}\, A, V^Q)$ *that extends* u.

$\triangleleft$ According to Proposition 3.8, there exists a section $u_1 \in C(Q, V^Q)$ such that $u_1 = u$ on A. Put $\overline{u} = u_1|_{\mathrm{cl}\,A}$.

Uniqueness of this extension is obvious. $\triangleright$

The section $\overline{u}$ of the statement of the extension principle is called the *closure* of u and denoted by $\mathrm{ext}(u)$.

3.9. It is easy to verify validity of the following assertion:

Theorem. *Consider a family $(u_\xi)_{\xi\in\Xi}$ of global continuous sections of V^Q and an antichain $(B_\xi)_{\xi\in\Xi}$ in the algebra* $\mathrm{Clop}(Q)$ *and put $B = (\bigvee_{\xi\in\Xi} B_\xi)^\perp$. The continuous section*

$$u = \mathrm{ext}\left(\bigcup_{\xi\in\Xi} u_\xi|_{B_\xi} \cup \varnothing^\wedge|_B\right)$$

is the mixing of the family $(u_\xi)_{\xi\in\Xi}$ with weights $(B_\xi)_{\xi\in\Xi}$. In particular, the mixing principle is valid for the Boolean-valued algebraic system $C(Q, V^Q)$.

Corollary. *The Boolean-valued algebraic system $C(Q, V^Q)$ satisfies the maximum principle.*

3.10. The pointwise truth-value theorem. *For an arbitrary formula $\varphi(t_1, \dots, t_n)$ and sections $u_1, \dots, u_n \in C(Q, V^Q)$, the following equality holds*:

$$\|\varphi(u_1, \dots, u_n)\| = \big\{q \in Q \;\big|\; V^q \models \varphi\big(u_1(q), \dots, u_n(q)\big)\big\}. \qquad (*)$$

$\triangleleft$ The proof is carried out by induction on the complexity of the formula φ.

If φ is atomic, i.e., has the form $t_1 \in t_2$ or $t_1 = t_2$; then $(*)$ follows from the definitions of $\|\cdot = \cdot\|$ and $\|\cdot \in \cdot\|$.

Assume that the claim is proven for formulas of smaller complexity. We restrict ourselves to the case in which the formula φ has the form $(\exists t_0)\ \varphi(t_0, \vec{t})$.

If $V^q \models (\exists t_0)\ \varphi\big(t_0, \vec{u}(q)\big)$ then there exists an element $x \in V^q$ such that $V^q \models \varphi\big(x, \vec{u}(q)\big)$. Choose a section $u_0 \in C(Q, V^Q)$ satisfying the equality $u_0(q) = x$. By the induction hypothesis, $q \in \|\varphi(u_0, \vec{u})\| \subset \|(\exists t_0)\ \varphi(t_0, \vec{u})\|$, which proves the inclusion "$\supset$" in $(*)$.

Show the reverse inclusion. Suppose that $q \in \|(\exists t_0)\ \varphi(t_0, \vec{u})\|$. By the maximum principle, there is a continuous section u_0 such that $\|\varphi(u_0, \vec{u})\| = \|(\exists t_0)\ \varphi(t_0, \vec{u})\|$. Therefore, by the induction hypothesis, $V^q \models \varphi\big(u_0(q), \vec{u}(q)\big)$ and, hence, $V^q \models (\exists t_0)\ \varphi\big(t_0, \vec{u}(q)\big)$. $\triangleright$

3.11. Lemma. *For every subset $X \subset V^Q$, the following relations hold*:

(1) $\sqcup \operatorname{cl} X \subset \operatorname{cl} \sqcup X$;
(2) $\sqcup \operatorname{int} X \subset \operatorname{int} \sqcup X$;
(3) *if* $X \in \operatorname{Clop}(V^Q)$ *then* $\sqcup X \in \operatorname{Clop}(V^Q)$;
(4) *if X is open then $\sqcup X$ is an open subset of V^Q*;
(5) *if X is open then $\sqcup \operatorname{cl} X = \operatorname{cl} \sqcup X$.*

$\lhd$ (1): Suppose that $x \in \sqcup \operatorname{cl} X$. Then $x \in y\!\downarrow$ for some $y \in \operatorname{cl} X$. Consider sections $u, v \in C(Q, V^Q)$ such that $u(q) = x$ and $v(q) = y$, where $q = \operatorname{pr}(x)$. For every $A \in \operatorname{Clop}(q)$, we have $v(A) \cap X \neq \varnothing$. Put $B = A \cap \|u \in v\| \sqsubset Q$. Since $q \in B$, there is a point $p \in B$ such that $v(p) \in X$. Obviously, $u(p) \in v(p)\!\downarrow \subset \sqcup X$ and, hence, $u(A) \cap (\sqcup X) \neq \varnothing$. Consequently, $x \in \operatorname{cl} \sqcup X$.

(2): Suppose that $x \in \sqcup \operatorname{int} X$ and consider $y \in \operatorname{int} X$ and $u, v \in C(Q, V^Q)$ such that $x \in y\!\downarrow$, $u(q) = x$, and $v(q) = y$, where $q = \operatorname{pr}(x)$. It is clear that the set $B = v^{-1}(X) \cap \|u \in v\|$ is a neighborhood about q and, hence, $u(B)$ is a neighborhood about x. Furthermore, $u(p) \in v(p)\!\downarrow \subset \sqcup X$ for all $p \in B$, i.e., $u(B) \subset \sqcup X$. Thus, $x \in \operatorname{int} \sqcup X$.

(3): According to Lemma 2.4, it suffices to consider an arbitrary section $v \in C(Q, V^Q)$ and show that the set $v^{-1}(\sqcup X)$ is clopen. Put $u = \ulcorner X \urcorner$. Obviously, $v(q) \in \sqcup X$ if and only if

$$V^q \models \big(\exists\, t \in u(q)\big)\, v(q) \in t.$$

By the pointwise truth-value theorem,

$$v^{-1}(X) = \big\{ q \in Q \,\big|\, V^q \models \big(\exists\, t \in u(q)\big)\ v(q) \in t \big\} = \|(\exists\, t \in u)\ v \in t\|$$

and, consequently, $v^{-1}(X) \sqsubset Q$.

(4): The claim follows readily from (2).

(5): Let the set X be open. Then its closure $\operatorname{cl} X$ is clopen and, according to (3), the set $\sqcup \operatorname{cl} X$ is clopen too. The obvious relation $\sqcup X \subset \sqcup \operatorname{cl} X$ implies $\operatorname{cl} \sqcup X \subset \sqcup \operatorname{cl} X$. The reverse inclusion holds by virtue of (1). $\rhd$

3.12. Theorem. *The Boolean-valued algebraic system $C(Q, V^Q)$ satisfies the ascent principle.*

$\lhd$ Let $(u_\xi)_{\xi \in \Xi}$ be a family of global continuous sections of V^Q and let $(B_\xi)_{\xi \in \Xi}$ be a family of clopen subsets of Q. Consider the clopen set $X = \operatorname{cl} \bigcup_{\xi \in \Xi} u_\xi(B_\xi)$ and put $u = \ulcorner X \urcorner$. Show that the section $u \in C(Q, V^Q)$ thus constructed is an ascent of $(u_\xi)_{\xi \in \Xi}$ with weights $(B_\xi)_{\xi \in \Xi}$. Indeed, for every section $v \in C(Q, V^Q)$, the following relations hold:

$$\|v \in u\| = v^{-1}(\llcorner u \lrcorner) = v^{-1}\Big(\mathrm{cl} \bigcup_{\xi\in\Xi} u_\xi(B_\xi)\Big) = \mathrm{cl}\, v^{-1}\Big(\bigcup_{\xi\in\Xi} u_\xi(B_\xi)\Big)$$

$$= \mathrm{cl} \bigcup_{\xi\in\Xi} v^{-1}\big(u_\xi(B_\xi)\big) = \mathrm{cl} \bigcup_{\xi\in\Xi} B_\xi \cap \|v = u_\xi\| = \bigvee_{\xi\in\Xi} B_\xi \wedge \|v = u_\xi\|.$$

Consider now an arbitrary section $u \in C(Q, V^Q)$ and show that it is an ascent of some family of elements in $C(Q, V^Q)$ with suitable weights. Put $X = \llcorner u \lrcorner$. For each $x \in X$, choose a section $u_x \in C(Q, V^Q)$ such that $x \in \mathrm{im}\, u_x$. Assign $B_x = \|u_x \in u\| = u_x^{-1}(X)$. Obviously, $x \in u_x(B_x) \subset X$ for all $x \in X$, whence $X = \bigcup_{x\in X} u_x(B_x) = \mathrm{cl} \bigcup_{x\in X} u_x(B_x)$. As in the first part of the proof, we can establish the equality $\|v \in u\| = \bigvee_{x\in X} B_x \wedge \|v = u_x\|$ for all $v \in C(Q, V^Q)$. Thus, u is an ascent of $(u_x)_{x\in X}$ with weights $(B_x)_{x\in X}$. ▷

3.13. Consider a $D \sqsubset Q$ and suppose that $\mathcal{U}$ is a subset of $C(D, V^Q)$. Given a point $q \in D$, denote by $\mathcal{U}(q)$ the totality $\{u(q) \mid u \in \mathcal{U}\}$.

Proposition. *Consider a $D \sqsubset Q$ and suppose that $\mathcal{U}$ is a nonempty subset of $C(D, V^Q)$. The following properties of a section $\overline{u} \in C(Q, V^Q)$ are equivalent:*

(1) $\overline{u} = \ulcorner \mathrm{cl} \bigcup_{u\in\mathcal{U}} \mathrm{im}\, u \urcorner$;

(2) $\|v \in \overline{u}\| = \mathrm{cl}\{q \in D \mid v(q) \in \mathcal{U}(q)\}$ *for all* $v \in C(Q, V^Q)$;

(3) $\|v \in \overline{u}\| = \mathrm{cl} \bigcup_{u\in\mathcal{U}} \{v = u\}$ *for all* $v \in C(Q, V^Q)$;

(4) $\overline{u}{\downarrow} = \Big\{\mathrm{ext}\Big(\bigcup_{u\in\mathcal{U}} u|_{D_u}\Big) \;\Big|\; (D_u)_{u\in\mathcal{U}}$ *is a partition of unity in the algebra* $\mathrm{Clop}(D)\Big\}$;

(5) $\overline{u}{\downarrow} = C(D, \mathrm{cl} \bigcup_{u\in\mathcal{U}} \mathrm{im}\, u)$.

(6) *$\overline{u}$ is pointwise the least section among $\tilde{u} \in C(Q, V^Q)$ satisfying the inclusion $\mathcal{U}(q) \subset \tilde{u}(q){\downarrow}$ for all $q \in D$.*

If $\mathcal{U} \subset C(Q, V^Q)$ then $\|v \in \overline{u}\| = \bigvee_{u\in\mathcal{U}} \|v = u\|$ for all $v \in C(Q, V^Q)$.

◁ (1)→(2): Put $X = \bigcup_{u\in\mathcal{U}} \mathrm{im}\, u$. Then $\llcorner \overline{u} \lrcorner = \mathrm{cl}\, X$ and, therefore, $\|v \in \overline{u}\| = v^{-1}(\llcorner u \lrcorner) = v^{-1}(\mathrm{cl}\, X) = \mathrm{cl}\, v^{-1}(X)$ for all $v \in C(Q, V^Q)$. It is easy to verify the relation $X = \bigcup_{q\in D} \mathcal{U}(q)$ and establish equivalence of the containments $v(q) \in \mathcal{U}(q)$ and $q \in v^{-1}\big(\bigcup_{q\in D} \mathcal{U}(q)\big)$.

(2)→(3): It suffices to show that $\{q \in D \mid v(q) \in \mathcal{U}(q)\} = \bigcup_{u\in\mathcal{U}} \{v = u\}$ for all $v \in C(Q, V^Q)$. Take an arbitrary point $q \in D$.

If $v(q) \in \mathcal{U}(q)$ then, for some element $u \in \mathcal{U}$, we have $v(q) = u(q)$ and, consequently, $q \in \{v = u\}$.

If $q \in \bigcup_{u\in\mathcal{U}}\{v = u\}$ then, for a suitable $u \in \mathcal{U}$, we have $q \in \{v = u\}$ and, hence, $v(q) = u(q) \in \mathcal{U}(q)$.

(3)→(4): Consider an arbitrary element $v \in C(D, V^Q)$ and define a section $\overline{v} \in C(Q, V^Q)$ as follows:

$$\overline{v}(q) = \begin{cases} v(q) & \text{if } q \in D, \\ \varnothing^\wedge(q) & \text{if } q \notin D. \end{cases}$$

Suppose that $v \in \overline{u}{\downarrow}$. Then $D = \{v \in \overline{u}\} \subset \|\overline{v} \in \overline{u}\| = \mathrm{cl}\bigcup_{u\in\mathcal{U}}\{\overline{v} = u\} \subset D$. For all $u \in \mathcal{U}$, the set $\{\overline{v} = u\} = u^{-1}(\mathrm{im}\,\overline{v})$ is clopen. According to the exhaustion principle, there is an antichain $(D_u)_{u\in\mathcal{U}}$ in the algebra $\mathrm{Clop}(Q)$ such that $D_u \subset \{\overline{v} = u\}$ and $\bigvee_{u\in\mathcal{U}} D_u = \mathrm{cl}\bigcup_{u\in\mathcal{U}}\{\overline{v} = u\} = D$. Obviously, the section $w = \bigcup_{u\in\mathcal{U}} u|_{D_u}$ is continuous, the set $\mathrm{dom}\, w$ is open, $D = \mathrm{cl}\,\mathrm{dom}\, w$, and $\{w = v\} = \{w = \overline{v}\} = \mathrm{dom}\, w$. It is clear that $\mathrm{ext}(w) \in C(D, V^Q)$ and $\{\mathrm{ext}(w) = v\} = D$. Therefore, $\mathrm{ext}(w) = v$ and, thus, the inclusion "$\subset$" holds.

We now establish the reverse inclusion. Let $(D_u)_{u\in\mathcal{U}}$ be a partition of unity in the algebra $\mathrm{Clop}(D)$ and let $v = \mathrm{ext}\big(\bigcup_{u\in\mathcal{U}} u|_{D_u}\big)$. Show that $v \in \overline{u}{\downarrow}$. Since $\mathrm{dom}\, v = D$, it suffices to establish the inclusion $\mathrm{im}\, v \subset \llcorner\overline{u}\lrcorner$. Obviously, $u(D_u) \subset \llcorner\overline{u}\lrcorner$ for all $u \in \mathcal{U}$ and, consequently, $\bigcup_{u\in\mathcal{U}} u(D_u) \subset \llcorner\overline{u}\lrcorner$. Observe that $\mathrm{im}\, v = \mathrm{cl}\bigcup_{u\in\mathcal{U}} u(D_u)$ and, hence, $\mathrm{im}\, v \subset \llcorner\overline{u}\lrcorner$.

(4)→(5): Put $X = \mathrm{cl}\bigcup_{u\in\mathcal{U}}\mathrm{im}\, u$. Let $(D_u)_{u\in\mathcal{U}}$ be a partition of unity in the algebra $\mathrm{Clop}(D)$ and let $v = \mathrm{ext}(\bigcup_{u\in\mathcal{U}} u|_{D_u})$. Obviously, $\mathrm{dom}\, v = D$. Show that $\mathrm{im}\, v \subset X$. The inclusion $u(D_u) \subset X$ implies $\bigcup_{u\in\mathcal{U}} u(D_u) \subset X$; whence, in view of the equality $\mathrm{im}\, v = \mathrm{cl}\bigcup_{u\in\mathcal{U}} u(D_u)$, the desired relation $\mathrm{im}\, v \subset X$ follows. Thus, $\overline{u}{\downarrow} \subset C(D, X)$.

For proving the reverse inclusion, consider an arbitrary section $v \in C(D, X)$ and establish the equality $v = \mathrm{ext}\big(\bigcup_{u\in\mathcal{U}} u|_{D_u}\big)$ for some partition of unity $(D_u)_{u\in\mathcal{U}}$ in the algebra $\mathrm{Clop}(D)$. Obviously, $v^{-1}(X) = D$. Since the section v is open, we have $D = \mathrm{cl}\, v^{-1}\big(\bigcup_{u\in\mathcal{U}}\mathrm{im}\, u\big)$. In addition, the set $A = v^{-1}\big(\bigcup_{u\in\mathcal{U}}\mathrm{im}\, u\big)$ is open and dense in D.

With each element $u \in \mathcal{U}$ we associate a clopen set $C_u = \{v = u\} = v^{-1}(\mathrm{im}\, u)$. The obvious equality $A = \bigcup_{u\in\mathcal{U}} C_u$ implies that $\bigvee_{u\in\mathcal{U}} C_u = D$. In view of the exhaustion principle, there is a partition of unity $(D_u)_{u\in\mathcal{U}}$ in the algebra $\mathrm{Clop}(D)$ such that $D_u \subset C_u$ for all $u \in \mathcal{U}$. Put $w = \bigcup_{u\in\mathcal{U}} u|_{D_u}$. It is clear that, for each $u \in \mathcal{U}$, the equalities $w|_{D_u} = u|_{D_u} = v|_{D_u}$ hold, since $D_u \subset \{v = u\}$. Consequently, by the extension principle, $\mathrm{ext}(w) = v$, which proves the desired inclusion.

(5)→(1): It is sufficient to observe that $D = \mathrm{pr}\big(\mathrm{cl}\bigcup_{u\in\mathcal{U}}\mathrm{im}\, u\big)$ and use Proposition 3.6 (3).

Equivalence of (1) and (6) is evident. ▷

Obviously, the section $\overline{u}$ of the statement of the proposition is unique. We call that section the *ascent* of the set $\mathcal{U}$ and denote it by $\mathcal{U}{\uparrow}$. In case $\mathcal{U}$ is a nonempty subset of $C(Q, V^Q)$, the notion of the ascent of $\mathcal{U}$ coincides with the eponymized notion of 1.8.

4. Function representation of a Boolean-valued universe

Throughout the section, we assume that Q is an extremally disconnected Hausdorff compact space and $\mathfrak{U}$ is a Boolean-valued universe over $\operatorname{Clop}(Q)$.

4.1. For the further considerations we need the notion of the quotient class $X/{\sim}$ where X is a class (that need not be a set) and $\sim$ is an equivalence relation on X. The traditional definition of quotient class, for the case in which X is a set, cannot be always applied to the case of a class, since the elements of X equivalent to a given $x \in X$ form a class that need not be a set. We can overcome this difficulty with the help of the following fact:

Theorem (Frege–Russell–Scott). *For every equivalence relation $\sim$ on a class X, there exists a function $F\colon X \to \mathbb{V}$ such that*

$$F(x) = F(y) \leftrightarrow x \sim y \quad \text{for all } x, y \in X. \qquad (**)$$

As F we can take the function defined as follows:

$$F(x) = \big\{ y \in X \mid y \sim x \ \& \ (\forall z \in X)\big(z \sim x \to \operatorname{rank}(y) \leqslant \operatorname{rank}(z)\big)\big\}.$$

This function F is conventionally called the *canonical projection* of the equivalence relation $\sim$. The relation $(**)$ allows us to regard $F(x)$ as an analog of the coset containing an element $x \in X$. In this connection, we denote $F(x)$ by ${\sim}(x)$.

4.2. For each point $q \in Q$, introduce the equivalence relation $\sim_q$ on the class $\mathfrak{U}$ as follows:

$$u \sim_q v \ \leftrightarrow \ q \in \|u = v\|.$$

Consider the bundle $V^Q = \big\{ \big(q, {\sim_q}(u)\big) \bigm| q \in Q,\ u \in \mathfrak{U} \big\}$ and make the convention to denote a pair $\big(q, {\sim_q}(u)\big)$ by $\widehat{u}(q)$. Obviously, for every element $u \in \mathfrak{U}$, the mapping $\widehat{u}\colon q \mapsto \widehat{u}(q)$ is a section of the bundle V^Q. Note that, for each $x \in V^Q$, there exist $u \in \mathfrak{U}$ and $q \in Q$ such that $\widehat{u}(q) = x$. In addition, the equality $\widehat{u}(q) = \widehat{v}(q)$ holds if and only if $q \in \|u = v\|$.

Make each stalk V^q of the bundle V^Q into an algebraic system of signature $\{\in\}$ by letting

$$V^q \models x \in y \ \leftrightarrow \ q \in \|u \in v\|,$$

where the elements $u, v \in \mathfrak{U}$ and such that $\widehat{u}(q) = x$ and $\widehat{v}(q) = y$. It is easy to verify that the above definition is sound. Indeed, if $\widehat{u}_1(q) = x$ and $\widehat{v}_1(q) = y$ for another pair u_1, v_1, then the containments $q \in \|u \in v\|$ and $q \in \|u_1 \in v_1\|$ are equivalent.

It is easily seen that the class $\{\widehat{u}(A) \mid u \in \mathfrak{U},\ A \sqsubset Q\}$ is a base of some open topology on V^Q, which allows us to regard V^Q as a continuous bundle.

4.3. Theorem. (1) *The bundle V^Q is a continuous polyverse.*

(2) *The mapping $u \mapsto \widehat{u}$ is an isomorphism between the Boolean-valued universes $\mathfrak{U}$ and $C(Q, V^Q)$.*

We divide the proof of the last theorem into several steps.

4.4. Lemma. *If $u \in \mathfrak{U}$ and $A \sqsubset Q$ then $\widehat{u}(A) \sqsubset V^Q$.*

$\lhd$ For every element $x \in V^Q \backslash \widehat{u}(A)$, there exist $v \in \mathfrak{U}$ and $q \in Q$ such that $x = \widehat{v}(q)$.

If $q \in A$ then $\widehat{u}(q) \neq x = \widehat{v}(q)$, $q \in \|u \neq v\|$, and, thus, the set $\widehat{v}(\|u \neq v\|)$ is a neighborhood about x disjoint from $\widehat{u}(A)$. If, otherwise, $q \notin A$, then the neighborhood $\widehat{v}(Q \backslash A)$ about x is disjoint from $\widehat{u}(A)$. $\rhd$

4.5. Lemma. *The classes $\{\widehat{u} \mid u \in \mathfrak{U}\}$ and $C(Q, V^Q)$ coincide.*

$\lhd$ Consider an arbitrary element $u \in \mathfrak{U}$ and show that the section $\widehat{u}$ is continuous. If $v \in \mathfrak{U}$ and $A \sqsubset Q$ then the set $\widehat{u}^{-1}\big(\widehat{v}(A)\big) = A \cap \|u = v\|$ is open. Arbitrariness of v and A allows us to conclude that $\widehat{u} \in C(Q, V^Q)$.

We now establish the reverse inclusion. Take an $f \in C(Q, V^Q)$. For each point $q \in Q$, choose an element $u_q \in \mathfrak{U}$ such that $\widehat{u}_q(q) = f(q)$ and assign $A_q := \{p \in Q \mid \widehat{u}_q(p) = f(p)\} = f^{-1}\big(\widehat{u}(Q)\big) \sqsubset Q$. Thus, $(A_q)_{q \in Q}$ is an open covering of the compact space Q from which we can refine a subcovering $(A_q)_{q \in F}$, where F is a finite subset of Q. By the exhaustion principle, there is an antichain $(B_q)_{q \in F}$ such that $B_q \subset A_q$ for all $q \in B$ and $\bigcup_{q \in F} B_q = Q$. Since the Boolean-valued algebraic system $\mathfrak{U}$ satisfies the mixing principle, we may consider $u = \operatorname{mix}_{q \in F} B_q u_q \in \mathfrak{U}$. It is easy to become convinced that $\widehat{u} = f$. $\rhd$

4.6. Lemma. *The topology of V^Q is extremally disconnected.*

$\lhd$ The claim follows from Lemmas 4.4 and 4.5 and Proposition 2.5. $\rhd$

4.7. Lemma. *The mapping $(u \mapsto \widehat{u})\colon \mathfrak{U} \to C(Q, V^Q)$ is bijective and, for all $u, v \in \mathfrak{U}$, the following equalities hold*:

$$\|u = v\|_{\mathfrak{U}} = \|\widehat{u} = \widehat{v}\|_{C(Q,V^Q)},$$

$$\|u \in v\|_{\mathfrak{U}} = \|\widehat{u} \in \widehat{v}\|_{C(Q,V^Q)}.$$

◁ It is easily seen that, for all $u, v \in \mathfrak{U}$ and $q \in Q$, we have:

$$V^q \models \widehat{u}(q) \in \widehat{v}(q) \ \leftrightarrow\ q \in \|u \in v\|,$$
$$V^q \models \widehat{u}(q) = \widehat{v}(q) \ \leftrightarrow\ q \in \|u = v\|.$$

The desired equalities are thus established. In Lemma 4.6, it is shown that the mapping $u \mapsto \widehat{u}$ is surjective. We are left with proving its injectivity. Let elements $u, v \in \mathfrak{U}$ be such that $\widehat{u} = \widehat{v}$. Then $\|u = v\| = \|\widehat{u} = \widehat{v}\| = Q$, which implies the equality $u = v$ due to the fact that the system $\mathfrak{U}$ is separated. ▷

Thus, the triple $\big(C(Q, V^Q),\ \|\cdot = \cdot\|,\ \|\cdot \in \cdot\|\big)$ is a Boolean-valued algebraic system over $\mathrm{Clop}(Q)$ isomorphic to $\mathfrak{U}$ and, hence, $C(Q, V^Q)$ is a Boolean-valued universe over $\mathrm{Clop}(Q)$.

4.8. Lemma. *If $u \in C(Q, V^Q)$ then $\llcorner u \lrcorner$ is a clopen subset of V^Q.*

◁ Take a $u \in C(Q, V^Q)$. Since $C(Q, V^Q)$ satisfies the ascent principle, $u = \mathrm{asc}_{\xi\in\Xi} B_\xi u_\xi$ for some family $(u_\xi)_{\xi\in\Xi}$ of continuous sections of V^Q and a family $(B_\xi)_{\xi\in\Xi}$ of clopen subsets of Q. For each $v \in C(Q, V^Q)$, the following relations hold:

$$v^{-1}\Big(\mathrm{cl}\bigcup_{\xi\in\Xi} u_\xi(B_\xi)\Big) = \mathrm{cl}\bigcup_{\xi\in\Xi} v^{-1}\big(u_\xi(B_\xi)\big) = \mathrm{cl}\bigcup_{\xi\in\Xi} B_\xi \cap \|v = u_\xi\|$$
$$= \bigvee_{\xi\in\Xi} B_\xi \wedge \|v = u_\xi\| = \|v \in u\| = v^{-1}(\llcorner u \lrcorner).$$

Thus, in view of Lemma 2.7, the equality $\llcorner u \lrcorner = \mathrm{cl}\bigcup_{\xi\in\Xi} u_\xi(B_\xi)$ is established. The set $\bigcup_{\xi\in\Xi} u_\xi(B_\xi)$ is open; therefore, by Lemma 4.6, the class $\llcorner u \lrcorner$ is a clopen set. ▷

4.9. Lemma. *For every subset $X \sqsubset V^Q$, there is a section $u \in C(Q, V^Q)$ such that $\llcorner u \lrcorner = X$.*

◁ With each element $x \in X$ we associate a section $u_x \in C(Q, V^Q)$ such that $x \in \mathrm{im}\, u_x$. Obviously, the set $B_x = u_x^{-1}(X)$ is clopen. Consider the ascent $u = \mathrm{asc}_{x\in X} B_x u_x$ and establish the equality $\llcorner u \lrcorner = X$. Since $x \in u_x(B_x) \subset X$ for all $x \in X$, we have $X = \bigcup_{x\in X} u_x(B_x) = \mathrm{cl}\bigcup_{x\in X} u_x(B_x)$. For an arbitrary section $v \in C(Q, V^Q)$, the following relations hold:

$$v^{-1}(X) = \bigcup_{x\in X} v^{-1}\big(u_x(B_x)\big) = \mathrm{cl}\bigvee_{x\in X} B_x \wedge \|v = u_x\| = \|v \in u\| = v^{-1}(\llcorner u \lrcorner).$$

In view of Lemma 2.7, the desired equality is established. ▷

4.10. Lemma. *For every formula* $\varphi(t_1,\dots,t_n)$ *and arbitrary sections* $u_1,\dots,u_n \in C(Q,V^Q)$, *the following equality holds*:

$$\|\varphi(u_1,\dots,u_n)\| = \bigl\{q \in Q \bigm| V^q \models \varphi\bigl(u_1(q),\dots,u_n(q)\bigr)\bigr\}.$$

◁ The proof of the lemma repeats that of the pointwise truth-value theorem (see 3.10). ▷

The last lemma implies in particular that the extensionality and regularity axioms are true in each stalk. Thus, Theorem 4.3 is completely proven.

In conclusion, we state a theorem that combines the basic results of Sections 3 and 4.

Theorem. *Let* Q *be the Stone space of a complete Boolean algebra* B.

(1) *The class* $C(Q,V^Q)$ *of continuous sections of a polyverse* V^Q *on* Q *is a Boolean-valued universe.*

(2) *For an arbitrary Boolean-valued universe* $\mathfrak{U}$ *over* B, *there exists a continuous polyverse* V^Q *on* Q *such that* $C(Q,V^Q)$ *is isomorphic to* $\mathfrak{U}$.

References

1. *Bourbaki N.* (1968) *General Topology. Fundamental Structures*, Nauka, Moscow (Russian).

2. *Arkhangel'skiĭ A. V. and Ponomarev V. I.* (1974) *Fundamentals of General Topology in Problems and Exercises*, Nauka, Moscow (Russian).

3. *Solovay R. and Tennenbaum S.* (1972) Iterated Cohen extensions and Souslin's problem, *Ann. Math.*, v. 94, N2, 201–245.

4. *Vladimirov D. A.* (1969) *Boolean Algebras*, Nauka, Moscow (Russian).

5. *Sikorski R.* (1964) *Boolean Algebras*, Springer-Verlag, Berlin, etc.

6. *Kusraev A. G. and Kutateladze S. S.* (1990) *Nonstandard Methods in Analysis*, Nauka, Novosibirsk (Russian) (English translation: (1994), Kluwer, Dordrecht).

7. *Gordon E. I. and Morozov S. F.* (1982) *Boolean-Valued Models of Set Theory*, Gor'kiĭ, Gor'kiĭ State University (Russian).