CHAPTER 3

Dual Banach Bundles

BY

A. E. Gutman and A. V. Koptev

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Bundles are traditionally employed for studying various algebraic systems in mathematical analysis. The technique of bundles is used in examining Banach spaces, Riesz spaces, C^*-algebras, Banach modules, etc. (see, for instance, [3, 6, 7, 13–15]). Representation of some objects of functional analysis as spaces of sections of corresponding bundles serves as a basis for some theories valuable in their own right. One of these theories in [8–12] is devoted to the notion of a continuous Banach bundle (CBB) and its applications to lattice normed spaces (LNSs). Within this theory, in particular, a representation is obtained for an arbitrary LNS as a space of sections of a suitable CBB.

In some sense, a CBB over a topological space Q formally reflects the intuitive notion of a family of Banach spaces $(X_q)_{q\in Q}$ varying continuously from point to point in the space Q. To be more precise, a Banach bundle $\mathscr{X}$ over Q is a mapping associating with each point $q \in Q$ a Banach space $\mathscr{X}(q)$ the so-called stalk of $\mathscr{X}$ at q. Furthermore, the bundle $\mathscr{X}$ is endowed with some structure that allows us to speak about continuity of sections of the bundle (a section is a function u defined on a subset of Q and taking values $u(q) \in \mathscr{X}(q)$ for all $q \in \operatorname{dom} u$). The notion of a section can be regarded as a generalization of the notion of a vector valued function: if X is a Banach space then X-valued functions are sections of the Banach bundle whose stalks are all equal to X.

In many questions of analysis, an essential role is played by duality theory, one of whose basic tools is the concept of a dual space (see, for instance, [17]). Existence of a functional representation for the initial space by means of sections of some bundle allows us to construct an analogous representation for the dual space. In particular, the problem of representing a dual LNS leads to the notion of a dual Banach bundle.

Which CBB $\mathscr{X}'$ should be considered dual to a given bundle $\mathscr{X}$ (discussed, for instance, in [7–9, 12, 19]) is a question closely connected with the notion of a homomorphism. A homomorphism v of a continuous Banach bundle $\mathscr{X}$ over Q is a functional valued mapping $v : q \mapsto v(q) \in \mathscr{X}(q)'$ taking every continuous section u of the bundle $\mathscr{X}$ into the continuous real-valued function $\langle u|v\rangle : q \mapsto \langle u(q)|v(q)\rangle$. When we try to define a dual CBB $\mathscr{X}'$, the following two requirements are worth to be imposed: first, homomorphisms should be continuous sections of the bundle $\mathscr{X}'$ and, second, all continuous sections of $\mathscr{X}'$ should be homomorphisms.

In the case of ample bundles over extremally disconnected compact spaces, the problem of defining a dual CBB is solved in [8] (see also [12]). However, the approach to the definition of a dual bundle presented in that article rests essentially on the specific properties of ample bundles and extremally disconnected compact spaces and, thus, cannot be extended to a wider class of bundles.

The natural intention to extend the domain of application for duality theory leads to the problem of constructing a dual CBB for an arbitrary Banach bundle over an arbitrary topological space. The study of this problem is the main topic of the present chapter, where, in particular, a definition of a dual bundle is presented, with the above-formulated requirements fulfilled, and a number of necessary and sufficient conditions is suggested for existence of a dual bundle.

In Section 3.1, auxiliary results are collected on topological spaces, Banach spaces, and functions acting in them.

Section 3.2 is devoted to studying the notion of a homomorphism of a Banach bundle. In particular, description of homomorphisms is suggested therein for a wide class of bundles and the question is examined of continuity of the pointwise norm of a homomorphism.

The question about the possibility of representing the space of all homomorphisms from a CBB $\mathscr{X}$ into a CBB $\mathscr{Y}$ as the space of continuous sections of some Banach bundle leads to the notion of an operator bundle $B(\mathscr{X},\mathscr{Y})$. In Section 3.3, some necessary and sufficient conditions are given for existence of such a bundle.

In Section 3.4, the notion of a dual Banach bundle is introduced and studied. This bundle is a particular case of an operator bundle (considered in the previous section). The definition of a dual bundle therein generalizes that of [8, 12] where the case is considered of an ample bundle over an extremally disconnected compact space. In the same articles it is established in particular that every ample CBB has the dual bundle. In the general case, dual bundles may fail to exist. Nevertheless, the above generalization is justified by the fact that new classes arise of CBBs that have dual bundles. In Section 3.4, various necessary and sufficient conditions are presented for existence of a dual bundle, the norming duality relations are established between the bundles $\mathscr{X}$ and $\mathscr{X}'$, and the questions are studied of existence of the second dual bundle and embedding of a bundle into its second dual.

In examining the notion of a dual bundle, one of the natural steps is consideration of weakly continuous sections (these are sections continuous with respect to the duality between a bundle and its dual). The notion of a weakly continuous section is introduced and studied in Section 3.5. In particular, the question is discussed about continuity of weakly continuous sections for various classes of Banach bundles and conditions are suggested for coincidence of the space of weakly continuous sections of a trivial CBB and the space of weakly continuous vector valued functions acting into the corresponding stalk.

When speaking about Banach bundles, we use the terminology and notation of [8] (see also [12]). In particular, we distinguish the notion of a Banach bundle and that of a continuous Banach bundle and employ the approach to the definition of continuity for sections by means of the notion of a continuity structure. All necessary information on the theory of Banach bundles can be found in [3, 7–12].

If $\mathscr{X}$ and $\mathscr{Y}$ are some CBBs over a topological space Q then we denote by $\mathrm{Hom}(\mathscr{X},\mathscr{Y})$ the set of all Q-homomorphisms from $\mathscr{X}$ into $\mathscr{Y}$ (which is denoted by

$\mathrm{Hom}_Q(\mathscr{X}, \mathscr{Y})$ in [8, 12]). As usual, the symbol $\mathrm{Hom}_D(\mathscr{X}, \mathscr{Y})$ is used for denoting the set of D-homomorphisms from $\mathscr{X}|_D$ into $\mathscr{Y}|_D$, where $D \subset Q$. Instead of "Q-homomorphism" we just say "homomorphism." Analogous convention is effective concerning the terms "Q-isometric embedding" and "Q-isometry."

In contrast to [8, 12], we use the symbol X_Q for denoting the trivial Banach bundle with stalk X over a topological space Q. The symbol $\mathscr{R}$ denotes the trivial CBB with stalk $\mathbb{R}$ over the topological space under consideration.

Let $\mathscr{X}$ be a continuous Banach bundle over a topological space Q, let u be a section of $\mathscr{X}$ defined on an $A \subset Q$, and let v be a section of $\mathscr{X}$ defined on a $B \subset Q$ such that $v(q) \in \mathscr{X}(q)'$ for all $q \in B$. The symbol $\langle u|v\rangle$ denotes the function acting from $A \cap B$ into $\mathbb{R}$ by the rule $\langle u|v\rangle(q) = \langle u(q)|v(q)\rangle$.

All vector spaces under consideration are assumed over $\mathbb{R}$, the field of reals.

3.1. Auxiliary Results

This section contains facts to be used in the sequel about topological and Banach spaces as well as functions acting in this spaces. The collected results are auxiliary and do not involve the notion of a Banach bundle.

3.1.1. *Lemma.* *Let X be a normed space and let x and y be norm-one vectors in X. Then either of the intervals $[x, y]$ or $[x, -y]$ does not intersect the open ball with radius $1/2$ centered at the origin, i.e..*

$$\inf_{\lambda\in[0,1]} \|\lambda x + (1-\lambda)y\| \geqslant 1/2 \quad \text{or} \quad \inf_{\lambda\in[0,1]} \|\lambda x + (1-\lambda)(-y)\| \geqslant 1/2.$$

$\triangleleft$ Assume that there are vectors $u = \lambda x + (1-\lambda)(-y)$ and $v = \mu x + (1-\mu)y$ such that $\|u\| < 1/2$ and $\|v\| < 1/2$. Obviously, $0 < \lambda, \mu < 1$ and $x \neq \pm y$. Moreover, the vectors u and v are linearly independent. Hence, $x = \alpha u + \beta v$ and $y = \gamma u + \delta v$ for some $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. Linear independence of (u, v) and (x, y), together with the equalities

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} \lambda & \lambda - 1 \\ \mu & 1-\mu \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix},$$

implies that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \lambda & \lambda - 1 \\ \mu & 1-\mu \end{pmatrix}^{-1},$$

i.e.,

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \frac{1}{\lambda + \mu - 2\lambda\mu} \begin{pmatrix} 1-\mu & 1-\lambda \\ -\mu & \lambda \end{pmatrix}.$$

The relations

$$1 = \|x\| \leqslant |\alpha|\|u\| + |\beta|\|v\| < \frac{|\alpha| + |\beta|}{2}$$

and

$$1 = \|y\| \leqslant |\gamma|\|u\| + |\delta|\|v\| < \frac{|\gamma| + |\delta|}{2}$$

allow us to conclude that

$$\frac{1}{|\alpha| + |\beta|} + \frac{1}{|\gamma| + |\delta|} < 1,$$

i.e., $|\alpha| + |\beta| + |\gamma| + |\delta| < (|\alpha| + |\beta|)(|\gamma| + |\delta|)$. It is easy to see that $\lambda + \mu - 2\lambda\mu > \lambda^2 + \mu^2 - 2\lambda\mu \geqslant 0$. Furthermore, $|\alpha| + |\beta| = (2 - \lambda - \mu)/(\lambda + \mu - 2\lambda\mu)$ and $|\gamma| + |\delta| = (\lambda + \mu)/(\lambda + \mu - 2\lambda\mu)$, whence

$$\frac{2}{\lambda + \mu - 2\lambda\mu} < \frac{2 - \lambda - \mu}{\lambda + \mu - 2\lambda\mu} \frac{\lambda + \mu}{\lambda + \mu - 2\lambda\mu}.$$

Consequently, $2(\lambda + \mu - 2\lambda\mu) < 2(\lambda + \mu) - (\lambda + \mu)^2$ and, finally, $(\lambda - \mu)^2 < 0$. This contradiction completes the proof. ▷

3.1.2. The following statement may be found, for instance, in [21, Proposition 1 (SP1)].

Lemma. *If a Banach space X possesses the Schur property then every weakly Cauchy sequence in X is norm convergent.*

◁ Consider a norm divergent sequence $(x_n) \subset X$ and show that it is not a weakly Cauchy sequence. There exist a number $\varepsilon > 0$ and a strictly increasing sequence $(n_k) \subset \mathbb{N}$ such that $\|x_{n_k} - x_{n_{k+1}}\| > \varepsilon$ for all odd $k \in \mathbb{N}$. Since the sequence $(x_{n_k} - x_{n_{k+1}})$ does not vanish in norm and X possesses the Schur property, there is a functional $x' \in X'$ such that the numerical sequence $\langle x_{n_k} - x_{n_{k+1}} \mid x' \rangle$ does not vanish. Consequently, the subsequence (x_{n_k}), together with the initial sequence (x_n), is not a weakly Cauchy sequence. ▷

3.1.3. ***Lemma.*** *Let X be an infinite-dimensional separable Banach space. Then every infinite-dimensional Banach subspace of X' includes a weakly* null sequence of norm-one functionals.*

◁ Let Y be an infinite-dimensional Banach subspace of X'. Consider a sequence (y_n) of norm-one vectors in Y such that $\|y_i - y_j\| \geqslant 1/2$ whenever $i \neq j$ (see, for instance, [18, 8.4.2]). By [4, XIII], from (y_n) we can extract a subsequence (y_{n_m}) convergent weakly* to an element $y \in X'$. It is clear that $y \in Y$. For every $m \in \mathbb{N}$, put $z_m := y_{n_m} - y$. Let $\varepsilon > 0$ and let (z_{m_k}) be a subsequence of (z_m) such that $\|z_{m_k}\| > \varepsilon$ for all $k \in \mathbb{N}$. Then $(z_{m_k} / \|z_{m_k}\|)$ is a sought sequence. ▷

3.1.4. Lemma. *Let X be an infinite-dimensional Banach space. Then there exist a weakly vanishing net $(x_\alpha)_{\alpha\in\aleph} \subset X$ and a norm vanishing net $(x'_\alpha)_{\alpha\in\aleph} \subset X'$ such that $\langle x_\alpha | x'_\alpha\rangle = 1$ for all $\alpha \in \aleph$.*

$\lhd$ As $\aleph$ we consider the set of all finite subsets of X' ordered by inclusion.

Fix an $\alpha = \{x'_1, \dots, x'_n\} \in \aleph$ and, employing the fact that X is infinite-dimensional, take an element $x_\alpha \in \bigcap_{i=1}^n \ker x'_i$ with norm $\|x_\alpha\| = n$. Next, choose a functional $x'_\alpha \in X'$ satisfying the equalities $\langle x_\alpha | x'_\alpha\rangle = 1$ and $\|x'_\alpha\| = 1/n$.

Obviously, the net $(x'_\alpha)_{\alpha\in\aleph}$ vanishes in norm. Show that the net $(x_\alpha)_{\alpha\in\aleph}$ is weakly vanishing. Let U be an arbitrary weak neighborhood about zero in X. Choose functionals $x'_1, \dots, x'_n \in X'$ so that $\bigcap_{i=1}^n \ker x'_i \subset U$. Then $x_\alpha \in \bigcap_{i=1}^n \ker x'_i \subset U$ for all $\alpha \in \aleph$, $\alpha \geqslant \{x'_1, \dots, x'_n\}$. $\rhd$

3.1.5. Let (x_n) be a sequence in a Banach space X.

Lemma. *The following are equivalent:*

(a) *for every sequence $(x'_n) \subset X'$ and every element $x' \in X'$, weak* convergence $x'_n \to x'$ implies $\langle x_n | x'_n\rangle \to 0$;*

(b) *for every sequence $(x'_m) \subset X'$ and every element $x' \in X'$, weak* convergence $x'_m \to x'$ implies $\langle x_n | x'_m\rangle \to 0$ as $n, m \to \infty$;*

(c) *(x_n) is weakly null and $\langle x_n | x'_n\rangle \to 0$ for every weakly* null sequence $(x'_n) \subset X'$;*

(d) *(x_n) is weakly null and $\langle x_n | x'_m\rangle \to 0$ as $n, m \to \infty$ for every weakly* null sequence $(x'_m) \subset X'$;*

(e) *$\sup_{m\in\mathbb{N}} |\langle x_n | x'_m\rangle| \to 0$ as $n \to \infty$ for every weakly* null sequence $(x'_m) \subset X'$;*

(f) *for every operator $T \in B(X, c_0)$, the sequence (Tx_n) vanishes in norm.*

The proof of equivalence of the above assertions is a routine and quite simple exercise.

DEFINITION. Say that a sequence is *w-w^*-vanishing* if (x_n) satisfies one of the conditions (a)–(f) of the above lemma. If $x \in X$ and the sequence $(x_n - x)$ is w-w^*-vanishing then we say that (x_n) *w-w^*-converges* to x.

A Banach space X is said to possesses the WS *property* (or the *weak Schur property*) if every w-w^*-convergent sequence in X converges in norm (or, which is the same, every w-w^*-vanishing sequence vanishes in norm).

We list some evident facts concerning the above notions.

Proposition. *The following are true:*

(1) *Each norm convergent sequence is w-w^*-convergent.*

(2) *Every subsequence of a w-w*-convergent sequence is also w-w*-convergent.*

(3) *If X and Y are Banach spaces, $T \in B(X,Y)$, and a sequence $(x_n) \subset X$ is w-w*-convergent to an $x \in X$, then the sequence (Tx_n) is w-w*-convergent to Tx.*

(4) *If a Banach space possesses the* WS *property then this property is also enjoyed by every Banach subspace.*

(5) *If a Banach space X possesses the* WS *property then this property is also enjoyed by every Banach space isomorphic to X.*

(6) *If a Banach space contains a copy of a space which does not possess the* WS *property, then the space does not possess the* WS *property either.*

3.1.6. *Lemma.* *If a Banach space X has weakly* sequentially compact dual ball then X possesses the* WS *property. The converse fails to be true.*

$\triangleleft$ Suppose that X does not possess the WS property. Then there exists a w-w^*-vanishing sequence $(x_n) \subset X$ which does not vanish in norm. Without loss of generality, we may assume that $\|x_n\| > \varepsilon$ for all $n \in \mathbb{N}$ and a suitable $\varepsilon > 0$. Since X has weakly* sequentially compact dual ball, from a sequence of functionals $(x'_n) \subset X'$ satisfying the conditions $\|x'_n\| = 1$ and $\langle x_n | x'_n \rangle > \varepsilon$ for all $n \in \mathbb{N}$ we can extract a weakly* convergent subsequence x'_{n_k}. However, $\langle x_{n_k} | x'_{n_k} \rangle > \varepsilon$, which contradicts the fact that (x_{n_k}) is w-w^*-vanishing.

The space $\ell^1(\mathbb{R})$ can be considered as a counterexample to the converse assertion. Indeed, this space possesses the Schur property and, therefore, the WS property. On the other hand, as is shown in [4, XIII], the dual ball of the space $\ell^1(\mathbb{R})$ is not weakly* sequentially compact. $\triangleright$

Each of the following properties of a Banach space X implies the WS property:

(1) X possesses the Schur property;
(2) X is separable;
(3) X' does not contain a copy of ℓ^1;
(4) X is reflexive;
(5) X is a subspace of a weakly compactly generated Banach space;
(6) for every separable subspace Y of X, the space Y' is separable.

Property (1) obviously implies the WS property, and the other properties guarantee that X has weakly* sequentially compact closed dual ball (see [4, XIII]), which allows us to apply the last lemma. Recall that a Banach space Y is said to be *weakly compactly generated* if Y contains a weakly compact absolutely convex set whose linear span is dense in Y.

3.1.7. A Banach space X is said to possess the *Dunford–Pettis property* if

$$\langle x_n | x'_n \rangle \to 0 \text{ for all weakly null sequences } (x_n) \subset X \text{ and } (x'_n) \subset X'.$$

In Section 3.5, within the study of weakly continuous sections of Banach bundles, the important role is clarified of the question whether a Banach space under consideration possesses the following property close to the Dunford–Pettis property.

DEFINITION. Say that a Banach space X possesses the DP* *property* if

$$\langle x_n | x'_n \rangle \to 0 \quad \begin{array}{l}\text{for every weakly null sequence } (x_n) \subset X \\ \text{and every weakly}^* \text{ null sequence } (x'_n) \subset X'.\end{array}$$

(Note that there is no reason to consider the analog of the DP* property for nets, since, in view of Lemma 3.1.4, only finite-dimensional spaces possess such a property.)

It is clear that X possesses the DP* property if and only if the sets of weakly convergent and w-w^*-convergent sequences in X coincide.

A Banach space X with the property that weakly* null sequences in X' are weakly null is called a *Grothendieck space* (see [4, VII, p. 121]). Obviously, every reflexive Banach space is a Grothendieck space.

The following assertions are easy to verify.

Lemma. *Let X be a Banach space.*

(1) *If X possesses the Schur property then X possesses the* DP* *property.*

(2) *If X possesses the* DP* *property then X possesses the Dunford–Pettis property.*

(3) *The space X possesses the* WS *and* DP* *properties if and only if X possesses the Schur property.*

(4) *For a Grothendieck space, the* DP* *property is equivalent to the Dunford–Pettis property.*

It is worth noting that assertion (2) does not admit conversion. Indeed, the space c_0 does not possess the Schur property and possesses the WS property, since c_0 is separable; therefore, by (3), c_0 does not possess the DP* property. At the same time, c_0 enjoys the Dunford–Pettis property, since $c'_0 \simeq \ell^1$ possesses the Schur property.

Recall that the intersection (union) of countably many open (closed) subsets of a topological space is called a *σ-open* (*σ-closed*) set.

Let K be a quasiextremally disconnected compact Hausdorff space (i.e. a compact Hausdorff space in which the closure of every open σ-closed subset is open). The spaces ℓ^∞ and $C(K)$ are Grothendieck spaces enjoying the Dunford–Pettis property and not the Schur property (see, for instance, [4, VII, Theorem 15, Exercise 1 (ii), XI, Exercise 4 (ii)], [1, Theorem 13.13], and [20, Theorem V.2.1]).

Corollary. *Let K be a quasiextremally disconnected compact Hausdorff space.*

(1) *The Banach spaces ℓ^∞ and $C(K)$ possess the* DP* *property.*
(2) *Every Banach space containing a copy of ℓ^∞ does not possess the* WS *property.*

◁ The claim follows immediately from the above-indicated properties of ℓ^∞ and $C(K)$, assertions (4) and (3) of the last lemma, and Proposition 3.1.5 (6). ▷

3.1.8. *Lemma.* *Given an arbitrary topological space Q, the following are equivalent:*

(a) *all functions in $C(Q)$ are locally constant;*
(b) *for every sequence of functions $(f_n) \subset C(Q)$ and every point $q \in Q$, there exists a neighborhood about q such that all functions f_n, $n \in \mathbb{N}$, are constant on the neighborhood;*
(c) *for every sequence of functions $(f_n) \subset C(Q)$, there is a partition of Q into clopen sets such that all functions f_n, $n \in \mathbb{N}$, are constant on every element of the partition.*

◁ (a)→(b): It is sufficient to find a neighborhood about q on which all functions $g_n = |f_n - f_n(q)| \wedge 1$, $n \in \mathbb{N}$, vanish. Since, the sum $g = \sum_{n=1}^{\infty} g_n/2^n$ is a continuous function and $g(q) = 0$, by (a) there is a neighborhood about q on which $g \equiv 0$. It is clear that all functions g_n, $n \in \mathbb{N}$, vanish too.

(b)→(c): According to (b), for every point $q \in Q$, the intersection $\bigcap_{n\in\mathbb{N}}\{f_n = f_n(q)\}$ of closed sets is a neighborhood about its every point; therefore, this intersection is clopen. All intersections of this kind form a sought partition of Q.

The implication (c)→(a) is evident. ▷

Definition. A topological space Q satisfying one of the equivalent conditions (a)–(c) of Lemma 3.1.8 is called *functionally discrete.*

3.1.9. A point of a topological space is *σ-isolated* or a *P-point* if the intersection of every sequence of neighborhoods about this point is again a neighborhood.

Remark. A Hausdorff topological space containing a single nonisolated point is a normal and Baire space.

Proposition. *Let Q be a completely regular topological space.*

(1) *The following are equivalent:*
  (a) *Q is functionally discrete;*
  (b) *all points in Q are σ-isolated;*
  (c) *every σ-open subset of Q is open;*
  (d) *every σ-closed subset of Q is closed.*

(2) *If Q is functionally discrete then all countable subsets of Q are closed.*

(3) *The converse of* (2) *is false.*

$\lhd$ (1): (a)$\to$(b): Consider an arbitrary point $q \in Q$, a sequence (U_n) of neighborhoods about q, and put $V = \bigcap_{n\in\mathbb{N}} U_n$. Since the space Q is completely regular; for every $n \in \mathbb{N}$, there is a continuous function $f_n : Q \to [0,1]$ such that $f_n(q) = 0$ and $f_n \equiv 1$ on $Q \backslash U_n$. The sum

$$f = \sum_{n=1}^{\infty} f_n/2^n : Q \to [0,1]$$

is a continuous function and, by (a), vanishes on some neighborhood U_0 about q. Since $f > 0$ outside V, the neighborhood U_0 is a subset of V; therefore, V is a neighborhood about q too.

(b)$\to$(c): By (b), the intersection of a sequence of open subsets of Q is a neighborhood about its every point and, hence, is open.

(c)$\to$(a): By (c), for every function $f \in C(Q)$ and a point $q \in Q$, the intersection

$$\bigcap_{n\in\mathbb{N}} \{p \in Q : |f(p) - f(q)| < 1/n\}$$

is a neighborhood about q on which the function f is constant.

Equivalence of the mutually dual assertions (c) and (d) is evident.

(2): It is sufficient to observe that countable subsets of Q are σ-closed and to apply (1).

(3): Construct a completely regular topological space Q whose all countable subsets are closed and choose a function in $C(Q)$ which is not locally constant.

Make the interval $[0,1]$ into a topological space Q by taking as a base for open sets all subsets of $(0,1]$ and all subsets of the form $[0,t] \backslash S$, where $t \in (0,1]$ and S is a countable subset of $(0,1]$. The topological space Q is constructed. It is clear that all countable subsets of Q are closed. Since Q is a Hausdorff space and contains a single nonisolated point, it is normal (see the remark above the proposition); therefore, Q is completely regular. It is easy to see that the identity mapping of $[0,1]$ is continuous and is not constant on every neighborhood about 0. $\rhd$

3.1.10. Recall that a topological space is *countably compact* if from every countable open cover of this space we can refine a finite subcover. A topological space is *perfectly normal* if it is normal and its every closed subset is σ-open.

Proposition. *Let Q be a completely regular topological space. Under each of the following conditions, the space Q includes a nonclosed countable subset (hence, Q is not functionally discrete):*

(1) *Q includes a nondiscrete countable compact subspace;*
(2) *Q includes an infinite compact subspace;*
(3) *Q includes a nondiscrete subspace that is a Fréchet–Urysohn space;*
(4) *Q includes a convergent sequence of pairwise distinct elements;*
(5) *Q contains a nonisolated point at which there is a countable base.*

Furthermore, a perfectly normal topological space is functionally discrete only if it is discrete.

$\triangleleft$ It is known (see, for example, [2, III, assertion 189]) that a topological space is countably compact if and only if its every infinite subset has a limit point. Using this criterion, we easily prove that condition (1) is sufficient for existence of a nonclosed countable subset of Q. Sufficiency of conditions (2), (4), and (5) is easily validated. Condition (3) is equivalent to (4).

For a nondiscrete perfectly normal topological space, existence of a not locally constant function follows from the Vedenisov Theorem (see [5, 1.5.19]). $\triangleright$

3.1.11. If a topological space Q is functionally discrete and completely regular then Q satisfies none of the conditions 3.1.10 (1)–(5). In particular, if Q is nondiscrete then Q cannot be compact, first-countable, or metrizable. These observations essentially restrict the class of topological spaces in which Q may fall. Therefore, it is worth verifying that a completely regular functionally discrete topological space need not be discrete.

First, for an arbitrary upward-directed set $\aleph$ without greatest element, define a nondiscrete normal topological space $\aleph^\bullet$. As the underlying set we take $\overline{\aleph} = \aleph \cup \{\infty\}$, where $\infty \notin \aleph$. Endow $\overline{\aleph}$ with an order, regarding $\aleph$ as an ordered subset of $\overline{\aleph}$ and assuming $\infty > \alpha$ for all $\alpha \in \aleph$. Consider open the subsets of $\aleph$ and all intervals of the form $(\alpha, \infty] := \{\beta \in \overline{\aleph} : \alpha < \beta \leqslant \infty\}$, where $\alpha \in \aleph$ to be open. Therefore, $\aleph^\bullet$ becomes a topological space. Since $\aleph$ has no greatest element, the point $\infty \in \aleph^\bullet$ is nonisolated; hence, the topology of $\aleph^\bullet$ is nondiscrete. The space $\aleph^\bullet$ is normal, since it is Hausdorff and contains a single nonisolated point (see Remark 3.1.9).

Remark. **(1)** If all countable subsets of $\aleph$ have upper bounds, every continuous function $f : \aleph^\bullet \to \mathbb{R}$ takes a constant value $f(\infty)$ on some neighborhood about ∞. (For instance, the intersection

$$\bigcap_{n \in \mathbb{N}} \{\alpha \in \aleph^\bullet : \ |f(\alpha) - f(\infty)| < 1/n\}$$

is such a neighborhood.)

(2) For an arbitrary topological space P, continuity of a function $f : \aleph^\bullet \to P$ is equivalent to the fact that the net $\big(f(\alpha)\big)_{\alpha\in\aleph}$ converges to $f(\infty)$.

EXAMPLE. There exists a functionally discrete normal topological space that is not discrete.

◁ Let $\aleph$ be an upward-directed set without greatest element and let all countable subsets of $\aleph$ have upper bounds. For instance, an arbitrary uncountable cardinal or the set of all countable subsets of an uncountable set (ordered by inclusion) is such an upward-directed set. Then, by the above remark, $\aleph^\bullet$ is a sought space. ▷

3.1.12. Lemma. *Let Y be a locally convex space and let a sequence $(y_n) \subset Y$ converge to some $y \in Y$. Suppose that a vector valued function $u : [0,1] \to Y$ satisfies the equality $u(0) = y$ and, for every $n \in \mathbb{N}$, maps the interval $[\frac{1}{n+1}, \frac{1}{n}]$ onto the interval $[y_{n+1}, y_n]$ by the formula*

$$u\left(\frac{\lambda}{n+1} + \frac{1-\lambda}{n}\right) = \lambda y_{n+1} + (1-\lambda)y_n, \quad 0 \leqslant \lambda \leqslant 1.$$

Then u is continuous.

◁ It is clear that u is continuous on the half-open interval $(0,1]$. Take an arbitrary neighborhood V about $y = u(0)$, take an arbitrary convex neighborhood $W \subset V$ about the same element, and consider a number n_0 such that $y_n \in W$ for $n \geqslant n_0$. Then, in view of convexity of W, the inclusion $u\big([0, \frac{1}{n_0}]\big) \subset W$ holds. ▷

3.1.13. Lemma. *Let X be an infinite-dimensional Banach space, whereas Q is not a functionally discrete topological space. Then there exists a weakly* continuous function from Q into X' whose pointwise norm is bounded and discontinuous.*

◁ By the Josefson–Nissenzweig Theorem [4, XII], there exists a weakly* null sequence (x'_n) of norm-one vectors in X'. Put $y_1 = x'_1$ and

$$y_{n+1} = \begin{cases} x'_{n+1}, & \|\lambda y'_n + (1-\lambda)x'_{n+1}\| \geqslant 1/2 \text{ for all } \lambda \in [0,1], \\ -x'_{n+1}, & \text{otherwise} \end{cases}$$

for every $n \in \mathbb{N}$. Obviously, the sequence (y_n) is weakly* null and, by Lemma 3.1.1, every interval $[y'_{n+1}, y'_n]$, $n \in \mathbb{N}$, does not intersect the open ball with radius 1/2 centered at the origin. Then the vector valued function $u : [0,1] \to X'$ defined in Lemma 3.1.12 (where Y is equal to the space X' endowed with the weak* topology and y equals to 0) is weakly* continuous. At the same time, $|\!|\!|u|\!|\!|(0) = 0$ and $|\!|\!|u|\!|\!|\big((0,1]\big) \subset [1/2, 1]$.

Now consider a function $f \in C(Q)$ such that f is not constant on each neighborhood about a point $q \in Q$ and put $g = |f - f(q)| \wedge 1$. It is clear that $g : Q \to [0,1]$, $g(q) = 0$, and $q \in \mathrm{cl}\{g > 0\}$. Consequently, the composition $u \circ g : Q \to X'$ is a sought vector valued function. ▷

3.1.14. Let X be a Banach space. A subset $F \subset X'$ is called *total* (or *separating*) if, for every nonzero element $x \in X$, there is a functional $x' \in F$ such that $\langle x | x' \rangle \neq 0$.

REMARK. In each of the following cases, the dual X' of a Banach space includes a countable total subset:

(1) X is separable;

(2) X is isomorphic to the dual of a separable Banach space.

◁ (1): Consider a set $\{x_n : n \in \mathbb{N}\}$ everywhere dense in X. With each number $n \in \mathbb{N}$, associate a norm-one functional $x'_n \in X'$ such that $\langle x_n | x'_n \rangle = \|x_n\|$. Then, for an arbitrary nonzero element $x \in X$, there is an $n \in \mathbb{N}$ for which $\|x - x_n\| \leqslant \|x\|/3$ and, consequently,

$$
\begin{aligned}
|\langle x | x'_n \rangle| &\geqslant |\langle x_n | x'_n \rangle| - |\langle x_n - x \mid x'_n \rangle| \\
&\geqslant \|x_n\| - \|x\|/3 \geqslant \|x\| - \|x\|/3 - \|x\|/3 > 0.
\end{aligned}
$$

(2): Without loss of generality, we may assume that $X = Y'$, where Y is a separable Banach space. It remains to observe that the image of a countable everywhere dense subset of Y under the canonical embedding of Y into Y'' is total. ▷

Given a topological space Q and a Banach space X, the symbol $C_w(Q, X)$ denotes the totality of all weakly continuous functions from Q into X.

Lemma. *Let X be a Banach space and let Q be a functionally discrete topological space. Suppose that X' includes a countable total subset. Then $C(Q, X) = C_w(Q, X)$.*

◁ Consider an arbitrary vector valued function $u \in C_w(Q, X)$. It is sufficient to show that, for some partition of Q into clopen subsets, the function u is constant on each element of the partition.

Let $\{x'_n : n \in \mathbb{N}\}$ be a total subset of X'. Since u is weakly continuous, $\langle u | x'_n \rangle \in C(Q)$ for all $n \in \mathbb{N}$. According to 3.1.8 (c), there is a partition of Q into clopen subsets such that all functions $\langle u | x'_n \rangle$, $n \in \mathbb{N}$, are constant on each element of the partition. Since the set $\{x'_n : n \in \mathbb{N}\}$ is total, the function u is constant on each element of the partition. ▷

3.2. Homomorphisms of Banach Bundles

The current section, as follows from its title, is devoted to studying homomorphisms of Banach bundles. Some of the facts below are of interest in their own right, but usefulness of the majority of the results in the section reveals itself later, in studying operator bundles (see Sections 3.3 and 3.4).

The first group of results, 3.2.1–3.2.4, suggests a number of conditions guaranteeing that continuous sections of a Banach bundle with operator stalks are homomorphisms.

Subsections 3.2.5–3.2.7 provide a repeatedly employed useful way of constructing sections, homomorphisms, and Banach bundles.

In 3.2.8 and 3.2.9, the notion of the dimension of a Banach bundle is studied. The results obtained, concerning domains of constancy for the dimension, are, to our opinion, of interest in their own right.

In 3.2.10, a description is given for homomorphisms of Banach bundles over a first-countable topological space. This result is supplied with examples (3.2.11) which justify essence of the restrictions imposed on the topological space.

Closing this section, we study the question of continuity for the pointwise norm of a homomorphism acting from a CBB with constant finite dimension into an arbitrary CBB (3.2.12). A number of examples (see 3.2.13) demonstrates that the constancy of dimension is an essential requirement.

3.2.1. *Proposition.* *Let $\mathscr{X}$, $\mathscr{Y}$, and $\mathscr{Z}$ be* CBB*s over a topological space Q, with $\mathscr{Z}(q) \subset B(\mathscr{X}(q), \mathscr{Y}(q))$ for all $q \in Q$, and let sets of sections $\mathscr{U} \subset C(Q, \mathscr{X})$ and $\mathscr{W} \subset C(Q, \mathscr{Z})$ be stalkwise dense in $\mathscr{X}$ and $\mathscr{Z}$. Suppose that the global section $w \otimes u$ of $\mathscr{Y}$ is continuous for every $u \in \mathscr{U}$ and $w \in \mathscr{W}$. Then, for every $D \subset Q$, the inclusion $C(D, \mathscr{Z}) \subset \mathrm{Hom}_D(\mathscr{X}, \mathscr{Y})$ holds.*

$\triangleleft$ Fix an arbitrary subset $D \subset Q$, elements $\overline{u} \in C(D, \mathscr{X})$ and $\overline{w} \in C(D, \mathscr{Z})$, and a point $q \in D$. We prove that the section $\overline{w} \otimes \overline{u}$ of $\mathscr{Y}$ is continuous at q. By [8, Proposition 1.3.2], it is sufficient to show upper semicontinuity of the function $|\!|\!|\overline{w} \otimes \overline{u} - v|\!|\!| : D \to \mathbb{R}$ at the point q for every $v \in C(D, \mathscr{Y})$. Let $\varepsilon > 0$ and $v \in C(D, \mathscr{Y})$. We find a neighborhood about q on which

$$|\!|\!|\overline{w} \otimes \overline{u} - v|\!|\!| < |\!|\!|\overline{w} \otimes \overline{u} - v|\!|\!|(q) + \varepsilon.$$

Take an element $u \in \mathscr{U}$ such that $|\!|\!|\overline{w}|\!|\!|(q)|\!|\!|\overline{u} - u|\!|\!|(q) < \varepsilon/8$. By continuity of the real-valued functions $|\!|\!|\overline{u} - u|\!|\!|$ and $|\!|\!|\overline{w}|\!|\!|$, we may find a neighborhood U_1 about q on which $|\!|\!|\overline{w}|\!|\!||\!|\!|\overline{u} - u|\!|\!| \leqslant \varepsilon/4$. Similarly, we take an element $w \in \mathscr{W}$ and a neighborhood U_2 about q such that $|\!|\!|\overline{w} - w|\!|\!|(q)|\!|\!|u|\!|\!|(q) < \varepsilon/8$ and $|\!|\!|\overline{w} - w|\!|\!||\!|\!|u|\!|\!| \leqslant \varepsilon/4$ on U_2. Then, on the intersection $U_1 \cap U_2$, the following hold:

$$\begin{aligned} |\!|\!|\overline{w} \otimes \overline{u} - w \otimes u|\!|\!| &\leqslant |\!|\!|\overline{w} \otimes \overline{u} - \overline{w} \otimes u|\!|\!| + |\!|\!|\overline{w} \otimes u - w \otimes u|\!|\!| \\ &\leqslant |\!|\!|\overline{w}|\!|\!||\!|\!|\overline{u} - u|\!|\!| + |\!|\!|\overline{w} - w|\!|\!||\!|\!|u|\!|\!| \leqslant \varepsilon/4 + \varepsilon/4 = \varepsilon/2. \end{aligned}$$

The same calculations yield the inequality $|||\overline{w}\otimes\overline{u}-w\otimes u|||(q)<\varepsilon/4$. Now we take a neighborhood U_3 about q, on which $|||w\otimes u-v|||\leqslant|||w\otimes u-v|||(q)+\varepsilon/4$. On the neighborhood $U_1\cap U_2\cap U_3$ about q, the following hold:

$$\begin{aligned}|||\overline{w}\otimes\overline{u}-v|||&\leqslant|||\overline{w}\otimes\overline{u}-w\otimes u|||+|||w\otimes u-v|||\\&\leqslant\varepsilon/2+|||w\otimes u-v|||(q)+\varepsilon/4\\&\leqslant\varepsilon/2+|||w\otimes u-\overline{w}\otimes\overline{u}|||(q)+|||\overline{w}\otimes\overline{u}-v|||(q)+\varepsilon/4\\&<\varepsilon/2+\varepsilon/4+|||\overline{w}\otimes\overline{u}-v|||(q)+\varepsilon/4\\&=|||\overline{w}\otimes\overline{u}-v|||(q)+\varepsilon,\end{aligned}$$

which completes the proof. $\triangleright$

3.2.2. Corollary. *Let $\mathscr{X}$, $\mathscr{Y}$, and $\mathscr{Z}$ be CBBs over a topological space Q, with $\mathscr{Z}(q)\subset B(\mathscr{X}(q),\mathscr{Y}(q))$ at every point $q\in Q$. Suppose that $C(Q,\mathscr{Z})\subset \mathrm{Hom}(\mathscr{X},\mathscr{Y})$. Then, for every $D\subset Q$, the inclusion $C(D,\mathscr{Z})\subset\mathrm{Hom}_D(\mathscr{X},\mathscr{Y})$ holds.*

$\triangleleft$ The claim follows from 3.2.1 with $\mathscr{U}=C(Q,\mathscr{X})$ and $\mathscr{W}=C(Q,\mathscr{Z})$. $\triangleright$

3.2.3. Corollary. *The inclusion $C(Q,B(X,Y))\subset\mathrm{Hom}(X_Q,Y_Q)$ holds for arbitrary Banach spaces X and Y.*

$\triangleleft$ Put $\mathscr{U}$ and $\mathscr{W}$ equal to the sets of all constant X-valued and $B(X,Y)$-valued functions and apply Proposition 3.2.1. $\triangleright$

One of the natural questions which may arise when considering the above corollary is as follows: When does the equality

$$C(Q,B(X,Y))=\mathrm{Hom}(X_Q,Y_Q)$$

hold? This question is addressed in Section 3.3.

3.2.4. Corollary. *Let $\mathscr{X}$, $\mathscr{Y}$, and $\mathscr{Z}$ be CBBs over a topological space Q and let $\mathscr{Z}(q)\subset B(\mathscr{X}(q),\mathscr{Y}(q))$ at every point $q\in Q$. Suppose that the space $\mathrm{Hom}(\mathscr{X},\mathscr{Y})$ includes a continuity structure for $\mathscr{Z}$. Then $C(Q,\mathscr{Z})\subset\mathrm{Hom}(\mathscr{X},\mathscr{Y})$.*

$\triangleleft$ Taking $C(Q,\mathscr{X})$ as $\mathscr{U}$, the above-mentioned continuity structure for $\mathscr{Z}$ as $\mathscr{W}$, and applying Proposition 3.2.1, we obtain the claim. $\triangleright$

3.2.5. In the sequel, we use the following auxiliary result.

Lemma. *Let Q be a completely regular topological space. Suppose that $q\in Q$ is a limit point for a countable discrete set $\{q_n:n\in\mathbb{N}\}$, with $q_i\neq q_j$ whenever $i\neq j$.*

(1) *There is a sequence (W_n) of open subsets of Q such that $q_n\in W_n$, $\mathrm{cl}\,W_n\cap\mathrm{cl}\bigcup_{k\neq n}W_k=\varnothing$, and $q\notin\mathrm{cl}\,W_n$ for all $n\in\mathbb{N}$.*

Consider continuous functions $f_n : Q \to [0,1]$, $f_n \equiv 0$ on $Q \backslash W_n$ for all $n \in \mathbb{N}$. Furthermore, let (ε_n) be a vanishing numerical sequence.

If at q there is a countable base then we may additionally stipulate that $\big(\mathrm{cl} \bigcup_{n\in\mathbb{N}} W_n\big) \backslash \bigcup_{n\in\mathbb{N}} \mathrm{cl}\, W_n = \{q\}$.

(2) *The function $f : Q \to [0,1]$ defined by the formula*

$$f(p) = \begin{cases} \varepsilon_n f_n(p), & p \in W_n, \\ 0, & p \notin \bigcup_{n\in\mathbb{N}} W_n \end{cases}$$

is continuous.

(3) *Let $\mathscr{X}$ be a CBB over Q. Given a sequence $(u_n)_{n\in\mathbb{N}} \subset C(Q, \mathscr{X})$ such that $|\!|\!| u_n |\!|\!| \leqslant M$ on W_n, from some index on, the section u over Q defined by the formula*

$$u(p) = \begin{cases} \varepsilon_n f_n(p) u_n(p), & p \in W_n, \\ 0, & p \notin \bigcup_{n\in\mathbb{N}} W_n \end{cases}$$

is continuous.

(4) *Let $\mathscr{X}$ and $\mathscr{Y}$ be CBBs over Q. If $(H_n)_{n\in\mathbb{N}} \subset \mathrm{Hom}(\mathscr{X}, \mathscr{Y})$ and $|\!|\!| H_n |\!|\!| \leqslant K$ on W_n for all n from some index on, then the mapping $H : p \in Q \mapsto H(p) \in B\big(\mathscr{X}(p), \mathscr{Y}(p)\big)$ defined by the formula*

$$H(p) = \begin{cases} \varepsilon_n f_n(p) H_n(p), & p \in W_n, \\ 0, & p \notin \bigcup_{n\in\mathbb{N}} W_n \end{cases}$$

is a homomorphism from $\mathscr{X}$ into $\mathscr{Y}$.

(5) *If X is a topological vector space and a sequence $(x_n) \subset X$ converges to an $x \in X$, then the vector valued function $u : Q \to X$ defined by the formula*

$$u(p) = \begin{cases} f_n(p) x_n + \big(1 - f_n(p)\big) x, & p \in W_n, \\ x, & p \notin \bigcup_{n\in\mathbb{N}} W_n \end{cases}$$

is continuous.

(6) *If X is a Banach space and a sequence of functionals $(x'_n) \subset X'$ converges weakly* to an $x' \in X'$, then the vector valued function $H : Q \to X'$ defined by the formula*

$$H(p) = \begin{cases} f_n(p) x'_n + \big(1 - f_n(p)\big) x', & p \in W_n, \\ x', & p \notin \bigcup_{n\in\mathbb{N}} W_n \end{cases}$$

is a homomorphism from X_Q into $\mathscr{R}$.

◁ (1): By induction, for every $n \in \mathbb{N}$, we construct open sets $W_n, V_n \subset Q$. Since the space Q is regular, the point q_1 and the closed set $\mathrm{cl}\{q_k : k \geqslant 2\}$ have disjoint open neighborhoods W_1 and V_1. We may assume that $\mathrm{cl}\, W_1 \cap \mathrm{cl}\, V_1 = \varnothing$. If W_k and V_k are chosen for all $k \leqslant n$ then we take W_{n+1} and V_{n+1} so that V_n contain W_{n+1} and V_{n+1}, and the sets $\mathrm{cl}\, W_{n+1}$ and $\mathrm{cl}\, V_{n+1}$ separate the point q_{n+1} and the closed set $\mathrm{cl}\{q_k : k \geqslant n+2\}$. It is easy to see that (W_n) is a sought sequence.

Finally, let (U_n) be a countable base for open neighborhoods about q, with $U_1 = Q$ and $U_n \supset U_{n+1}$ for all $n \in \mathbb{N}$. Then, when constructing the sequence of sets W_n, we may take $W_n \subset U_{k(n)}$, where $k(n) = \max\{k \in \mathbb{N} : q_n \in U_k\}$. This provides the desired relation, $\big(\mathrm{cl} \bigcup_{n\in\mathbb{N}} W_n\big) \setminus \bigcup_{n\in\mathbb{N}} \mathrm{cl}\, W_n = \{q\}$.

(2): It is obvious that the function f is the pointwise sum of the uniformly convergent series $\sum_{n=1}^{\infty} \varepsilon_n f_n$; therefore, f is continuous.

Assertions (3)–(5) may be proven in much the same way by using Proposition [8, 1.3.6] for (3) and [8, 1.4.11] for (4).

(6): By (5) the function H is weakly* continuous; therefore, $H \otimes u \in C(Q)$ for all constant functions $u : Q \to X$. It remains to observe that the pointwise norm of H is bounded by construction and to apply [8, Theorem 1.4.9]. ▷

3.2.6. Corollary. *Let $\mathscr{X}$ and $\mathscr{Y}$ be* CBB*s over a completely regular topological space Q. Suppose that a sequence $(q_n)_{n\in\mathbb{N}}$, $q_i \neq q_j$ $(i \neq j)$ converges to a point q and $q \neq q_k$ for all $k \in \mathbb{N}$.*

(1) *Let $x_n \in \mathscr{X}(q_n)$ $(n \in \mathbb{N})$, let $x \in \mathscr{X}(q)$, and let the convergence $(q_n, x_n) \to (q, x)$ as $n \to \infty$ hold in the topological space $Q \otimes \mathscr{X}$* (see [8, 1.1.4]). *(For $x = 0$, this is equivalent to the equality $\lim_{n\to\infty} \|x_n\| = 0$.) Then there exists a bounded section $u \in C(Q, \mathscr{X})$ such that $u(q_n) = x_n$ for all $n \in \mathbb{N}$ and $u(q) = x$.*

(2) *Let $H_n \in \mathrm{Hom}(\mathscr{X}, \mathscr{Y})$ $(n \in \mathbb{N})$ and let the sequence $(|\!|\!|H_n|\!|\!|)_{n\in\mathbb{N}}$ be uniformly vanishing. Then there exists a bounded homomorphism $H \in \mathrm{Hom}(\mathscr{X}, \mathscr{Y})$ such that $H(q_n) = H_n(q_n)$ for all $n \in \mathbb{N}$ and $H(q) = 0$.*

(3) *Let X be a topological vector space. Suppose that the sequence $(x_n) \subset X$ converges to an $x \in X$. Then there is a continuous vector valued function $u : Q \to X$ such that $u(q_n) = x_n$ for all $n \in \mathbb{N}$ and $u(q) = x$.*

(4) *Let X be a Banach space. Suppose that the sequence $(x'_n) \subset X'$ is convergent weakly* to an $x' \in X'$. Then there exists a homomorphism $H \in \mathrm{Hom}(X_Q, \mathscr{R})$ such that $H(q_n) = x'_n$ for all $n \in \mathbb{N}$ and $H(q) = x'$.*

◁ We only need to explain assertion (1). If $x = 0$, this assertion follows directly from Lemma 3.2.5 (3) and Dupré's Theorem (see [8, 1.3.5]). Dealing with the general case, use Dupré's Theorem again and consider a bounded section $v \in C(Q, \mathscr{X})$ taking the value x at q. From [8, Proposition 1.3.8] it follows that $\|x_n - v(q_n)\| \to 0$ as $n \to \infty$. Since the assertion under proof is true for the case $x = 0$, there is a bounded section $w \in C(Q, \mathscr{X})$ satisfying the equalities $w(q_n) = x_n - v(q_n)$ $(n \in \mathbb{N})$ and $w(q) = 0$. It remains to put $u = v + w$. $\rhd$

3.2.7. Lemma. *Let $X_1 \subset X_2 \subset \cdots$ be Banach spaces, let Q be a completely regular topological space, and let $(U_n)_{n\in\mathbb{N}}$ be a partition of Q such that the sets $U_1 \cup \cdots \cup U_n$ are closed for all $n \in \mathbb{N}$. Then there is a* CBB $\mathscr{X}$ *over Q satisfying the following conditions:*

(a) $\mathscr{X}|_{U_n} \equiv X_n$ *for all $n \in \mathbb{N}$;*

(b) *if the sequence of functionals $x'_n \in X'_n$ $(n \in \mathbb{N})$ is such that x'_{n+1} extends x'_n and $\|x'_n\| \leqslant 1$ for all $n \in \mathbb{N}$, then the mapping H satisfying the relations $H|_{U_n} \equiv x'_n$ $(n \in \mathbb{N})$ belongs to* $\mathrm{Hom}(\mathscr{X}, \mathscr{R})$.

$\lhd$ Consider a (discrete) Banach bundle $\mathscr{X}$ satisfying condition (a) and define a continuity structure in $\mathscr{X}$ as follows: Put

$$C_0 = C(Q);$$
$$C_n = \big\{ f \in C(Q) : f \equiv 0 \text{ on } U_1 \cup \cdots \cup U_n \big\}, \quad n \in \mathbb{N}.$$

It is clear that the set of sections

$$\mathscr{C} = \big\{ f_1 x_1 + \cdots + f_n x_n : f_i \in C_i,\ x_i \in X_i,\ i = 1, \dots, n,\ n \in \mathbb{N} \big\}$$

of the bundle $\mathscr{X}$ is a subspace of the space of all global sections of $\mathscr{X}$. Moreover, the set $\mathscr{C}$ is stalkwise dense in $\mathscr{X}$. Indeed, let $q \in Q$, $x \in \mathscr{X}(q)$, and let a number $n \in \mathbb{N}$ be such that $q \in U_n$. Since the space Q is completely regular, there is a function $f \in C_{n-1}$ such that $f(q) = 1$. Therefore, fx belongs to $\mathscr{C}$ and passes through x at q. Consequently, $\mathscr{C}$ is a continuity structure in $\mathscr{X}$ which makes $\mathscr{X}$ a CBB.

Let H satisfy condition (b). Verify that $H \in \mathrm{Hom}(\mathscr{X}, \mathscr{R})$. By Theorem [8, 1.4.9], it is sufficient to show that $H \otimes u \in C(Q)$ for all $u \in \mathscr{C}$. If $u = f_1 x_1 + \cdots + f_n x_n \in \mathscr{C}$, where $f_i \in C_i$, $x_i \in X_i$, $i = 1, \dots, n$, then, for all $q \in Q$, the equality $(H \otimes u)(q) = \langle u(q) | x'_n \rangle$ holds. Next,

$$\langle u(q) | x'_n \rangle = f_1(q) \langle x_1 | x'_n \rangle + \cdots + f_n(q) \langle x_n | x'_n \rangle.$$

Therefore, the function $H \otimes u$ is continuous. $\rhd$

3.2.8. DEFINITION. Let $\mathscr{X}$ be an arbitrary Banach bundle over a set Q. The function $\dim \mathscr{X}$ which, with every point $q \in Q$, associates the dimension $\dim \mathscr{X}(q)$ of the stalk $\mathscr{X}(q)$ is the *dimension of* $\mathscr{X}$.

We say that $\mathscr{X}$ *has constant dimension* n if $\dim \mathscr{X}(q) = n$ for all $q \in Q$.

Lemma. *Let $\mathscr{X}$ be a* CBB *with finite-dimensional stalks over an arbitrary topological space. For every $n = 0, 1, 2, \dots,$ consider the following conditions:*

(a) *the set $\{\dim \mathscr{X} = n\}$ is open;*
(b) *the set $\{\dim \mathscr{X} < n\}$ is open;*
(c) *the set $\{\dim \mathscr{X} \leqslant n\}$ is open;*
(d) *the set $\{\dim \mathscr{X} > n\}$ is closed;*
(e) *the set $\{\dim \mathscr{X} \geqslant n\}$ is closed.*

If one of the conditions (a)–(e) *holds for every $n = 0, 1, 2, \dots,$ then each of the conditions holds for every $n = 0, 1, 2, \dots$. In this case, all sets mentioned in* (a)–(e) *are clopen.*

$\lhd$ It suffices to observe that, due to [7, 18.1], the sets of the form $\{\dim \mathscr{X} > n\}$ and $\{\dim \mathscr{X} \geqslant n\}$ are open and, therefore, the sets of the form $\{\dim \mathscr{X} < n\}$ and $\{\dim \mathscr{X} \leqslant n\}$ are closed. $\rhd$

3.2.9. ***Proposition.*** *The following hold:*

(1) *Let Q be a Baire topological space. Then, for every* CBB $\mathscr{X}$ *over Q with finite-dimensional stalks, the union $\bigcup_{n \geqslant 0} \operatorname{int} \{\dim \mathscr{X} = n\}$ is everywhere dense in Q.*

(2) *If the space Q is completely regular and, for every* CBB $\mathscr{X}$ *over Q with finite-dimensional stalks, the set $\bigcup_{n \geqslant 0} \operatorname{int} \operatorname{cl} \{\dim \mathscr{X} = n\}$ is everywhere dense, then Q is a Baire space.*

$\lhd$ (1): For proving that the union under consideration is everywhere dense, it is sufficient, given a nonempty open set $U \subset Q$, to find an open nonempty subset $W \subset U$ such that the dimension of $\mathscr{X}$ is constant on W.

Since Q is a Baire space, there is a number $n \geqslant 0$ such that

$$V := \operatorname{int} \operatorname{cl} \{\dim \mathscr{X} = n\} \neq \varnothing.$$

Consequently, from [7, 18.1] we easily infer that the set $\{\dim \mathscr{X} \leqslant n\}$ is closed; therefore, $V \subset \operatorname{cl} \{\dim \mathscr{X} = n\} \subset \{\dim \mathscr{X} \leqslant n\}$, i.e., $\dim \mathscr{X} \leqslant n$ on V. The relation $V \subset \operatorname{cl} \{\dim \mathscr{X} = n\}$ and the fact that the set V is open imply that there exists a point $q \in V \cap \{\dim \mathscr{X} = n\}$. Since the set $\{\dim \mathscr{X} \geqslant n\}$ is open, $\dim \mathscr{X} \geqslant n$ on some open neighborhood $W \subset V$ about q. Thus, the dimension of $\mathscr{X}$ is constant on the open nonempty set $W \subset V \subset U$.

(2): Let Q be a completely regular space that is not a Baire space. We will construct a CBB $\mathscr{X}$ over Q such that $\mathscr{X}$ has finite-dimensional stalks while the set $\bigcup_{n\geqslant 0}\operatorname{int}\operatorname{cl}\{\dim\mathscr{X}=n\}$ is not everywhere dense.

Since Q is not a Baire space, there exist an open nonempty set $U\subset Q$ and a cover $(V_n)_{n\in\mathbb{N}}$ consisting of nowhere dense subsets $V_n\subset U$. Put $U_1=Q\backslash U$ and $U_{n+1}=\operatorname{cl}V_n\setminus(U_1\cup\cdots\cup U_n)$ for all $n\in\mathbb{N}$. It is clear that, for all $n\in\mathbb{N}$, the set U_n is nowhere dense, the union $U_1\cup\cdots\cup U_n$ is closed, and $\bigcup_{n\in\mathbb{N}}U_n=Q$.

Consider a sequence $X_1\subset X_2\subset\cdots$ of finite-dimensional Banach spaces with strictly increasing dimensions: $\dim X_n<\dim X_{n+1}$ for all $n\in\mathbb{N}$. By Lemma 3.2.7, there exists a CBB $\mathscr{X}$ over Q such that $\mathscr{X}|_{U_n}\equiv X_n$ for all $n\in\mathbb{N}$. It is easy to see that

$$\bigcup_{n\geqslant 0}\operatorname{int}\operatorname{cl}\{\dim\mathscr{X}=n\}=\bigcup_{m\geqslant 0}\operatorname{int}\operatorname{cl}U_m=\operatorname{int}U_1,$$

where the latter set is not everywhere dense. ▷

Corollary. *If $\mathscr{X}$ is a* CBB *with finite-dimensional stalks over a Baire space Q then, for every $m=0,1,2,\dots$, the equality holds*

$$\operatorname{cl}\{\dim\mathscr{X}\geqslant m\}=\operatorname{cl}\bigcup_{n\geqslant m}\operatorname{int}\{\dim\mathscr{X}=n\}.$$

◁ Fix a number $0\leqslant m\in\mathbb{Z}$. The inclusion $\supset$ is obvious. Prove the reverse inclusion. Let $q\in Q$ and $\dim\mathscr{X}(q)\geqslant m$. The union $\bigcup_{n\geqslant 0}\operatorname{int}\{\dim\mathscr{X}=n\}$ is everywhere dense by Proposition 3.2.9 (1),

$$\bigcup_{n<m}\operatorname{int}\{\dim\mathscr{X}=n\}\subset\{\dim\mathscr{X}<m\},$$

and the latter set is closed.

Hence, the point q belongs to the closure of $\bigcup_{n\geqslant m}\operatorname{int}\{\dim\mathscr{X}=n\}$. Therefore,

$$\{\dim\mathscr{X}\geqslant m\}\subset\operatorname{cl}\bigcup_{n\geqslant m}\operatorname{int}\{\dim\mathscr{X}=n\},$$

which implies the required inclusion. ▷

3.2.10. The following assertion differs from [8, Theorem 1.4.7] only in the conditions on Q.

Theorem. *Let $\mathscr{X}$ and $\mathscr{Y}$ be* CBBs *over a first-countable completely regular topological space Q. A mapping $H: q\in Q\mapsto H(q)\in B(\mathscr{X}(q),\mathscr{Y}(q))$ is a homomorphism from $\mathscr{X}$ into $\mathscr{Y}$ if and only if $H\otimes u\in C(Q,\mathscr{Y})$ for all $u\in C(Q,\mathscr{X})$.*

◁ Necessity follows from [8, Theorem 1.4.4]. Prove sufficiency. In view of [8, Theorem 1.4.4], it is enough to prove that H is locally bounded. Suppose that the function $|\!|\!|H|\!|\!|$ is not bounded in any neighborhood about a point $q \in Q$. In this case, since Q is first-countable, there is a sequence $(q_n) \subset Q \setminus \{q\}$, $q_i \neq q_j$ $(i \neq j)$, convergent to q such that $|\!|\!|H|\!|\!|(q_n) > (|\!|\!|H|\!|\!|(q) + n)^2$ for all $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, we take an element $x_n \in \mathscr{X}(q_n)$ so that $\|H(q_n)x_n\| = \|H(q_n)\|$ and $\|x_n\| \leqslant 2$. By Corollary 3.2.6 (1) there exists a bounded section $u \in C(Q, \mathscr{X})$ taking values $u(q_n) = \frac{1}{n} x_n$ for all $n \in \mathbb{N}$ and $u(q) = 0$. Then

$$|\!|\!|H \otimes u|\!|\!|(q_n) = \tfrac{1}{n}\,\|H(q_n)\| \geqslant \tfrac{1}{n}(|\!|\!|H|\!|\!|(q) + n)^2 > n.$$

This contradicts continuity of $H \otimes u$, since $q_n \to q$ and $(H \otimes u)(q) = 0$. ▷

Remark. From the above proof and the proof of 3.2.5 (3), it is clear that, in the last theorem, the condition $H \otimes u \in C(Q, \mathscr{Y})$ for all $u \in C(Q, \mathscr{X})$ can be replaced by a "weaker" condition: $H \otimes u \in C(Q, \mathscr{Y})$ for all u in a stalkwise dense $C^b(Q)$-submodule of $C^b(Q, \mathscr{X})$ closed with respect to the uniform norm. For instance, we may take as such a submodule $C^b(Q, \mathscr{X})$.

3.2.11. Thus, Theorem 3.2.10 is stated for the case of a first-countable topological space Q. In the literature, the class of Fréchet–Urysohn spaces is usually the smallest class of topological spaces under consideration which includes the class of first-countable spaces (cf. [5, 1.6.14]). (Recall that a topological space Q is said to be a *Fréchet–Urysohn space* if, for every point $q \in Q$ and every $P \subset Q$, the condition $p \in \operatorname{cl} P$ implies existence of a sequence in P convergent to q.) Show that Theorem 3.2.10 cannot be generalized to the class of Fréchet–Urysohn spaces Q.

Example. We construct a topological space Q with the following properties:

(a) Q is a Fréchet–Urysohn space;
(b) Q is a normal space;
(c) Q is not first-countable;
(d) Q is not locally pseudocompact;
(e) Q is a Baire space;
(f) there exist a CBB $\mathscr{X}$ over Q with finite-dimensional stalks and a mapping $H : q \in Q \mapsto H(q) \in \mathscr{X}(q)'$ such that $H \otimes u \in C(Q)$ for all $u \in C(Q, \mathscr{X})$, but $H \notin \operatorname{Hom}(\mathscr{X}, \mathscr{R})$;
(g) for every infinite-dimensional Banach space X, there is a mapping $H : Q \to X'$ such that $H \otimes u \in C(Q)$ for all $u \in C(Q, \mathscr{X})$, but $H \notin \operatorname{Hom}(X_Q, \mathscr{R})$.

Consider the set $Q = (\mathbb{N} \times \mathbb{N}) \cup \{\infty\}$, where $\infty \notin \mathbb{N} \times \mathbb{N}$, and endow Q with a topology in the following way. We regard all elements of $\mathbb{N} \times \mathbb{N}$ as isolated points and all subsets $U \subset Q$, for which $\infty \in U$ and

$$(\forall m \in \mathbb{N})\ (\exists n_m \in \mathbb{N})\ (\forall n \geqslant n_m)\ \ (m,n) \in U,$$

as neighborhoods of ∞. It is clear that

$$C(Q) = \Big\{ f : Q \to \mathbb{R}\ :\ \lim_{n\to\infty} f\big((m,n)\big) = f(\infty) \text{ for all } m \in \mathbb{N} \Big\}. \qquad (1)$$

Verify that the topological space Q possesses properties (a)–(g).

(a): It is sufficient to consider a subset $P \subset Q$ that does not contain a sequence convergent to ∞ and show that $\infty \notin \operatorname{cl} P$. Obviously, for every $m \in \mathbb{N}$, there is a number n_m such that $\big\{(m,n) \in P : n \in \mathbb{N}\big\} \subset \big\{(m,1),\dots,(m,n_m)\big\}$. Hence, the set P and the neighborhood $\big\{(m,n) : m \in \mathbb{N},\ n > n_m\big\} \cup \{\infty\}$ about ∞ are disjoint; therefore, $\infty \notin \operatorname{cl} P$.

(b), (e): See Remark 3.1.9.

Conditions (c) and (d) immediately follow from assertion (f) proven below and Theorems 3.2.10 and [8, 1.4.7] respectively.

(f): Consider a CBB $\mathscr{X}$ over Q such that $\mathscr{X}(q) = \mathbb{R}$ for all $q \in \mathbb{N}\times\mathbb{N}$, $\mathscr{X}(\infty) = \{0\}$, and $C(Q,\mathscr{X}) = \{u \in C(Q) : u(\infty) = 0\}$. Define an H by the equalities $H(\infty) = 0$ and $H\big((m,n)\big) = m$ for all $(m,n) \in \mathbb{N}\times\mathbb{N}$. It is easy to verify that $H \otimes u \in C(Q)$ for all $u \in C(Q,\mathscr{X})$ $\big($see (1)$\big)$. Nevertheless, the pointwise norm of H is not locally bounded; therefore, by [8, Theorem 1.4.4], $H \notin \operatorname{Hom}(\mathscr{X},\mathscr{R})$.

(g): By the Josefson–Niessenzweig Theorem [4, XII], there is a weakly* null sequence (x'_n) of norm-one vectors in X'. Define $H(\infty) = 0 \in X'$ and $H\big((m,n)\big) = m x'_n$ for all $(m,n) \in \mathbb{N}\times\mathbb{N}$. Then $H \otimes u \in C(Q)$ for an arbitrary section $u \in C(Q, X_Q)$. Indeed, for every $m \in \mathbb{N}$, the relation $\lim_{n\to\infty}(H\otimes u)\big((m,n)\big) = 0$ holds, since $\big(H\big((m,n)\big)\big)_{n\in\mathbb{N}}$ is a weakly* null sequence and $\|u\big((m,n)\big) - u(\infty)\| \to 0$ as $n \to \infty$. It remains to observe that the pointwise norm of H is not locally bounded and to apply [8, Theorem 1.4.4].

3.2.12. *Theorem.* *Let a* CBB $\mathscr{X}$ *over a topological space* Q *have constant finite dimension, let* $\mathscr{Y}$ *be an arbitrary* CBB *over* Q*, and let* $\mathscr{U}$ *be a subset of* $C(Q,\mathscr{X})$ *stalkwise dense in* $\mathscr{X}$*. If a mapping* $H : p \in Q \mapsto H(p) \in B\big(\mathscr{X}(p),\mathscr{Y}(p)\big)$ *is such that* $H \otimes u \in C(Q,\mathscr{Y})$ *for every* $u \in \mathscr{U}$*, then* $H \in \operatorname{Hom}(\mathscr{X},\mathscr{Y})$ *and the pointwise norm* $|\!|\!| H |\!|\!|$ *is continuous.*

$\lhd$ Fix an arbitrary point $q \in Q$ and prove continuity of $|\!|\!| H |\!|\!|$ at this point. Due to the relation

$$\begin{aligned}\|H(p)\| &= \sup\Big\{ \Big\| H(p)\Big(\frac{1}{\max\{\|u(p)\|,1\}}\,u(p)\Big)\Big\| : u \in \mathscr{U} \Big\}\\ &= \sup\Big\{ \Big(\frac{1}{|\!|\!| u |\!|\!| \vee 1} |\!|\!| H \otimes u |\!|\!|\Big)(p) : u \in \mathscr{U} \Big\}\end{aligned}$$

valid for all $p \in Q$, the function $|\!|\!|H|\!|\!|$ is lower-semicontinuous. It remains to prove that the function $|\!|\!|H|\!|\!|$ is upper-semicontinuous at q. Take an arbitrary $\varepsilon > 0$ and prove that, in some neighborhood U about q, the inequality $|\!|\!|H|\!|\!| \leqslant |\!|\!|H|\!|\!|(q) + \varepsilon$ holds.

Since the stalk $\mathscr{X}(q)$ is finite-dimensional, there is a collection of sections $\mathbf{u} = (u_1, \dots, u_n) \subset \operatorname{lin}\mathscr{U}$ such that the values $u_1(q), \dots, u_n(q)$ lie on the unit sphere and constitute a basis for $\mathscr{X}(q)$. Since the set

$$\Lambda = \{\boldsymbol{\lambda} \in \mathbb{R}^n : |\!|\!|\boldsymbol{\lambda}\mathbf{u}|\!|\!|(q) = 1\}$$

is bounded in $\mathbb{R}^n$, the number

$$\|\Lambda\|_1 := \sup\{|\lambda_1| + \dots + |\lambda_n| : (\lambda_1, \dots, \lambda_n) \in \Lambda\}$$

is finite. (Here and in the sequel, we denote by $\boldsymbol{\lambda}\mathbf{u}$ the sum $\lambda_1 u_1 + \dots + \lambda_n u_n$.) Choose some number $\delta \in (0,1)$ such that $\frac{1}{1-\delta}\big(\delta + |\!|\!|H|\!|\!|(q)\big) < |\!|\!|H|\!|\!|(q) + \varepsilon$.

By [16, Lemma 7], there exists a neighborhood U_δ about q, where $1 - \delta \leqslant |\!|\!|\boldsymbol{\lambda}\mathbf{u}|\!|\!| \leqslant 1 + \delta$ for all $\boldsymbol{\lambda} \in \Lambda$. Without loss of generality, we may assume that the collection $\mathbf{u}(p) = \big(u_1(p), \dots, u_n(p)\big)$ is linearly independent for every element $p \in U_\delta$ (see [7, 18.1]). In particular, an arbitrary vector $x \in \mathscr{X}(p)$ can be represented as

$$x = \frac{\|x\|}{|\!|\!|\boldsymbol{\lambda}_x\mathbf{u}|\!|\!|(p)}(\boldsymbol{\lambda}_x\mathbf{u})(p)$$

with a suitable $\boldsymbol{\lambda}_x \in \Lambda$. Since the sections $H \otimes u_i$, $i = 1, \dots, n$, are continuous, there exists a neighborhood $U \subset U_\delta$ about q such that

$$\|\Lambda\|_1 \max\big\{\big||\!|\!|H \otimes u_i|\!|\!|(p) - |\!|\!|H \otimes u_i|\!|\!|(q)\big| \ : \ i = 1, \dots, n\big\} < \delta$$

for all $p \in U$. At a point $p \in U$, the value of the norm $\|H(p)\|$ is attained at some vector $x(p) \in \mathscr{X}(p)$, $\|x(p)\| = 1$. Hence,

$$\begin{aligned}
|\!|\!|H|\!|\!|(p) = \|H(p)x(p)\| &= \frac{1}{|\!|\!|\boldsymbol{\lambda}_{x(p)}\mathbf{u}|\!|\!|(p)} |\!|\!|H \otimes \big(\boldsymbol{\lambda}_{x(p)}\mathbf{u}\big)|\!|\!|(p) \\
&\leqslant \frac{1}{1-\delta}\Big(\Big||\!|\!|H \otimes \big(\boldsymbol{\lambda}_{x(p)}\mathbf{u}\big)|\!|\!|(p) - |\!|\!|H \otimes \big(\boldsymbol{\lambda}_{x(p)}\mathbf{u}\big)|\!|\!|(q)\Big| \\
&\qquad\qquad + |\!|\!|H \otimes \big(\boldsymbol{\lambda}_{x(p)}\mathbf{u}\big)|\!|\!|(q)\Big) \\
&\leqslant \frac{1}{1-\delta}\Big(\|\Lambda\|_1 \max\big\{\big||\!|\!|H \otimes u_i|\!|\!|(p) - |\!|\!|H \otimes u_i|\!|\!|(q)\big| : \\
&\qquad\qquad i = 1, \dots, n\big\} + |\!|\!|H|\!|\!|(q)\Big) \\
&\leqslant \frac{1}{1-\delta}\big(\delta + |\!|\!|H|\!|\!|(q)\big) < |\!|\!|H|\!|\!|(q) + \varepsilon.
\end{aligned}$$

The fact that $H \in \operatorname{Hom}(\mathscr{X}, \mathscr{Y})$ now follows from continuity of $|\!|\!|H|\!|\!|$ and [8, Theorem 1.4.4]. $\triangleright$

Corollary. *Let $\mathscr{X}$ and $\mathscr{Y}$ be* CBBs *over the same topological space. If $\mathscr{X}$ has constant finite dimension then the pointwise norm of every homomorphism from $\mathscr{X}$ into $\mathscr{Y}$ is continuous.*

3.2.13. As we see from the examples below, the constant dimension requirement for a bundle $\mathscr{X}$ in Corollary 3.2.12 is essential.

Intending to emphasize diversity of situations in which a homomorphism $H \in \mathrm{Hom}(\mathscr{X}, \mathscr{R})$ with a discontinuous norm arises for a CBB $\mathscr{X}$ with finite-dimensional stalks, we give three different examples. In the first case, the dimension of $\mathscr{X}$ is equal to 0 at a unique discontinuity point of the function $|\!|\!|H|\!|\!|$ and the dimension of $\mathscr{X}$ is equal to 1 at other points. In the second case, the dimension of $\mathscr{X}$ takes two distinct (possibly, nonzero) values and, in the third case, the dimension of $\mathscr{X}$ takes infinitely many distinct values and the function $|\!|\!|H|\!|\!|$ is discontinuous at every point.

Examples. **(1)** Let $Q = [0, 1]$. Define $\mathscr{X}(q) = \mathbb{R}$ whenever $0 < q \leqslant 1$ and $\mathscr{X}(0) = \{0\}$. Consider the set $\{u \in C[0, 1] : u(0) = 0\}$ as a continuity structure in $\mathscr{X}$. Then the pointwise norm of the homomorphism H identically equal to values $\mathrm{id}_{\mathbb{R}}$ on the half-open interval $(0, 1]$ is not continuous at the point $0 \in Q$. It is easy to verify that, in this case, $\mathrm{Hom}(\mathscr{X}, \mathscr{R})$ can be identified in a natural way with the space of real-valued continuous functions defined on the interval $[0, 1]$ bounded on the half-open interval $(0, 1]$ and vanishing at the point $0 \in Q$. However, such functions are far from being always continuous.

(2) This time, consider a completely regular topological space Q and let q be a nonisolated point of Q. Define $U_1 = \{q\}$, $U_2 = Q \backslash U_1$, and $U_3 = U_4 = \dots = \varnothing$. Let X be a finite-dimensional Banach space, let X_1 be a proper subspace of X, and let $X_2 = X_3 = \dots = X$. Fix a norm-one functional $x' \in X'$ vanishing on X_1 and define $x'_1 = 0$, $x'_2 = x'_3 = \dots = x'$. Consider the CBB $\mathscr{X}$ of Lemma 3.2.7 and a homomorphism H satisfying condition (b) of the lemma. It is clear that $|\!|\!|H|\!|\!|(q) = 0$ and $|\!|\!|H|\!|\!| \equiv 1$ outside $\{q\}$. Therefore, since the point q is nonisolated, the function $|\!|\!|H|\!|\!|$ is discontinuous.

(3) Let $Q = \mathbb{Q}$ be the set of rationals with the natural topology and let $n \mapsto q_n$ be an arbitrary bijection from $\mathbb{N}$ onto Q. Define $U_n = \{q_n\}$ for all $n \in \mathbb{N}$ and consider an arbitrary sequence of Banach spaces $X_1 \subset X_2 \subset \cdots$ and an arbitrary sequence of functionals x'_n satisfying condition 3.2.7 (b). We additionally require that the dimensions of X_n and the norms of x'_n be strictly monotone increasing. Let $\mathscr{X}$ be the CBB of Lemma 3.2.7 and let H be a homomorphism satisfying condition 3.2.7 (b). It is obvious that the stalks of $\mathscr{X}$ have pairwise distinct dimensions and the pointwise norm of H is discontinuous at every point of Q.

The authors are unaware of an answer to the following question: Given a bundle, is the requirement that the dimension be constant on some neighborhood about

q sufficient for continuity of the pointwise norms of all homomorphisms at q? Theorem 3.3.5 (2) in the next section gives a positive answer to this question in some particular case.

3.3. An Operator Bundle

In this section, we suggest a number of necessary and sufficient conditions for existence of a Banach bundle $B(\mathscr{X},\mathscr{Y})$ whose continuous sections are homomorphisms from a CBB $\mathscr{X}$ into a CBB $\mathscr{Y}$. Separately treated are the cases of arbitrary bundles $\mathscr{X}$ and $\mathscr{Y}$, bundles with finite-dimensional stalks, and the case of trivial CBBs and CBBs with constant finite dimension.

3.3.1. Let $\mathscr{X}$, $\mathscr{Y}$, and $\mathscr{Z}$ be CBBs over a topological space Q, with $\mathscr{Z}(q) \subset B(\mathscr{X}(q),\mathscr{Y}(q))$ for all $q \in Q$.

Lemma. *The following assertions are equivalent:*

(a) $C(Q,\mathscr{Z}) = \mathrm{Hom}(\mathscr{X},\mathscr{Y})$;

(b) $\mathrm{Hom}(\mathscr{X},\mathscr{Y})$ *is a stalkwise dense subset of* $C(Q,\mathscr{Z})$ *in* $\mathscr{Z}$ (*in other words,* $\mathrm{Hom}(\mathscr{X},\mathscr{Y})$ *is a continuity structure in* $\mathscr{Z}$).

◁ Equivalence of (a) and (b) follows immediately from Corollary 3.2.4. ▷

Obviously, a bundle $\mathscr{Z}$ satisfying condition (a) or (b) of the lemma is unique. This allows us to introduce the following notion.

Definition. The Banach bundle $\mathscr{Z}$ satisfying condition (a) or (b) of the above lemma (if such a bundle exists) is called the *operator bundle* for the CBBs $\mathscr{X}$ and $\mathscr{Y}$ and denoted by the symbol $B(\mathscr{X},\mathscr{Y})$.

The above definition of operator bundle generalizes the analogous notion introduced in [8, 1.2.3] for the case of bundles over extremally disconnected compact Hausdorff spaces.

3.3.2. The following result, repeatedly used throughout the article, provides the basic criterion for existence of an operator bundle.

Theorem. *Let* $\mathscr{X}$ *and* $\mathscr{Y}$ *be* CBBs *over a topological space* Q. *For existence of the bundle* $B(\mathscr{X},\mathscr{Y})$, *it is necessary and sufficient that the pointwise norm of every homomorphism from* $\mathscr{X}$ *into* $\mathscr{Y}$ *be continuous.*

◁ Necessity for continuity of pointwise norms is evident. Sufficiency of this condition may be explained by using the equivalent definition 3.3.1 (b) of an operator bundle. The stalk $B(\mathscr{X},\mathscr{Y})(q)$ for each point $q \in Q$ is the closure of the subspace $\{H(q) : H \in \mathrm{Hom}(\mathscr{X},\mathscr{Y})\}$ in the Banach space $B(\mathscr{X}(q),\mathscr{Y}(q))$. ▷

By [8, Corollary 2.2.2], in the case of an ample CBB $\mathscr{X}$ over an extremally disconnected compact Hausdorff space Q, the pointwise norm of every homomorphism from $\mathscr{X}$ into an arbitrary CBB $\mathscr{Y}$ over Q is continuous. Proven by using this lemma, by [8, 2.2.3] we see that, in the case indicated, the operator bundle $B(\mathscr{X},\mathscr{Y})$ exists. This allows us to regard criterion 3.3.2 as a generalization of [8, Theorem 2.2.3] to the case of an arbitrary CBB over an arbitrary topological space.

3.3.3. Proposition. *If a* CBB $\mathscr{X}$ *over a topological space* Q *has constant finite dimension then, for every* CBB $\mathscr{Y}$ *over* Q*, the operator bundle* $B(\mathscr{X},\mathscr{Y})$ *exists.*

$\lhd$ The claim follows from Corollary 3.2.12 and Theorem 3.3.2. $\rhd$

Examples 3.2.13, with 3.3.2 taken into account, demonstrate that the constant dimension requirement for a bundle $\mathscr{X}$ in the last proposition is essential.

3.3.4. Proposition. *Suppose that a* CBB $\mathscr{X}$ *over a completely regular topological space* Q *has constant finite dimension. Then, for every* CBB $\mathscr{Y}$ *over* Q*, the equality* $B(\mathscr{X},\mathscr{Y})(q) = B\big(\mathscr{X}(q),\mathscr{Y}(q)\big)$ *holds at every point* $q \in Q$*. In particular, if* $\mathscr{X}$ *and* $\mathscr{Y}$ *have constant finite dimension, then* $B(\mathscr{X},\mathscr{Y})$ *has the same property.*

$\lhd$ Fix a point $q \in Q$ and a linear operator $S \in B\big(\mathscr{X}(q),\mathscr{Y}(q)\big)$. If we construct a homomorphism $H \in \operatorname{Hom}(\mathscr{X},\mathscr{Y})$ such that $H(q) = S$ then the claim will be proven.

First, observe that if W is a closed neighborhood about q, a section w over W is continuous (locally bounded), and a function $f \in C(Q)$ vanishes outside W, then the global section $f * w$, defined by the formula

$$(f * w)(p) = \begin{cases} f(p)w(p), & p \in W, \\ 0, & p \notin W, \end{cases}$$

is continuous (locally bounded). Hence, in view of [8, Theorem 1.4.4], given a homomorphism $G \in \operatorname{Hom}_W(\mathscr{X},\mathscr{Y})$, the mapping

$$H = f * G : p \in Q \mapsto H(p) \in B\big(\mathscr{X}(p),\mathscr{Y}(p)\big)$$

is a homomorphism from $\mathscr{X}$ into $\mathscr{Y}$ because the pointwise norm of H is locally bounded and $H \otimes u = f * (G \otimes u) \in C(Q,\mathscr{Y})$ for all $u \in C(Q,\mathscr{X})$.

Recalling the fact that the space Q is completely regular, we can require $f(q) = 1$. Then $H(q) = G(q)$. Therefore, for proving the claim, it suffices to define a homomorphism $G \in \operatorname{Hom}_W(\mathscr{X},\mathscr{Y})$ on any closed neighborhood W about q taking value S at the point q. By [16, Lemma 7], there exists a linear operator $T : \mathscr{X}(q) \to C(Q,\mathscr{X})$ such that, for every $x \in \mathscr{X}(q)$, the inequality $\|x\| \leqslant |\!|\!|Tx|\!|\!|$ holds on some neighborhood U about q. Since, for every point $p \in U$, the operator

$$T_p : x \in \mathscr{X}(q) \mapsto (Tx)(p) \in \mathscr{X}(p)$$

is invertible and the dimension of $\mathscr{X}$ is constant, we conclude that the range of T is stalkwise dense in $\mathscr{X}$ on U. By Dupré's Theorem (see [8, 1.3.5]), there exists a collection of sections $\mathscr{V} \subset C(Q, \mathscr{Y})$ such that $\{v(q) : v \in \mathscr{V}\}$ is a basis for the subspace $\operatorname{Im} S \subset \mathscr{Y}(q)$ on the unit sphere. Therefore, by [16, Lemma 7], there is a linear operator $R : \operatorname{Im} S \to C(Q, \mathscr{Y})$ such that the range of R coincides with the linear span of $\mathscr{V}$ and $|||Ry||| \leqslant 2\,\|y\|$ for every $y \in \operatorname{Im} S$ on some neighborhood V about q. By analogy to the definition of the operators T_p, we consider a linear operator $R_r : \operatorname{Im} S \to \mathscr{Y}(p)$ for every point $r \in V$. It is obvious that the operator R_q is invertible and $\|R_r\| \leqslant 2$ for all $r \in V$. At the same time, for all $p \in U$, the estimate $\|T_p^{-1}\| \leqslant 1$ holds.

Finally, take a closed neighborhood $W \subset U \cap V$ about q and, with each element $p \in W$, associate the linear operator

$$G(p) = R_p \circ R_q^{-1} \circ S \circ T_q \circ T_p^{-1} : \mathscr{X}(p) \to \mathscr{Y}(p).$$

By [8, Theorem 1.4.9], the mapping $G : p \in W \mapsto G(p) \in B\big(\mathscr{X}(p), \mathscr{Y}(p)\big)$ thus obtained is a sought homomorphism, because $G(q) = S$, $|||G||| \leqslant 2\,\|R_q^{-1}\|\,\|S\|\,\|T_q\|$, and $G \otimes u \in C(W, \mathscr{Y})$ for all $u \in \operatorname{Im} T$. $\triangleright$

3.3.5. Assertion (1) of the following theorem under the assumption $\mathscr{Y} = \mathscr{R}$ presents a particular answer to G. Gierz's question [7, 19, Problem 1, p. 231].

Theorem. *Let $\mathscr{X}$ be a* CBB *with finite-dimensional stalks over a completely regular Baire space Q and let $\mathscr{Y}$ be a* CBB *over Q.*

(1) *Given a point q of the everywhere dense set $\bigcup_{n \geqslant 0} \operatorname{int}\{\dim \mathscr{X} = n\}$* (see Proposition 3.2.9) *and an operator $T \in B\big(\mathscr{X}(q), \mathscr{Y}(q)\big)$, there exists a homomorphism $H \in \operatorname{Hom}(\mathscr{X}, \mathscr{Y})$ such that $H(q) = T$ and $|||H||| \leqslant \|T\|$.*

(2) *Suppose that there is a countable base at a point $q \in Q$ and the bundle $\mathscr{Y}$ has nonzero stalks on an everywhere dense set. The pointwise norms of all elements in $\operatorname{Hom}(\mathscr{X}, \mathscr{Y})$ are continuous at q if and only if the dimension of $\mathscr{X}$ is constant on some neighborhood about q.*

$\triangleleft$ (1): Let $0 \leqslant n \in \mathbb{Z}$, $q \in \operatorname{int}\{\dim \mathscr{X} = n\}$, let $U \subset \operatorname{int}\{\dim \mathscr{X} = n\}$ be a closed neighborhood about q, and let $T \in B\big(\mathscr{X}(q), \mathscr{Y}(q)\big)$. From Proposition 3.3.4 and [8, Lemma 1.3.9] we easily infer that there is a homomorphism $G \in \operatorname{Hom}_U(\mathscr{X}, \mathscr{Y})$ such that $G(q) = T$ and $|||G||| \leqslant \|T\|$. Since the space Q is completely regular, there exists a continuous function $f : Q \to [0, 1]$ satisfying the equalities $f(q) = 1$ and $f \equiv 0$ on $Q \backslash U$. It remains to put $H = f * G$ (see the proof of 3.3.4).

(2): Theorem 3.2.12 implies the sufficiency part of the assertion. For proving necessity, suppose that, in every neighborhood about q, there are points at which the dimension of $\mathscr{X}$ is greater than $\dim \mathscr{X}(q) =: m$, and construct a homomorphism $H \in \mathrm{Hom}(\mathscr{X}, \mathscr{Y})$ with discontinuous pointwise norm at q.

By Corollary 3.2.9, the point q belongs to the closure of the open set

$$\bigcup_{n>m} \mathrm{int}\,\{\dim \mathscr{X} = n\};$$

moreover, the hypotheses imply that the set $\{\dim \mathscr{Y} > 0\}$ is open. Since at the point q there is a countable base, we may take a sequence $(q_n) \subset \bigcup_{n>m} \mathrm{int}\,\{\dim \mathscr{X} = n\} \cap \{\dim \mathscr{Y} > 0\}$, $q_i \neq q_j$ $(i \neq j)$, convergent to q.

According to Dupré's Theorem [8, 1.3.5], there exist bounded sections $u_1, \dots, u_m \in C(Q, \mathscr{X})$ with linearly independent values $u_1(q), \dots, u_m(q)$. From [7, Proposition 18.1] it follows that the sections are pointwise linearly independent on an open neighborhood U about q. Without loss of generality, we may assume that $q_n \in U$ for all $n \in \mathbb{N}$.

For every $n \in \mathbb{N}$, the inequality $\dim \mathscr{X}(q_n) > m$ and nondegeneracy of the stalk $\mathscr{Y}(q_n)$ allow us to find an operator $T_n \in B(\mathscr{X}(q_n), \mathscr{Y}(q_n))$ such that $T_n \equiv 0$ on $\mathrm{lin}\{u_1(q_n), \dots, u_m(q_m)\}$ and $\|T_n\| = 1$. By (1), for every number $n \in \mathbb{N}$, there is a homomorphism $H_n \in \mathrm{Hom}(\mathscr{X}, \mathscr{Y})$ satisfying the relations $H_n(q_n) = T_n$ and $|\!|\!| H_n |\!|\!| \leqslant 1$.

Let $\mathscr{X}_0$ be the CBB over U with continuity structure $\mathrm{lin}\{u_1|_U, \dots, u_m|_U\}$, let $\mathscr{Y}_0 = \mathscr{Y}\big|_U$, and let $n \in \mathbb{N}$. By Theorem 3.2.12, the mapping $p \in U \mapsto H_n(p)\big|_{\mathrm{lin}\{u_1(p), \dots, u_m(p)\}} \in B(\mathscr{X}_0(p), \mathscr{Y}_0(p))$ has continuous pointwise norm. Therefore, we can take an open neighborhood $V_n \subset U$ about q_n such that

$$\|H_n(p)\big|_{\mathrm{lin}\{u_1(p), \dots, u_m(p)\}}\| < 1/n$$

for all $p \in V_n$.

By Lemma 3.2.5 (1), there exists a sequence (W_n) of open subsets of Q satisfying the conditions $\mathrm{cl}\, W_n \cap \mathrm{cl} \bigcup_{k \neq n} W_k = \varnothing$, $q_n \in W_n$, and

$$\Big(\mathrm{cl} \bigcup_{n \in \mathbb{N}} W_n\Big) \setminus \bigcup_{n \in \mathbb{N}} \mathrm{cl}\, W_n = \{q\}.$$

We additionally require that $W_n \subset V_n$ for all $n \in \mathbb{N}$. Moreover, consider a sequence of continuous functions $f_n : Q \to [0,1]$ such that $f_n(q_n) = 1$ and $f_n \equiv 0$ in $Q \setminus W_n$. Define

$$H(p) = \begin{cases} f_n(p) H_n(p), & p \in W_n, \\ 0, & p \notin \bigcup_{n \in \mathbb{N}} W_n \end{cases}$$

for all $p \in Q$. It is obvious that $|\!|\!| H |\!|\!| \leqslant 1$. Since the space Q is completely regular, the set $N_q = \{u \in C(Q, \mathscr{X}) : u(q) = 0\}$ enlarges the linear span of $\mathrm{lin}\{u_1, \dots, u_m\}$

to a subset of $C(Q, \mathscr{X})$ stalkwise dense in $\mathscr{X}$. By applying [8, Theorem 1.4.9] to this subset, we show that H is a homomorphism from $\mathscr{X}$ into $\mathscr{Y}$.

If $u \in \operatorname{lin}\{u_1, \dots, u_m\}$ then the series $\sum_{n=1}^{\infty} f_n H_n \otimes u$ uniformly converges. Indeed, its terms have disjoint supports, the pointwise norm of u is bounded, and $|\!|\!| f_n H_n \otimes u |\!|\!| \leqslant \frac{1}{n} |\!|\!| u |\!|\!|$ for all $n \in \mathbb{N}$. Then, by [8, Theorem 1.3.6], the section $H \otimes u$ is continuous as the sum of the series.

Now let $u \in N_q$. The section $H \otimes u$ is continuous on every set $\operatorname{cl} W_n$, $n \in \mathbb{N}$, since $\operatorname{cl} W_n$ is a subset of an open set $Q \setminus \operatorname{cl} \bigcup_{k \neq n} W_k$, and $H \otimes u = f_n H_n \otimes u$ on this subset. If

$$p \in \Big(\operatorname{cl} \bigcup_{n \in \mathbb{N}} W_n\Big) \setminus \bigcup_{n \in \mathbb{N}} \operatorname{cl} W_n$$

then $p = q$ and the section $H \otimes u$ is continuous at p, since $|\!|\!| H |\!|\!| \leqslant 1$ and the function $|\!|\!| u |\!|\!|$ is continuous and vanishes at q. Finally, the set $Q \setminus \operatorname{cl} \bigcup_{n \in \mathbb{N}} W_n$ is open and the equality $|\!|\!| H \otimes u |\!|\!| \equiv 0$ holds on this set.

Thus, $H \in \operatorname{Hom}(\mathscr{X}, \mathscr{Y})$. Furthermore, $|\!|\!| H |\!|\!|(q) = 0$, $|\!|\!| H |\!|\!|(q_n) = 1$ for all $n \in \mathbb{N}$, and $q_n \to q$; therefore, the function $|\!|\!| H |\!|\!|$ is discontinuous at q. $\rhd$

3.3.6. *Theorem.* *Let $\mathscr{X}$ and $\mathscr{Y}$ be* CBBs *over a first-countable completely regular Baire space Q. Suppose that all stalks of $\mathscr{X}$ are finite-dimensional and the bundle $\mathscr{Y}$ has nonzero stalks on an everywhere dense subset of Q. Then the operator bundle $B(\mathscr{X}, \mathscr{Y})$ exists if and only if the sets $\{\dim \mathscr{X} = n\}$ are clopen for all $n = 0, 1, 2, \dots$.*

$\lhd$ Sufficiency of the indicated condition for existence of the bundle $B(\mathscr{X}, \mathscr{Y})$ follows from Proposition 3.3.3.

For proving necessity, observe that, by Theorem 3.3.2 and assertion (2) of Theorem 3.3.5, existence of the bundle $B(\mathscr{X}, \mathscr{Y})$ implies that the sets $\{\dim \mathscr{X} = n\}$ are open for all $n = 0, 1, 2, \dots$. It remains to use Lemma 3.2.8. $\rhd$

3.3.7. The following assertion follows immediately from Theorem 3.3.6.

Corollary. *Let $\mathscr{X}$ be a* CBB *with finite-dimensional stalks over a first-countable connected completely regular Baire topological space Q and let $\mathscr{Y}$ be a* CBB *over Q with nonzero stalks on an everywhere dense subset of Q. Then existence of the bundle $B(\mathscr{X}, \mathscr{Y})$ is equivalent to the fact that the dimension of $\mathscr{X}$ is constant.*

Observe that the space Q satisfying the hypotheses of the above corollary may fail to be metrizable. It is easy to verify that the Nemytskiĭ plane is such a nonmetrizable space (see [5, 1.2.4, 1.4.5, 2.1.10]).

3.3.8. In the rest of this section, we mainly deal with trivial CBBs. For these CBBs, the existence of the bundle $B(X_Q, Y_Q)$ is closely connected with the question whether the inclusion

$$C(Q, B(X, Y)) \subset \mathrm{Hom}(X_Q, Y_Q)$$

is strict (considered in 3.2.3).

Proposition. *Given Banach spaces X and Y, the bundle $B(\mathscr{X}, \mathscr{Y})$ exists if and only if $C(Q, B(X, Y)) = \mathrm{Hom}(X_Q, Y_Q)$. Moreover, if the bundle $B(X_Q, Y_Q)$ exists then it is equal to the trivial* CBB *with stalk $B(X, Y)$.*

◁ We first prove the second assertion. Let $B(X_Q, Y_Q)$ exist. Since the relations $B(X_Q, Y_Q)(q) \subset B(X_Q(q), Y_Q(q)) = B(X, Y)$ are true at each point $q \in Q$ and the relation $C(Q, B(X, Y)) \subset \mathrm{Hom}(X_Q, Y_Q) = C(Q, B(X_Q, Y_Q))$ holds, every stalk of $B(X_Q, Y_Q)$ coincides with the space $B(X, Y)$. In this case, $C(Q, B(X, Y))$ is a continuity structure for both $B(X, Y)_Q$ and $B(X_Q, Y_Q)$; therefore, these two CBBs coincide (see [8, 2.1.8, 2.1.9]). Whence it is immediate that the equality $C(Q, B(X, Y)) = \mathrm{Hom}(X_Q, Y_Q)$ is necessary for existence of $B(X_Q, Y_Q)$. Sufficiency is evident by Theorem 3.3.2. ▷

3.3.9. Corollary. *Let X and Y be Banach spaces and let X be finite-dimensional. Then the bundle $B(X_Q, Y_Q)$ exists and, moreover, $B(X_Q, Y_Q) = B(X, Y)_Q$ and $\mathrm{Hom}(X_Q, Y_Q) = C(Q, B(X, Y))$.*

◁ The claim follows from 3.3.2 and 3.3.8. ▷

3.3.10. Theorem. *Let X be an infinite-dimensional Banach space and let Q be a topological space. Suppose that, for some* CBB *$\mathscr{Y}$ with nonzero stalks, the bundle $B(X_Q, \mathscr{Y})$ exists. Then the space Q is functionally discrete.*

◁ Assume that there exists a not locally constant function in $C(Q)$ and construct a homomorphism H from X_Q into $\mathscr{Y}$ with discontinuous pointwise norm. By the theorem of 3.3.2, the theorem will be thus proven.

Due to Lemma 3.1.13, there exists a weakly* continuous vector valued function $w : Q \to X'$ with bounded and discontinuous pointwise norm. Let q be a discontinuity point of $|\!|\!|w|\!|\!|$. Consider a section $v \in C(Q, \mathscr{Y})$ with nonzero value $v(q)$ and define a mapping $H : q \in Q \mapsto H(q) \in B(X, \mathscr{Y}(q))$ by the rule $H(q) : x \in X \mapsto \langle x | w(q) \rangle v(q)$ for all $q \in Q$. Then, for every constant section $u \in C(Q, X_Q)$, the equality $H \otimes u = \langle u | w \rangle v \in C(Q, \mathscr{Y})$ holds. Moreover, $|\!|\!|H|\!|\!| = |\!|\!|w|\!|\!| |\!|\!|v|\!|\!|$. Boundedness of $|\!|\!|w|\!|\!|$ implies local boundedness of $|\!|\!|H|\!|\!|$. Therefore, $H \in \mathrm{Hom}(X_Q, \mathscr{Y})$ by [8, Theorem 1.4.9]. Finally, since the function $|\!|\!|w|\!|\!|$ is discontinuous at q, and the function $|\!|\!|v|\!|\!|$ is continuous and nonzero at this point, $|\!|\!|H|\!|\!| = |\!|\!|w|\!|\!| |\!|\!|v|\!|\!| \notin C(Q)$. ▷

Below (see 3.3.13) we show that, in the last theorem, the necessary condition for existence of the operator bundle $B(X_Q, \mathscr{Y})$ (namely, functional discreteness of Q) is also sufficient in case the Banach space X is separable. In general, this condition is not sufficient (cf. Proposition 3.3.14 as applied to the Banach space X' and the bundle $\mathscr{Y} = \mathscr{R}$).

3.3.11. Proposition. *Let X and Y be Banach spaces, $Y \neq \{0\}$, and let Q be a topological space that is not functionally discrete. The following are equivalent:*

(a) *the Banach bundle $B(X_Q, Y_Q)$ exists;*
(b) $B(X,Y)_Q = B(X_Q, Y_Q)$;
(c) $\operatorname{Hom}(X_Q, Y_Q) = C(Q, B(X,Y))$;
(d) *X is finite-dimensional.*

$\lhd$ Equivalence of (a), (b), and (c) is proven in 3.3.8, (d) follows from (a) by 3.3.10, and (a) follows from (d) by 3.3.9. $\rhd$

3.3.12. Proposition. *Let $\mathscr{X}$ be a* CBB *over a functionally discrete topological space Q. Suppose that $C(Q, \mathscr{X})$ includes a countable subset stalkwise dense in $\mathscr{X}$. Then, for every* CBB *$\mathscr{Y}$ over Q, the bundle $B(\mathscr{X}, \mathscr{Y})$ exists.*

$\lhd$ Let $\mathscr{U} \subset C(Q, \mathscr{X})$ be a countable subset stalkwise dense in $\mathscr{X}$. Consider an arbitrary CBB $\mathscr{Y}$ over Q, a homomorphism $H \in \operatorname{Hom}(\mathscr{X}, \mathscr{Y})$, and a point $q \in Q$ and prove continuity for the pointwise norm of H at q. Since the space Q is functionally discrete, there is a neighborhood U about q on which all functions $|\!|\!|u|\!|\!|$, $|\!|\!|H \otimes u|\!|\!|$, $u \in \mathscr{U}$, are constant. In view of stalkwise denseness of $\mathscr{U}$ in $\mathscr{X}$, the equality $|\!|\!|H|\!|\!|(p) = \sup\{|\!|\!|H \otimes u|\!|\!|(p) : u \in \mathscr{U},\ |\!|\!|u|\!|\!|(p) \leqslant 1\}$ holds for every point $p \in Q$; therefore, the function $|\!|\!|H|\!|\!|$ is constant on U and, in particular, $|\!|\!|H|\!|\!|$ is continuous at q. It remains to use Theorem 3.3.2. $\rhd$

3.3.13. Corollary. *Let Q be an arbitrary topological space and let X be a separable infinite-dimensional Banach space. The following are equivalent:*

(a) *for every* CBB *$\mathscr{Y}$ over Q, the bundle $B(X_Q, \mathscr{Y})$ exists;*
(b) *the bundle $B(X_Q, \mathscr{R})$ exists;*
(c) *the space Q is functionally discrete.*

$\lhd$ The implication (a)$\to$(b) is evident, (c) follows from (b) by 3.3.10, and (a) follows from (c) by 3.3.12. $\rhd$

3.3.14. Proposition. *Let X be a nonseparable Banach space. There exists a functionally discrete normal topological space Q such that, for every* CBB *$\mathscr{Y}$ over Q with nonzero stalks, the bundle $B(X_Q, \mathscr{Y})$ does not exist.*

$\lhd$ Given a subset $F \subset X$, denote by the symbol $F^\perp$ the annihilator of F, i.e., $F^\perp = \{x' \in X' : \langle x | x' \rangle = 0 \text{ for all } x \in F\}$. Consider the set

$$\aleph = \{F^\perp : F \text{ is a countable subset of } X\}$$

ordered by the rule

$$F_1^{\perp} \leqslant F_2^{\perp} \;\Leftrightarrow\; F_1^{\perp} \supset F_2^{\perp}.$$

It is easy to see that all countable subsets of $\aleph$ have upper bounds. Moreover, $\aleph$ has no greatest element. Indeed, since the space X is nonseparable, for every annihilator $F^{\perp} \in \aleph$, there exists a nonzero element $x \in X$ outside the closure of the linear span of F. On the other hand, there is a functional in $F^{\perp}$ with nonzero value at x. Whence, $F^{\perp} < (F \cup \{x\})^{\perp}$.

As is shown in 3.1.11, the space $Q := \aleph^{\bullet}$ is normal and functionally discrete.

Let $\mathscr{Y}$ be an arbitrary CBB over Q with nonzero stalks. Construct a homomorphism $H \in \operatorname{Hom}(X_Q, \mathscr{Y})$ with discontinuous pointwise norm. To this end, consider a section $v \in C(Q, \mathscr{Y})$ taking nonzero value at the point $\infty \in Q$. Since $\{0\} \notin \aleph$, for every element $\alpha \in \aleph$, we may take a norm-one functional $x'_{\alpha} \in \alpha$. Let $H(\alpha) = v(\alpha) \otimes x'_{\alpha}$ for all $\alpha \in \aleph$ and let $H(\infty) = 0$. Then, by [8, Theorem 1.4.9], the mapping H is a homomorphism, since, for every constant section $u_x \equiv x$, $x \in X$, the section $H \otimes u_x$ vanishes on the interval $(\{x\}^{\perp}, \infty]$; therefore, $H \otimes u_x$ is continuous. At the same time, the pointwise norm of H is discontinuous at ∞. Consequently, by Theorem 3.3.2 the bundle $B(X_Q, \mathscr{Y})$ does not exist. $\rhd$

3.3.15. Lemma. *Let $\aleph$ be an upward-directed set without greatest element and let $\mathscr{X}$ be a* CBB *over $\aleph^{\bullet}$ (see 3.1.11). Suppose that in $C(\aleph^{\bullet}, \mathscr{X})$ there is a stalkwise dense subset such that every subset of $\aleph$ of the same cardinality has an upper bound. Then, for every* CBB *$\mathscr{Y}$ over $\aleph^{\bullet}$, the bundle $B(\mathscr{X}, \mathscr{Y})$ exists.*

$\lhd$ Let $\mathscr{U}$ be a subset of $C(\aleph^{\bullet}, \mathscr{X})$ satisfying the hypotheses of the lemma.

Consider an arbitrary CBB $\mathscr{Y}$ over $\aleph^{\bullet}$ and verify continuity for the pointwise norm of an arbitrary homomorphism $H \in \operatorname{Hom}(\mathscr{X}, \mathscr{Y})$. Hence, by Theorem 3.3.2, the assertion will be proven.

For every element $u \in \mathscr{U}$, take an $\alpha_u \in \aleph$ such that $|\!|\!| u |\!|\!|(\alpha) = |\!|\!| u |\!|\!|(\infty)$ and $|\!|\!| H \otimes u |\!|\!|(\alpha) = |\!|\!| H \otimes u |\!|\!|(\infty)$ for all $\alpha \geqslant \alpha_u$ $\big($see Remark 3.1.11 (1)$\big)$. Then, for every $u \in \mathscr{U}$, the two latter equalities hold for $\alpha \geqslant \beta$, where β is an upper bound for the set $\{\alpha_u : u \in \mathscr{U}\}$. Since $\mathscr{U}$ is stalkwise dense in $\mathscr{X}$, the value of the norm $|\!|\!| H |\!|\!|$ can be calculated at every point $\alpha \in \aleph^{\bullet}$ by the formula $|\!|\!| H |\!|\!|(\alpha) = \sup\{ |\!|\!| H \otimes u |\!|\!|(\alpha) : u \in \mathscr{U},\ |\!|\!| u |\!|\!|(\alpha) \leqslant 1\}$. From this formula we readily see that, for $\alpha \geqslant \beta$, the pointwise norm of H takes the value $|\!|\!| H |\!|\!|(\alpha) = |\!|\!| H |\!|\!|(\infty)$ and, therefore, is continuous. $\rhd$

Corollary. *Given a Banach space X, there is a nondiscrete normal topological space Q such that, for every* CBB *$\mathscr{Y}$ over Q, the bundle $B(X_Q, \mathscr{Y})$ exists.*

$\lhd$ It is sufficient to take $Q = \aleph^{\bullet}$, where $\aleph$ is a cardinal greater than the cardinality of X, and use Lemma 3.3.15. $\rhd$

3.4. The Dual of a Banach Bundle

In this section, we consider the problem of existence and the properties of the bundle $\mathscr{X}'$ dual to a Banach bundle $\mathscr{X}$.

In 3.4.2, we state various necessary and sufficient conditions for existence of a dual bundle. All assertions in the subsection are direct consequences of results of the preceding section. Proposition 3.4.3 asserts existence for a dual bundle of a CBB with Hilbert stalks.

One of the natural steps in studying the notion of a dual bundle is establishing norming duality relations between the bundles $\mathscr{X}$ and $\mathscr{X}'$. Item 3.4.5 is devoted to this subject. As a preliminary, in 3.4.4, we discuss the condition that the stalks of a CBB are stalkwise normed by the values of the corresponding homomorphisms. Unfortunately, we have to leave open the question whether this condition always holds, restricting ourselves to listing certain situations in which the condition is satisfied.

In 3.4.6–3.4.9, the interrelation is considered between separability of a distinguished stalk of a CBB and finiteness of the dimension of the stalks of the bundle or of the stalks of its dual.

The rest of the section (3.4.10–3.4.15) is devoted to studying the second dual bundle, $\mathscr{X}''$. Among the topics considered here, are existence of $\mathscr{X}''$, isometry between the bundles under study, and embedding of a Banach bundle into its second dual.

3.4.1. Definition. Let $\mathscr{X}$ be a continuous Banach bundle. The bundle $B(\mathscr{X},\mathscr{R})$ (whenever the latter exists) is called the *dual* of $\mathscr{X}$ and denoted by the symbol $\mathscr{X}'$. If the bundle $\mathscr{X}'$ exists then we say that $\mathscr{X}$ *has the dual bundle.*

By Theorem 3.3.2, the dual $\mathscr{X}'$ exists if and only if the pointwise norms of all homomorphisms from $\mathscr{X}$ into $\mathscr{R}$ are continuous.

3.4.2. *Proposition.* *The following are true:*

(1) *Every* CBB $\mathscr{X}$ *with constant finite dimension over a topological space Q has the dual bundle. Moreover, if Q is completely regular then $\mathscr{X}'(q) = \mathscr{X}(q)'$ for all $q \in Q$.*

(2) *A* CBB $\mathscr{X}$ *with finite-dimensional stalks over a first-countable completely regular Baire topological space has the dual bundle if and only if* $\{\dim \mathscr{X} = n\}$ *is a clopen set for every* $n = 0, 1, 2, \dots$.

(3) *Suppose that a trivial* CBB *with stalk X has the dual bundle. Then the latter is the trivial* CBB *with stalk X'.*

(4) *If a trivial* CBB *with infinite-dimensional stalk over a topological space Q has the dual bundle then Q is functionally discrete (if, in addition, Q is completely regular then all of its countable subsets are closed).*

(5) *For every nonseparable Banach space X, there exists a functionally discrete topological space Q such that the* CBB X_Q *has no dual bundle.*

(6) *A trivial* CBB *with infinite-dimensional separable stalk over a topological space Q has the dual bundle if and only if Q is functionally discrete.*

(7) *For every Banach space X, there exists a nondiscrete normal topological space Q such that the* CBB X_Q *has the dual bundle.*

(8) *If a topological space Q is not functionally discrete then, for every Banach space X, the following are equivalent:*
- **(a)** *the dual $(X_Q)'$ exists;*
- **(b)** $(X')_Q = (X_Q)'$;
- **(c)** $C(Q, X') = \mathrm{Hom}(X_Q, \mathscr{R})$;
- **(d)** X *is finite-dimensional.*

◁ Assertions (1)–(8) follow directly from 3.3.3 and 3.3.4, 3.3.6, 3.3.8, 3.3.10, 3.3.14, 3.3.13, Corollary 3.3.15, and 3.3.11. ▷

Remark. Examples 3.2.13 (1)–(3), with 3.3.2 taken into account, imply that the constant dimension requirement in assertion (1) of the above proposition is essential for existence of a dual bundle.

3.4.3. Lemma. *Let Q be a topological space and let $\mathscr{X}$ be a* CBB *over Q with Hilbert stalks (i.e., all stalks of $\mathscr{X}$ are Hilbert spaces). For every global section u of $\mathscr{X}$ and every point $q \in Q$ put*

$$h(u)(q) = \langle \cdot, u(q) \rangle \in \mathscr{X}(q)'.$$

Then $h[C(Q, \mathscr{X})] \subset \mathrm{Hom}(\mathscr{X}, \mathscr{R})$. Moreover, $h[C(Q, \mathscr{X})]$ is a continuity structure in the (discrete) Banach bundle with stalks $\mathscr{X}(q)'$ $(q \in Q)$.

◁ By [8, 1.4.4], the inclusion $h[C(Q, \mathscr{X})] \subset \mathrm{Hom}(\mathscr{X}, \mathscr{R})$ follows from the relations

$$\langle u_1 | h(u_2) \rangle = \langle u_1(\cdot), u_2(\cdot) \rangle = \frac{1}{2}\Big(|\!|\!| u_1 |\!|\!|^2 + |\!|\!| u_2 |\!|\!|^2 - |\!|\!| u_1 - u_2 |\!|\!|^2 \Big) \in C(Q),$$
$$|\!|\!| h(u_2) |\!|\!| = |\!|\!| u_2 |\!|\!|$$

valid for all $u_1, u_2 \in C(Q, \mathscr{X})$. The second assertion follows from the Riesz Theorem. ▷

Proposition. *Let $\mathscr{X}$ be a CBB with Hilbert stalks. If the dual $\mathscr{X}'$ exists then $\mathscr{X}'$ is isometric to $\mathscr{X}$* (see [8, 1.4.12]).

◁ Let Q be a topological space and let $\mathscr{X}$ be a CBB over Q with Hilbert stalks. Consider a CBB $\mathscr{Y}$ with stalks $\mathscr{Y}(q) = \mathscr{X}(q)'$ $(q \in Q)$ and continuity structure $\mathscr{C} = h[C(Q, \mathscr{X})]$ (see the previous lemma). By [8, Theorem 1.4.12 (3)], the bundles $\mathscr{X}$ and $\mathscr{Y}$ are isometric. Stalkwise denseness of $\mathscr{C}$ in $\mathscr{Y}$ and the relations $\mathscr{C} \subset \operatorname{Hom}(\mathscr{X}, \mathscr{R}) = C(Q, \mathscr{X}')$ imply that, at every point $q \in Q$, the stalks $\mathscr{X}'(q)$ and $\mathscr{Y}(q)$ coincide and $\mathscr{C}$ is a continuity structure in $\mathscr{X}'$, i.e., $\mathscr{X}' = \mathscr{Y}$. ▷

3.4.4. DEFINITION. Let $\mathscr{X}$ be a CBB over a topological space Q. Say that $\operatorname{Hom}(\mathscr{X}, \mathscr{R})$ *norms* $\mathscr{X}$ *on a subset* $D \subset Q$ if, for every point $q \in D$ and every $x \in \mathscr{X}(q)$, the equality $\|x\| = \sup\{|\langle x | H(q)\rangle| : H \in \operatorname{Hom}(\mathscr{X}, \mathscr{R}),\ |\!|\!|H|\!|\!| \leqslant 1\}$ holds. Say that $\operatorname{Hom}(\mathscr{X}, \mathscr{R})$ *norms* $\mathscr{X}$ if $\operatorname{Hom}(\mathscr{X}, \mathscr{R})$ norms $\mathscr{X}$ on Q.

We are not aware of an example of a CBB $\mathscr{X}$ for which $\operatorname{Hom}(\mathscr{X}, \mathscr{R})$ does not norm $\mathscr{X}$. (Moreover, we do not know if there exists a nonzero Banach bundle whose dual is zero.) At present, we can only indicate some classes of Banach bundles $\mathscr{X}$ for which $\operatorname{Hom}(\mathscr{X}, \mathscr{R})$ does norm $\mathscr{X}$. The following bundles fall in such a class:

(1) a CBB $\mathscr{X}$ over a topological space Q such that, for every $q \in Q$, the set $\{H(q) : H \in \operatorname{Hom}(\mathscr{X}, \mathscr{R})\} \subset \mathscr{X}(q)'$ norms $\mathscr{X}(q)$ and, for every homomorphism $H \in \operatorname{Hom}(\mathscr{X}, \mathscr{R})$, there is a homomorphism $G \in \operatorname{Hom}(\mathscr{X}, \mathscr{R})$ such that $G(q) = H(q)$ and $|\!|\!|G|\!|\!| \in C(Q)$;

(2) a CBB $\mathscr{X}$ over a completely regular topological space Q satisfying the following conditions: for every $q \in Q$, the set $\{H(q) : H \in \operatorname{Hom}(\mathscr{X}, \mathscr{R})\} \subset \mathscr{X}(q)'$ norms $\mathscr{X}(q)$ and, for every homomorphism $H \in \operatorname{Hom}(\mathscr{X}, \mathscr{R})$, there is a homomorphism $G \in \operatorname{Hom}(\mathscr{X}, \mathscr{R})$ such that $G(q) = H(q)$ and the pointwise norm of G is continuous at q;

(3) a trivial CBB;

(4) a CBB with constant finite dimension over a completely regular topological space;

(5) a CBB over a compact topological space or a locally compact Hausdorff topological space which admits a countable stalkwise dense set of continuous sections;

(6) a CBB with finite-dimensional stalks over a metrizable locally compact space;

(7) a CBB with Hilbert stalks;

(8) a CBB over a regular extremally disconnected topological space;

(9) a CBB over $\overline{\mathbb{N}}$ with separable stalk at ∞;

(10) a CBB $\mathscr{X}$ over a Hausdorff topological space with finitely many nonisolated points such that the stalks of $\mathscr{X}$ at these points are separable;

(11) the dual of a CBB.

$\lhd$ A proof of the fact that $\mathrm{Hom}(\mathscr{X},\mathscr{R})$ norms $\mathscr{X}$ in cases (1) and (2) can be easily obtained by multiplying the homomorphism G by a suitable element of $C(Q)$.

Cases (3), (4), and (7) are easily reduces to case (1) with the help of Corollary 3.2.3, Proposition 3.4.2 (1), and Lemma 3.4.3 respectively.

Case (5) for a compact topological space is considered in [7, 19.16], and the case of a locally compact Hausdorff (and, hence, completely regular) space is reduced to the case of a compact space by employing a compact neighborhood about an arbitrary point q and multiplying the homomorphism by a continuous real-valued function equal to unity at q and vanishing outside the neighborhood under consideration. By analogous reasoning, case (6) can be reduced to (5) with the help of assertion [7, 19.5 (iii)].

(8): Let $\mathscr{X}$ be a CBB over a regular extremally disconnected topological space D. Consider an extremally disconnected compact space Q that includes D as an everywhere dense subset, and let $\beta\mathscr{X}$ be the Stone–Čech extension of $\mathscr{X}$ onto Q (see [8, 1.1.4, 2.5.10]). Denote by $\overline{\beta\mathscr{X}}$ the ample hull of $\beta\mathscr{X}$ (see [8, 2.1.5]). With every homomorphism $\overline{H} \in \mathrm{Hom}(\overline{\beta\mathscr{X}},\mathscr{R})$ associate the mapping $H : q \in Q \mapsto \overline{H}(q)|_{\beta\mathscr{X}(q)}$, $q \in Q$. From [8, 1.4.4] it follows that $H \in \mathrm{Hom}(\beta\mathscr{X},\mathscr{R})$. Applying [8, Theorem 2.3.3 (1)] to the bundle $\overline{\beta\mathscr{X}}$, we conclude that $\mathrm{Hom}(\beta\mathscr{X},\mathscr{R})$ norms $\beta\mathscr{X}$. It remains to observe that $\{H|_D : H \in \mathrm{Hom}(\beta\mathscr{X},\mathscr{R})\} \subset \mathrm{Hom}_D(\mathscr{X},\mathscr{R})$.

(10): If a Hausdorff topological space Q has finitely many nonisolated points then, as is easily seen, each of these points is separated from the other nonisolated points by a clopen neighborhood. Consequently, without loss of generality, we may assume that Q has a single nonisolated point q.

Let $\mathscr{X}$ be a CBB over Q with the stalk $\mathscr{X}(q)$ separable. It is sufficient, given an $x' \in \mathscr{X}(q)'$, $\|x'\| < 1$, to construct a homomorphism $H \in \mathrm{Hom}(\mathscr{X},\mathscr{R})$ taking the value $H(q) = x'$ and satisfying the inequality $|\!|\!| H |\!|\!| \leqslant 1$.

Consider a countable system $\{x_n : n \in \mathbb{N}\}$ of linearly independent elements in $\mathscr{X}(q)$ whose linear span is everywhere dense in $\mathscr{X}(q)$ and, employing Dupré's Theorem (see [8, 1.3.5]), with each number $n \in \mathbb{N}$ associate a section $u_n \in C(Q, \mathscr{X})$ passing through x_n at q. By [7, Proposition 18.1], for every $n \in \mathbb{N}$, there exists a neighborhood U_n about q such that the sections $u_1, \dots, u_n$ are pointwise linearly independent over U_n. For all $n \in \mathbb{N}$ and $p \in U_n$, define a functional $y_n(p) : \mathrm{lin}\{u_1(p), \dots, u_n(p)\} \to \mathbb{R}$ by the formula $\langle u_i(p)|y_n(p)\rangle = \langle u_i(q)|x'\rangle$, $i = 1, \dots, n$.

Since $\|x'\| < 1$, in view of [16, Lemma 7], each neighborhood U_n about q can be replaced by a smaller neighborhood V_n so that the inequalities $\|y_n(p)\| \leqslant 1$ be valid for all $p \in V_n$. Without loss of generality, we may assume that $V_n \supset V_{n+1}$ for all $n \in \mathbb{N}$.

The fact that the set $\{u_n : n \in \mathbb{N}\}$ is pointwise linearly independent over

$$V_\infty = \bigcap_{n\in\mathbb{N}} V_n$$

allows us, for every point $p \in V_\infty$, to define a functional $y_\infty(p) : \operatorname{lin}\{u_n(p) : n \in \mathbb{N}\} \to \mathbb{R}$ as a common extension of the functionals $y_n(p)$, $n \in \mathbb{N}$, i.e., to put $\langle u_n(p)|y_\infty(p)\rangle = \langle u_n(q)|x'\rangle$ for all $n \in \mathbb{N}$. Observe that $\|y_\infty(p)\| \leqslant 1$ for $p \in V_\infty$.

Define

$$H(p) := \begin{cases} 0, & p \notin V_1; \\ \overline{y}_n(p), & p \in V_n \backslash V_{n+1}; \\ \overline{y}_\infty(p), & p \in V_\infty, \end{cases}$$

where $\overline{y}_n(p)$, $1 \leqslant n \leqslant \infty$, is an arbitrary extension of $y_n(p)$ onto the entire stalk $\mathscr{X}(p)$ with norm preserved. It is clear that $H(q) = x'$ and $|\!|\!| H |\!|\!| \leqslant 1$.

Denote by $\mathscr{U}$ the set $\operatorname{lin}\{u_n : n \in \mathbb{N}\}$ complemented by all sections with singleton supports. Obviously, the set $\mathscr{U}$ is stalkwise dense in $\mathscr{X}$ and, for each $u \in \mathscr{U}$, the function $\langle u|H\rangle$ is constant on some neighborhood about q and, hence, continuous. Consequently, by Theorem [8, 1.4.4], the mapping H is a homomorphism.

(9): This is a particular case of (10).

(11): Let Q be a topological space and let $\mathscr{X}$ be a CBB over Q which has the dual bundle. From [8, 1.3.9] it follows that, for every point $q \in Q$ and every functional $x' \in \mathscr{X}'(q)$, the relation

$$\|x'\| = \sup\big\{\langle u(q)|x'\rangle : u \in C(Q, \mathscr{X})\big\}$$

holds. On the other hand, by [8, Theorem 1.4.4], for each section $u \in C(Q, \mathscr{X})$, the mapping

$$u'' : q \in Q \mapsto u(q)|_{\mathscr{X}'(q)}$$

belongs to $\operatorname{Hom}(\mathscr{X}', \mathscr{R})$ and, moreover, $\|u''\| \leqslant \|u\|$. Consequently, $\operatorname{Hom}(\mathscr{X}', \mathscr{R})$ norms $\mathscr{X}'$. ⊳

3.4.5. Assertion (3) of the following proposition gives a positive answer to G. Gierz's question [7, 19, Problem 2, p. 231] for the bundles 3.4.4 (1)–(11) as well as for bundles with finite-dimensional stalks over completely regular Baire spaces (see Theorem 3.3.5 (1)).

Proposition. *Let $\mathscr{X}$ be a* CBB *over a topological space Q.*

(1) *Suppose that $\mathscr{X}$ has the dual bundle. Then, for every point $q \in Q$ and every element $x' \in \mathscr{X}'(q)$, the equality*

$$\|x'\| = \sup\left\{\left|\langle u(q)|x'\rangle\right| : u \in C(Q, \mathscr{X}),\ \|\!|u|\!\| \leqslant 1\right\}$$

holds. In particular, for every section $u' \in C(Q, \mathscr{X}')$, the relation

$$\|\!|u'|\!\| = \sup\left\{\left|\langle u|u'\rangle\right| : u \in C(Q, \mathscr{X}),\ \|\!|u|\!\| \leqslant 1\right\}$$

holds in the vector lattice $C(Q)$.

Suppose that $\mathrm{Hom}(\mathscr{X}, \mathscr{R})$ *norms $\mathscr{X}$ on an everywhere dense subset of Q.*

(2) *For every section $u \in C(Q, \mathscr{X})$, the relation*

$$\|\!|u|\!\| = \sup\left\{\left|\langle u|H\rangle\right| : H \in \mathrm{Hom}(\mathscr{X}, \mathscr{R}),\ \|\!|H|\!\| \leqslant 1\right\}$$

holds in the vector lattice $C(Q)$.

(3) *The uniform norm of every section $u \in C^b(Q, \mathscr{X})$ is calculated by the formula*

$$\|u\|_\infty = \sup\left\{\left\|\langle u|H\rangle\right\|_\infty : H \in \mathrm{Hom}(\mathscr{X}, \mathscr{R}),\ \|\!|H|\!\| \leqslant 1\right\}.$$

$\lhd$ (1): Since $x' \in \mathscr{X}(q)'$ and the set $C(Q, \mathscr{X})$ is stalkwise dense in $\mathscr{X}$, there is a sequence of sections $(u_n) \subset C(Q, \mathscr{X})$ such that $\|\!|u_n|\!\|(q) \leqslant 1$ and $\|x'\| - 1/n \leqslant \langle u_n(q)|x'\rangle \leqslant \|x'\|$ for all $n \in \mathbb{N}$. It remains to observe that, by [8, Lemma 1.3.9], for every n, there is a section $v_n \in C(Q, \mathscr{X})$ satisfying the relations $v_n(q) = u_n(q)$ and $\|\!|v_n|\!\| \leqslant 1$.

(2): Let D be an everywhere dense subset of Q on which $\mathrm{Hom}(\mathscr{X}, \mathscr{R})$ norms $\mathscr{X}$ and consider an arbitrary section $u \in C(Q, \mathscr{X})$ and put

$$\mathscr{F} = \left\{\langle u|H\rangle : H \in \mathrm{Hom}(\mathscr{X}, \mathscr{R}),\ \|\!|H|\!\| \leqslant 1\right\}.$$

It is clear that $\|\!|u|\!\|$ is an upper bound for $\mathscr{F}$. If $g \in C(Q)$ is an arbitrary upper bound of $\mathscr{F}$ then it is easy to see that, for every point $q \in D$,

$$g(q) \geqslant \sup_{f \in \mathscr{F}} f(q) = \|\!|u|\!\|(q);$$

hence, $g \geqslant \|\!|u|\!\|$.

(3): Let $u \in C^b(Q, \mathscr{X})$. It is clear that

$$\|u\|_\infty \geqslant \sup\left\{\left\|\langle u|H\rangle\right\|_\infty : H \in \mathrm{Hom}(\mathscr{X}, \mathscr{R}),\ \|\!|H|\!\| \leqslant 1\right\}.$$

To prove the assertion, for an arbitrary $\varepsilon > 0$, find a homomorphism H belonging to $\mathrm{Hom}(\mathscr{X},\mathscr{R})$ with $|||H||| \leqslant 1$ and such that $\|u\|_\infty - \varepsilon < \|\langle u|H\rangle\|_\infty$.

Consider a point $q \in Q$ satisfying the inequality $|||u|||(q) > \|u\|_\infty - \varepsilon$ and a neighborhood U about this point on which $|||u||| > \|u\|_\infty - \varepsilon$. Since $\mathrm{Hom}(\mathscr{X},\mathscr{R})$ norms $\mathscr{X}$ on an everywhere dense subset of Q, there is a $p \in U$ such that

$$|||u|||(p) = \|u(p)\| = \sup\left\{\left|\langle u(p)|H(p)\rangle\right| : H \in \mathrm{Hom}(\mathscr{X},\mathscr{R}),\ |||H||| \leqslant 1\right\};$$

therefore, $\|u\|_\infty - \varepsilon < \left|\langle u(p)|H(p)\rangle\right|$ for some homomorphism $H \in \mathrm{Hom}(\mathscr{X},\mathscr{R})$, $|||H||| \leqslant 1$. Consequently, $\|u\|_\infty - \varepsilon < \|\langle u|H\rangle\|_\infty$. $\triangleright$

3.4.6. Theorem. *Let Q be a completely regular topological space and let $q \in Q$ be a nonisolated point at which there is a countable base. Suppose that a* CBB *$\mathscr{X}$ over Q has the dual bundle. Then separability of the stalk $\mathscr{X}(q)$ implies that the stalk $\mathscr{X}'(q)$ is finite-dimensional.*

$\triangleleft$ Suppose that the stalk $\mathscr{X}(q)$ is separable and the stalk $\mathscr{X}'(q)$ is infinite-dimensional. We will construct a homomorphism H from $\mathscr{X}$ into $\mathscr{R}$ with discontinuous norm and thus, according to Theorem 3.3.2, obtain a contradiction with the hypotheses.

Let a set $\{x_n : n \in \mathbb{N}\}$ be everywhere dense in $\mathscr{X}(q)$ and let (x'_n) be a weakly* null sequence of elements in $\mathscr{X}'(q)$ such that $\|x'_n\| = 1$ for every $n \in \mathbb{N}$ (see 3.1.3). We assume that $|\langle x_i|x'_n\rangle| < 1/n$ for $i = 1,\dots,n$, since this can be fulfilled by passing to a subsequence. Making use of Dupré's Theorem (see [8, 1.3.5]), for every $n \in \mathbb{N}$, consider sections $u_n \in C(Q,\mathscr{X})$ and $v_n \in C(Q,\mathscr{X}')$ such that $u_n(q) = x_n$ and $v_n(q) = x'_n$.

Let $(U_n)_{n\in\mathbb{N}}$ be a neighborhood base at q. Since Q is a Hausdorff space, by induction we can construct a new neighborhood base $(V_n)_{n\in\mathbb{N}}$ at q such that, for every $n \in \mathbb{N}$, the following conditions hold: $V_{n+1} \subset V_n \cap U_1 \cap \dots \cap U_n$, the difference $V_n \backslash V_{n+1}$ contains a point q_n together with an open neighborhood W_n about q_n, and the estimates $1/2 < |||v_n||| < 2$ and $|\langle u_i|v_n\rangle| < 1/n$, $i = 1,\dots,n$, hold on V_n. Show that, for every continuous section $u \in C(Q,\mathscr{X})$ and an arbitrary $\varepsilon > 0$, for n large enough, the inequality $|\langle u|v_n\rangle| < \varepsilon$ holds on V_n. Indeed, let $\|u(q) - x_k\| < \varepsilon/4$ and $1/l < \varepsilon/2$ for some $k, l \in \mathbb{N}$. Take an element V_m of the constructed neighborhood base about q on which $|||u - u_k||| < \varepsilon/4$. Then, for every $n \geqslant \max\{k, l, m\}$, the following relations hold on V_n:

$$\begin{aligned}|\langle u|v_n\rangle| &\leqslant |\langle u - u_k \mid v_n\rangle| + |\langle u_k|v_n\rangle| \\ &< |||u - u_k|||\,|||v_n||| + 1/n < \frac{\varepsilon}{4}\cdot 2 + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Now define a mapping $H : p \in Q \mapsto H(p) \in \mathscr{X}'(p)$. Put $H(p) = 0 \in \mathscr{X}(p)'$ whenever $p \notin \bigcup_{n\in\mathbb{N}} W_n$ and, for every $n \in \mathbb{N}$, put $H\big|_{W_n} = (f_n v_n)\big|_{W_n}$, where $f_n : Q \to [0,1]$ is a continuous function equal to 1 at q_n and vanishing outside W_n.

The function $\langle u|v\rangle : Q \to \mathbb{R}$ is continuous as the pointwise sum of the series $\sum_{n=1}^{\infty} f_n \langle u|v_n\rangle$ that uniformly converges due to pairwise disjointness of the sets W_n ($n \in \mathbb{N}$) and the relations $\operatorname{supp} f_n \subset W_n$ and $\sup_{W_n} |\langle u|v_n\rangle| \leqslant \sup_{V_n} |\langle u|v_n\rangle| \to 0$ as $n \to \infty$.

Thus, H is a homomorphism, since $|\!|\!|H|\!|\!| \leqslant 2$ (see [8, 1.4.4]). At the same time, $|\!|\!|H|\!|\!|(q_n) = |f_n(q_n)| |\!|\!|v_n|\!|\!|(q_n) = |\!|\!|v_n|\!|\!|(q_n) > 1/2$ for every $n \in \mathbb{N}$. Moreover, $q_n \to q$ and $|\!|\!|H|\!|\!|(q) = 0$. Consequently, the homomorphism H has discontinuous pointwise norm. $\triangleright$

3.4.7. Corollary. *Let Q be a completely regular topological space and let $q \in Q$ be a nonisolated point at which there is a countable base. Suppose that a* CBB *$\mathscr{X}$ over Q with Hilbert stalks has the dual bundle. Then the stalk $\mathscr{X}(q)$ is separable if and only if it is finite-dimensional.*

Thus, if a CBB $\mathscr{X}$ with Hilbert stalks over a completely regular topological space has the dual bundle, then the stalk of $\mathscr{X}$ at a nonisolated point with a countable base cannot be isometric to ℓ^2.

3.4.8. Proposition. *Let $Q = \overline{\mathbb{N}}$ be the one-point compactification of the set of naturals. A* CBB *$\mathscr{X}$ over Q with the stalk $\mathscr{X}(\infty)$ separable has the dual bundle if and only if the dimension of $\mathscr{X}$ is finite and constant on some neighborhood about ∞.*

$\triangleleft$ Sufficiency follows from Proposition 3.4.2 (2). Establish necessity. Suppose that the bundle $\mathscr{X}$ under consideration has dual bundle. Then, due to 3.4.6, the space $\mathscr{X}'(\infty)$ is finite-dimensional, whence, in view of 3.4.4 (10), it follows that the stalk $\mathscr{X}(\infty)$ is finite-dimensional too. Put $m = \dim \mathscr{X}(\infty)$ and consider sections $u_1, \dots, u_m \in C(Q, \mathscr{X})$ with linearly independent values $u_1(\infty), \dots, u_m(\infty)$ which exist by the Dupré Theorem (see [8, 1.3.5]). According to [7, 18.1], the sections $u_1, \dots, u_m$ are pointwise linearly independent over some neighborhood U about ∞ and, hence, $\dim \mathscr{X} \geqslant m$ on U.

Assume that there is no neighborhood about ∞ on which the dimension of $\mathscr{X}$ is constant. Then there exists a strictly increasing sequence of naturals n_k such that $\dim \mathscr{X}(n_k) > m$ for all $k \in \mathbb{N}$. Given a $k \in \mathbb{N}$, choose a functional $x'_k \in \mathscr{X}(n_k)'$ satisfying the equalities $\|x'_k\| = 1$ and $\langle u_1(n_k)|x'_k\rangle = \dots = \langle u_m(n_k)|x'_k\rangle = 0$. Introduce a mapping $H : q \in Q \mapsto H(q) \in \mathscr{X}(q)'$ as follows:

$$H(q) = \begin{cases} x'_k, & q = n_k; \\ 0, & q \notin \{n_k : k \in \mathbb{N}\}. \end{cases}$$

It is clear that $|\!|\!|H|\!|\!| \leqslant 1$. Denote by $\mathscr{U}$ the set $\operatorname{lin}\{u_1, \dots, u_m\}$ supplemented by all sections with singleton supports. Obviously, $\mathscr{U}$ is stalkwise dense in $\mathscr{X}$ and, for every $u \in \mathscr{U}$, the function $\langle u|H\rangle$ vanishes on a neighborhood about ∞

and, hence, continuous. Consequently, by [8, Theorem 1.4.4], the mapping H is a homomorphism, which, with 3.3.2 taken into account, contradicts existence of $\mathscr{X}'$ in view of the fact that the pointwise norm of H is discontinuous. ▷

3.4.9. The one-point compactification Q of the set of naturals can be regarded as the simplest topological space which is, on the one hand, classical (completely regular, metrizable, compact, etc.) and, on the other hand, nontrivial (nondiscrete, not antidiscrete, etc.). As Proposition 3.4.8 asserts, a CBB $\mathscr{X}$ over Q with the stalk $\mathscr{X}(\infty)$ separable has the dual bundle if and only if the dimension of $\mathscr{X}$ is finite and constant on some neighborhood about ∞. Moreover, due to Proposition 3.4.2 (4), every trivial bundle over Q with infinite-dimensional stalk has no dual bundle. Show that, nevertheless, there exists a CBB over Q with infinite-dimensional stalk at ∞ which has the dual bundle.

EXAMPLE. We construct a CBB $\mathscr{X}$ over $Q = \overline{\mathbb{N}}$ possessing the following properties:

(a) all stalks of $\mathscr{X}$ on $\mathbb{N}$ are finite-dimensional and $\mathscr{X}(\infty)$ is nonseparable;
(b) $\mathscr{X}'$ exists;
(c) the inclusion $\mathscr{X}'(\infty) \subset \mathscr{X}(\infty)'$ is strict;
(d) $\mathrm{Hom}(\mathscr{X}, \mathscr{R}) = C(Q, \mathscr{X}')$ norms $\mathscr{X}$.

For every natural n, consider the element $e_n = \chi_{\{n\}} \in \ell^\infty$ and the coordinate functional $\delta_n \in (\ell^\infty)'$, $\langle x | \delta_n \rangle = x(n)$ for all $x \in \ell^\infty$.

Denote by $\bar{\ell}^1$ the image of ℓ^1 under the natural isometric embedding of this space into $(\ell^\infty)'$. It is clear that $\delta_n \in \bar{\ell}^1$ for all $n \in \mathbb{N}$. Put $\mathscr{X}(\infty) = \ell^\infty$ and $\mathscr{X}(n) = \mathrm{lin}\{e_1, \dots, e_n\}$, $n \in \mathbb{N}$.

Given an element $x \in \ell^\infty$, define a section u_x of $\mathscr{X}$ as follows:

$$u_x(q) = \begin{cases} \big(x(1), \dots, x(q), 0, 0 \dots\big), & q \in \mathbb{N}, \\ x, & q = \infty. \end{cases}$$

It is easy to see that the totality $\mathscr{C} = \{u_x : x \in \ell^\infty\}$ is a continuity structure in $\mathscr{X}$ which makes $\mathscr{X}$ a CBB.

By construction it is immediate that $\mathscr{X}$ possesses property (a).

(b), (c): For all $n \in \mathbb{N}$ and $f \in \mathscr{X}(n)'$, put

$$\langle x \mid \overline{f} \rangle = \big\langle (x(1), \dots, x(n), 0, 0, \dots) \mid f \big\rangle, \quad x \in \ell^\infty.$$

It is clear that, for each $n \in \mathbb{N}$, the correspondence $f \mapsto \overline{f}$ performs an isometric embedding of $\mathscr{X}(n)'$ into $\bar{\ell}^1$.

Let H be an arbitrary homomorphism from $\mathscr{X}$ into $\mathscr{R}$. For every $x \in \ell^\infty$, the following relations hold:

$$\begin{aligned}\langle x \mid \overline{H(n)}\,\rangle &= \langle (x(1),\dots,x(n),0,0,\dots) \mid H(n)\,\rangle \\ &= (H \otimes u_x)(n) \to (H \otimes u_x)(\infty) = \langle x \mid H(\infty)\rangle\end{aligned}$$

as $n \to \infty$. Therefore, $\big(\overline{H(n)}\big) \subset \ddot{\ell}^1$ is a weakly Cauchy sequence and, hence, converges in norm, since the space $\ddot{\ell}^1$ possesses the Schur property (see Lemma 3.1.2). Whence it follows that $H(\infty)$ is the norm limit of the sequence $\big(\overline{H(n)}\big)$; in particular, $H(\infty) \in \ddot{\ell}^1$ and $|\!|\!|H|\!|\!| \in C(Q)$. Thus, the CBB $\mathscr{X}$ has the dual bundle and $\mathscr{X}'(\infty) \neq \mathscr{X}(\infty)'$ due to the inclusion $\mathscr{X}'(\infty) \subset \ddot{\ell}^1$.

(d): According to 3.4.4 (1), it is sufficient, given an arbitrary functional $y \in \ddot{\ell}^1$, to present a homomorphism $H_y \in \operatorname{Hom}(\mathscr{X},\mathscr{R})$ such that $H_y(\infty) = y$. The sought homomorphism can be defined as follows:

$$H_y(q) = \begin{cases} y|_{\mathscr{X}(q)}, & q \in \mathbb{N}, \\ y, & q = \infty. \end{cases}$$

The containment $H_y \in \operatorname{Hom}(\mathscr{X},\mathscr{R})$ is justified by [8, Theorem 1.4.9] (with $\mathscr{V} = \mathscr{C}$).

3.4.10. The CBB $\mathscr{X}'' = (\mathscr{X}')'$ (if the latter exists) is called the *second dual* of a continuous Banach bundle $\mathscr{X}$.

It is clear that, for every CBB over a discrete topological space, the second dual exists. Ample CBBs over extremally disconnected compact Hausdorff spaces (see [8, 1.3]) form an important available class of continuous Banach bundles for which the second dual bundles exist.

First of all, we note that existence of $\mathscr{X}'$ does not imply existence of $\mathscr{X}''$.

Proposition. *Let X be a separable Banach space with nonseparable dual (for instance, $X = \ell^1$). Then there exists a topological space Q such that the trivial* CBB *X_Q has the dual bundle and has no second dual bundle.*

$\lhd$ By Proposition 3.4.2 (5), there exists a functionally discrete topological space Q such that the CBB $(X')_Q$ has no dual bundle. By 3.4.2 (6), the CBB X_Q has the dual bundle. By assertion 3.4.2 (3), the bundle $(X_Q)'$ coincides with $(X')_Q$ and, thereby, $(X_Q)'$ has no dual bundle, i.e., the bundle $(X_Q)''$ does not exist. $\rhd$

Remark. The CBB $\mathscr{X}$ constructed in 3.4.9 is also an example of a Banach bundle which has the dual but not the second dual bundle. Indeed, with each element $n \in \mathbb{N}$ associate the functional $e_n'' \in \mathscr{X}'(n)'$ related to the element $e_n \in \mathscr{X}(n)$ by the rule $\langle x' | e_n''\rangle = \langle e_n | x'\rangle$ for all $x' \in \mathscr{X}'(n)$. Put $G(n) = e_n''$ for all

$n \in \mathbb{N}$ and $G(\infty) = 0 \in \mathscr{X}'(\infty)'$. It is clear that the set $\mathscr{D} = \{H_y : y \in \ell^1\}$ is stalkwise dense in $\mathscr{X}'$. By applying [8, Theorem 1.4.9] (with $\mathscr{V} = \mathscr{D}$), we obtain the containment $G \in \mathrm{Hom}(\mathscr{X}', \mathscr{R})$. Furthermore, $|\!|\!|G|\!|\!| \notin C(Q)$ and, hence, in view of 3.3.2, the CBB $\mathscr{X}'$ has no dual bundle, i.e., $\mathscr{X}''$ does not exist.

3.4.11. *Proposition.* *The following are true:*

(1) *Suppose that a trivial* CBB *with stalk X has the second dual bundle. Then the latter is the trivial* CBB *with stalk X''.*

(2) *If a trivial* CBB *over a topological space Q with infinite-dimensional stalk has the second dual bundle, then Q is functionally discrete.*

(3) *Let X be an infinite-dimensional Banach space with separable dual. Then existence of the second dual for the bundle X_Q is equivalent to functional discreteness of Q.*

(4) *For every Banach space X, there exists a nondiscrete normal topological space Q such that the* CBB *X_Q has the second dual bundle.*

(5) *If a topological space Q is not functionally discrete then, for every Banach space X, the following are equivalent:*
   (a) *$(X_Q)''$ exists;*
   (b) *$(X'')_Q = (X_Q)''$;*
   (c) *$(X_Q)'$ exists and $C(Q, X'') = \mathrm{Hom}\big((X_Q)', \mathscr{R}\big)$;*
   (d) *X is finite-dimensional.*

◁ Assertions (1), (2), and (5) are simple consequences of Proposition 3.4.2.

A proof of assertion (4) can be obtained by a simple modification of the proof of Corollary 3.3.15 with Q a nondiscrete normal topological space such that the constant CBBs X_Q and $(X')_Q$ both have dual bundles.

Prove assertion (3). Necessity holds due to (2). Proceeding with sufficiency, observe first that the space X is itself separable. From 3.4.2 (6) it follows that the dual $(X_Q)'$ exists and, in view of 3.4.2 (3), the latter coincides with $(X')_Q$. Applying 3.4.2 (6) again, we complete the proof. ▷

3.4.12. In contrast to the situation described in Proposition 3.4.10, existence of $\mathscr{X}'$ in the following case implies existence of $\mathscr{X}''$.

Proposition. *If a* CBB *with Hilbert stalks over a topological space Q has the dual bundle then it has the second dual bundle. Moreover, the bundles $\mathscr{X}$, $\mathscr{X}'$, and $\mathscr{X}''$ are pairwise isometric.*

◁ Obviously, if two CBBs are isometric and one of them has the dual bundle then the other has the dual bundle too and these duals are isometric. This fact and Proposition 3.4.3 imply the claim. ▷

3.4.13. Let Q be a topological space and let $\mathscr{X}$ be a CBB over Q which has the dual bundle. The mapping $\imath$ that associates with every point $q \in Q$ the operator

$$\imath(q) : x \in \mathscr{X}(q) \mapsto x''|_{\mathscr{X}'(q)}$$

is called the *double prime mapping* for $\mathscr{X}$. (Here $x \mapsto x''$ is the canonical embedding into the second dual.)

Proposition. *Let Q be a topological space and let $\mathscr{X}$ be a* CBB *over Q which has the dual bundle. Suppose that* $\mathrm{Hom}(\mathscr{X},\mathscr{R})$ *norms $\mathscr{X}$ and let $\imath$ be the double prime mapping for $\mathscr{X}$.*

(1) *For every point $q \in Q$, the operator $\imath(q)$ is an isometric embedding of $\mathscr{X}(q)$ into $\mathscr{X}'(q)'$.*

(2) *Assume that $\mathscr{X}$ has the second dual bundle. Then the mapping $\imath$ is an isometric embedding of $\mathscr{X}$ into $\mathscr{X}''$.*

$\lhd$ (1): For $q \in Q$ and $x \in \mathscr{X}(q)$, we have

$$\begin{aligned}
\|x''|_{\mathscr{X}'(q)}\| &= \sup\big\{\langle x'|x''\rangle : x' \in \mathscr{X}'(q),\ \|x'\| \leqslant 1\big\}\\
&= \sup\big\{\langle x|x'\rangle : x' \in \mathscr{X}'(q),\ \|x'\| \leqslant 1\big\}\\
&= \sup\big\{\langle x|v(q)\rangle : v \in C(Q,\mathscr{X}'),\ \|v(q)\| \leqslant 1\big\}\\
&= \sup\big\{\langle x|v(q)\rangle : v \in C(Q,\mathscr{X}'),\ |\!|\!|v|\!|\!| \leqslant 1\big\}\\
&= \sup\big\{\langle x|H\rangle : H \in \mathrm{Hom}(\mathscr{X},\mathscr{R}),\ \|H(q)\| \leqslant 1\big\}\\
&= \|x\| \quad \text{(cf. [8, 1.3.9])}.
\end{aligned}$$

(2): In view of (1), the mapping $u \mapsto \imath \otimes u$ embeds the space $C(Q,\mathscr{X})$ into $\mathrm{Hom}(\mathscr{X}',\mathscr{R}) = C(Q,\mathscr{X}'')$ with pointwise norm preserved. It remains to employ [8, Theorem 1.4.4]. $\rhd$

3.4.14. *Proposition.* *Let $\mathscr{X}$ be a* CBB *with constant finite dimension over a completely regular topological space. Then the bundle $\mathscr{X}''$ exists,* $\mathrm{Hom}(\mathscr{X},\mathscr{R})$ *norms $\mathscr{X}$, and the double prime mapping for $\mathscr{X}$ performs an isometry of $\mathscr{X}$ onto $\mathscr{X}''$.*

$\lhd$ By assertion 3.4.2 (1), in the situation under consideration, the dual bundle $\mathscr{X}'$ exists and $\dim \mathscr{X}' = \dim \mathscr{X}$. The same assertion implies that $\mathscr{X}''$ exists and the equality $\dim \mathscr{X}'' = \dim \mathscr{X}'$ holds. Hence, for every point q, the stalks $\mathscr{X}(q)$ and $\mathscr{X}''(q)$ have the same finite dimension. It remains to apply Proposition 3.4.13 (2) and [8, Theorem 1.4.12]. $\rhd$

3.4.15. Let Q be a topological space and let $\mathscr{X}$ be a CBB over Q which has the second dual bundle. In the following cases, the double prime mapping for $\mathscr{X}$ is an isometry of $\mathscr{X}$ onto $\mathscr{X}''$:

(1) $\mathscr{X}$ is a trivial CBB with reflexive stalk;

(2) $\mathscr{X}$ has constant finite dimension and the topological space Q is completely regular;

(3) $\mathscr{X}$ is a CBB with Hilbert stalks;

(4) $\mathscr{X}$ is an ample CBB over an extremally disconnected compact Hausdorff space Q and all stalks of $\mathscr{X}$ at nonisolated points are reflexive.

$\lhd$ Assertions (1)–(4) are easy from 3.4.11 (1), 3.4.14, 3.4.12, and [8, 2.3.5 (1), 2.3.7]. $\rhd$

Observe that conditions (2) and (4) imply existence of $\mathscr{X}''$ without additional assumptions.

3.5. Weakly Continuous Sections

In this section, we introduce and study the notion of a weakly continuous section of a Banach bundle.

Since weakly continuous sections are closely connected with homomorphisms of the dual bundle (which are known to have locally bounded pointwise norms), the problem is natural of finding conditions that guarantee local boundedness for weakly continuous sections. Subsections 3.5.3–3.5.5 are devoted to this subject.

In 3.5.6–3.5.12, we study the question of continuity of weakly continuous sections for various classes of Banach bundles.

The remaining part of this section (3.5.13–3.5.18) is devoted to finding conditions for coincidence of the space of weakly continuous sections of a trivial Banach bundle and the space of weakly continuous vector valued functions acting into the corresponding stalk.

3.5.1. Let $\mathscr{X}$ be a CBB over a topological space Q and let $D \subset Q$.

Definition. A section u over D of a bundle $\mathscr{X}$ is called *weakly continuous* if $\langle u|H\rangle \in C(D)$ for all $H \in \operatorname{Hom}(\mathscr{X}, \mathscr{R})$. The totality of all these sections is denoted by $C_w(D, \mathscr{X})$.

If $\mathscr{X}$ has the dual bundle then $\operatorname{Hom}(\mathscr{X}, \mathscr{R}) = C(Q, \mathscr{X}')$ and, in this case, weak continuity of a section u is equivalent to continuity of the functions $\langle u|u'\rangle$ for all $u' \in C(Q, \mathscr{X}')$.

It is clear that $C_w(D, \mathscr{X})$ is a vector subspace of the space of all sections over D of the bundle $\mathscr{X}$ and includes $C(D, \mathscr{X})$ as a vector subspace.

Note that a weakly continuous section need not be continuous. Indeed, considering the CBB $\mathscr{X}$ constructed in 3.4.9 and putting $u(n) = e_n$, $n \in \mathbb{N}$, and $u(\infty) = 0$, we obtain a weakly continuous (see Remark 3.4.10) but, obviously, discontinuous section of $\mathscr{X}$.

3.5.2. Lemma. *Let X be a Banach space and let Q be a topological space. Suppose that $D \subset Q$ and a sequence $(q_n) \subset D$ converges to a point $q \in D$.*

(1) *If Q is completely regular and $u \in C_w(D, X_Q)$ then the sequence $\big(u(q_n)\big)$ w-w*-converges to $u(q)$.*

(2) *For every $H \in \operatorname{Hom}(X_Q, \mathscr{R})$, the sequence $(H(q_n))$ is weakly* convergent to $H(q)$.*

(3) *If Q is a completely regular Fréchet–Urysohn space and the points q_n are pairwise distinct and distinct from q then, for every w-w*-vanishing sequence $(x_n) \subset X$, there exists a section $u \in C_w(D, X_Q)$ taking the values $u(q_n) = x_n$ for all $n \in \mathbb{N}$ and $u(q) = 0$.*

(4) *If $u \in C_w(D, X)$ then the sequence $(u(q_n))$ converges weakly to $u(q)$.*

$\lhd$ (1): As is easily seen, we do not restrict generality by assuming that the points q_n are pairwise distinct and distinct from q. From 3.2.6 (4) it follows that, for every sequence $(x'_n) \subset X'$ convergent weakly* to an element $x' \in X'$, there exists a homomorphism $H \in \operatorname{Hom}(X_Q, \mathscr{R})$ taking the values $H(q_n) = x'_n$ for all $n \in \mathbb{N}$ and $H(q) = x'$. Hence, $\langle u(q_n) | x'_n \rangle = \langle u|H \rangle(q_n) \to \langle u|H \rangle(q) = \langle u(q) | x' \rangle$.

Assertions (2) and (4) are evident.

(3): Let (W_n) and (f_n) be sequences of open subsets of Q and of continuous functions from Q into $[0,1]$ presented in Lemma 3.2.5. Then the section u over D defined by the formula

$$u(p) = \begin{cases} f_n(p)x_n, & p \in D \cap W_n, \\ 0, & p \in D \setminus \bigcup_{n \in \mathbb{N}} W_n \end{cases}$$

is weakly continuous. Indeed, consider an arbitrary $H \in \operatorname{Hom}(X_Q, \mathscr{R})$. The function $\langle u|H \rangle$ is continuous on each set $D \cap \operatorname{cl} W_n$, since $\operatorname{cl} W_n$ is included in

$$Q \setminus \operatorname{cl} \bigcup_{k \neq n} W_k,$$

the latter difference is open, and $\langle u|H \rangle$ and $\langle x_n|H \rangle f_n$ coincide on the intersection of D and the difference.

Assume that the function $\langle u|H \rangle$ is discontinuous at some point

$$p \in \Big(\operatorname{cl} \bigcup_{n \in \mathbb{N}} W_n\Big) \setminus \bigcup_{n \in \mathbb{N}} \operatorname{cl} W_n.$$

Then there exist an $\varepsilon > 0$, a sequence $(p_m) \subset D$, and a strictly increasing sequence $(n_m) \subset \mathbb{N}$ such that p belongs to $\mathrm{cl}\{p_m : m \in \mathbb{N}\}$, $p_m \in W_{n_m}$, and $|\langle u|H\rangle(p_m)| > \varepsilon$ for all $m \in \mathbb{N}$. Since Q is a Fréchet–Urysohn space, we can extract a subsequence (p_{m_k}) convergent to p. It is easy to verify that the sequence $(u(p_m))$ is w-w^*-vanishing; therefore, the subsequence $u(p_{m_k})$ of $(u(p_m))$ is w-w^*-vanishing too. At the same time, by (2), the sequence $(H(p_{m_k}))$ converges weakly* to $H(p)$. Consequently, $\varepsilon < |\langle u|H\rangle(p_{m_k})| \to |\langle u|H\rangle(p)| = 0$. The assumption that $\langle u|H\rangle$ is discontinuous at p yields a contradiction. It remains to observe that the function $\langle u|H\rangle$ vanishes on the set $Q \setminus \mathrm{cl}\bigcup_{n\in\mathbb{N}} W_n$. ▷

3.5.3. EXAMPLE. There exist a Fréchet–Urysohn space Q, a Banach space X, and a section $u \in C_w(Q, X_Q)$ that is not locally bounded.

◁ Consider the space Q constructed in Example 3.2.11.

As follows from Corollary 3.1.7 (2), the space ℓ^∞ contains a sequence (x_n) which is w-w^*-vanishing and does not converge in norm. Without loss of generality, we may assume that $\|x_n\| \geqslant 1$ for all $n \in \mathbb{N}$ (this may be fulfilled by extracting a subsequence and multiplying the latter by an appropriate constant elementwise). Put $u((m,n)) := mx_n$ for every $(m,n) \in \mathbb{N} \times \mathbb{N}$ and put $u(\infty) := 0 \in \ell^\infty$. Obviously, the section u is not locally bounded. Show that $H \otimes u \in C(Q)$ for an arbitrary homomorphism $H \in \mathrm{Hom}\left((\ell^\infty)_Q, \mathscr{R}\right)$. By Lemma 3.5.2 (2), for every m, the sequence $\left(H((m,n))\right)_{n\in\mathbb{N}}$ is weakly* convergent, whence $(H \otimes u)((m,n)) = m\langle x_n | H((m,n))\rangle \to 0$ as $n \to \infty$. The latter relation implies continuity of the function $H \otimes u$ (see the description (1) of the elements of $C(Q)$ in Example 3.2.11). ▷

3.5.4. ***Proposition.*** *Let $\mathscr{X}$ be a CBB over a topological space Q. Suppose that* $\mathrm{Hom}(\mathscr{X}, \mathscr{R})$ *norms $\mathscr{X}$ and the space Q satisfies one of the following conditions:*

(a) *Q is first-countable and completely regular;*

(b) *Q is locally pseudocompact. Then every weakly continuous global section of $\mathscr{X}$ is locally bounded.*

◁ First suppose that Q satisfies condition (a). Assume that there is a weakly continuous and not locally bounded global section u of $\mathscr{X}$. In this case, the pointwise norm $|\!|\!|u|\!|\!|$ is unbounded on every neighborhood about some point $q \in Q$. By Dupré's Theorem (see [8, 1.3.5]), we may find a bounded continuous global section taking the value $u(q)$ at q and, next, subtract this section from u; therefore, we may assume that $|\!|\!|u|\!|\!|(q) = 0$.

Since Q is first-countable, there is a sequence $(q_n) \subset Q$ such that $|\!|\!|u|\!|\!|(q_n) > n^2$, $q_i \neq q_j$ for $i \neq j$, and $q_n \to q$. Using the hypotheses, for every number $n \in \mathbb{N}$, take a homomorphism $H_n \in \mathrm{Hom}(\mathscr{X}, \mathscr{R})$ satisfying the relations $\langle u|H_n\rangle(q_n) = \|u(q_n)\|$ and $|\!|\!|H_n|\!|\!| \leqslant 2$.

Corollary 3.2.6 (2) implies existence of an $H \in \mathrm{Hom}(\mathscr{X},\mathscr{R})$ such that $H(q)=0$ and $H(q_n)=\frac{1}{n}H_n(q_n)$ for all $n \in \mathbb{N}$. On the other hand,

$$\langle u|H\rangle(q_n) = \frac{1}{n}\langle u|H_n\rangle(q_n) = \frac{1}{n}\|u(q_n)\| > n,$$

which contradicts weak continuity of u, since $q_n \to q$ and $\langle u|H\rangle(q)=0$.

Now suppose that Q satisfies condition (b). Denote by $\mathrm{Hom}^b(\mathscr{X},\mathscr{R})$ the space of all bounded homomorphisms from $\mathscr{X}$ into $\mathscr{R}$. Fix an arbitrary weakly continuous section u of $\mathscr{X}$ and, for every point $q \in Q$, define a linear functional $T_q : \mathrm{Hom}^b(\mathscr{X},\mathscr{R}) \to \mathbb{R}$ by the formula $T_q(H) = \langle u(q)|H(q)\rangle$. Endowing the space $\mathrm{Hom}^b(\mathscr{X},\mathscr{R})$ with the uniform norm and considering an arbitrary pseudocompact subset $U \subset Q$, we conclude that $\|T_q\| \leqslant \|u(q)\|$; moreover,

$$\sup_{q\in U}\|T_q(H)\| = \sup_{q\in U}\left|\langle u|H\rangle(q)\right| < \infty$$

for all $H \in \mathrm{Hom}^b(\mathscr{X},\mathscr{R})$. By [8, 1.4.11], $\mathrm{Hom}^b(\mathscr{X},\mathscr{R})$ is a Banach space. Therefore, $\sup_{q\in U}\|T_q\| < \infty$ in view of the uniform boundedness principle. It remains to employ the relations

$$\|u(q)\| = \sup\left\{\left|\langle u(q)|H(q)\rangle\right| : H \in \mathrm{Hom}(\mathscr{X},\mathscr{R}),\ |||H||| \leqslant 1\right\} = \|T_q\|. \ \triangleright$$

Observe that, in the last proposition, conditions (a) and (b) are essential even if the CBB $\mathscr{X}$ is trivial (see 3.5.3).

3.5.5. *Corollary.* *Let X be a Banach space and let Q be a topological space satisfying (a) or (b) of* 3.5.4. *Then every weakly global continuous section of X_Q is locally bounded.*

$\triangleleft$ The claim follows immediately from 3.5.4 and 3.4.4 (3). $\triangleright$

3.5.6. Remark. By the definition of continuity for sections (see [8, 1.1.2]), if $\mathscr{U}$ is a vector space of sections over $D \subset Q$ of a CBB $\mathscr{X}$ over a topological space Q and all elements of $\mathscr{U}$ have continuous pointwise norms, then the inclusion $C(D,\mathscr{X}) \subset \mathscr{U}$ implies the equality $C(D,\mathscr{X}) = \mathscr{U}$.

Proposition. *Let $\mathscr{X}$ be a* CBB *over a topological space Q.*

(1) *Suppose that $\mathscr{X}$ has the dual bundle and let $\imath$ be the double prime mapping for $\mathscr{X}$. For every subset $D \subset Q$, the mapping $u \mapsto \imath \otimes u$ performs a linear embedding of the space of locally bounded sections $u \in C_w(D,\mathscr{X})$ into* $\mathrm{Hom}_D(\mathscr{X}',\mathscr{R})$. *If, in addition,* $\mathrm{Hom}(\mathscr{X},\mathscr{R})$ *norms $\mathscr{X}$ then the embedding preserves the pointwise norm.*

(2) *Suppose that* $\mathscr{X}$ *has the second dual bundle and* $\operatorname{Hom}(\mathscr{X},\mathscr{R})$ *norms* $\mathscr{X}$. *If a section* $u \in C_w(Q,\mathscr{X})$ *is locally bounded then* $u \in C(Q,\mathscr{X})$.

◁ (1): The containment $\imath \otimes u \in \operatorname{Hom}_D(\mathscr{X}',\mathscr{R})$ holds in view of [8, Theorem 1.4.9]. Furthermore, if $\operatorname{Hom}(\mathscr{X},\mathscr{R})$ norms $\mathscr{X}$ then the equality $|\!|\!|\imath \otimes u|\!|\!| = |\!|\!|u|\!|\!|$ follows from 3.4.13 (1).

(2): Let a section $u \in C_w(Q,\mathscr{X})$ be locally bounded. Then, in view of assertion (1), the containment $\imath \otimes u \in \operatorname{Hom}(\mathscr{X}',\mathscr{R})$ holds which, together with the equality $\operatorname{Hom}(\mathscr{X}',\mathscr{R}) = C(Q,\mathscr{X}'')$, yields continuity of the pointwise norm of the homomorphism $\imath \otimes u$. Since, due to (1), the functions $|\!|\!|\imath \otimes u|\!|\!|$ and $|\!|\!|u|\!|\!|$ coincide, the latter function is continuous too. Therefore, the vector space $\mathscr{U}$ of locally bounded sections $u \in C_w(Q,\mathscr{X})$ consists of sections with continuous pointwise norms and contains $C(Q,\mathscr{X})$. The above Remark allows us to conclude that $\mathscr{U} = C(Q,\mathscr{X})$. ▷

3.5.7. Corollary. *Let* $\mathscr{X}$ *be a* CBB *with constant finite dimension over a completely regular topological space* Q. *For every subset* $D \subset Q$, *the equality* $C_w(D,\mathscr{X}) = C(D,\mathscr{X})$ *holds.*

◁ The claim may be derived from Theorem 3.2.12, Proposition 3.5.6 (1), and Remark 3.5.6. ▷

3.5.8. Corollary. *Suppose that a topological space* Q *and a* CBB $\mathscr{X}$ *over* Q *satisfy the conditions of Proposition* 3.5.4. *Then existence of* $\mathscr{X}''$ *implies continuity of all weakly continuous sections of* $\mathscr{X}$.

◁ This claims follows from Propositions 3.5.4 and 3.5.6 (2). ▷

3.5.9. Proposition. *Let* $\mathscr{X}$ *be a* CBB *with Hilbert stalks over an arbitrary topological space. If a global section of* $\mathscr{X}$ *is locally bounded and weakly continuous then it is continuous.*

◁ Let Q be a topological space and let $\mathscr{X}$ be a CBB with Hilbert stalks over Q. Fix a locally bounded section $v \in C_w(Q,\mathscr{X})$ and use the mapping h of Lemma 3.4.3 which asserts that

$$h[C(Q,\mathscr{X})] \subset \operatorname{Hom}(\mathscr{X},\mathscr{R}).$$

Thus, the relations $\langle c|h(u)\rangle = \langle u|h(c)\rangle \in C(Q)$ are valid for all $c \in C(Q,\mathscr{X})$. By [8, 1.4.4] these relations imply $h(u) \in \operatorname{Hom}(\mathscr{X},\mathscr{R})$. Therefore,

$$|\!|\!|u|\!|\!|^2 = \langle u|h(u)\rangle \in C(Q).$$

Finally, since

$$|\!|\!|u - c|\!|\!|^2 = |\!|\!|u|\!|\!|^2 - 2\langle c|h(u)\rangle + |\!|\!|u|\!|\!|^2 \in C(Q)$$

for every $c \in C(Q,\mathscr{X})$, the section u is continuous. ▷

3.5.10. *Corollary.* *Let $\mathscr{X}$ be a* CBB *with Hilbert stalks over a topological space Q satisfying* (a) *or* (b) *of* 3.5.4. *Then $C_w(Q, \mathscr{X}) = C(Q, \mathscr{X})$.*

◁ The claim follows immediately from Propositions 3.5.4 and 3.5.9, and Lemma 3.4.3. ▷

3.5.11. *Lemma.* *Suppose that a* CBB *$\mathscr{X}$ over a topological space Q has the dual bundle. For arbitrary sections $u \in C(Q, \mathscr{X})$ and $v \in C_w(Q, \mathscr{X}')$, the real-valued function $\langle u|v\rangle$ is continuous.*

◁ Let $\imath$ be the double-prime mapping for $\mathscr{X}$. Then $\imath \otimes u$ is an element of $\mathrm{Hom}(\mathscr{X}', \mathscr{R})$ according to Proposition 3.5.6 (1). Consequently, $\langle u|v\rangle = \langle v \mid \imath \otimes u\rangle \in C(Q)$. ▷

Proposition. *Suppose that a* CBB *$\mathscr{X}$ over a topological space Q has the dual bundle.*

(1) *If $v \in C_w(Q, \mathscr{X}')$ is locally bounded then $v \in C(Q, \mathscr{X}')$.*

(2) *If Q satisfies* (a) *or* (b) *of* 3.5.4 *then $C_w(Q, \mathscr{X}') = C(Q, \mathscr{X}')$.*

◁ (1): Let $v \in C_w(Q, \mathscr{X}')$ be a locally bounded section. In view of the above lemma, $\langle u|v\rangle \in C(Q)$ for all $u \in C(Q, \mathscr{X})$. Consequently, $v \in \mathrm{Hom}(\mathscr{X}, \mathscr{R})$ due to [8, Theorem 1.4.9] and local boundedness of v. It remains to recall that $\mathrm{Hom}(\mathscr{X}, \mathscr{R}) = C(Q, \mathscr{X}')$.

(2): It suffices to prove the inclusion $C_w(Q, \mathscr{X}') \subset C(Q, \mathscr{X}')$. Suppose that $v \in C_w(Q, \mathscr{X}')$. In view of the above lemma, $\langle u|v\rangle \in C(Q)$ for all $u \in C(Q, \mathscr{X})$. If Q satisfies 3.5.4 (a) then $v \in \mathrm{Hom}(\mathscr{X}, \mathscr{R})$ due to Theorem 3.2.10; if Q satisfies 3.5.4 (b) then $v \in \mathrm{Hom}(\mathscr{X}, \mathscr{R})$ due to [8, Theorem 1.4.7]. Therefore, in both cases, $v \in \mathrm{Hom}(\mathscr{X}, \mathscr{R}) = C(Q, \mathscr{X}')$. ▷

3.5.12. *Theorem.* *Let X be a Banach space and let Q be a completely regular Fréchet–Urysohn space.*

(1) *If X possesses the* WS *property then $C_w(D, X_Q) = C(D, X_Q)$ for all subsets $D \subset Q$.*

(2) *If $C_w(D, X_Q) = C(D, X_Q)$ for some subset $D \subset Q$ which contains one of its limit points (in particular, if $D = Q$ and Q is nondiscrete), then X possesses the* WS *property.*

For instance, if Q is nondiscrete then the equality $C_w(Q, X_Q) = C(Q, X_Q)$ is equivalent to the fact that X possesses the WS *property.*

◁ (1): Suppose that the inclusion $C_w(D, X_Q) \supset C(D, X_Q)$ is strict for a subset $D \subset Q$ and show that X does not possess the WS property. Consider a section $u \in C_w(D, X_Q)$ discontinuous at a point $q \in D$. We may assume that $u(q) = 0$, since, otherwise, we can subtract from u the constant section taking the value $u(q)$.

Since Q is a Fréchet–Urysohn space, we may find a sequence $(q_n) \subset D$ convergent to q such that $|\!|\!|u|\!|\!|(q_n) > \varepsilon > 0$ for all $n \in \mathbb{N}$. By Lemma 3.5.2 (1), the sequence $\big(u(q_n)\big)$ is w-w^*-convergent to $u(q) = 0$. Consequently, X does not possess the WS property.

(2): Suppose that X does not enjoy the WS property and establish the inequality $C_w(D, X_Q) \neq C(D, X_Q)$ for every subset $D \subset Q$ that contains one of its limit point. Let $q \in D$ be a limit point of D. Since Q is a Fréchet–Urysohn space, there is a sequence $(q_n) \subset D \backslash \{q\}$ convergent to q. Without loss of generality, we may assume that $q_i \neq q_j$ whenever $i \neq j$. Since X does not possess the WS property, we may take a sequence $(x_n) \subset X$ which is w-w^*-vanishing and does not vanish in norm. By Lemma 3.5.2 (3), there is a section $u \in C_w(D, X_Q)$ taking the values $u(q_n) = x_n$ for all $n \in \mathbb{N}$ and $u(q) = 0$. It is clear that $u \notin C(D, X_Q)$. ▷

3.5.13. *Proposition.* *For every infinite-dimensional Banach space X, there exists a normal topological space Q such that the inclusion $C_w(Q, X_Q) \subset C_w(Q, X)$ is strict.*

◁ Let $(x_\alpha)_{\alpha \in \aleph}$ and $(x'_\alpha)_{\alpha \in \aleph}$ be the nets existent by Lemma 3.1.4. Put $Q = \aleph^\bullet$ (see 3.1.11) and consider vector valued functions $u : Q \to X$ and $H : Q \to X'$ satisfying the equalities $u(\alpha) = x_\alpha$, $H(\alpha) = x'_\alpha$ for all $\alpha \in \aleph$, $u(\infty) = 0$, and $H(\infty) = 0$.

In view of Remark 3.1.11 (2), the function u is weakly continuous and, in addition, $H \in C(Q, X')$. In particular, $H \in \operatorname{Hom}\big(X_Q, \mathscr{R}\big)$. Furthermore, $\langle u|H \rangle \equiv 1$ on $\aleph$ and $\langle u|H\rangle(\infty) = 0$, whence $u \notin C_w(Q, X_Q)$. ▷

3.5.14. *Corollary.* *Let X be a Banach space and let Q be an arbitrary topological space. The equality $C_w(Q, X_Q) = C_w(Q, X)$ holds for every topological space Q if and only if X is finite-dimensional.*

Observe that, in case X is finite-dimensional, we have

$$C(Q, X_Q) = C_w(Q, X_Q) = C_w(Q, X) = C(Q, X).$$

3.5.15. *Theorem.* *Let X be a Banach space and let Q be an arbitrary topological space.*

(1) *If Q is a Fréchet–Urysohn space and X possesses the* DP* *property, then $C_w(D, X_Q) = C_w(D, X)$ for every subset $D \subset Q$.*

(2) *Let a subset $D \subset Q$ be such that $C(Q)$ contains a function which is not locally constant on D. If $C_w(D, X_Q) = C_w(D, X)$ then X possesses the* DP* *property.*

In particular, if Q is a nondiscrete completely regular Fréchet–Urysohn space then the equality $C_w(Q, X_Q) = C_w(Q, X)$ is equivalent to the fact that X possesses the DP* *property.*

$\triangleleft$ (1): Suppose $C_w(D, X_Q) \neq C_w(D, X)$ for some subset $D \subset Q$. Show that X does not possess the DP* property. Take a vector valued function $u \in C_w(D, X) \setminus C_w(D, X_Q)$ and consider a homomorphism $H \in \operatorname{Hom}(X_Q, \mathscr{R})$ such that the function $\langle u|H\rangle$ is discontinuous at some point $q \in D$. Then the function $\langle u - u_q \mid H - H_q\rangle$ is discontinuous at q, where u_q and H_q are constant functions with values $u(q)$ and $H(q)$. (This is so due to the fact that the functions $\langle u|H_q\rangle$, $\langle u_q|H\rangle$, and $\langle u_q|H_q\rangle$ are continuous.) Since Q is a Fréchet–Urysohn space, there is a sequence $(q_n) \subset D \setminus \{q\}$ which converges to q and satisfies the condition $|\langle u(q_n) - u(q) \mid H(q_n) - H(q)\rangle| > \varepsilon$ for some $\varepsilon > 0$ and all $n \in \mathbb{N}$. Furthermore, in view of 3.5.2 (2), (4), the sequence $\big(u(q_n) - u(q)\big)$ is weakly vanishing and the sequence $\big(H(q_n) - H(q)\big)$ is weakly* vanishing. Consequently, X does not possess the DP* property.

(2): Suppose that X does not possess the DP* property. Consider a weakly null sequence $(x_n) \subset X$ and a weakly* null sequence $(x'_n) \subset X'$ such that $\langle x_n|x'_n\rangle$ does not vanish. By passing to a subsequence and multiplying all elements of one of them by $\pm\delta$ for a suitable $\delta \in \mathbb{R}$, we may achieve validity of the inequalities $\langle x_n|x'_n\rangle \geqslant 1$ for all $n \in \mathbb{N}$. We additionally require that $\langle x_{n+1}|x'_n\rangle + \langle x_n|x'_{n+1}\rangle \geqslant 0$ for all $n \in \mathbb{N}$, which in turn can be fulfilled by pairwise multiplication of the elements x_2 and x'_2, x_3 and x'_3, etc. by ± 1. Let vector valued functions $u : [0,1] \to X$ and $u' : [0,1] \to X'$ satisfy the equalities $u(0) = 0$, $u'(0) = 0$,

$$u\big(\lambda \tfrac{1}{n+1} + (1-\lambda)\tfrac{1}{n}\big) = \lambda x_{n+1} + (1-\lambda)x_n,$$
$$u'\big(\lambda \tfrac{1}{n+1} + (1-\lambda)\tfrac{1}{n}\big) = \lambda x'_{n+1} + (1-\lambda)x'_n$$

for all $\lambda \in [0,1]$ and $n \in \mathbb{N}$. By Lemma 3.1.12, the function u is weakly continuous and u' is weakly* continuous. Consider the function $\langle u|u'\rangle : [0,1] \to \mathbb{R}$. Given arbitrary $n \in \mathbb{N}$ and $0 \leqslant \lambda \leqslant 1$, we have

$$\begin{aligned}
\langle u|u'\rangle\big(\lambda \tfrac{1}{n+1} + (1-\lambda)\tfrac{1}{n}\big) &= \big\langle \lambda x_{n+1} + (1-\lambda)x_n \;\big|\; \lambda x'_{n+1} + (1-\lambda)x'_n\big\rangle \\
&= \lambda^2 \langle x_{n+1}|x'_{n+1}\rangle + (1-\lambda)^2 \langle x_n|x'_n\rangle \\
&\qquad + \lambda(1-\lambda)\big(\langle x_{n+1}|x'_n\rangle + \langle x_n|x'_{n+1}\rangle\big) \\
&\geqslant \lambda^2 + (1-\lambda)^2 + 0 \\
&= 2(\lambda - 1/2)^2 + 1/2 \\
&\geqslant 1/2.
\end{aligned}$$

Thus, $\langle u|u'\rangle(0) = 0$ and, in addition, $\langle u|u'\rangle \geqslant 1/2$ on $(0,1]$. Next, take a continuous function $g \in C(Q)$ such that the restriction $g|_D$ is not constant on any neighborhood about a point $q \in D$. Without loss of generality, we may assume that $g : Q \to [0,1]$ and $g(q) = 0$ (see the proof of 3.1.13). As is easily seen, $u \circ g|_D \in C_w(D, X)$ and $u' \circ g \in \operatorname{Hom}(X_Q, \mathscr{R})$. It is clear that the function $\big\langle (u \circ g)|_D \;\big|\; u' \circ g\big\rangle = \langle u|u'\rangle \circ g|_D$

vanishes at q and, in addition, the image of the function on each neighborhood about q intersects the interval $[1/2, \infty)$. Consequently, $(u \circ g)|_D \notin C_w(D, X_Q)$.

The last assertion of the theorem follows from (1) and (2) and 3.1.10 (3). ▷

3.5.16. Corollary. *Let X be a Banach space and let Q be a topological space that is not functionally discrete. In each of the following cases, the inclusion $C_w(Q, X_Q) \subset C_w(Q, X)$ is strict:*

(1) *X is infinite-dimensional and reflexive;*

(2) *X is separable and does not possess the Schur property;*

(3) *X is a Banach space which does not possess the Schur property and satisfies one of the conditions* 3.1.6 (3), (5), *or* (6).

◁ In view of assertion (2) of Theorem 3.5.15, it suffices to show that, in each of the cases under consideration, X does not possess the DP* property. In cases (2) and (3) the latter is provided by Lemma 3.1.7 (3) and, in case (1), we can employ the Josefson–Niessenzweig Theorem [4, XII] according to which there exists a weakly* null sequence of norm-one vectors in X''. ▷

3.5.17. Proposition. *Let X be a Banach space and let Q be a functionally discrete topological space. If X' includes a countable total subspace then $C(Q, X) = C(Q, X_Q) = C_w(Q, X_Q) = C_w(Q, X)$.*

◁ The claim follows from Lemma 3.1.14, since the relations

$$C(Q, X) = C(Q, X_Q) \subset C_w(Q, X_Q) \subset C_w(Q, X)$$

are always true. ▷

3.5.18. Corollary. *Let Q be a topological space and let X be a separable Banach space that does not possess the Schur property. The equality $C_w(Q, X_Q) = C_w(Q, X)$ holds if and only if Q is functionally discrete.*

◁ Necessity follows from 3.1.6 (2), Lemma 3.1.7 (3), and Theorem 3.5.15 (2); sufficiency is justified by Proposition 3.5.17. ▷

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