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MATHEMATICS

Completeness Criterion for the Nonstandard Hull of a Normed Space in a Boolean-Valued Universe

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In this work, some results in infinitesimal analysis concerning normed spaces and the field of real numbers are extended to a functional representation of the Boolean-valued universe. Specifically, for an arbitrary polyverse over Q, the following conditions are proved to be equivalent: $q \in Q$ is not a σ-isolated point, the stalk at q of a polyverse is countably saturated, and the nonstandard hull of any normed space in the stalk at q of a polyverse is complete.

One of the most important concepts in infinitesimal analysis as applied to the theory of normed spaces is the concept of the nonstandard hull of a normed space, i.e., the set of its bounded elements factorized by the infinite closeness relation (see, e.g., [3]).

For an arbitrary Boolean-valued universe, a convenient functional analogue—the class of sections of the corresponding continuous polyverse—was suggested in [1]. Such a functional model makes it possible, in particular, to introduce the concept of the infinite closeness of elements in a normed space inside a stalk of a polyverse and, thus, offers an additional possibility of synthesizing methods in infinitesimal and Boolean-valued analyses.

A natural task arising in this connection is to extend the main results obtained in classical infinitesimal analysis to the Boolean-valued universe. A key question in the framework of this subject is whether or not the nonstandard hull of a normed space inside a stalk of a polyverse is complete.

A positive answer to this question is given here in the case when the point of the compact set corresponding to the stalk under consideration is not σ-isolated. The completeness of a nonstandard hull is proved by applying the criterion for the countable saturation of a stalk of a polyverse established here.

PRELIMINARIES

Let Q be an extremally disconnected compact set. Assume that points $q \in Q$ are associated with pairwise disjoint classes ${}^{q}\mathbb{V}$ that are ZFC models. Denote by ${}^{Q}\mathbb{V}$ the union $\bigcup\limits_{q \in Q} {}^{q}\mathbb{V}$. The class ${}^{q}\mathbb{V}$ is called the stalk of ${}^{Q}\mathbb{V}$ at the point $q \in Q$. Suppose that a class-topology is defined on ${}^{Q}\mathbb{V}$ (see [1]). A continuous function $u\colon Q \to {}^{Q}\mathbb{V}$ is called a continuous section of ${}^{Q}\mathbb{V}$ if $u(q) \in {}^{q}\mathbb{V}$ for all $q \in Q$. The symbol $C(Q, {}^{Q}\mathbb{V})$ stands for the class of all continuous sections of ${}^{Q}\mathbb{V}$.

We impose on ${}^{Q}\mathbb{V}$ some additional conditions, specifically, those ensuring the open-closedness of the set

$$\{q \in Q\colon {}^{q}\mathbb{V} \models \varphi(u_1(q), u_2(q), \ldots, u_n(q))\}$$

for an arbitrary formula $\varphi(t_1, t_2, \ldots, t_n)$ of the signature of set theory and for sections $u_1, u_2, \ldots, u_n \in C(Q, {}^{Q}\mathbb{V})$. This set is called the truth of $\varphi(u_1, u_2, \ldots, u_n)$ in $C(Q, {}^{Q}\mathbb{V})$ and is denoted by $\|\varphi(u_1, u_2, \ldots, u_n)\|$. Under the conditions imposed, the class ${}^{Q}\mathbb{V}$ is called a (continuous) polyverse over Q.

A rigorous and complete description of the basic facts about a continuous polyverse can be found in [1], where it is shown, in particular, that the class of continuous sections of a polyverse is a general Boolean-valued universe.

Let $\mathbb{R}$ and $\mathbb{N}$ denote the sets of reals and positive integers ($0 \notin \mathbb{N}$), and let $\mathcal{R}$ and $\mathcal{N}$ designate the elements of $C(Q, {}^{Q}\mathbb{V})$ that are the indicated sets in $C(Q, {}^{Q}\mathbb{V})$. We also introduce the notation ${}^{q}\mathbb{R} = \mathcal{R}(q)$ and ${}^{q}\mathbb{N} = \mathcal{N}(q)$. For a number $\alpha \in \mathbb{R}$, the symbol ${}^{q}\alpha$ stands for the element $\alpha^{\wedge}(q) \in {}^{q}\mathbb{V}$, where $(\cdot)^{\wedge}$ is the canonical embedding of $\mathbb{V}$ in $C(Q, {}^{Q}\mathbb{V})$ (see [2]). Additionally, ${}^{q}\varnothing$ denotes the element $\varnothing^{\wedge}(q) \in {}^{q}\mathbb{V}$.

For any element $X \in {}^{q}\mathbb{V}$, the descent of X is the class $X{\downarrow} := \{x \in {}^{q}\mathbb{V}\colon {}^{q}\mathbb{V} \models x \in X\}$. The equalities $X{\downarrow} = Y{\downarrow}$ and $X = Y$ are equivalent for all $X, Y \in {}^{q}\mathbb{V}$.

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If $X \in {}^q\mathbb{V}$ is a field or a partially ordered set inside ${}^q\mathbb{V}$, then field operations or an order relation, respectively, can be naturally introduced on $X{\downarrow}$. For example, for $\alpha, \beta \in {}^q\mathbb{R}{\downarrow}$, the sum $\alpha + \beta$ is defined as an element $\gamma \in {}^q\mathbb{R}{\downarrow}$ such that ${}^q\mathbb{V} \models (\gamma = \alpha + \beta)$. The descent of a field is a field. If an element $X \in {}^q\mathbb{V}$ is a vector space over a field $F \in {}^q\mathbb{V}$ inside ${}^q\mathbb{V}$, then $X{\downarrow}$ is a vector space over $F{\downarrow}$. For any number $\alpha \in \mathbb{R}$, the element ${}^q\alpha \in {}^q\mathbb{V}$ is a number inside ${}^q\mathbb{V}$; i.e., ${}^q\alpha \in {}^q\mathbb{R}{\downarrow}$. The function ${}^q(\cdot)$: $\mathbb{R} \to {}^q\mathbb{R}{\downarrow}$ is injective and preserves the order relation and the addition and multiplication operations. Moreover, $\mathbb{N}^\wedge(q) = {}^q\mathbb{N}$ and $\mathbb{R}^\wedge(q) \subset {}^q\mathbb{R}$ inside ${}^q\mathbb{V}$.

In what follows, we identify the elements $\alpha \in \mathbb{R}$ and ${}^q\alpha \in {}^q\mathbb{R}{\downarrow}$ and, thus, assume that $\mathbb{R} \subset {}^q\mathbb{R}{\downarrow}$.

Let P be an element of ${}^q\mathbb{V}$ that is a relation (i.e., a set of pairs) inside ${}^q\mathbb{V}$. Consider the following property of a sequence $(x_n)_{n \in N}$ of elements of $(\operatorname{dom} P){\downarrow}$: if for any $n \in \mathbb{N}$, there is an element $y_n \in {}^q\mathbb{V}$ such that

$$ {}^q\mathbb{V} \models ((x_1, y_n), (x_2, y_n), \ldots, (x_n, y_n) \in P), $$

then there exists an element $y \in {}^q\mathbb{V}$ such that

$$ {}^q\mathbb{V} \models ((x_n, y) \in P) \text{ for all } n \in \mathbb{N}. $$

A relation P is called countably saturated if any sequence of elements of $(\operatorname{dom} P){\downarrow}$ has this property. A stalk ${}^q\mathbb{V}$ is countably saturated if any relation in ${}^q\mathbb{V}$ is countably saturated.

A point in a topological space is called σ-isolated if the intersection of any countable set of its neighborhoods is a neighborhood of that point.

MAIN RESULTS

Theorem 1. *The class ${}^q\mathbb{V}$ is countably saturated if and only if the point $q \in Q$ is not σ-isolated.*

By convention, the elements of ${}^q\mathbb{R}{\downarrow}$ are called internal numbers. An internal number λ is called standard if there exists a number $\alpha \in \mathbb{R}$ such that ${}^q\alpha = \lambda$. Note that, in view of the above assumption (on the embedding $\mathbb{R} \subset {}^q\mathbb{R}{\downarrow}$), we identify the standard numbers and the elements of $\mathbb{R}$. A bounded number is an internal number whose modulus is less than some standard number. The symbol $\mathbb{O}({}^q\mathbb{R})$ stands for the set of all bounded numbers. Numbers that are not bounded are called infinitely large. An internal number λ is called infinitesimal if $|\lambda| < \alpha$ for any standard number $\alpha > 0$. Two internal numbers are said to be infinitely close if their difference is infinitesimal. The infinite closeness relation is denoted by $\approx$. This relation is an equivalence relation on the set of internal numbers. Let $[\lambda]$ denote the equivalence class containing the internal number λ.

Proposition 1. *For any bounded number λ, there exists a unique standard number that is infinitely close to λ.*

Thus, the mapping $\alpha \mapsto [\alpha]$ is a bijection of $\mathbb{R}$ onto $\mathbb{O}({}^q\mathbb{R})/\approx$. For any bounded number λ, its infinitely close standard number is called the standard part of λ and is denoted by ${}^\circ\lambda$.

Let $X \in {}^q\mathbb{V}$ be a normed space over ${}^q\mathbb{R}$ inside ${}^q\mathbb{V}$. An element $x \in X{\downarrow}$ is called bounded if its norm inside ${}^q\mathbb{V}$ is a bounded number. Denote by $\mathbb{O}(X)$ the set of all bounded elements of $X{\downarrow}$. Elements $x, y \in X{\downarrow}$ are called infinitely close (designated as $x \approx y$) if the norm of their difference inside ${}^q\mathbb{V}$ is infinitesimal. The relation of infinite closeness is an equivalence relation on $X{\downarrow}$. The equivalence class containing an element $x \in X{\downarrow}$ is designated as $[x]$. Denote the quotient set $\mathbb{O}(X)/\approx$ by $\mathbf{X}$, and set

$$ \mathbf{x} + \mathbf{y} = [x + y], \quad \lambda\mathbf{x} = [\lambda x], \quad \|\mathbf{x}\| = {}^\circ\|x\| $$

for any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and $\lambda \in \mathbb{R}$, where x and y are arbitrary representatives of the classes $\mathbf{x}$ and $\mathbf{y}$ and $\|x\|$ is the norm of an element x inside ${}^q\mathbb{V}$. It is easy to show that the operations introduced are well defined and convert $\mathbf{X}$ into a normed space over $\mathbb{R}$, denoted by $\hat{X}$ and called the nonstandard hull of X.

Theorem 2. *Let $q \in Q$ not be a σ-isolated point and $X \in {}^q\mathbb{V}$ be a normed space inside ${}^q\mathbb{V}$. Then, the normed space $\hat{X}$ is complete.*

If $q \in Q$ is a σ-isolated point, then all real numbers are standard in ${}^q\mathbb{V}$. This means that, when q is σ-isolated, we have ${}^q\mathbb{R}{\downarrow} = \mathbb{R}$ and ${}^q\mathbb{N}{\downarrow} = \mathbb{N}$ (in view of the above assumption that $\mathbb{R} \subset {}^q\mathbb{R}{\downarrow}$).

Proposition 2. *Let $q \in Q$ be a σ-isolated point. Then, there exists an element $X \in {}^q\mathbb{V}$ that is a normed space inside ${}^q\mathbb{V}$ and is such that the normed space $\hat{X}$ is incomplete.*

To conclude, we state a theorem combining the main results of this work.

Theorem 3. *Let ${}^q\mathbb{V}$ be the stalk at $q \in Q$ of a continuous polyverse over an extremally disconnected compact set Q. The following statements are equivalent*:

(a) *q is not a σ-isolated point;*

(b) *The model ${}^q\mathbb{V}$ is countably saturated*;

(c) *The nonstandard hull of any normed space inside ${}^q\mathbb{V}$ is complete.*

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