CHAPTER 2

Functional Representation of a Boolean Valued Universe

BY

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S.S. Kutateladze (ed.), Nonstandard Analysis and Vector Lattices, 81–104.

The methods of Boolean valued analysis rest on multivalued nonstandard models of set theory. More exactly, the truth value of an assertion in such a model acts into some complete Boolean algebra.

At present, Boolean valued analysis is a rather powerful theory rich in deep results and various applications mostly to set theory. As regards functional analysis, the methods of Boolean valued analysis found successful applications in such domains as the theory of vector lattices and lattice normed spaces, the theory of positive and dominated operators, the theory of von Neumann algebras, convex analysis, and the theory of vector measures.

Contemporary methods of Boolean valued analysis, due to their nature, involve a rather bulky logical technique. From a pragmatic viewpoint, this technique might distract the user-analyst from a concrete aim: to apply the results of Boolean valued analysis for solving analytical problems.

Various function spaces reside in functional analysis, and so the intention is natural of replacing an abstract Boolean valued system by some function analog, a model whose elements are functions and in which the basic logical operations are calculated "pointwise." An example of such a model is given by the class $\mathbf{V}^Q$ of all functions defined on a fixed nonempty set Q and acting into the class $\mathbf{V}$ of all sets. Truth values in the model $\mathbf{V}^Q$ are various subsets of Q and, in addition, the truth value $[\![\varphi(u_1, \dots, u_n)]\!]$ of an assertion $\varphi(t_1, \dots, t_n)$ at functions $u_1, \dots, u_n \in \mathbf{V}^Q$ is calculated as follows:

$$[\![\varphi(u_1, \dots, u_n)]\!] = \big\{q \in Q : \varphi\big(u_1(q), \dots, u_n(q)\big)\big\}.$$

In the present article, a solution is proposed to the above problem. To this end, we introduce and study a new notion of continuous polyverse which is a continuous bundle of models of set theory. It is shown that the class of continuous sections of a continuous polyverse is a Boolean valued system satisfying all basic principles of Boolean valued analysis and, conversely, every Boolean valued algebraic system can be represented as the class of sections of a suitable continuous polyverse.

2.1. Preliminaries

2.1.1. Let X and Y be topological spaces. A mapping $f : X \to Y$ is called *open* if f satisfies one (and hence all) of the following equivalent conditions:

(1) for every open subset $A \subset X$, the image $f(A)$ is open in Y;

(2) for every point $x \in X$ and every neighborhood $A \subset X$ about x, the image $f(A)$ is a neighborhood about $f(x)$ in Y;

(3) $f^{-1}(\operatorname{cl} B) \subset \operatorname{cl} f^{-1}(B)$ for every subset $B \subset Y$. Observe that the equality $f^{-1}(\operatorname{cl} B) = \operatorname{cl} f^{-1}(B)$ holds for all subsets $B \subset Y$ if and only if f is a continuous and open mapping.

A mapping $f : X \to Y$ is called *closed* if f satisfies one (and hence all) of the following equivalent conditions:

(1) for every closed subset $A \subset X$, the image $f(A)$ is closed in Y;

(2) $\operatorname{cl} f(A) \subset f(\operatorname{cl} A)$ for every subset $A \subset X$. The equality $\operatorname{cl} f(A) = f(\operatorname{cl} A)$ holds for every subset $A \subset X$ if and only if $f : X \to Y$ is a continuous and closed mapping.

2.1.2. Given a class X, the symbol $\mathscr{P}(X)$ denotes the class of all subsets of X. Let X be a class. A subclass $\tau \subset \mathscr{P}(X)$ is called a *topology* on X whenever

(1) $\cup\tau = X$;

(2) $U \cap V \in \tau$ for all $U, V \in \tau$;

(3) $\cup\mathscr{U} \in \tau$ for every subset $\mathscr{U} \subset \tau$. As usual, a class X endowed with a topology is called a *topological space*.

All basic topological concepts (such as neighborhood about a point, closed set, interior, closure, continuous function, Hausdorff space, etc.) can be introduced by analogy to the case of a topology on a set. However, observe that not all classical approaches to the definition of these concepts remain formally valid in the case of a class-topology. For instance, considering the two definitions of a closed set

(a) as a subset of X whose complement belongs to τ,

(b) as a subset of X whose complement, together with each point of it, contains an element of τ,

we should choose the second.

Defining the closure of a set A as the smallest closed subset of X that contains A, we take a risk: some sets may turn out to have no closure. However, the problem disappears if the topology τ is Hausdorff. (Indeed, in the case of a Hausdorff topology, every convergent filter has a unique limit and, hence, the totality of all limits of convergent filters over a given set makes a set rather than a proper class.)

The symbol $\operatorname{Clop}(X)$ denotes the class of all clopen subsets of X (i.e., subsets that are closed and open simultaneously). Henceforth the notation $U \sqsubset X$ means that $U \in \operatorname{Clop}(X)$. The class $\{A \sqsubset X : x \in A\}$ is denoted by $\operatorname{Clop}(x)$.

A topology is called *extremally disconnected* if the closure of every open set is again open.

Most of the necessary information about topological spaces can be found, for instance, in [1, 2].

2.1.3. Let B be a complete Boolean algebra. A triple $(\mathfrak{U}, [\![\cdot = \cdot]\!], [\![\cdot \in \cdot]\!])$ is called a *Boolean valued algebraic system* over B (or a *B-valued algebraic system*) if the classes $[\![\cdot = \cdot]\!]$ and $[\![\cdot \in \cdot]\!]$ are class-functions from $\mathfrak{U} \times \mathfrak{U}$ to B satisfying the following conditions:

(1) $[\![u = u]\!] = \mathbf{1}$;

(2) $[\![u = v]\!] = [\![v = u]\!]$;

(3) $[\![u = v]\!] \wedge [\![v = w]\!] \leqslant [\![u = w]\!]$;

(4) $[\![u = v]\!] \wedge [\![v \in w]\!] \leqslant [\![u \in w]\!]$;

(5) $[\![u = v]\!] \wedge [\![w \in v]\!] \leqslant [\![w \in u]\!]$ for all $u, v, w \in \mathfrak{U}$.

The class-functions $[\![\cdot = \cdot]\!]$ and $[\![\cdot \in \cdot]\!]$ are called the Boolean valued (B-valued) *truth values* of equality and membership.

Instead of $(\mathfrak{U}, [\![\cdot = \cdot]\!], [\![\cdot \in \cdot]\!])$, we usually write simply $\mathfrak{U}$, furnishing the symbols of truth values with the index: $[\![\cdot = \cdot]\!]_{\mathfrak{U}}$ and $[\![\cdot \in \cdot]\!]_{\mathfrak{U}}$ if need be.

A Boolean valued system $\mathfrak{U}$ is called *separated* whenever, for all $u, v \in \mathfrak{U}$, the equality $[\![u = v]\!] = \mathbf{1}$ implies $u = v$.

2.1.4. Consider Boolean valued algebraic systems $\mathfrak{U}$ and $\mathfrak{V}$ over complete Boolean algebras B and C and assume that there is a Boolean isomorphism $\jmath : B \to C$. By an *isomorphism between the Boolean valued algebraic systems* $\mathfrak{U}$ *and* $\mathfrak{V}$ *(associated with the isomorphism $\jmath$)* we mean a bijective class-function $\imath : \mathfrak{U} \to \mathfrak{V}$ that satisfies the following relations:

$$\jmath([\![u_1 = u_2]\!]_{\mathfrak{U}}) = [\![\imath(u_1) = \imath(u_2)]\!]_{\mathfrak{V}},$$
$$\jmath([\![u_1 \in u_2]\!]_{\mathfrak{U}}) = [\![\imath(u_1) \in \imath(u_2)]\!]_{\mathfrak{V}}$$

for all $u_1, u_2 \in \mathfrak{U}$. Boolean valued systems are said to be *isomorphic* if there is an isomorphism between them. In case $\mathfrak{U}$ and $\mathfrak{V}$ are Boolean valued algebraic systems over the same algebra B, each isomorphism $\imath : \mathfrak{U} \to \mathfrak{V}$ is assumed by default to be associated with the identity isomorphism: $[\![u_1 = u_2]\!]_{\mathfrak{U}} = [\![\imath(u_1) = \imath(u_2)]\!]_{\mathfrak{V}}$, $[\![u_1 \in u_2]\!]_{\mathfrak{U}} = [\![\imath(u_1) \in \imath(u_2)]\!]_{\mathfrak{V}}$. For emphasizing this convention, whenever necessary, we call such an isomorphism *B-isomorphism* and refer to the corresponding systems as *B-isomorphic.*

2.1.5. In what follows, using an expression like $\varphi(t_1, \dots, t_n)$, we assume that φ is a set-theoretic formula with all free variables listed in $(t_1, \dots, t_n)$.

An arbitrary tuple $(u_1, \dots, u_n)$ of elements of a system $\mathfrak{U}$ is called a *valuation* of the list of variables $(t_1, \dots, t_n)$. By recursion on the length of a formula, the (Boolean) *truth value* $[\![\varphi(u_1, \dots, u_n)]\!]$ of a formula $\varphi(t_1, \dots, t_n)$ can be defined by assignment $(u_1,\dots,u_n)$ to the variables $(t_1,\dots,t_n)$. If a formula φ is atomic, i.e., has the form $t_1=t_2$ or $t_1\in t_2$; then the truth value of φ by assignment (u_1,u_2) is defined to be $[\![u_1=u_2]\!]$ or $[\![u_1\in u_2]\!]$. Considering compound formulas, we define their truth values as follows:

$$\begin{aligned}
[\![\varphi(u_1,\dots,u_n)\,\&\,\psi(u_1,\dots,u_n)]\!] &:= [\![\varphi(u_1,\dots,u_n)]\!]\wedge[\![\psi(u_1,\dots,u_n)]\!],\\
[\![\varphi(u_1,\dots,u_n)\vee\psi(u_1,\dots,u_n)]\!] &:= [\![\varphi(u_1,\dots,u_n)]\!]\vee[\![\psi(u_1,\dots,u_n)]\!],\\
[\![\varphi(u_1,\dots,u_n)\to\psi(u_1,\dots,u_n)]\!] &:= [\![\varphi(u_1,\dots,u_n)]\!]\Rightarrow[\![\psi(u_1,\dots,u_n)]\!],\\
[\![\neg\varphi(u_1,\dots,u_n)]\!] &:= [\![\varphi(u_1,\dots,u_n)]\!]^{\perp},\\
[\![(\forall t)\,\varphi(t,u_1,\dots,u_n)]\!] &:= \bigwedge_{u\in\mathfrak{U}}[\![\varphi(u,u_1,\dots,u_n)]\!],\\
[\![(\exists t)\,\varphi(t,u_1,\dots,u_n)]\!] &:= \bigvee_{u\in\mathfrak{U}}[\![\varphi(u,u_1,\dots,u_n)]\!],
\end{aligned}$$

where the symbol $b^{\perp}$ denotes the complement of b in the Boolean algebra B. A formula $\varphi(t_1,\dots,t_n)$ is said to be *true* in an algebraic system $\mathfrak{U}$ by assignment $(u_1,\dots,u_n)$ if the equality $[\![\varphi(u_1,\dots,u_n)]\!]=\mathbf{1}$ holds. In this case, we write $\mathfrak{U}\models\varphi(u_1,\dots,u_n)$.

2.1.6. *Proposition.* *If a formula $\varphi(t_1,\dots,t_n)$ is provable in the first-order predicate calculus then $[\![\varphi(u_1,\dots,u_n)]\!]=\mathbf{1}$ for all $u_1,\dots,u_n\in\mathfrak{U}$.*

$\lhd$ It is easy to verify that all axioms of the first-order predicate calculus are true in $\mathfrak{U}$ and the rules of inference increase the truth value. The latter means that derivability (in the first-order predicate calculus) of a formula φ from formulas $\varphi_1,\dots,\varphi_n$ ensures the inequality $[\![\varphi_1\wedge\dots\wedge\varphi_n]\!]\leqslant[\![\varphi]\!]$. $\rhd$

In particular, the last proposition implies that, for all $\varphi(t,t_1,\dots,t_n)$ and $u,v,w_1,\dots,w_n\in\mathfrak{U}$, we have the inequality

$$[\![u=v]\!]\wedge[\![\varphi(u,w_1,\dots,w_n)]\!]\leqslant[\![\varphi(v,w_1,\dots,w_n)]\!].$$

2.1.7. Let $u\in\mathfrak{U}$ be such that $\mathfrak{U}\models u\neq\varnothing$. The *descent* of u is the class $\{v\in\mathfrak{U}:\mathfrak{U}\models v\in u\}$ denoted by $u{\downarrow}$.

2.1.8. Let $(u_\xi)_{\xi\in\Xi}$ be a family of elements in $\mathfrak{U}$ and let $(b_\xi)_{\xi\in\Xi}$ be a family of elements in the Boolean algebra B. An element $u\in\mathfrak{U}$ is called an *ascent of the family* $(u_\xi)_{\xi\in\Xi}$ *by* (*weights*) $(b_\xi)_{\xi\in\Xi}$, if $[\![v\in u]\!]=\bigvee_{\xi\in\Xi}b_\xi\wedge[\![v=u_\xi]\!]$ for all $v\in\mathfrak{U}$.

Let $\mathscr{U}$ be a subset of $\mathfrak{U}$. An element $\overline{u}\in\mathfrak{U}$ is called an *ascent of the set* $\mathscr{U}$, if $[\![v\in\overline{u}]\!]=\bigvee_{u\in\mathscr{U}}[\![v=u]\!]$ for all $v\in\mathfrak{U}$, i.e., $\overline{u}$ is an ascent of the family $(u)_{u\in\mathscr{U}}$ by unit weights.

Assume that $(b_\xi)_{\xi\in\Xi}$ is an antichain in the algebra B. An element $u \in \mathfrak{U}$ is called a *mixing* of the family $(u_\xi)_{\xi\in\Xi}$ by $(b_\xi)_{\xi\in\Xi}$, if $[\![u = u_\xi]\!] \geqslant b_\xi$ for all $\xi \in \Xi$, and $[\![u = \varnothing]\!] \geqslant (\bigvee_{\xi\in\Xi} b_\xi)^\perp$.

If the system $\mathfrak{U}$ is separated and the axiom of extensionality is true in $\mathfrak{U}$, then an ascent (mixing) of a family $(u_\xi)_{\xi\in\Xi}$ by $(b_\xi)_{\xi\in\Xi}$ is uniquely determined. In this case, whenever the ascent (mixing) exists we denote it by $\mathrm{asc}_{\xi\in\Xi}\, b_\xi u_\xi$ ($\mathrm{mix}_{\xi\in\Xi}\, b_\xi u_\xi$). For the ascent of a set $\mathscr{U} \subset \mathfrak{U}$, we use the notation $\mathscr{U}\!\uparrow$.

2.1.9. A key role in Boolean valued analysis is played by the three basic principles: the maximum principle, the mixing principle, and the ascent principle. The reason behind this is the fact that, in algebraic systems satisfying these principles, there is a possibility of constructing new elements from those available.

In the current section, we state the above-mentioned principles and study interrelations between them, leaving aside verification of the principles for concrete algebraic systems.

Let B be a complete Boolean algebra, and let $\mathfrak{U}$ be a B-valued algebraic system.

Maximum Principle. *For every formula $\varphi(t, t_1, \dots, t_n)$ and arbitrary elements $u_1, \dots, u_n \in \mathfrak{U}$, there exists an element $u \in \mathfrak{U}$ such that $[\![(\exists t)\, \varphi(t, u_1, \dots, u_n)]\!] = [\![\varphi(u, u_1, \dots, u_n)]\!]$.*

Mixing Principle. *For every family $(u_\xi)_{\xi\in\Xi}$ of elements in $\mathfrak{U}$ and every antichain $(b_\xi)_{\xi\in\Xi}$ in the algebra B, there exists a mixing $(u_\xi)_{\xi\in\Xi}$ by $(b_\xi)_{\xi\in\Xi}$.*

Ascent Principle. *The following hold:*

(1) *For every family $(u_\xi)_{\xi\in\Xi}$ of elements in $\mathfrak{U}$ and every family $(b_\xi)_{\xi\in\Xi}$ of elements in the algebra B, there exists an ascent $(u_\xi)_{\xi\in\Xi}$ by $(b_\xi)_{\xi\in\Xi}$.*

(2) *For every element $u \in \mathfrak{U}$, there exist a family $(u_\xi)_{\xi\in\Xi}$ of elements in $\mathfrak{U}$ and a family $(b_\xi)_{\xi\in\Xi}$ of elements in the algebra B such that u is an ascent of $(u_\xi)_{\xi\in\Xi}$ by $(b_\xi)_{\xi\in\Xi}$.*

2.1.10. ***Theorem.*** *If a B-valued system $\mathfrak{U}$ satisfies the mixing principle then $\mathfrak{U}$ satisfies the maximum principle.*

$\lhd$ Considering a formula $\varphi(t, t_1, \dots, t_n)$, denote by $\vec{u}$ a tuple of arbitrary elements $u_1, \dots, u_n \in \mathfrak{U}$ and put $b = [\![(\exists t)\, \varphi(t, \vec{u})]\!]$. By the definition of truth value, $b = \bigvee_{v\in\mathfrak{U}} [\![\varphi(v, \vec{u})]\!]$. According to the exhaustion principle, there exist an antichain $(b_\xi)_{\xi\in\Xi}$ in the algebra B and a family $(v_\xi)_{\xi\in\Xi}$ of elements in $\mathfrak{U}$ such that $\bigvee_{\xi\in\Xi} b_\xi = b$ and $b_\xi \leqslant [\![\varphi(v_\xi, \vec{u})]\!]$. By the hypothesis of the theorem, there exists a mixing $v \in \mathfrak{U}$ of the family $(v_\xi)_{\xi\in\Xi}$ by $(b_\xi)_{\xi\in\Xi}$. In particular, $[\![v = v_\xi]\!] \geqslant b_\xi$. In view of Proposition 2.1.6, the following inequalities hold: $[\![\varphi(v, \vec{u})]\!] \geqslant [\![v = v_\xi]\!] \wedge [\![\varphi(v_\xi, \vec{u})]\!] \geqslant b_\xi$. Consequently, $[\![\varphi(v, \vec{u})]\!] \geqslant \bigvee_{\xi\in\Xi} b_\xi = b$. The inequality $[\![\varphi(v, \vec{u})]\!] \leqslant b$ is obvious. $\rhd$

2.1.11. Theorem. *Let a B-valued algebraic system $\mathfrak{U}$ satisfy the ascent principle and let the axiom of extensionality be true in $\mathfrak{U}$. Then the mixing principle is valid for $\mathfrak{U}$.*

$\lhd$ Let $(u_\xi)_{\xi\in\Xi}$ be a family of elements in $\mathfrak{U}$ and let $(b_\xi)_{\xi\in\Xi}$ be an antichain in the algebra B. By the hypothesis of the theorem, for every $\xi \in \Xi$, there exist a family $(u_\xi^\alpha)_{\alpha\in A(\xi)}$ of elements in $\mathfrak{U}$ and a family $(b_\xi^\alpha)_{\alpha\in A(\xi)}$ of elements in the algebra B such that

$$[\![v \in u_\xi]\!] = \bigvee_{\alpha\in A(\xi)} b_\xi^\alpha \wedge [\![v = u_\xi^\alpha]\!] \quad \text{for all } v \in \mathfrak{U}.$$

Consider the set $\Gamma = \{(\xi,\alpha) : \xi \in \Xi,\ \alpha \in A(\xi)\}$ and, for each pair $\gamma = (\xi,\alpha) \in \Gamma$, put $c_\gamma = b_\xi \wedge b_\xi^\alpha$ and $v_\gamma = u_\xi^\alpha$. Let $u \in \mathfrak{U}$ be an ascent of the family $(v_\gamma)_{\gamma\in\Gamma}$ by $(c_\gamma)_{\gamma\in\Gamma}$. Using straightforward calculation and employing definitions, we obtain:

$$\begin{aligned}
[\![v \in u]\!] &= \bigvee_{\gamma\in\Gamma} c_\gamma \wedge [\![v = v_\gamma]\!] \\
&= \bigvee_{\xi\in\Xi} \ \bigvee_{\alpha\in A(\xi)} b_\xi \wedge b_\xi^\alpha \wedge [\![v = u_\xi^\alpha]\!] \\
&= \bigvee_{\xi\in\Xi} b_\xi \wedge [\![v \in u_\xi]\!].
\end{aligned}$$

Show that u is a mixing of the family $(u_\xi)_{\xi\in\Xi}$ by $(b_\xi)_{\xi\in\Xi}$. We begin with establishing the inequality $[\![u = u_\xi]\!] \geqslant b_\xi$. Since the axiom of extensionality is true, it is sufficient to show that $\big([\![v \in u]\!] \Leftrightarrow [\![v \in u_\xi]\!]\big) \geqslant b_\xi$ or, which is equivalent, $b_\xi \wedge [\![v \in u]\!] = b_\xi \wedge [\![v \in u_\xi]\!]$. Since $b_\xi \wedge b_\eta = \mathbf{0}$ for $\xi \neq \eta$, we have:

$$b_\xi \wedge [\![v \in u]\!] = \bigvee_{\eta\in\Xi} b_\xi \wedge b_\eta \wedge [\![v \in u_\eta]\!] = b_\xi \wedge [\![v \in u_\xi]\!].$$

We now show that $[\![u \neq \varnothing]\!] \leqslant \bigvee_{\xi\in\Xi} b_\xi$. Indeed,

$$[\![u \neq \varnothing]\!] = [\![(\exists t)\, t \in u]\!] = \bigvee_{v\in\mathfrak{U}} [\![v \in u]\!] = \bigvee_{v\in\mathfrak{U}} \ \bigvee_{\xi\in\Xi} b_\xi \wedge [\![v \in u_\xi]\!] \leqslant \bigvee_{\xi\in\Xi} b_\xi. \ \rhd$$

2.1.12. Theorem. *If a B-valued algebraic system $\mathfrak{U}$ satisfies the maximum and ascent principles then $\mathfrak{U}$ satisfies the mixing principle.*

$\lhd$ Let $\varnothing^\wedge \in \mathfrak{U}$ be an ascent of the empty subset of $\mathfrak{U}$. It is easy to verify that $[\![\varnothing^\wedge = \varnothing]\!] = \mathbf{1}$. (Here and in the sequel, the notation $u = \varnothing$ means $(\forall t)\, t \notin u$.)

Consider a family $(u_\xi)_{\xi\in\Xi}$ of elements in $\mathfrak{U}$ and an antichain $(b_\xi)_{\xi\in\Xi}$ in the algebra B. Put $b = (\bigvee_{\xi\in\Xi} b_\xi)^\perp$. Define a family $(v_\xi)_{\xi\in\Xi'}$ and a partition of unity

$(c_\xi)_{\xi\in\Xi'}$ as follows: $\Xi' = \Xi \cup \{\Xi\}$, $v_\xi = u_\xi$, $c_\xi = b_\xi$ for $\xi \in \Xi$, and $v_\Xi = \varnothing^\wedge$, $c_\Xi = b$. Let $u \in \mathfrak{U}$ be an ascent of the family $(v_\xi)_{\xi\in\Xi'}$ by $(c_\xi)_{\xi\in\Xi'}$. It is easily seen that $[\![u \neq \varnothing]\!] = \mathbf{1}$. Indeed, $[\![v_\xi \in u]\!] \geqslant c_\xi$ for $\xi \in \Xi'$, which implies

$$[\![u \neq \varnothing]\!] = \bigvee_{v\in\mathfrak{U}} [\![v \in u]\!] \geqslant \bigvee_{\xi\in\Xi'} c_\xi = \mathbf{1}.$$

Thus, $[\![(\exists t)\, t \in u]\!] = \mathbf{1}$. According to the maximum principle, there exists an element $v \in \mathfrak{U}$ such that $[\![v \in u]\!] = \mathbf{1}$. Then, by the definition of ascent,

$$c_\xi = \mathbf{1} \wedge c_\xi = \bigvee_{\eta\in\Xi'} c_\eta \wedge [\![v = v_\eta]\!] \wedge c_\xi = [\![v = v_\xi]\!] \wedge c_\xi$$

and, hence, $[\![v = v_\xi]\!] \geqslant c_\xi$ for all $\xi \in \Xi'$. In particular, for $\xi \in \Xi$, we have $[\![v = u_\xi]\!] \geqslant b_\xi$. In addition, by Proposition 2.1.6, the following relations hold:

$$\Big(\bigvee_{\xi\in\Xi} b_\xi\Big)^{\perp} \leqslant [\![v = \varnothing^\wedge]\!] = [\![v = \varnothing^\wedge]\!] \wedge [\![\varnothing^\wedge = \varnothing]\!] \leqslant [\![v = \varnothing]\!].$$

Consequently, v is a mixing of the family $(u_\xi)_{\xi\in\Xi}$ by $(b_\xi)_{\xi\in\Xi}$. $\triangleright$

2.1.13. Let B be a complete Boolean algebra and let $\mathfrak{U}$ be a B-valued algebraic system. The system $\mathfrak{U}$ is called a *Boolean valued universe over B* (a *B-valued universe*) if it satisfies the following three conditions:

(1) $\mathfrak{U}$ is separated;

(2) $\mathfrak{U}$ satisfies the ascent principle;

(3) the axioms of extensionality and regularity are true in $\mathfrak{U}$.

Theorem ([6]). *For every complete Boolean algebra B, there is a unique B-valued universe up to isomorphism.*

A detailed presentation of the theories of Boolean algebras and Boolean valued algebraic systems can be found in [3–5, 7].

2.2. The Concept of Continuous Bundle

2.2.1. Let Q be an arbitrary nonempty set and let $V^Q \subset Q \times \mathbf{V}$ be a class-correspondence. As usual, $\mathbf{V}$ denotes the class of all sets. Given $q \in Q$, denote the class

$$\{q\} \times V^Q(q) = \big\{(q, x) : (q, x) \in V^Q\big\}$$

by V^q. Obviously, $V^p \cap V^q = \varnothing$ for $p \neq q$. The correspondence V^Q is called a *bundle* on Q and the class V^q is called the *stalk* of V^Q at a point q.

Let $D \subset Q$. A function $u : D \to V^Q$ is called a *section* of the bundle V^Q on D if $u(q) \in V^q$ for all $q \in D$. The class of all sections of V^Q on D is denoted by $S(D, V^Q)$. The sections defined on Q are called *global*. If X is a subset of V^Q then the symbol $S(D, X)$ stands for the set of all sections of X on D.

A point $q \in Q$ is called the *projection of an element* $x \in V^Q$ and denoted by $\mathrm{pr}(x)$ if $x \in V^q$. The *projection of a set* $X \subset V^Q$ is defined to be $\{\mathrm{pr}(x) : x \in X\}$ and denoted by $\mathrm{pr}(X)$.

2.2.2. Assume now Q to be a topological space and suppose that some topology is given on a class $V^Q \subset Q \times \mathbf{V}$. In this case, we call V^Q a *continuous bundle* on Q.

By a *continuous section* of the bundle V^Q we mean a section that is a continuous function. Given a subset $D \subset Q$, the symbol $C(D, V^Q)$ stands for the class of all continuous sections of V^Q on D. Analogously, if X is a subset of V^Q then $C(D, X)$ stands for the totality of all continuous sections of X on D. Obviously, $C(D, X) = C(D, V^Q) \cap S(D, X)$.

Henceforth we suppose that Q is an extremally disconnected Hausdorff compact space and assume satisfied the following conditions:

(1) $(\forall\, q \in Q)\ (\forall\, x \in V^q)\ (\exists\, u \in C(Q, V^Q))\ u(q) = x$;

(2) $(\forall\, u \in C(Q, V^Q))\ (\forall\, A \sqsubset Q)\ u(A) \sqsubset V^Q$.

2.2.3. Proposition. *The continuous bundle V^Q possesses the following properties:*

(1) *the topology of V^Q is Hausdorff;*

(2) *for every $u \in C(Q, V^Q)$ and $q \in Q$, the family $\{u(A) : A \in \mathrm{Clop}(q)\}$ is a neighborhood base of the point $u(q)$;*

(3) *all elements of $C(Q, V^Q)$ are open and closed mappings (see 2.1.1).*

$\triangleleft$ Let x and y be different elements of V^Q. Put $p = \mathrm{pr}(x)$ and $q = \mathrm{pr}(y)$. In view of 2.2.2 (1), there are sections $u, v \in C(Q, V^Q)$ such that $u(p) = x$ and $v(q) = y$.

Suppose first that $p = q$. The set

$$A = \{q \in Q : u(q) \neq v(q)\} = Q \setminus u^{-1}\big(v(Q)\big)$$

is clopen in view of 2.2.2 (2). Then $u(A)$ and $v(A)$ are disjoint neighborhoods about the points x and y.

Suppose now that $p \neq q$. In this case, there exist $A, B \sqsubset Q$ such that $A \cap B = \varnothing$, $p \in A$, and $q \in B$. Then $u(A)$ and $v(B)$ are disjoint neighborhoods about the points x and y.

Assertion (2) follows readily from 2.2.2 (2).

Assertion (3) is equivalent to 2.2.2 (2) due to the fact that $\mathrm{Clop}(Q)$ is a base both for the open and close topologies of Q. $\triangleright$

2.2.4. Lemma. *A subset $X \subset V^Q$ is clopen if and only if $u^{-1}(X) \sqsubset Q$ for all $u \in C(Q, V^Q)$.*

$\triangleleft$ Only sufficiency requires some comments. Consider an arbitrary element $x \in V^Q$. Let a section $u \in C(Q, V^Q)$ and a point $q \in Q$ be such that $u(q) = x$.

Suppose first that $x \in X$. The set $A = u^{-1}(X)$ is clopen in Q and, therefore, $u(A)$ is a neighborhood about x lying in X. Since x is arbitrary, we conclude that X is open.

If $x \notin X$ then the set $A = Q \setminus u^{-1}(X)$ is clopen in Q and, hence, $u(A)$ is a neighborhood about x disjoint from X. Since x is arbitrary, we conclude that X is closed. $\triangleright$

2.2.5. Proposition. *The topology of V^Q is extremally disconnected.*

$\triangleleft$ Let X be an open subset of V^Q. Since the topology of V^Q is Hausdorff, the closure $\mathrm{cl}\, X$ is a set (see 2.1.2). Furthermore, for every section $u \in C(Q, V^Q)$, the set $u^{-1}(\mathrm{cl}\, X) = \mathrm{cl}\, u^{-1}(X)$ is clopen. In view of Lemma 2.2.4, the set $\mathrm{cl}\, X$ is open. $\triangleright$

2.2.6. Lemma. *For every subset $X \subset V^Q$ the following hold:*

$$X = \bigcup_{u \in C(Q, V^Q)} u\big(u^{-1}(X)\big);$$

$$\mathrm{int}\, X = \bigcup_{u \in C(Q, V^Q)} u\big(\mathrm{int}\, u^{-1}(X)\big);$$

$$\mathrm{cl}\, X = \bigcup_{u \in C(Q, V^Q)} u\big(\mathrm{cl}\, u^{-1}(X)\big).$$

$\triangleleft$ The claim is an obvious consequence of 2.2.2 (1) and the fact that all continuous sections are open. $\triangleright$

2.2.7. Lemma. *Let X and Y be subclasses of V^Q. The equality $X = Y$ holds if and only if $u^{-1}(X) = u^{-1}(Y)$ for all $u \in C(Q, V^Q)$.*

$\triangleleft$ Take arbitrary $q \in Q$ and $x \in V^q$ and consider a section $u \in C(Q, V^Q)$ such that $u(q) = x$. If $x \in X$ then $q \in u^{-1}(X) = u^{-1}(Y)$ and, consequently, $x = u(q) \in Y$. The reverse inclusion can be established similarly. $\triangleright$

2.2.8. Proposition. *A section $u \in S(D, V^Q)$ defined on an open subset $D \subset Q$ is continuous if and only if* $\operatorname{im} u$ *is an open subset of V^Q.*

$\lhd$ Suppose that a section u is continuous. For every $q \in D$, choose a section $u_q \in C(Q, V^Q)$ such that $u_q(q) = u(q)$. The set $D_q = \{p \in D : u(p) = u_q(p)\} = u^{-1}(\operatorname{im} u_q)$ is open in D and, hence, it is also open in Q. Therefore, the image $u(D_q) = u_q(D_q)$ is open in view of the fact that global continuous sections are open. Obviously, $D = \bigcup_{q \in D} D_q$, since $q \in D_q$. Thus, $\operatorname{im} u = u(D) = u\big(\bigcup_{q \in D} D_q\big) = \bigcup_{q \in D} u(D_q)$ is an open set.

Suppose now that $\operatorname{im} u$ is an open set. Consider an arbitrary point $q \in D$ and choose a section $u_q \in C(Q, V^Q)$ such that $u(q) = u_q(q)$. The open set $\{p \in D : u(p) = u(p)\} = u^{-1}(\operatorname{im} u)$ is a neighborhood about q, whence it follows that u is continuous at q. $\rhd$

2.2.9. Lemma. *For every subset $X \subset V^Q$, the following hold:*

(1) $\operatorname{pr}(\operatorname{cl} X) \subset \operatorname{cl} \operatorname{pr}(X)$;

(2) $\operatorname{pr}(\operatorname{int} X) \subset \operatorname{int} \operatorname{pr}(X)$.

$\lhd$ Consider an arbitrary section $u \in C(Q, V^Q)$. In view of the properties of the closure, we have the relations $u^{-1}(\operatorname{cl} X) = \operatorname{cl} u^{-1}(X) \subset \operatorname{cl} \operatorname{pr}(X)$, whence, due to the equality $\operatorname{pr}(X) = \bigcup_{u \in C(Q, V^Q)} u^{-1}(X)$, it follows that $\operatorname{pr}(\operatorname{cl} X) \subset \operatorname{cl} \operatorname{pr}(X)$.

Relation (2) can be established similarly. $\rhd$

2.3. A Continuous Polyverse

2.3.1. Consider a nonempty set Q and a bundle $V^Q \subset Q \times \mathbf{V}$. Suppose that, for each point $q \in Q$, the class V^q is an algebraic system of signature $\{\in\}$.

Given an arbitrary formula $\varphi(t_1, \dots, t_n)$ and sections $u_1, \dots, u_n$ of the bundle V^Q, we denote by $\{\varphi(u_1, \dots, u_n)\}$ the set

$$\big\{q \in \operatorname{dom} u_1 \cap \dots \cap \operatorname{dom} u_n : V^q \models \varphi\big(u_1(q), \dots, u_n(q)\big)\big\}.$$

For every element $x \in V^q$, put $x{\downarrow} = \{y \in V^q : V^q \models y \in x\}$. Obviously, if the axiom of extensionality is true in the system V^q, then $x{\downarrow} = y{\downarrow} \leftrightarrow x = y$ for all $x, y \in V^q$. If X is a subset of V^Q then the symbol $\sqcup X$ denotes the union $\bigcup_{x \in X} x{\downarrow}$.

Henceforth we assume that Q is an extremally disconnected Hausdorff compact space and V^Q is a continuous bundle on Q.

For an arbitrary section $u \in C(Q, V^Q)$, the class $\bigcup_{q \in Q} u(q){\downarrow}$ is called the *unpack* of the section u and denoted by $\llcorner u \lrcorner$.

2.3.2. A continuous bundle V^Q is called a *continuous polyverse* on Q, if the axioms of extensionality and regularity are true in each stalk V^q $(q \in Q)$ and, in addition, the following conditions hold:

(1) $(\forall\, q \in Q)\ (\forall\, x \in V^q)\ (\exists\, u \in C(Q, V^Q))\ u(q) = x$;

(2) $(\forall\, u \in C(Q, V^Q))\ (\forall\, A \in \mathrm{Clop}(Q))\ u(A) \in \mathrm{Clop}(V^Q)$;

(3) $(\forall\, u \in C(Q, V^Q))\ \llcorner u \lrcorner \in \mathrm{Clop}(V^Q)$;

(4) $(\forall\, X \in \mathrm{Clop}(V^Q))\ (\exists\, u \in C(Q, V^Q))\ \llcorner u \lrcorner = X$.

2.3.3. For arbitrary sections $u, v \in C(Q, V^Q)$, the equalities $\{u = v\} = u^{-1}(\mathrm{im}\, v)$ and $\{u \in v\} = u^{-1}(\llcorner v \lrcorner)$ imply that the sets $\{u = v\}$ and $\{u \in v\}$ are clopen, which allows us to introduce the two class-functions

$$[\![\cdot = \cdot]\!], [\![\cdot \in \cdot]\!] : C(Q, V^Q) \times C(Q, V^Q) \to \mathrm{Clop}(Q)$$

by letting $[\![u = v]\!] = \{u = v\}$ and $[\![u \in v]\!] = \{u \in v\}$.

It is easy to verify that the triple $\big(C(Q, V^Q),\ [\![\cdot = \cdot]\!],\ [\![\cdot \in \cdot]\!]\big)$ is a separated $\mathrm{Clop}(Q)$-valued algebraic system (see 2.1.3).

The definition 2.3.2 (4) of continuous polyverse implies that there exists a continuous section $\varnothing^\wedge$ satisfying the condition $\llcorner \varnothing^\wedge \lrcorner = \varnothing$. Obviously, this section is unique. It is easy that $V^q \models \varnothing^\wedge(q) = \varnothing$, $[\![\varnothing^\wedge = \varnothing]\!] = Q$, and, in addition, $[\![u = \varnothing^\wedge]\!] = [\![u = \varnothing]\!]$ for all $u \in C(Q, V^Q)$.

2.3.4. Lemma. *For every subset $X \subset V^Q$, the following hold:*

(1) *if $X \sqsubset V^Q$ then* $\mathrm{pr}(X) \sqsubset Q$;

(2) *if X is open then* $\mathrm{pr}(\mathrm{cl}\, X) = \mathrm{cl}\, \mathrm{pr}(X)$.

$\lhd$ (1): If $X \sqsubset V^Q$ then there is a section $u \in C(Q, V^Q)$ such that $\bigsqcup \mathrm{im}\, u = \llcorner u \lrcorner = X$. Obviously, $\mathrm{pr}\big(\bigsqcup \mathrm{im}\, u\big) = [\![u \neq \varnothing]\!]$, whence $\mathrm{pr}(X)$ is clopen.

(2): Let X be an open subset of V^Q. Then the closure $\mathrm{cl}\, X$ is clopen, the same is true of its projection $\mathrm{pr}(\mathrm{cl}\, X)$. The obvious inclusion $\mathrm{pr}(X) \subset \mathrm{pr}(\mathrm{cl}\, X)$ implies $\mathrm{cl}\, \mathrm{pr}(X) \subset \mathrm{pr}(\mathrm{cl}\, X)$. The reverse inclusion is established in 2.2.9. $\rhd$

2.3.5. The *support* $\mathrm{supp}\, u$ of a section $u \in S(D, V^Q)$ on $D \subset Q$ is defined to be the set $\{q \in D : V^q \models u(q) \neq \varnothing\}$. Obviously, $\mathrm{supp}\, u = \{u \neq \varnothing\} = \{u \neq \varnothing^\wedge\}$. So, if $u \in C(Q, V^Q)$ then $\mathrm{supp}\, u$ is a clopen set.

Let u be a continuous section of V^Q and let D be a subset of $\mathrm{supp}\, u$. The symbol $C(D, u)$ denotes the class

$$\big\{v \in C(D, V^Q) : (\forall q \in D)\ V^q \models v(q) \in u(q)\big\}.$$

Obviously, $C(D, u) = C(D, \llcorner u \lrcorner)$.

By the *descent* of a section u we mean the class $C(\mathrm{supp}\, u, u)$ and denote the latter by $u{\downarrow}$. It is easily seen that $u{\downarrow} = C(\mathrm{supp}\, u, \llcorner u \lrcorner)$. Obviously, in case $[\![u \neq \varnothing]\!] = Q$, the descent of u is the descent of the section u regarded as an element of a Boolean valued algebraic system (see 2.1.7).

2.3.6. Proposition. *For arbitrary $X \sqsubset V^Q$ and $u \in C(Q, V^Q)$, the following are equivalent:*

(1) $\llcorner u \lrcorner = X$;

(2) $u(q)\downarrow = X \cap V^q$ *for all* $q \in Q$;

(3) $\operatorname{supp} u = \operatorname{pr}(X)$ *and* $u\downarrow = C(\operatorname{pr}(X), X)$;

(4) $[\![v \in u]\!] = v^{-1}(X)$ *for all* $v \in C(Q, V^Q)$.

$\lhd$ (1)$\to$(3): It suffices to observe that $\operatorname{supp} u = [\![u \neq \varnothing]\!] = \operatorname{pr}(\llcorner u \lrcorner)$ and employ the equality $u\downarrow = C(\operatorname{supp} u, \llcorner u \lrcorner)$.

(3)$\to$(2): Put $A = \operatorname{supp} u$. It is clear that $X \cap V^q = \varnothing = u(q)\downarrow$ for all $q \in Q \backslash A$.

Given an arbitrary point $q \in A$, there are $x \in u(q)\downarrow$ and $v_q \in C(Q, V^Q)$ such that $v_q(q) = x$. Put $B_q = [\![v_q \in u]\!]$. The family $(B_q)_{q \in A}$ is an open cover of the compact set A; therefore, we can refine a subcover $(B_q)_{q \in F}$, where $F \subset A$ is finite. By the exhaustion principle, there is an antichain $(C_q)_{q \in F}$ such that $C_q \subset B_q$ for $q \in F$ and $\bigcup_{q \in F} C_q = \bigvee_{q \in F} C_q = \bigvee_{q \in F} B_q = A$. Construct a section $v \in S(A, V^Q)$ by putting $v(p) = v_q(p)$ for each point $p \in A$, where q is a (unique) element of F such that $p \in C_q$. The section v is continuous, since $v = v_q$ on C_q ($q \in F$). It is easily seen that $v \in u\downarrow = C(A, X)$.

Let q be an arbitrary element of A.

Consider an $x \in u(q)\downarrow$, choose a section $w \in C(Q, V^Q)$ such that $w(q) = x$, and construct a section $\overline{w} \in S(A, V^Q)$ as follows:

$$\overline{w}(p) = \begin{cases} w(p) & \text{if } p \in [\![w \in u]\!], \\ v(p) & \text{if } p \in A \backslash [\![w \in u]\!]. \end{cases}$$

Obviously, the section $\overline{w}$ is continuous and $\overline{w} \in u\downarrow = C(A, X)$, whence $x = \overline{w}(q) \in X$ in view of the containment $q \in [\![w \in u]\!]$.

Now let $x \in X \cap V^q$. As before, choose a section $w \in C(Q, V^Q)$ such that $w(q) = x$. Consider the section $\overline{w} \in S(A, V^Q)$ defined as follows:

$$\overline{w}(p) = \begin{cases} w(p) & \text{if } p \in w^{-1}(X), \\ v(p) & \text{if } p \in A \backslash w^{-1}(X). \end{cases}$$

The obvious relations $\overline{w} \in C(A, X) = u\downarrow$ and $q \in w^{-1}(X)$ imply that $x = w(q) = \overline{w}(q) \in u(q)\downarrow$.

(2)$\to$(4): Consider an arbitrary section $v \in C(Q, V^Q)$. If $q \in [\![v \in u]\!] = v^{-1}(\llcorner u \lrcorner)$ then $v(q) \in \llcorner u \lrcorner$; consequently, $v(q) \in u(q)\downarrow = X \cap V^q$, i.e., $q \in v^{-1}(X)$.

If $q \in v^{-1}(X)$ then $v(q) \in X \cap V^q = u(q)\downarrow$ and, hence, $V^q \models v(q) \in u(q)$ and $q \in [\![v \in u]\!]$.

(4)→(1): Observe that $v^{-1}(\llcorner u \lrcorner) = [\![v \in u]\!] = v^{-1}(X)$ for all $v \in C(Q, V^Q)$. Therefore, in view of Lemma 2.2.7, the equality $X = \llcorner u \lrcorner$ holds. ▷

Obviously, for every $X \sqsubset V^Q$, a section u satisfying conditions (1)–(4) is unique. We call this section the *pack* of the set X and denote it by $\ulcorner X \urcorner$.

It is easy to verify validity of the following

Proposition. *Let X be an open subset of V^Q. A section $\overline{u} \in C(Q, V^Q)$ coincides with $\ulcorner \operatorname{cl} X \urcorner$ if and only if $\overline{u}$ is pointwise the least section among $u \in C(Q, V^Q)$ satisfying the inclusion $X \cap V^q \subset u(q){\downarrow}$ for all $q \in Q$.*

2.3.7. Lemma. *If $u \in C(Q, V^Q)$ and $A \in \operatorname{Clop}(Q)$ then $\sqcup u(A) \in \operatorname{Clop}(V^Q)$.*

◁ For every section $v \in C(Q, V^Q)$, the set $v^{-1}\big(\sqcup u(A)\big) = A \cap [\![v \in u]\!]$ is clopen; whence, in view of 2.2.4, the set $\sqcup u(A)$ is clopen too. ▷

2.3.8. Proposition. *Every continuous section of V^Q defined on an open or closed subset of Q is extendible to a global continuous section.*

◁ Consider $A \subset Q$ and $u \in C(A, V^Q)$. For every point $q \in A$, there exist a section $u_q \in C(Q, V^Q)$ and a set $B_q \sqsubset Q$ such that $q \in B_q$ and $u_q = u$ on $B_q \cap A$.

Suppose that A is an open set. Without loss of generality, we may assume that $B_q \subset A$. Consider the open set $X = \bigcup_{q \in Q} u(q){\downarrow} = \bigcup_{q \in A} \sqcup u_q(B_q)$ and show that $(\operatorname{cl} X) \cap V^q = u(q){\downarrow}$ for all $q \in A$. We only establish the inclusion $(\operatorname{cl} X) \cap V^q \subset u(q){\downarrow}$ (the reverse inclusion follows from the obvious properties of closure). Take an $x \in \operatorname{cl} X \cap V^q$. There is a section $v \in C(Q, V^Q)$ such that $v(q) = x$. Evidently, for each neighborhood $B \sqsubset Q$ about q, the intersection $v(B) \cap X$ is nonempty and, thus, there exists a point $p \in B \cap B_q$ such that $v(p) \in u(p){\downarrow}$. On the other hand, $u(p) = u_q(p)$; consequently, $v(B) \cap \sqcup u_q(B_q) \neq \varnothing$. The set $\sqcup u_q(B_q)$ is closed and, therefore, $x \in \sqcup u_q(B_q)$, whence $x \in u_q(q){\downarrow} = u(q){\downarrow}$. Put $\overline{u} = \ulcorner \operatorname{cl} X \urcorner$. From what was established above it follows that $\overline{u}(q){\downarrow} = u(q){\downarrow}$ for all $q \in A$. Thus, $\overline{u}$ is a sought global extension of the section u.

Suppose now that the set A is closed. The family $(B_q)_{q \in A}$ forms an open cover of the compact set A and, therefore, we can refine a subcover $(B_q)_{q \in F}$, where F is a finite subset of A. Without loss of generality, we may assume that $\bigcup_{q \in F} B_q = Q$. By the exhaustion principle, there is an antichain $(C_q)_{q \in F}$ such that $C_q \subset B_q$ for all $q \in F$ and $\bigcup_{q \in F} C_q = Q$. Construct a section $\overline{u} \in S(Q, V^Q)$ by putting $\overline{u}(p) = u_q(p)$ for each point $p \in Q$, where q is a (unique) element of F such that $p \in C_q$. The section $\overline{u}$ is continuous, since $\overline{u} = u_q$ on C_q $(q \in F)$. Obviously, $\overline{u} = u$ on A. ▷

Corollary. *If A is an open or closed subset of Q then $C(A, V^Q) = \{u|_A : u \in C(Q, V^Q)\}$.*

Extension Principle. *For every section $u \in C(A, V^Q)$ defined on an open subset $A \subset Q$, there is a unique section $\overline{u} \in C(\mathrm{cl}\, A, V^Q)$ extending u.*

$\lhd$ Due to Proposition 2.3.8, there exists a section $u_1 \in C(Q, V^Q)$ such that $u_1 = u$ on A. Put $\overline{u} = u_1|_{\mathrm{cl}\, A}$.

Uniqueness of this extension is obvious. $\rhd$

The section $\overline{u}$ of the extension principle is called the *closure* of u and denoted by $\mathrm{ext}(u)$.

2.3.9. It is easy to verify validity of the following

Theorem. *Consider a family $(u_\xi)_{\xi\in\Xi}$ of global continuous sections of V^Q and an antichain $(B_\xi)_{\xi\in\Xi}$ in the algebra $\mathrm{Clop}(Q)$ and put $B = (\bigvee_{\xi\in\Xi} B_\xi)^\perp$. The continuous section*

$$u = \mathrm{ext}\left(\bigcup_{\xi\in\Xi} u_\xi|_{B_\xi} \cup \varnothing^\wedge|_B\right)$$

is the mixing of the family $(u_\xi)_{\xi\in\Xi}$ by $(B_\xi)_{\xi\in\Xi}$. In particular, the mixing principle is valid for the Boolean valued algebraic system $C(Q, V^Q)$.

Corollary. *The Boolean valued algebraic system $C(Q, V^Q)$ satisfies the maximum principle.*

2.3.10. *Pointwise Truth Value Theorem.* *The following equality holds*

$$[\![\varphi(u_1, \dots, u_n)]\!] = \big\{q \in Q : V^q \models \varphi\big(u_1(q), \dots, u_n(q)\big)\big\} \qquad (*)$$

for an arbitrary formula $\varphi(t_1, \dots, t_n)$ and sections $u_1, \dots, u_n \in C(Q, V^Q)$.

$\lhd$ The proof is carried out by induction on the length of φ.

If φ is atomic, i.e., has the form $t_1 \in t_2$ or $t_1 = t_2$; then $(*)$ follows from the definitions of $[\![\cdot = \cdot]\!]$ and $[\![\cdot \in \cdot]\!]$.

Assume that the claim is proven for formulas of a "smaller" length. We restrict ourselves to the case in which the formula φ has the form $(\exists t_0)\ \varphi(t_0, \vec{t})$.

If $V^q \models (\exists t_0)\ \varphi\big(t_0, \vec{u}(q)\big)$ then there exists an element $x \in V^q$ such that $V^q \models \varphi\big(x, \vec{u}(q)\big)$. Choose a section $u_0 \in C(Q, V^Q)$ satisfying the equality $u_0(q) = x$. By the induction hypothesis, $q \in [\![\varphi(u_0, \vec{u})]\!] \subset [\![(\exists t_0)\ \varphi(t_0, \vec{u})]\!]$, which proves the inclusion $\supset$ in $(*)$.

Show the reverse inclusion. Suppose that $q \in [\![(\exists t_0)\ \varphi(t_0, \vec{u})]\!]$. By the maximum principle, there is a continuous section u_0 such that $[\![\varphi(u_0, \vec{u})]\!] = [\![(\exists t_0)\ \varphi(t_0, \vec{u})]\!]$. Therefore, by the induction hypothesis, $V^q \models \varphi\big(u_0(q), \vec{u}(q)\big)$ and, hence, $V^q \models (\exists t_0)\ \varphi\big(t_0, \vec{u}(q)\big)$. $\rhd$

2.3.11. *Lemma.* *For every subset $X \subset V^Q$, the following hold:*

(1) $\sqcup \operatorname{cl} X \subset \operatorname{cl} \sqcup X$;

(2) $\sqcup \operatorname{int} X \subset \operatorname{int} \sqcup X$;

(3) *if* $X \in \operatorname{Clop}(V^Q)$ *then* $\sqcup X \in \operatorname{Clop}(V^Q)$;

(4) *if* X *is open then* $\sqcup X$ *is an open subset of* V^Q;

(5) *if* X *is open then* $\sqcup \operatorname{cl} X = \operatorname{cl} \sqcup X$.

$\lhd$ (1): Suppose that $x \in \sqcup \operatorname{cl} X$. Then $x \in y{\downarrow}$ for some $y \in \operatorname{cl} X$. Consider sections $u, v \in C(Q, V^Q)$ such that $u(q) = x$ and $v(q) = y$, where $q = \operatorname{pr}(x)$. For every $A \in \operatorname{Clop}(q)$, we have $v(A) \cap X \neq \varnothing$. Put $B = A \cap [\![u \in v]\!] \sqsubset Q$. Since $q \in B$, there is a point $p \in B$ such that $v(p) \in X$. Obviously, $u(p) \in v(p){\downarrow} \subset \sqcup X$ and, hence, $u(A) \cap (\sqcup X) \neq \varnothing$. Consequently, $x \in \operatorname{cl} \sqcup X$.

(2): Suppose that $x \in \sqcup \operatorname{int} X$ and consider $y \in \operatorname{int} X$ and $u, v \in C(Q, V^Q)$ such that $x \in y{\downarrow}$, $u(q) = x$, and $v(q) = y$, where $q = \operatorname{pr}(x)$. It is clear that the set $B = v^{-1}(X) \cap [\![u \in v]\!]$ is a neighborhood about q and, hence, $u(B)$ is a neighborhood about x. Furthermore, $u(p) \in v(p){\downarrow} \subset \sqcup X$ for all $p \in B$, i.e., $u(B) \subset \sqcup X$. Thus, $x \in \operatorname{int} \sqcup X$.

(3): According to Lemma 2.2.4, it suffices to consider an arbitrary section $v \in C(Q, V^Q)$ and show that the set $v^{-1}(\sqcup X)$ is clopen. Put $u = \ulcorner X \urcorner$. Obviously, $v(q) \in \sqcup X$ if and only if

$$V^q \models \big(\exists\, t \in u(q)\big)\, v(q) \in t.$$

By the Pointwise Truth Value Theorem,

$$v^{-1}(X) = \big\{q \in Q : V^q \models \big(\exists\, t \in u(q)\big)\ v(q) \in t\big\} = [\![(\exists\, t \in u)\ v \in t]\!]$$

and, consequently, $v^{-1}(X) \sqsubset Q$.

(4): The claim follows readily from (2).

(5): Let X be an open set. Then its closure $\operatorname{cl} X$ is clopen and, according to (3), the set $\sqcup \operatorname{cl} X$ is clopen too. The obvious relation $\sqcup X \subset \sqcup \operatorname{cl} X$ implies $\operatorname{cl} \sqcup X \subset \sqcup \operatorname{cl} X$. The reverse inclusion holds by (1). $\rhd$

2.3.12. Theorem. *The Boolean valued algebraic system* $C(Q, V^Q)$ *satisfies the ascent principle.*

$\lhd$ Let $(u_\xi)_{\xi \in \Xi}$ be a family of global continuous sections of V^Q and let $(B_\xi)_{\xi \in \Xi}$ be a family of clopen subsets of Q. Consider the clopen set $X = \operatorname{cl} \bigcup_{\xi \in \Xi} u_\xi(B_\xi)$ and put $u = \ulcorner X \urcorner$. Show that the section $u \in C(Q, V^Q)$ thus constructed is an ascent of $(u_\xi)_{\xi\in\Xi}$ by $(B_\xi)_{\xi\in\Xi}$. Indeed, for every section $v \in C(Q, V^Q)$, the following hold:

$$[\![v \in u]\!] = v^{-1}(\llcorner u \lrcorner) = v^{-1}\Big(\operatorname{cl} \bigcup_{\xi\in\Xi} u_\xi(B_\xi)\Big) = \operatorname{cl} v^{-1}\Big(\bigcup_{\xi\in\Xi} u_\xi(B_\xi)\Big)$$
$$= \operatorname{cl} \bigcup_{\xi\in\Xi} v^{-1}\big(u_\xi(B_\xi)\big) = \operatorname{cl} \bigcup_{\xi\in\Xi} B_\xi \cap [\![v = u_\xi]\!] = \bigvee_{\xi\in\Xi} B_\xi \wedge [\![v = u_\xi]\!].$$

Consider now an arbitrary section $u \in C(Q, V^Q)$ and show that u is an ascent of some family of elements in $C(Q, V^Q)$ by suitable weights. Put $X = \llcorner u \lrcorner$. For each $x \in X$, choose a section $u_x \in C(Q, V^Q)$ such that $x \in \operatorname{im} u_x$. Assign $B_x = [\![u_x \in u]\!] = u_x^{-1}(X)$. Obviously, $x \in u_x(B_x) \subset X$ for all $x \in X$, whence $X = \bigcup_{x\in X} u_x(B_x) = \operatorname{cl} \bigcup_{x\in X} u_x(B_x)$. As in the first part of the proof, we can establish the equality $[\![v \in u]\!] = \bigvee_{x\in X} B_x \wedge [\![v = u_x]\!]$ for all $v \in C(Q, V^Q)$. Thus, u is an ascent of $(u_x)_{x\in X}$ by $(B_x)_{x\in X}$. $\rhd$

2.3.13. Consider a $D \sqsubset Q$ and suppose that $\mathscr{U}$ is a subset of $C(D, V^Q)$. Given a point $q \in D$, denote by $\mathscr{U}(q)$ the totality $\{u(q) : u \in \mathscr{U}\}$.

Proposition. *Consider a $D \sqsubset Q$ and suppose that $\mathscr{U}$ is a nonempty subset of $C(D, V^Q)$. The following properties of a section $\overline{u} \in C(Q, V^Q)$ are equivalent:*

(1) $\overline{u} = \ulcorner \operatorname{cl} \bigcup_{u\in\mathscr{U}} \operatorname{im} u \urcorner$;

(2) $[\![v \in \overline{u}]\!] = \operatorname{cl}\{q \in D : v(q) \in \mathscr{U}(q)\}$ *for all* $v \in C(Q, V^Q)$;

(3) $[\![v \in \overline{u}]\!] = \operatorname{cl} \bigcup_{u\in\mathscr{U}} \{v = u\}$ *for all* $v \in C(Q, V^Q)$;

(4) $\overline{u}{\downarrow} = \Big\{\operatorname{ext}\Big(\bigcup_{u\in\mathscr{U}} u|_{D_u}\Big) : (D_u)_{u\in\mathscr{U}}$ *is a partition of unity in the algebra* $\operatorname{Clop}(D)\Big\}$;

(5) $\overline{u}{\downarrow} = C(D, \operatorname{cl} \bigcup_{u\in\mathscr{U}} \operatorname{im} u)$;

(6) *$\overline{u}$ is pointwise the least section among $u \in C(Q, V^Q)$ satisfying the inclusion $\mathscr{U}(q) \subset u(q){\downarrow}$ for all $q \in D$. If $\mathscr{U} \subset C(Q, V^Q)$ then* $[\![v \in \overline{u}]\!] = \bigvee_{u\in\mathscr{U}} [\![v = u]\!]$ *for all* $v \in C(Q, V^Q)$.

$\lhd$ (1)→(2): Put $X = \bigcup_{u\in\mathscr{U}} \operatorname{im} u$. Then $\llcorner \overline{u} \lrcorner = \operatorname{cl} X$ and, therefore, $[\![v \in \overline{u}]\!] = v^{-1}(\llcorner u \lrcorner) = v^{-1}(\operatorname{cl} X) = \operatorname{cl} v^{-1}(X)$ for all $v \in C(Q, V^Q)$. It is easy to verify the relation $X = \bigcup_{q\in D} \mathscr{U}(q)$ and establish equivalence of the containments $v(q) \in \mathscr{U}(q)$ and $q \in v^{-1}\big(\bigcup_{q\in D} \mathscr{U}(q)\big)$.

(2)→(3): It suffices to show that $\{q \in D : v(q) \in \mathscr{U}(q)\} = \bigcup_{u\in\mathscr{U}} \{v = u\}$ for all $v \in C(Q, V^Q)$. Take an arbitrary point $q \in D$.

If $v(q) \in \mathscr{U}(q)$ then, for some element $u \in \mathscr{U}$, we have $v(q) = u(q)$ and, consequently, $q \in \{v = u\}$.

If $q \in \bigcup_{u\in\mathscr{U}}\{v = u\}$ then, for a suitable $u \in \mathscr{U}$, we have $q \in \{v = u\}$ and, hence, $v(q) = u(q) \in \mathscr{U}(q)$.

(3)→(4): Consider an arbitrary element $v \in C(D, V^Q)$ and define a section $\overline{v} \in C(Q, V^Q)$ as follows:

$$\overline{v}(q) = \begin{cases} v(q) & \text{if } q \in D, \\ \varnothing^\wedge(q) & \text{if } q \notin D. \end{cases}$$

Suppose that $v \in \overline{u}{\downarrow}$. Then

$$D = \{v \in \overline{u}\} \subset [\![\overline{v} \in \overline{u}]\!] = \operatorname{cl} \bigcup_{u\in\mathscr{U}} \{\overline{v} = u\} \subset D.$$

For all $u \in \mathscr{U}$, the set $\{\overline{v} = u\} = u^{-1}(\operatorname{im}\overline{v})$ is clopen. According to the exhaustion principle, there is an antichain $(D_u)_{u\in\mathscr{U}}$ in the algebra Clop(Q) such that $D_u \subset \{\overline{v} = u\}$ and $\bigvee_{u\in\mathscr{U}} D_u = \operatorname{cl}\bigcup_{u\in\mathscr{U}}\{\overline{v} = u\} = D$. Obviously, the section $w = \bigcup_{u\in\mathscr{U}} u|_{D_u}$ is continuous, the set $\operatorname{dom} w$ is open, $D = \operatorname{cl}\operatorname{dom} w$, and $\{w = v\} = \{w = \overline{v}\} = \operatorname{dom} w$. It is clear that $\operatorname{ext}(w) \in C(D, V^Q)$ and $\{\operatorname{ext}(w) = v\} = D$. Therefore, $\operatorname{ext}(w) = v$ and, thus, the inclusion $\subset$ holds.

We now establish the reverse inclusion. Let $(D_u)_{u\in\mathscr{U}}$ be a partition of unity in the algebra Clop(D) and let $v = \operatorname{ext}\big(\bigcup_{u\in\mathscr{U}} u|_{D_u}\big)$. Show that $v \in \overline{u}{\downarrow}$. Since $\operatorname{dom} v = D$, it suffices to establish the inclusion $\operatorname{im} v \subset \llcorner\overline{u}\lrcorner$. Obviously, $u(D_u) \subset \llcorner\overline{u}\lrcorner$ for all $u \in \mathscr{U}$ and, consequently, $\bigcup_{u\in\mathscr{U}} u(D_u) \subset \llcorner\overline{u}\lrcorner$. Observe that $\operatorname{im} v = \operatorname{cl}\bigcup_{u\in\mathscr{U}} u(D_u)$ and, hence, $\operatorname{im} v \subset \llcorner\overline{u}\lrcorner$.

(4)→(5): Put $X = \operatorname{cl}\bigcup_{u\in\mathscr{U}} \operatorname{im} u$. Let $(D_u)_{u\in\mathscr{U}}$ be a partition of unity in the algebra Clop(D) and let $v = \operatorname{ext}(\bigcup_{u\in\mathscr{U}} u|_{D_u})$. Obviously, $\operatorname{dom} v = D$. Show that $\operatorname{im} v \subset X$. The inclusion $u(D_u) \subset X$ implies $\bigcup_{u\in\mathscr{U}} u(D_u) \subset X$; whence, in view of the equality $\operatorname{im} v = \operatorname{cl}\bigcup_{u\in\mathscr{U}} u(D_u)$, the desired relation $\operatorname{im} v \subset X$ follows. Thus, $\overline{u}{\downarrow} \subset C(D, X)$.

For the reverse inclusion, consider an arbitrary section $v \in C(D, X)$ and prove that $v = \operatorname{ext}\big(\bigcup_{u\in\mathscr{U}} u|_{D_u}\big)$ for some partition of unity $(D_u)_{u\in\mathscr{U}}$ in the algebra Clop(D). Obviously, $v^{-1}(X) = D$. Since the section v is open, we have $D = \operatorname{cl} v^{-1}\big(\bigcup_{u\in\mathscr{U}} \operatorname{im} u\big)$. In addition, the set $A = v^{-1}\big(\bigcup_{u\in\mathscr{U}} \operatorname{im} u\big)$ is open and dense in D.

With each element $u \in \mathscr{U}$ we associate a clopen set $C_u = \{v = u\} = v^{-1}(\operatorname{im} u)$. The obvious equality $A = \bigcup_{u\in\mathscr{U}} C_u$ implies that $\bigvee_{u\in\mathscr{U}} C_u = D$. In view of the exhaustion principle, there is a partition of unity $(D_u)_{u\in\mathscr{U}}$ in the algebra Clop(D) such that $D_u \subset C_u$ for all $u \in \mathscr{U}$. Put $w = \bigcup_{u\in\mathscr{U}} u|_{D_u}$. It is clear that, for each $u \in \mathscr{U}$, the equalities $w|_{D_u} = u|_{D_u} = v|_{D_u}$ hold, since $D_u \subset \{v = u\}$. Consequently, by the extension principle, $\operatorname{ext}(w) = v$, which proves the desired inclusion.

(5)→(1): It is sufficient to observe that $D = \operatorname{pr}\big(\operatorname{cl}\bigcup_{u\in\mathscr{U}} \operatorname{im} u\big)$ and use Proposition 2.3.6 (3).

Equivalence of (1) and (6) is evident. ▷

Obviously, the section $\overline{u}$ of the proposition is unique. We call $\overline{u}$ the *ascent* of the set $\mathscr{U}$ and denote it by $\mathscr{U}\!\uparrow$. In case $\mathscr{U}$ is a nonempty subset of $C(Q, V^Q)$, the notion of the ascent of $\mathscr{U}$ coincides with the eponymized notion of 2.1.8.

2.4. Functional Representation

Throughout the section, we assume that Q is an extremally disconnected Hausdorff compact space and $\mathfrak{U}$ is a Boolean valued universe over $\operatorname{Clop}(Q)$.

2.4.1. For the further considerations we need the notion of the quotient class $X/{\sim}$ where X is a class (that need not be a set) and $\sim$ is an equivalence relation on X. The traditional definition of quotient class, for the case in which X is a set, cannot be always applied to the case of a class, since the elements of X equivalent to a given $x \in X$ form a class that need not be a set. We can overcome this difficulty with the help of the following fact:

Theorem (Frege–Russell–Scott). *To every equivalence $\sim$ on a class X, there is a function $F : X \to \mathbf{V}$ such that*

$$F(x) = F(y) \leftrightarrow x \sim y \quad \textit{for all } x, y \in X. \tag{**}$$

As F we can take the function

$$F(x) = \big\{y \in X : y \sim x \ \&\ (\forall z \in X)\big(z \sim x \to \operatorname{rank}(y) \leqslant \operatorname{rank}(z)\big)\big\}.$$

This function F is conventionally called the *canonical projection* of $\sim$. In view of (**) we may regard $F(x)$ as an analog of the coset containing an element $x \in X$. In this connection, we denote $F(x)$ by ${\sim}(x)$.

2.4.2. For each point $q \in Q$, introduce the equivalence relation $\sim_q$ on the class $\mathfrak{U}$ as follows:

$$u \sim_q v \ \leftrightarrow\ q \in [\![u = v]\!].$$

Consider the bundle

$$V^Q = \big\{\big(q, {\sim_q}(u)\big) : q \in Q,\ u \in \mathfrak{U}\big\}$$

and make the convention to denote a pair $\big(q, {\sim_q}(u)\big)$ by $\widehat{u}(q)$. Obviously, for every element $u \in \mathfrak{U}$, the mapping

$$\widehat{u} : q \mapsto \widehat{u}(q)$$

is a section of the bundle V^Q. Note that, for each $x \in V^Q$, there exist $u \in \mathfrak{U}$ and $q \in Q$ such that $\widehat{u}(q) = x$. In addition, the equality $\widehat{u}(q) = \widehat{v}(q)$ holds if and only if $q \in [\![u = v]\!]$.

Make each stalk V^q of the bundle V^Q into an algebraic system of signature $\{\in\}$ by letting

$$V^q \models x \in y \;\leftrightarrow\; q \in [\![u \in v]\!],$$

where the elements $u, v \in \mathfrak{U}$ are such that $\widehat{u}(q) = x$ and $\widehat{v}(q) = y$. It is easy to verify that the above definition is sound. Indeed, if $\widehat{u}_1(q) = x$ and $\widehat{v}_1(q) = y$ for another pair u_1, v_1, then the containments $q \in [\![u \in v]\!]$ and $q \in [\![u_1 \in v_1]\!]$ are equivalent.

It is easily seen that the class $\{\widehat{u}(A) : u \in \mathfrak{U},\ A \sqsubset Q\}$ is a base for some open topology on V^Q, which allows us to regard V^Q as a continuous bundle.

2.4.3. *Theorem.* *The following hold:*

(1) *The bundle V^Q is a continuous polyverse.*

(2) *The mapping $u \mapsto \widehat{u}$ is an isomorphism between the Boolean valued universes $\mathfrak{U}$ and $C(Q, V^Q)$.*

We divide the proof of the last theorem into several steps.

2.4.4. Lemma. *If $u \in \mathfrak{U}$ and $A \sqsubset Q$ then $\widehat{u}(A) \sqsubset V^Q$.*

$\lhd$ For every element $x \in V^Q \setminus \widehat{u}(A)$, there exist $v \in \mathfrak{U}$ and $q \in Q$ such that $x = \widehat{v}(q)$.

If $q \in A$ then

$$\widehat{u}(q) \neq x = \widehat{v}(q), \quad q \in [\![u \neq v]\!],$$

and, thus, the set $\widehat{v}([\![u \neq v]\!])$ is a neighborhood about x disjoint from $\widehat{u}(A)$. If, otherwise, $q \notin A$, then the neighborhood $\widehat{v}(Q \setminus A)$ about x is disjoint from $\widehat{u}(A)$. $\rhd$

2.4.5. Lemma. *The classes $\{\widehat{u} : u \in \mathfrak{U}\}$ and $C(Q, V^Q)$ coincide.*

$\lhd$ Consider an arbitrary element $u \in \mathfrak{U}$ and show that the section $\widehat{u}$ is continuous. If $v \in \mathfrak{U}$ and $A \sqsubset Q$ then the set $\widehat{u}^{-1}\big(\widehat{v}(A)\big) = A \cap [\![u = v]\!]$ is open. Arbitrariness of v and A allows us to conclude that $\widehat{u} \in C(Q, V^Q)$.

We now establish the reverse inclusion. Take an $f \in C(Q, V^Q)$. For each point $q \in Q$, choose an element $u_q \in \mathfrak{U}$ such that $\widehat{u}_q(q) = f(q)$ and assign

$$A_q := \{p \in Q : \widehat{u}_q(p) = f(p)\} = f^{-1}\big(\widehat{u}(Q)\big) \sqsubset Q.$$

Thus, $(A_q)_{q \in Q}$ is an open cover of the compact space Q from which we can refine a subcover $(A_q)_{q \in F}$, where F is a finite subset of Q. By the exhaustion principle, there is an antichain $(B_q)_{q \in F}$ such that $B_q \subset A_q$ for all $q \in B$ and $\bigcup_{q \in F} B_q = Q$.

Since the Boolean valued algebraic system $\mathfrak{U}$ satisfies the mixing principle, we may consider

$$u = \operatorname*{mix}_{q\in F} B_q u_q \in \mathfrak{U}.$$

It is easy to become convinced that $\widehat{u} = f$. ▷

2.4.6. Lemma. *The topology of V^Q is extremally disconnected.*

◁ The claim follows from Lemmas 2.4.4 and 2.4.5 and Proposition 2.2.5. ▷

2.4.7. Lemma. *The mapping $(u \mapsto \widehat{u}) : \mathfrak{U} \to C(Q, V^Q)$ is bijective and, for all $u, v \in \mathfrak{U}$, the following hold:*

$$[\![u = v]\!]_{\mathfrak{U}} = [\![\widehat{u} = \widehat{v}]\!]_{C(Q,V^Q)},$$
$$[\![u \in v]\!]_{\mathfrak{U}} = [\![\widehat{u} \in \widehat{v}]\!]_{C(Q,V^Q)}.$$

◁ It is easily seen that, for all $u, v \in \mathfrak{U}$ and $q \in Q$, we have:

$$V^q \models \widehat{u}(q) \in \widehat{v}(q) \ \leftrightarrow\ q \in [\![u \in v]\!],$$
$$V^q \models \widehat{u}(q) = \widehat{v}(q) \ \leftrightarrow\ q \in [\![u = v]\!].$$

The desired equalities are thus established. In Lemma 2.4.6, it is shown that the mapping $u \mapsto \widehat{u}$ is surjective. We are left with proving its injectivity. Let elements $u, v \in \mathfrak{U}$ be such that $\widehat{u} = \widehat{v}$. Then $[\![u = v]\!] = [\![\widehat{u} = \widehat{v}]\!] = Q$, which implies the equality $u = v$ due to the fact that the system $\mathfrak{U}$ is separated. ▷

Thus, the triple

$$\big(C(Q, V^Q),\ [\![\cdot = \cdot]\!],\ [\![\cdot \in \cdot]\!]\big)$$

is a Boolean valued algebraic system over $\operatorname{Clop}(Q)$ isomorphic to $\mathfrak{U}$ and, hence, $C(Q, V^Q)$ is a Boolean valued universe over $\operatorname{Clop}(Q)$.

2.4.8. Lemma. *If $u \in C(Q, V^Q)$ then $\llcorner u \lrcorner$ is a clopen subset of V^Q.*

◁ Take a $u \in C(Q, V^Q)$. Since $C(Q, V^Q)$ satisfies the ascent principle, $u = \operatorname{asc}_{\xi\in\Xi} B_\xi u_\xi$ for some family $(u_\xi)_{\xi\in\Xi}$ of continuous sections of V^Q and a family $(B_\xi)_{\xi\in\Xi}$ of clopen subsets of Q. For each $v \in C(Q, V^Q)$, the following relations hold:

$$v^{-1}\Big(\operatorname{cl}\bigcup_{\xi\in\Xi} u_\xi(B_\xi)\Big) = \operatorname{cl}\bigcup_{\xi\in\Xi} v^{-1}\big(u_\xi(B_\xi)\big) = \operatorname{cl}\bigcup_{\xi\in\Xi} B_\xi \cap [\![v = u_\xi]\!]$$
$$= \bigvee_{\xi\in\Xi} B_\xi \wedge [\![v = u_\xi]\!] = [\![v \in u]\!] = v^{-1}(\llcorner u \lrcorner).$$

Thus, in view of Lemma 2.2.7, the equality

$$\llcorner u \lrcorner = \operatorname{cl} \bigcup_{\xi \in \Xi} u_\xi(B_\xi)$$

is established. The set

$$\bigcup_{\xi \in \Xi} u_\xi(B_\xi)$$

is open; therefore, by Lemma 2.4.6, the class $\llcorner u \lrcorner$ is a clopen set. $\rhd$

2.4.9. Lemma. *For every subset $X \sqsubset V^Q$, there exists a section $u \in C(Q, V^Q)$ such that $\llcorner u \lrcorner = X$.*

$\lhd$ With each element $x \in X$ we associate a section $u_x \in C(Q, V^Q)$ such that $x \in \operatorname{im} u_x$. Obviously, the set $B_x = u_x^{-1}(X)$ is clopen. Consider the ascent

$$u = \operatorname*{asc}_{x \in X} B_x u_x$$

and establish the equality $\llcorner u \lrcorner = X$. Since $x \in u_x(B_x) \subset X$ for all $x \in X$, we have

$$X = \bigcup_{x \in X} u_x(B_x) = \operatorname{cl} \bigcup_{x \in X} u_x(B_x).$$

For an arbitrary section $v \in C(Q, V^Q)$, the following relations hold:

$$v^{-1}(X) = \bigcup_{x \in X} v^{-1}\big(u_x(B_x)\big) = \operatorname{cl} \bigvee_{x \in X} B_x \wedge [\![v = u_x]\!] = [\![v \in u]\!] = v^{-1}(\llcorner u \lrcorner).$$

In view of Lemma 2.2.7, the desired equality is established. $\rhd$

2.4.10. Lemma. *For every formula $\varphi(t_1, \dots, t_n)$ and arbitrary sections $u_1, \dots, u_n \in C(Q, V^Q)$, the following holds:*

$$[\![\varphi(u_1, \dots, u_n)]\!] = \big\{q \in Q : V^q \models \varphi\big(u_1(q), \dots, u_n(q)\big)\big\}.$$

$\lhd$ The proof of the lemma repeats that of the Pointwise Truth Value Theorem (see 2.3.10). $\rhd$

The last lemma implies in particular that the axioms of extensionality and regularity are true in each stalk. Thus, Theorem 2.4.3 is completely proven.

In conclusion, we state a theorem that combines the basic results of Sections 2.3 and 2.4.

Theorem. *Let Q be the Stone space of a complete Boolean algebra B.*

(1) *The class $C(Q, V^Q)$ of continuous sections of a polyverse V^Q on Q is a Boolean valued universe.*

(2) *For an arbitrary Boolean valued universe $\mathfrak{U}$ over B, there exists a continuous polyverse V^Q on Q such that $C(Q, V^Q)$ is isomorphic to $\mathfrak{U}$.*

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