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ORDER PROPERTIES OF THE SPACE OF FINITELY ADDITIVE TRANSITION FUNCTIONS

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Abstract: The basic order properties, as well as some metric and algebraic properties, are studied of the set of finitely additive transition functions on an arbitrary measurable space, as endowed with the structure of an ordered normed algebra, and some connections are revealed with the classical spaces of linear operators, vector measures, and measurable vector-valued functions. In particular, the question is examined of splitting the space of transition functions into the sum of the subspaces of countably additive and purely finitely additive transition functions.

Keywords: transition function, purely finitely additive measure, lifting of a measure space, vector measure, measurable vector-valued function, ordered vector space, vector lattice, Riesz space, K-space, Banach lattice, ordered Banach algebra

§ 1. Introduction

Let X be a nonempty set and let Σ be a σ-algebra of its subsets. The pair (X,Σ) is called a *measurable space* and the elements of Σ, *measurable sets*.

Denote by $B(X,\Sigma)$ or, in concise form, $B(X)$ the Banach space of all bounded Σ-measurable functions $f\colon X\to\mathbb{R}$ with the norm $\|f\|=\sup_{x\in X}|f(x)|$.

Throughout, the term *measure* stands for a finitely additive function acting from a σ-algebra into $\mathbb{R}$.

Following [1], we denote by $ba(X,\Sigma)$ or $ba(\Sigma)$ the vector space of all bounded measures from Σ into $\mathbb{R}$ and by $ca(X,\Sigma)$ or $ca(\Sigma)$ the subspace of $ba(\Sigma)$ constituted by all countably additive bounded measures. (In the case of an infinite σ-algebra Σ the inclusion $ca(\Sigma)\subset ba(\Sigma)$ is known to be proper.)

In probability theory, a Markov chain is uniquely determined by a *transition probability* that is an arbitrary function $p\colon X\times\Sigma\to\mathbb{R}$ satisfying the following conditions:

(a) $p(\cdot,E)\in B(X)$ for all $E\in\Sigma$;
(b) $p(x,\cdot)\in ca(\Sigma)$ for all $x\in X$;
(c) $p(x,E)\geqslant 0$ for all $x\in X$ and $E\in\Sigma$;
(d) $p(x,X)=1$ for all $x\in X$.

In [2–4] A. I. Zhdanok introduced and studied the finitely additive Markov chains whose transition probability satisfies a weaker version of (b): $p(x,\cdot)\in ba(\Sigma)$ for all $x\in X$. Intending to make the set of functions under consideration into a vector space, we reject their positivity and normalization and arrive at the following definition of a transition function:

Definition 1.1. Let (X,Σ) be a measurable space. A *transition function* on (X,Σ) is a function $p\colon X\times\Sigma\to\mathbb{R}$ satisfying the following conditions:

(1) $p(\cdot,E)\in B(X)$ for all $E\in\Sigma$;
(2) $p(x,\cdot)\in ba(\Sigma)$ for all $x\in X$.

It is worth noting that the term “transition function” is sometimes used as a synonym for “transition probability.” We distinguish between the two terms and use the latter only for the functions satisfying the above conditions (a)–(d).

The totality of all transition functions on a measurable space (X,Σ) is denoted by $\mathscr{P}(X,\Sigma)$.

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In this article we study the set $\mathscr{P}(X,\Sigma)$ which is endowed with the structure of an ordered normed algebra and investigate its connections with the classical spaces of linear operators, vector measures, and measurable vector-valued functions.

In probability theory, with each transition probability p on a measurable space (X,Σ) the two so-called *Markov operators* are associated: $T_p\colon B(X)\to B(X)$ and $A_p\colon ca(\Sigma)\to ca(\Sigma)$ which are defined as

$$(T_pf)(x)=\int\limits_X f\,dp(x,\cdot),\quad (A_p\mu)(E)=\int\limits_X p(\cdot,E)\,d\mu,$$

where $f\in B(X)$, $x\in X$, $\mu\in ca(\Sigma)$, $E\in\Sigma$.

As follows from the results of § 4, in the case of arbitrary transition functions, the analogs of Markov operators of the first kind constitute the space $\mathscr{L}(B(X))$ of all bounded linear operators on $B(X)$, while the analogs of Markov operators of the second kind form a certain subspace of $\mathscr{L}(ba(\Sigma))$, namely, the space $\mathscr{L}_w(ba(\Sigma))$ of all weakly* continuous linear operators on $ba(\Sigma)$. Furthermore, it is shown that the mappings $p\mapsto T_p$ and $p\mapsto A_p$ are isomorphisms of the ordered normed algebra $\mathscr{P}(X,\Sigma)$ onto $\mathscr{L}(B(X))$ and $\mathscr{L}_w(ba(\Sigma))$.

Note that the examination of Markov operators for finitely additive transition probabilities was initiated in [2–4]. Moreover, one of the questions we consider in § 5, of splitting $\mathscr{P}(X,\Sigma)$ into the sum of the subspaces of countably additive and purely finitely additive transition functions, was first raised in [2].

§ 2. Preliminaries to the Theory of Ordered Vector Spaces

In this section we give some definitions and facts from the theory of ordered vector spaces and positive operators which are necessary for further exposition.

We assume the reader familiar with such notions as ordered vector space, positive and negative parts of an element (u^+, u^-), vector lattice (= Riesz space), normed lattice, K-space (= Dedekind complete Riesz space), order limit of a sequence and that of a net ($o\text{-}\lim_{n\to\infty}u_n$, $o\text{-}\lim_{\alpha\in\mathrm{A}}u_\alpha$, $u_\alpha\xrightarrow{o}u$), order sum of a family ($o\text{-}\sum_{\xi\in\Xi}u_\xi$), disjoint elements and subsets of a vector lattice ($u\perp v$, $U\perp V$, $u\perp U$), disjoint and second disjoint complement ($U^\perp$, $U^{\perp\perp}$), band (= component) of a vector lattice, Boolean algebra, complete Boolean algebra, atom of a Boolean algebra, and positive linear operator. (All necessary details can be found in [5–14].)

The term "operator" always stands for "linear operator." Throughout the article, all vector spaces are assumed over the field $\mathbb{R}$ of reals and all vector lattices are assumed Archimedean.

A set U is said to be *hereditarily embedded* in an ordered space V whenever $U\subset V$ and, for every subset $U_0\subset U$, existence of $\sup_U U_0$ implies existence of $\sup_V U_0$ and the equality $\sup_V U_0=\sup_U U_0$.

Say that U is a *minorizing* subset of a vector lattice V (or U *minorizes* V) if $U\subset V$ and for each element $0<v\in V$ there exists a $u\in U$ such that $0<u\leqslant v$.

The disjointness $\perp$ on a vector lattice possesses all properties of an "abstract" disjointness introduced as follows:

Definition 2.1. Let V be a vector space and let d be a relation on V, i.e., $\mathrm{d}\subset V^2$. Given an arbitrary subset $U\subset V$, put $U^{\mathrm{d}}=\{v\in V : u\ \mathrm{d}\ v \text{ for all } u\in U\}$. For $(U^{\mathrm{d}})^{\mathrm{d}}$ we use the shorter notation U^{dd}. Say that d is a *disjointness* on V if, for all $v,v_1,v_2\in V$ and $\lambda\in\mathbb{R}$, the following hold:

(1) $v\ \mathrm{d}\ 0$;
(2) if $v\ \mathrm{d}\ v$ then $v=0$;
(3) if $v_1\ \mathrm{d}\ v_2$ then $v_2\ \mathrm{d}\ v_1$;
(4) if $\{v_1\}^{\mathrm{dd}}\cap\{v_2\}^{\mathrm{dd}}=\{0\}$ then $v_1\ \mathrm{d}\ v_2$;
(5) if $v_1\ \mathrm{d}\ v$ then $\lambda v_1\ \mathrm{d}\ v$;
(6) if $v_1\ \mathrm{d}\ v$ and $v_2\ \mathrm{d}\ v$ then $(v_1+v_2)\ \mathrm{d}\ v$.

The notion of disjointness on a vector space V is in a sense a particular case of the disjointness introduced in [13] for an arbitrary set V.

Let d be a symmetric relation on a set V. A subset $W \subset V$ is called a d-*band* if $W^{\mathrm{dd}} = W$. Note that W is a d-band if and only if $W = U^{\mathrm{d}}$ for some subset $U \subset V$. If d is a disjointness on a vector space then every d-band is a vector subspace.

Theorem 2.2. *Suppose that $\overline{V}$ is a vector lattice and V is a minorizing vector subspace of $\overline{V}$. Introduce a relation* d *on V by letting $v_1 \mathrel{\mathrm{d}} v_2$ whenever $v_1 \perp v_2$ (i.e., $v_1, v_2 \in V$ are disjoint as elements of $\overline{V}$). Then the following hold*:

(1) *V is hereditarily embedded in $\overline{V}$;*
(2) $\sup_{\overline{V}}\{v \in V : 0 \leqslant v \leqslant \overline{v}\} = \overline{v}$ *for all* $0 \leqslant \overline{v} \in \overline{V}$;
(3) $U^{\mathrm{dd}} = U^{\perp\perp} \cap V$ *for every subset* $U \subset V$;
(4) d *is a disjointness on* V.

Proof. (1): Let $U \subset V$ and $v = \sup_V U$. Show that $v = \sup_{\overline{V}} U$. Consider an arbitrary upper bound $\overline{v} \in \overline{V}$ of U and establish the inequality $v \leqslant \overline{v}$. Denote $\overline{v} \wedge v$ by $\overline{v}_0$ and assume to the contrary that $\overline{v}_0 < v$. Since V minorizes $\overline{V}$, there exists a $w \in V$ such that $0 < w \leqslant v - \overline{v}_0$. For each $u \in U$ we have $u \leqslant \overline{v}_0 = \overline{v}_0 + w - w \leqslant \overline{v}_0 + (v - \overline{v}_0) - w = v - w$. Hence, $v = \sup_V U \leqslant v - w$, which contradicts the inequality $w > 0$.

(2): Let $0 \leqslant \overline{v} \in \overline{V}$. Put $U = \{v \in V : 0 \leqslant v \leqslant \overline{v}\}$ and demonstrate that $\sup_{\overline{V}} U = \overline{v}$. Consider an arbitrary upper bound $\overline{v}_1 \in \overline{V}$ of U and show the inequality $\overline{v} \leqslant \overline{v}_1$. Denote $\overline{v}_1 \wedge \overline{v}$ by $\overline{v}_0$ and assume to the contrary that $\overline{v}_0 < \overline{v}$. Since V minorizes $\overline{V}$, there exists a $v \in V$ such that $0 < v \leqslant \overline{v} - \overline{v}_0$. By the Archimedean property, the inequality $nv \leqslant \overline{v}$ cannot hold for all $n \in \mathbb{N}$. Let n be the greatest natural satisfying $nv \leqslant \overline{v}$. Then $nv \in U$, whence $(n+1)v = nv + v \leqslant \overline{v}_0 + v \leqslant \overline{v}_0 + (\overline{v} - \overline{v}_0) = \overline{v}$, which contradicts the choice of n.

(3): The inclusion $U^{\perp\perp} \cap V \subset U^{\mathrm{dd}}$ is obvious. Show the reverse inclusion. Let $v \in U^{\mathrm{dd}}$. Then, as is easy to see, $v \perp U^{\perp} \cap V$. To prove the required relation $v \perp U^{\perp}$ it suffices to consider an arbitrary positive element $\overline{v} \in U^{\perp}$ and demonstrate that $v \perp \overline{v}$. Put $W = \{w \in V : 0 \leqslant w \leqslant \overline{v}\}$. The relation $\overline{v} \perp U$ implies $W \perp U$, whence $W \subset U^{\perp} \cap V$ and, therefore, $v \perp W$. The latter, together with assertion (2) above, yields $v \perp \sup_{\overline{V}} W = \overline{v}$.

(4): We only have to prove condition (4) of Definition 2.1. Assume that $u, v \in V$, $\{u\}^{\mathrm{dd}} \cap \{v\}^{\mathrm{dd}} = \{0\}$, but nevertheless $\overline{w} = |u| \wedge |v| \neq 0$. Since V minorizes $\overline{V}$, there is a $w \in V$ such that $0 < w \leqslant \overline{w}$. By (3) we conclude that $w \in \{u\}^{\perp\perp} \cap \{v\}^{\perp\perp} \cap V = \{u\}^{\mathrm{dd}} \cap \{v\}^{\mathrm{dd}}$. This contradiction completes the proof. □

A normed space with an order making it into an ordered vector space is called an *ordered normed space* (no agreement of the norm with the order is imposed).

Remarks 2.3. Let (X, Σ) be an arbitrary measurable space. Endow the normed space $B(X)$ with the pointwise order thus making $B(X)$ into a Banach lattice.

(1) Norm boundedness is equivalent to order boundedness in the normed lattice $B(X)$. Therefore, we use the term "boundedness" for either of this equivalent properties.

(2) A sequence $(f_n)_{n\in\mathbb{N}}$ of elements in $B(X)$ is bounded if and only if $\sup_{n\in\mathbb{N}} f_n$ and $\inf_{n\in\mathbb{N}} f_n$ exist in $B(X)$. Furthermore, for all $x \in X$ we have $(\sup_{n\in\mathbb{N}} f_n)(x) = \sup_{n\in\mathbb{N}} f_n(x)$ and $(\inf_{n\in\mathbb{N}} f_n)(x) = \inf_{n\in\mathbb{N}} f_n(x)$.

(3) A sequence in $B(X)$ o-converges to a function $f \in B(X)$ if and only if the sequence is bounded and converges pointwise to f.

An *ordered algebra* is an algebra V that is an ordered vector space and possesses the following property: if $u, v \in V$ and $u, v \geqslant 0$ then $uv \geqslant 0$.

A normed algebra with an order making it into an ordered algebra is called an *ordered normed algebra* (no agreement of the norm with the order is imposed). The space $\mathscr{L}(V, V)$ of all norm bounded operators acting in a normed lattice V is an example of an ordered normed algebra.

A norm-complete ordered normed algebra is called an *ordered Banach algebra*.

If V is a normed space then V' stands for the dual of V, i.e., the Banach space of all bounded linear functionals from V into $\mathbb{R}$. A subset of V' which is dense in V' relative to the weak* topology is called *weakly* dense.* The symbol T' denotes the operator in $\mathscr{L}(W', V')$ that is adjoint to an operator $T \in \mathscr{L}(V, W)$.

Given normed spaces V and W, the symbol $\mathscr{L}_w(V', W')$ stands for the set of weakly* continuous operators from V' into W', i.e., operators that are continuous in the weak* topology. Observe that $\mathscr{L}_w(V', W') \subset \mathscr{L}(V', W')$ (which follows, for instance, from the closed graph theorem).

An operator $T\colon V \to W$ between ordered vector spaces V and W is called *sequentially o-continuous* (or *σ-o-continuous*) if $v_n \xrightarrow{o} v$ implies $Tv_n \xrightarrow{o} Tv$ for each sequence $(v_n)_{n\in\mathbb{N}}$ in V and each $v \in V$. The set of sequentially o-continuous operators from V into W is denoted by $\mathscr{L}_o(V, W)$.

If V and W coincide then we conventionally omit the symbol for the second space in the notations $\mathscr{L}(V, W)$, $\mathscr{L}_w(V', W')$, $\mathscr{L}_o(V, W)$ and write, for instance, $\mathscr{L}(V)$ instead of $\mathscr{L}(V, V)$.

The following assertion is an easy consequence of the fact that the image of a normed space V under the natural embedding of V into V'' coincides with the set of all weakly* continuous functionals on V' (see, for instance, [1, V.3.9]).

Proposition 2.4. *Let V and W be normed spaces. An operator from W' into V' is adjoint to some bounded operator from V into W if and only if it is weakly* continuous.*

§ 3. Preliminaries to Measure Theory

In this section we present basic information from measure theory (particularly, on finitely additive measures [15–19], vector measures [1, 20, 21], and measurable vector-valued functions [12, 21]) and establish some auxiliary facts about measure spaces and liftings (see [12, 20, 22, 23]).

Definition 3.1. By a *measure space* we mean a triple $(X, \Sigma, |\cdot|)$, where (X, Σ) is a measurable space and $|\cdot|$ is a positive countably additive function from Σ into $\overline{\mathbb{R}}$ (conventionally called a measure) satisfying the following conditions:

(1) if $E \subset X$ and $E \cap F \in \Sigma$ for all $F \in \Sigma$ of finite measure then $E \in \Sigma$;
(2) if $E \in \Sigma$ and $|E| = \infty$ then there exists an $E_0 \in \Sigma$ such that $E_0 \subset E$ and $0 < |E_0| < \infty$;
(3) if $E \in \Sigma$, $|E| = 0$, and $E_0 \subset E$ then $E_0 \in \Sigma$;
(4) $|X| \neq 0$.

(Note that, to within condition (4), our definition of a measure space coincides, for instance, with that in [9, I.6.2].)

Sets $E, F \in \Sigma$ are called *equivalent* if $|E \vartriangle F| = 0$. The equivalence class containing a set $E \in \Sigma$ is denoted by $E^{\sim}$ and the quotient of Σ relative to the equivalence, by $\widetilde{\Sigma}$. Endowed with the natural order (induced by the inclusion relation), $\widetilde{\Sigma}$ is a Boolean algebra; it is called the *quotient algebra* of the measure space $(X, \Sigma, |\cdot|)$.

As usual, we say that a condition holds *almost everywhere* if it holds for all elements of X but some set of measure zero. The symbol $\mathscr{L}^\infty(X)$ stands for the totality of essentially bounded real-valued measurable functions defined almost everywhere; $L^\infty(X)$ denotes the space (lattice-ordered Banach algebra) of equivalence classes of such functions relative to the equality almost everywhere. The equivalence class in $L^\infty(X)$ containing a function $f \in \mathscr{L}^\infty(X)$ is denoted by $f^{\sim}$.

A mapping $\rho\colon L^\infty(X) \to \mathscr{L}^\infty(X)$ is called a *lifting of the measure space* $(X, \Sigma, |\cdot|)$ or a *lifting of the space* $L^\infty(X)$ if, for all $\alpha, \beta \in \mathbb{R}$ and $\mathbf{f}, \mathbf{g} \in L^\infty(X)$, the following hold:

(1) $\rho(\mathbf{f}) \in \mathbf{f}$ and $\operatorname{dom} \rho(\mathbf{f}) = X$;
(2) if $\mathbf{f} \leqslant \mathbf{g}$ then $\rho(\mathbf{f}) \leqslant \rho(\mathbf{g})$ everywhere on X;
(3) $\rho(\alpha\mathbf{f}+\beta\mathbf{g}) = \alpha\rho(\mathbf{f})+\beta\rho(\mathbf{g})$, $\rho(\mathbf{fg}) = \rho(\mathbf{f})\rho(\mathbf{g})$, $\rho(\mathbf{f}\vee\mathbf{g}) = \rho(\mathbf{f})\vee\rho(\mathbf{g})$, and $\rho(\mathbf{f}\wedge\mathbf{g}) = \rho(\mathbf{f})\wedge\rho(\mathbf{g})$;
(4) $\rho(0^{\sim}) = 0$ and $\rho(1^{\sim}) = 1$ everywhere on X.

(Some of the above conditions are consequences of the rest.)

If $f \in \mathscr{L}^\infty(X)$ then for the function $\rho(f^\sim)$ a shorter notation is accepted, $\rho(f)$. Since a lifting is a right inverse to the mapping $f \mapsto f^\sim$, we sometimes write $f_\sim$ instead of $\rho(f)$.

A mapping $\rho\colon \widetilde{\Sigma} \to \Sigma$ is called a *lifting of the quotient algebra* $\widetilde{\Sigma}$ if for all classes $\mathbf{E}, \mathbf{F} \in \widetilde{\Sigma}$ the following hold:

(1) $\rho(\mathbf{E}) \in \mathbf{E}$;
(2) if $\mathbf{E} \leqslant \mathbf{F}$ then $\rho(\mathbf{E}) \subset \rho(\mathbf{F})$;
(3) $\rho(\mathbf{E} \vee \mathbf{F}) = \rho(\mathbf{E}) \cup \rho(\mathbf{F})$, $\rho(\mathbf{E} \wedge \mathbf{F}) = \rho(\mathbf{E}) \cap \rho(\mathbf{F})$;
(4) $\rho((X \backslash E)^\sim) = X \backslash \rho(E^\sim)$ for all $E \in \Sigma$;
(5) $\rho(\varnothing^\sim) = \varnothing$ and $\rho(X^\sim) = X$.

By analogy with a lifting of the space $L^\infty(X)$ we sometimes write $\rho(E)$ or $E_\sim$ instead of $\rho(E^\sim)$ and thus assume that $\rho\colon \Sigma \to \Sigma$.

The following rather obvious observation often makes it possible to simplify verification of the fact that a certain mapping possesses all properties of a lifting:

Proposition 3.2. *Let $(X, \Sigma, |\cdot|)$ be a measure space. A mapping $\rho\colon \widetilde{\Sigma} \to \Sigma$ is a lifting of the quotient algebra $\widetilde{\Sigma}$ if and only if it satisfies the following conditions:*

(a) *$\rho(\mathbf{E}) \in \mathbf{E}$ for all $\mathbf{E} \in \widetilde{\Sigma}$;*
(b) *for each $x \in X$ the set $\{\mathbf{E} \in \widetilde{\Sigma} : x \in \rho(\mathbf{E})\}$ is an ultrafilter of the Boolean algebra $\widetilde{\Sigma}$.*

If $\rho\colon L^\infty(X) \to \mathscr{L}^\infty(X)$ is a lifting of $L^\infty(X)$ then the mapping $\mathbf{E} \mapsto \{x \in X : \rho(1_{\mathbf{E}})(x) \neq 0\}$ is a lifting of the quotient algebra $\widetilde{\Sigma}$ (here $1_{\mathbf{E}} \in L^\infty(X)$ is the class containing the characteristic functions 1_E of the sets $E \in \mathbf{E}$). Conversely, for every lifting of the quotient algebra $\widetilde{\Sigma}$ there exists a unique lifting of $L^\infty(X)$ related to the lifting of $\widetilde{\Sigma}$ in the way indicated above (see [20, § 11]).

It is known (see [20, 23]) that a measure space $(X, \Sigma, |\cdot|)$ has a lifting if and only if it possesses the so-called *direct sum property*: there exists a family $(E_\xi)_{\xi \in \Xi}$ of pairwise disjoint measurable sets of finite measure such that $\sup_{\xi \in \Xi} E_\xi^\sim = X^\sim$ in the Boolean algebra $\widetilde{\Sigma}$ (i.e., for each measurable set E, if $|E| > 0$ then $|E \cap E_\xi| > 0$ for some $\xi \in \Xi$). In particular, every measure space with finite or σ-finite measure has a lifting (the latter fact was first obtained in [22]).

Henceforth we need the following easy result:

Lemma 3.3. *Suppose that $(X, \Sigma, |\cdot|)$ is a measure space and the Boolean algebra $\widetilde{\Sigma}$ has no atoms.*

(1) *If $x \in E \in \Sigma$ and $|E| < \infty$ then there exists an $F \in \Sigma$ such that $x \in F \subset E$ and $|F| = \frac{1}{2}|E|$.*

(2) *If $(X, \Sigma, |\cdot|)$ has a lifting, $x \in E \in \Sigma$, $E_\sim = E$, and $|E| < \infty$ then there exists an $F \in \Sigma$ such that $x \in F \subset E$, $F_\sim = F$, and $|F| = \frac{1}{2}|E|$.*

Proof. Assertion (1) is immediate from the classical theorem by P. Halmos on the image of a measure (see [24]), while (2) follows from (1) and elementary properties of a lifting. □

Corollary 3.4. *If the quotient algebra $\widetilde{\Sigma}$ of a measure space $(X, \Sigma, |\cdot|)$ is atomless then all singletons of X are measurable and have measure zero.*

Proof. It suffices to observe that due to 3.1 (1), (2) each point belongs to a measurable set of finite measure, and employ assertion (1) of Lemma 3.3. □

Note that the converse to Corollary 3.4 does not hold. Indeed, if X is uncountable, Σ is the σ-algebra constituted by all countable subsets of X and their complements, and $|\cdot|$ vanishes on countable sets and equals 1 on their complements, then all singletons of X have measure zero, while $X^\sim$ is an atom of the quotient algebra $\widetilde{\Sigma}$ of the measure space $(X, \Sigma, |\cdot|)$.

Let (X, Σ) be an arbitrary measurable space. For $x \in X$ and $E \in \Sigma$ put $\delta_x(E) = 1$ whenever $x \in E$ and $\delta_x(E) = 0$ otherwise. The so-defined measure δ_x is conventionally called a *delta-measure* or *Dirac measure* (concentrated at x). Obviously, $\delta_x \in ca(\Sigma)$ for all $x \in X$.

Say that a set $E \subset X$ *separates* points $x, y \in X$ (or x and y are *separated* by E) if either $x \in E$ and $y \notin E$, or $x \notin E$ and $y \in E$. Points $x, y \in X$ are said to be Σ-*separable* if x and y are separated by some element of Σ, and Σ-*inseparable* otherwise. It is evident that x and y are Σ-inseparable whenever $\delta_x = \delta_y$.

It is easy to see that Σ-inseparability is an equivalence relation. The equivalence classes of X relative to Σ-inseparability are called *lumps* of (X, Σ).

It is obvious that the measurable lumps of (X, Σ) are exactly the atoms of the Boolean algebra Σ.

REMARK 3.5. Note that a lump need not be a measurable set. Indeed, let $X = [0,1]$, consider a Lebesgue nonmeasurable subset $G \subset X$, and put

$$\Sigma = \{E \in \mathscr{L} : \ E \supset G \text{ or } E \cap G = \varnothing\},$$

where $\mathscr{L}$ is the σ-algebra of Lebesgue measurable subsets of X. It is obvious that (X, Σ) is a measurable space and G is a nonmeasurable lump of it. (All other lumps are measurable and have the form $\{x\}$, where $x \in X \backslash G$.)

A subset $Y \subset X$ is called Σ-*compatible* whenever Y does not separate Σ-inseparable points or, which is the same, Y is the union of some set of lumps. The totality of all Σ-compatible subsets of X is denoted by $\overline{\Sigma}$. It is easily seen that $\overline{\Sigma}$ is a σ-algebra and an atomic complete Boolean algebra extending Σ (the question of coincidence of Σ and $\overline{\Sigma}$ is considered below in Proposition 3.6).

A function $f \colon X \to \mathbb{R}$ is called Σ-*compatible* if $f(x) = f(y)$ for all Σ-inseparable points $x, y \in X$, i.e., whenever $\delta_x = \delta_y$ implies $f(x) = f(y)$ for all $x, y \in X$. It is clear that a function is Σ-compatible if and only if it is measurable relative to the σ-algebra $\overline{\Sigma}$. Observe that the space $B(X, \overline{\Sigma})$ of all Σ-compatible bounded functions is a Banach K-space.

A measurable space (X, Σ) is called *pointwise measurable* if $\{x\} \in \Sigma$ for all $x \in X$.

A measurable space (X, Σ) is *atomic* if all its lumps are measurable (i.e., they all are atoms of Σ) or, which is the same, the union of atoms of Σ coincides with X.

As is easy to see, if (X, Σ) is an atomic measurable space then Σ is an atomic Boolean algebra. The converse is not true. For instance, the measurable space (X, Σ) defined in Remark 3.5 is not atomic, while Σ is an atomic Boolean algebra.

Proposition 3.6. *The following properties of a measurable space (X, Σ) are equivalent:*

(1) *(X, Σ) is atomic and Σ is a complete Boolean algebra;*
(2) *all Σ-compatible subsets of X are measurable (i.e., $\Sigma = \overline{\Sigma}$);*
(3) *$\Sigma = \{\bigcup_{j \in J} X_j : J \subset I\}$ for some partition $(X_i)_{i \in I}$ of X.*

PROOF. The implications (3)$\Rightarrow$(2) and (3)$\Rightarrow$(1) are obvious. To prove the implication (2)$\Rightarrow$(3) it suffices to take as $(X_i)_{i \in I}$ the family of all lumps of (X, Σ).

Prove the implication (1)$\Rightarrow$(3). Let $(X_i)_{i \in I}$ be the family of all atoms of Σ.

Consider an $E \in \Sigma$ and put $J = \{i \in I : X_i \subset E\}$. Assume that $E \neq \bigcup_{j \in J} X_j$, i.e., there is a point $x \in E \backslash \bigcup_{j \in J} X_j$. Since $\bigcup_{i \in I} X_i = X$, the point x belongs to X_i for some $i \in I \backslash J$ and therefore X_i is not included in E but has a common point with E, which contradicts Σ-compatibility of E.

Now let J be an arbitrary subset of I. By completeness of the Boolean algebra Σ, the supremum $E = \sup_{j \in J} X_j \in \Sigma$ exists. Using the first part of the proof of the implication (1)$\Rightarrow$(3), it is easy to show that $E = \bigcup_{j \in J} X_j$ and hence $\bigcup_{j \in J} X_j \in \Sigma$. $\square$

A measurable space possessing one (and hence all) of the properties (1), (2), or (3) in Proposition 3.6 is called *discrete*.

Observe that pointwise measurable and discrete measurable spaces are atomic. An example of a nondiscrete pointwise measurable space is the real line $\mathbb{R}$ with the Borel σ-algebra $\mathscr{B}(\mathbb{R})$. The space $(\mathbb{R}^2, \{A \times \mathbb{R} : A \in \mathscr{B}(\mathbb{R})\})$ is an example of an atomic measurable space that is neither pointwise measurable nor discrete.

A family of sets $(E_i)_{i \in I}$ is called a *measurable partition* of a set $E \in \Sigma$ if $E_i \cap E_j = \varnothing$ whenever $i \neq j$, $\bigcup_{i \in I} E_i = E$, and $E_i \in \Sigma$ for all $i \in I$.

Recall that the *variation* of a measure $\mu \in ba(\Sigma)$ is the measure $|\mu| \in ba(\Sigma)$ defined as

$$|\mu|(E) = \sup\left\{ \sum_{i=1}^{n} |\mu(E_i)| \,:\, (E_1, \dots, E_n) \text{ is a measurable partition of } E \right\}.$$

Observe that the following equalities hold for all $E \in \Sigma$ (see [18]):

$$|\mu|(E) = \sup_{\substack{F,\,G\in\Sigma \\ F,\,G\subset E}} (\mu(F) - \mu(G)) = \sup_{\substack{F\in\Sigma \\ F\subset E}} (\mu(F) - \mu(X\backslash F)).$$

It is known that the vector spaces $ba(\Sigma)$ and $ca(\Sigma)$, endowed with the natural order ($\mu_1 \leqslant \mu_2$ whenever $\mu_1(E) \leqslant \mu_2(E)$ for all $E \in \Sigma$) and norm $\|\mu\| = |\mu|(X)$, are Banach lattices (and even Banach K-spaces, see [18]). Furthermore, the variation of a measure coincides with its modulus in the corresponding vector lattice.

A positive measure $\mu \in ba(\Sigma)$ is said to be *purely finitely additive* if $0 \leqslant \nu \leqslant \mu$ implies $\nu = 0$ for each $\nu \in ca(\Sigma)$. An arbitrary measure $\mu \in ba(\Sigma)$ is called *purely finitely additive* if the measures μ^+ and μ^- are purely finitely additive. The subspace of $ba(\Sigma)$ constituted by all bounded purely finitely additive measures is denoted by $pfa(X, \Sigma)$ or $pfa(\Sigma)$.

Remark 3.7. It is known (see [18]) that $pfa(\Sigma)$, as well as $ca(\Sigma)$, is a Banach K-space; furthermore, $ca(\Sigma)$ and $pfa(\Sigma)$ are mutually complementary bands in $ba(\Sigma)$, i.e., $ca(\Sigma)^{\perp} = pfa(\Sigma)$ and $pfa(\Sigma)^{\perp} = ca(\Sigma)$. In particular, $ba(\Sigma) = ca(\Sigma) \oplus pfa(\Sigma)$.

Henceforth we need the following fact:

Theorem 3.8. *Let $(X, \Sigma, |\cdot|)$ be a measure space possessing a lifting $\rho\colon \Sigma \to \Sigma$ and suppose that the Boolean algebra $\widetilde{\Sigma}$ has no atoms. Then there exists a set $X_0 \in \Sigma$ such that $|X\backslash X_0| = 0$ and $\delta_x \circ \rho \in pfa(\Sigma)$ for $x \in X_0$. If, in addition, the measure $|\cdot|$ is σ-finite then $\delta_x \circ \rho \in pfa(\Sigma)$ for all $x \in X$.*

Proof. For each $x \in X$ the containment $\delta_x \circ \rho \in ba(\Sigma)$ is immediate from the properties of a lifting.

Since the measure space $(X, \Sigma, |\cdot|)$ possesses the direct sum property (see above), there is a family $(E_\xi)_{\xi\in\Xi}$ of pairwise disjoint measurable sets of finite nonzero measure such that $\sup_{\xi\in\Xi} E_\xi^{\sim} = X^{\sim}$. Put $X_0 = \bigcup_{\xi\in\Xi} \rho(E_\xi)$. According to [12, 1.2.12; 23, Chapter I] the set X_0 is measurable and $|X\backslash X_0| = 0$. With each point $x \in X_0$ associate the index $\xi_x \in \Xi$ for which $x \in \rho(E_{\xi_x})$.

Show that $\delta_x \circ \rho \in pfa(\Sigma)$ for all $x \in X_0$. To this end, fix an arbitrary point $x \in X_0$, consider a measure $\mu \in ca(\Sigma)$ satisfying the conditions $0 \leqslant \mu \leqslant \delta_x \circ \rho$, and establish the equality $\mu = 0$.

Note that $\mu(E) = 0$ whenever $E \in \Sigma$ and $|E| = 0$. Consequently, the measure μ is absolutely continuous with respect to $|\cdot|$.

Put $F_0^x = \rho(E_{\xi_x})$. By way of induction, using Lemma 3.3, construct a decreasing sequence of sets $F_n^x \in \Sigma$ ($n \in \mathbb{N}$) satisfying the following conditions: $x \in F_n^x$, $\rho(F_n^x) = F_n^x$, $|F_n^x| = \frac{1}{2^n}|F_0^x|$. For all $n \in \mathbb{N}$ we have $0 \leqslant \mu(X\backslash F_n^x) \leqslant \delta_x\big(\rho(X\backslash F_n^x)\big) = \delta_x(X\backslash F_n^x) = 0$. On the other hand, tending n to infinity and noting that $|F_n^x| \to 0$, we infer $\mu(X\backslash F_n^x) = \mu(X) - \mu(F_n^x) \to \mu(X)$ due to the fact that μ is absolutely continuous with respect to $|\cdot|$. Hence, $\mu = 0$.

Now assume additionally that the measure $|\cdot|$ is σ-finite and demonstrate that the containment $\delta_x \circ \rho \in pfa(\Sigma)$ holds not only for the above-considered case $x \in X_0$ but also for $x \in X\backslash X_0$. To this end, fix an arbitrary $x \in X\backslash X_0$, take a measure $\mu \in ca(\Sigma)$ satisfying $0 \leqslant \mu \leqslant \delta_x \circ \rho$, and prove that $\mu = 0$.

Since E_ξ are pairwise disjoint and $|E_\xi| > 0$ for all $\xi \in \Xi$, from σ-finiteness of $|\cdot|$ it easily follows that Ξ is countable (see, for instance, [9, X.1.6]). Consequently, $\mu(X_0) = \mu(\bigcup_{\xi\in\Xi} \rho(E_\xi)) = \sum_{\xi\in\Xi} \mu(\rho(E_\xi)) \leqslant \sum_{\xi\in\Xi} \delta_x(\rho(E_\xi)) = 0$. In addition, $\mu(X\backslash X_0) \leqslant \delta_x(\rho(X\backslash X_0)) = \delta_x(\varnothing) = 0$. Therefore, $\mu = 0$. $\square$

Consideration of transition functions on (X, Σ) and operators in $B(X)$ and $ba(\Sigma)$ is closely connected with the theory of vector measures. Below we present the basic definitions and facts of this theory.

Let (X, Σ) be a measurable space and let V be a normed space. A function $m\colon \Sigma \to V$ is called a *bounded vector* (V*-valued*) *measure* if

(1) $m(E \cup F) = m(E) + m(F)$ for all $E, F \in \Sigma$, $E \cap F = \varnothing$;
(2) the image of m is norm bounded.

Denote by $ba(X, \Sigma, V)$ or $ba(\Sigma, V)$ the vector space of all bounded V-valued measures on Σ. It is easy to observe that if V is a normed lattice then $ba(\Sigma, V)$ is an ordered vector space with respect to the following order: $m_1 \leqslant m_2$ whenever $m_1(E) \leqslant m_2(E)$ for all $E \in \Sigma$.

Note that $\varphi \circ m \in ba(\Sigma)$ for all $\varphi \in V'$ and $m \in ba(\Sigma, V)$, and $ba(\Sigma, V)$ is a normed space with respect to the norm

$$\|m\| = \sup_{\substack{\varphi \in V' \\ \|\varphi\| \leqslant 1}} \|\varphi \circ m\| = \sup_{\substack{\varphi \in V' \\ \|\varphi\| \leqslant 1}} |\varphi \circ m|(X).$$

Remark 3.9. In case $V = B(X)$, the formula for the norm of a vector measure can be simplified. Namely, given a vector measure $m \in ba(\Sigma, B(X))$, the following holds:

$$\|m\| = \sup_{x \in X} \|m(\cdot)(x)\|,$$

i.e., $\|m\| = \sup_{x \in X} \|\varphi_x \circ m\|$, where the functional $\varphi_x \in B(X)'$ is defined as $\varphi_x(f) = f(x)$. Indeed,

$$\begin{aligned}
\|m\| &= \sup_{\substack{\varphi \in B(X)' \\ \|\varphi\| \leqslant 1}} \|\varphi \circ m\| = \sup_{\substack{\varphi \in B(X)' \\ \|\varphi\| \leqslant 1}} \sup_{E, F \in \Sigma} (\varphi(m(E)) - \varphi(m(F))) \\
&= \sup_{E, F \in \Sigma} \sup_{\substack{\varphi \in B(X)' \\ \|\varphi\| \leqslant 1}} (\varphi(m(E) - m(F))) = \sup_{E, F \in \Sigma} \|m(E) - m(F)\| \\
&= \sup_{E, F \in \Sigma} \sup_{x \in X} |m(E)(x) - m(F)(x)| = \sup_{x \in X} \sup_{E, F \in \Sigma} |m(E)(x) - m(F)(x)| \\
&= \sup_{x \in X} \|m(\cdot)(x)\|.
\end{aligned}$$

Let V be a normed lattice. A vector measure $m \in ba(\Sigma, V)$ is called *order continuous* (or *o-continuous*) if, for every sequence of measurable sets $(E_n)_{n \in \mathbb{N}}$, from $E_n \downarrow \varnothing$ it follows that $m(E_n) \xrightarrow{o} 0$. A vector measure $m \in ba(\Sigma, V)$ is said to be *order countably additive* (or *o-countably additive*) if, for every sequence $(E_n)_{n \in \mathbb{N}}$ of pairwise disjoint measurable sets, the equality $m(\bigcup_{n=1}^{\infty} E_n) = o\text{-}\sum_{n=1}^{\infty} m(E_n)$ holds. In exactly the same way as in the scalar case (see, for instance, [7, IV.1]) it can be proven that a vector measure is o-continuous if and only if it is o-countably additive. The set of all o-countably additive vector measures is denoted by $o\text{-}ca(X, \Sigma, V)$ or $o\text{-}ca(\Sigma, V)$.

Let V be a Banach space and let $m \in ba(\Sigma, V)$. We denote by $St(X, \Sigma)$ or $St(X)$ the normed subspace of $B(X)$ constituted by all step functions (i.e., measurable functions with finite images). Define an operator $I_m\colon St(X) \to V$ by putting $I_m s = \sum_{i=1}^{n} \alpha_i m(E_i)$ for each step function $s = \sum_{i=1}^{n} \alpha_i 1_{E_i}$, where the sets $E_i \in \Sigma$ are pairwise disjoint (correctness of this definition is quite obvious). It is easy to verify that the operator I_m is norm bounded. Since $St(X)$ is a dense subspace of $B(X)$ and V is a Banach space, there exists a unique extension of I_m to a bounded operator from $B(X)$ into V. The value of this extension at an element $f \in B(X)$ is called the *integral of the function f with respect to the vector measure m* and denoted by $\int_X f\,dm$ or $\langle f, m \rangle$ (see [20, Chapter II.7]).

From the construction of the integral it readily follows that $\|\langle f, m \rangle\| \leqslant \|f\|\,\|m\|$ for all $f \in B(X)$ and $m \in ba(\Sigma, V)$.

If $V = \mathbb{R}$ and $\mu \in ba(\Sigma)$ then, using the above construction, we arrive at the *integral of a function $f \in B(X)$ with respect to a finitely additive measure μ* which is denoted, as above, by $\int_X f\,d\mu$ or $\langle f, \mu \rangle$ (see [25, Chapter VII]). Note that the so-defined integral coincides on $B(X)$ with the so-called generalized Radon integral (see [26, XI.3; 1]).

Many properties of the integral with respect to a finitely additive measure reflect the analogous properties of the Lebesgue integral and those of the integral with respect to a countably additive (signed) measure (see, for instance, [26, XI.3]). In particular, for all $f \in B(X)$ and $\mu \in ba(\Sigma)$ we have $|\langle f, \mu\rangle| \leqslant \langle |f|, |\mu|\rangle \leqslant \|f\|\,\|\mu\|$.

REMARK 3.10. It is known (see [26, XI.4]) that every bounded linear functional $\varphi\colon B(X) \to \mathbb{R}$ is uniquely representable as the integral with respect to some measure $\mu_\varphi \in ba(\Sigma)$. Furthermore, the correspondence $\varphi \mapsto \mu_\varphi$ is a linear isometry of $B(X)'$ onto $ba(\Sigma)$. With this fact taken into account, the space $ba(\Sigma)$ can be regarded as the dual to $B(X)$. For instance, when talking about the weak* topology on $ba(\Sigma)$, we always regard $ba(\Sigma)$ as the dual to $B(X)$.

Denote by $\Delta(\Sigma)$ the vector subspace of $ba(\Sigma)$ constituted by all measures of the form $\sum_{i=1}^{n} \alpha_i \delta_{x_i}$, where $\alpha_i \in \mathbb{R}$ and $x_i \in X$.

REMARK 3.11. It is known that the set $\Delta(\Sigma)$ is weakly* dense in $ba(\Sigma)$ (see [18, 4.9]), whence due to the inclusion $\Delta(\Sigma) \subset ca(\Sigma)$ it follows that $ca(\Sigma)$ is also weakly* dense in $ba(\Sigma)$.

Endow the space $ba(\Sigma, B(X))$ with a multiplication by putting

$$(m_1 * m_2)(E)(x) = \langle m_2(E), m_1(\cdot)(x)\rangle$$

for all $x \in X$ and $E \in \Sigma$. Below (see Corollary 4.9) we will show that $ba(\Sigma, B(X))$ is an ordered Banach algebra.

Let V be a Banach space. A vector-valued function $w\colon X \to V$ is *bounded* if $\sup_{x\in X} \|w(x)\| < \infty$.

A vector-valued function $w\colon X \to V'$ is called *weakly* measurable* if for every $v \in V$ the function $\langle v, w(\cdot)\rangle\colon X \to \mathbb{R}$ is measurable, i.e., $\{x \in X : \langle v, w(x)\rangle < \alpha\} \in \Sigma$ for all $v \in V$ and $\alpha \in \mathbb{R}$. Denote by $\ell_w^\infty(X, \Sigma, V')$ or $\ell_w^\infty(X, V')$ the set of all bounded weakly* measurable functions from X into V'. The set $\ell_w^\infty(X, V')$ is a normed space with respect to the pointwise linear operations and norm $\|w\| = \sup_{x\in X} \|w(x)\|$.

In case $V = B(X)$, the space $\ell_w^\infty(X, V') = \ell_w^\infty(X, ba(\Sigma))$ can be endowed with a multiplication by putting

$$(w_1 * w_2)(x)(E) = \langle w_2(\cdot)(E), w_1(x)\rangle$$

for all $x \in X$ and $E \in \Sigma$. As is shown below (see Corollary 4.9), this multiplication and the pointwise order make $\ell_w^\infty(X, ba(\Sigma))$ into an ordered Banach algebra.

Denote by $\ell_w^\infty(X, ca(\Sigma))$ the subspace of $\ell_w^\infty(X, ba(\Sigma))$ constituted by functions with images in $ca(\Sigma)$.

Proposition 3.12. *If (X, Σ) is a measurable space with Σ an infinite σ-algebra then the inclusion $\mathscr{L}_w(ba(\Sigma)) \subset \mathscr{L}(ba(\Sigma))$ is proper.*

PROOF. Since $ca(\Sigma)$ is a proper closed subspace of $ba(\Sigma)$, there is a nonzero bounded linear functional $\varphi\colon ba(\Sigma) \to \mathbb{R}$ such that $\varphi \equiv 0$ on $ca(\Sigma)$.

Let ν be an arbitrary nonzero element of $ba(\Sigma)$. Define an operator $A\colon ba(\Sigma) \to ba(\Sigma)$ by putting $A\mu = \varphi(\mu)\nu$ for all $\mu \in ba(\Sigma)$. Obviously, $A \in \mathscr{L}(ba(\Sigma))$, $A \neq 0$, and $A \equiv 0$ on $ca(\Sigma)$. Therefore, the nonzero operator A vanishes on a weakly* dense subset of $ba(\Sigma)$ and hence is not weakly* continuous, i.e., $A \notin \mathscr{L}_w(ba(\Sigma))$. $\square$

Given an arbitrary operator $A \in \mathscr{L}(ba(\Sigma))$ and a set $E \in \Sigma$, denote by A_E the function from X into $\mathbb{R}$ defined as $A_E(x) = (A\delta_x)(E)$ for all $x \in X$.

REMARK 3.13. If $A \in \mathscr{L}_w(ba(\Sigma))$ then $A_E \in B(X)$ for all $E \in \Sigma$. Indeed, given a set $E \in \Sigma$, consider the function $\varphi_E\colon ba(\Sigma) \to \mathbb{R}$, defined by the rule $\varphi_E(\mu) = (A\mu)(E)$, $\mu \in ba(\Sigma)$. It is clear that φ_E is a weakly* continuous linear functional. By Remark 3.10 there is a function $f \in B(X)$ such that $\varphi_E(\mu) = \langle f, \mu\rangle$ for all $\mu \in ba(\Sigma)$. Consequently, $A_E(x) = (A\delta_x)(E) = \varphi_E(\delta_x) = \langle f, \delta_x\rangle = f(x)$ for all $x \in X$.

The converse fails in general: the operator $A \in \mathscr{L}(ba(\Sigma))$ in the proof of Proposition 3.12 does not belong to $\mathscr{L}_w(ba(\Sigma))$ but satisfies the relation $A_E \in B(X)$ for all $E \in \Sigma$.

Observe that, by Proposition 2.4 and Remark 3.10, we have $\mathscr{L}_w(ba(\Sigma)) = \{T' : T \in \mathscr{L}(B(X))\}$. Since $(T_1T_2)' = T_2'T_1'$; therefore, it is natural to endow the space $\mathscr{L}_w(ba(\Sigma))$ with the multiplication $A_1 * A_2 = A_2A_1$, under which $\mathscr{L}_w(ba(\Sigma))$ is an ordered Banach algebra.

DEFINITION 3.14. Say that an operator $A \in \mathscr{L}_w(ba(\Sigma))$ is *ca-invariant* if $A\,ca(\Sigma) \subset ca(\Sigma)$. The set of all ca-invariant weakly* continuous operators is denoted by $\mathscr{L}_{wc}(ba(\Sigma))$.

It is clear that $\mathscr{L}_{wc}(ba(\Sigma))$ is an ordered Banach subalgebra of $\mathscr{L}_w(ba(\Sigma))$.

§ 4. Isomorphisms Between the Space of Transition Functions and Other Classical Spaces

In this section we endow the set of transition functions $\mathscr{P}(X,\Sigma)$ with the structure of an ordered normed algebra and study its connections with spaces of linear operators, vector measures, and measurable vector-valued functions. We also show that the ordered vector space of transition functions is not in general a vector lattice.

Henceforth (in § 5) we need the following generalization of the notion of a transition function:

DEFINITION 4.1. Let Σ_1 and Σ_2 be σ-algebras of subsets of a set X. Denote by $\mathscr{P}(X,\Sigma_1,\Sigma_2)$ the set of all functions $p\colon X \times \Sigma_1 \to \mathbb{R}$ satisfying the following conditions:

(1) $p(\cdot,E) \in B(X,\Sigma_2)$ for all $E \in \Sigma_1$;
(2) $p(x,\cdot) \in ba(\Sigma_1)$ for all $x \in X$.

Theorem 4.2. *If $p \in \mathscr{P}(X,\Sigma_1,\Sigma_2)$ then the function p is uniformly bounded and, moreover, $\sup_{x\in X}\|p(x,\cdot)\| < \infty$.*

PROOF. For each $x \in X$ put $\mu_x = p(x,\cdot)$. According to Definition 4.1 we have $\mu_x \in ba(\Sigma_1)$ and $\sup_{x\in X}|\mu_x(E)| < \infty$ for all $E \in \Sigma_1$. The Nikodým boundedness theorem (see [21, I.3.1]) implies that $\sup_{x\in X}\|p(x,\cdot)\| = \sup_{x\in X}\|\mu_x\| = C < \infty$. In particular, $|p(x,E)| \leqslant \|p(x,\cdot)\| \leqslant C$ for all $x \in X$ and $E \in \Sigma_1$. $\square$

Given a function $p \in \mathscr{P}(X,\Sigma_1,\Sigma_2)$, denote by T_p the function from $B(X,\Sigma_1)$ into $\mathbb{R}^X$ defined by the equality $(T_pf)(x) = \langle f, p(x,\cdot)\rangle$ for all $f \in B(X,\Sigma_1)$ and $x \in X$.

Lemma 4.3. *If $p \in \mathscr{P}(X,\Sigma_1,\Sigma_2)$ then $T_p \in \mathscr{L}(B(X,\Sigma_1),B(X,\Sigma_2))$ and $\|T_p\| \leqslant \sup_{x\in X}\|p(x,\cdot)\|$.*

PROOF. Obviously, T_p is a linear operator from $B(X,\Sigma_1)$ into $\mathbb{R}^X$. Show that T_p acts into $B(X,\Sigma_2)$. For each $E \in \Sigma_1$ we have $T_p1_E = p(\cdot,E) \in B(X,\Sigma_2)$. Since T_p is linear, we conclude that $T_ps \in B(X,\Sigma_2)$ for every step function $s \in St(X,\Sigma_1)$. Now let f be an arbitrary element of $B(X,\Sigma_1)$ and let $s_n \in St(X,\Sigma_1)$ ($n \in \mathbb{N}$) be a sequence of step functions uniformly convergent to f. Note that $(T_p(\cdot))(x) \in B(X,\Sigma_1)'$ for all $x \in X$. Consequently, $(T_ps_n)(x) \to (T_pf)(x)$ as $n \to \infty$ for all $x \in X$ and hence the function T_pf is Σ_2-measurable as a limit of Σ_2-measurable functions. In addition, for all $x \in X$ we have

$$|(T_pf)(x)| = |\langle f, p(x,\cdot)\rangle| \leqslant \|f\|\,\|p(x,\cdot)\| \leqslant \|f\| \sup_{y\in X}\|p(y,\cdot)\|. \qquad (*)$$

By Theorem 4.2, we conclude that $T_pf \in B(X,\Sigma_2)$. Thus, T_p is a linear operator from $B(X,\Sigma_1)$ into $B(X,\Sigma_2)$. The estimate for $\|T_p\|$ follows from $(*)$. $\square$

Let $\mathscr{P}(X,\Sigma)$ be the set of all transition functions (see § 1) on a measurable space (X,Σ). Theorem 4.2 immediately implies that every function $p \in \mathscr{P}(X,\Sigma)$ is uniformly bounded and $\sup_{x\in X}\|p(x,\cdot)\| < \infty$.

Endow the set $\mathscr{P}(X,\Sigma)$ with the structure of a vector space with pointwise linear operations. Furnish the resultant vector space with the following norm

$$\|p\| = \sup_{x\in X}\|p(x,\cdot)\| = \sup_{x\in X}|p(x,\cdot)|(X).$$

Endow $\mathscr{P}(X,\Sigma)$ with the pointwise order:

$$p_1 \leqslant p_2 \text{ whenever } p_1(x,E) \leqslant p_2(x,E) \text{ for all } x \in X \text{ and } E \in \Sigma.$$

Define the product of transition functions $p_1, p_2 \in \mathscr{P}(X,\Sigma)$ as follows:

$$(p_1 * p_2)(x,E) = \langle p_2(\cdot,E), p_1(x,\cdot)\rangle \quad \text{for all } x \in X \text{ and } E \in \Sigma.$$

It is easy to verify that the product of transition functions is a transition function and under the above operations $\mathscr{P}(X,\Sigma)$ is an ordered normed algebra.

Below we will show isomorphy of the ordered normed algebras $\mathscr{P}(X,\Sigma)$, $\mathscr{L}(B(X))$, $\mathscr{L}_w(ba(\Sigma))$, $ba(\Sigma, B(X))$, and $\ell_w^\infty(X, ba(\Sigma))$.

DEFINITION 4.4. Given arbitrary $p \in \mathscr{P}(X,\Sigma)$, $T \in \mathscr{L}(B(X))$, $A \in \mathscr{L}_w(ba(\Sigma))$, $m \in ba(\Sigma, B(X))$, $v \in \ell_w^\infty(X, ba(\Sigma))$, define the functions $p_T, p_A, p_m, p_v \colon X \times \Sigma \to \mathbb{R}$, $T_p, T_A, T_m, T_v \colon B(X) \to B(X)$, $A_p, A_T, A_m, A_v \colon ba(\Sigma) \to ba(\Sigma)$, $m_p, m_T, m_A, m_v \colon \Sigma \to B(X)$, $v_p, v_T, v_A, v_m \colon X \to ba(\Sigma)$ as follows:

$$\begin{array}{lll} p_T(x,E) = (T1_E)(x), & (T_p f)(x) = \langle f, p(x,\cdot)\rangle, & (A_p\mu)(E) = \langle p(\cdot,E),\mu\rangle, \\ p_A(x,E) = (A\delta_x)(E), & (T_A f)(x) = \langle f, A\delta_x\rangle, & (A_T\mu)(E) = \langle T1_E,\mu\rangle, \\ p_m(x,E) = m(E)(x), & (T_m f)(x) = \langle f, m\rangle(x), & (A_m\mu)(E) = \langle m(E),\mu\rangle, \\ p_v(x,E) = v(x)(E), & (T_v f)(x) = \langle f, v(x)\rangle, & (A_v\mu)(E) = \langle v(\cdot)(E),\mu\rangle, \end{array}$$

$$\begin{array}{ll} m_p(E)(x) = p(x,E), & v_p(x)(E) = p(x,E), \\ m_T(E)(x) = (T1_E)(x), & v_T(x)(E) = (T1_E)(x), \\ m_A(E)(x) = (A\delta_x)(E), & v_A(x)(E) = (A\delta_x)(E), \\ m_v(E)(x) = v(x)(E), & v_m(x)(E) = m(E)(x), \end{array}$$

where $x \in X$, $E \in \Sigma$, $f \in B(X)$, $\mu \in ba(\Sigma)$.

Lemma 4.5. *For all $T \in \mathscr{L}(B(X))$, $f \in B(X)$, and $\mu \in ba(\Sigma)$ the equality*

$$\langle Tf, \mu\rangle = \langle f, A_T\mu\rangle$$

holds. In other words, under the natural identification of $B(X)'$ and $ba(\Sigma)$ (see Remark 3.10) *we have $A_T = T'$.*

PROOF. Put $\varphi = \langle T(\cdot), \mu\rangle$. It is obvious that $\varphi \in B(X)'$ and hence $\varphi = \langle \cdot, \nu\rangle$ for some measure $\nu \in ba(\Sigma)$ (see Remark 3.10). For all $E \in \Sigma$ we have $(A_T\mu)(E) = \langle T1_E, \mu\rangle = \varphi(1_E) = \langle 1_E, \nu\rangle = \nu(E)$. Consequently, $\langle Tf, \mu\rangle = \varphi(f) = \langle f, \nu\rangle = \langle f, A_T\mu\rangle$. □

Proposition 4.6. *Under the conditions of Definition* 4.4, *the following hold:* $p_T, p_A, p_m, p_v \in \mathscr{P}(X,\Sigma)$, $T_p, T_A, T_m, T_v \in \mathscr{L}(B(X))$, $A_p, A_T, A_m, A_v \in \mathscr{L}_w(ba(\Sigma))$, $m_p, m_T, m_A, m_v \in ba(\Sigma, B(X))$, $v_p, v_T, v_A, v_m \in \ell_w^\infty(X, ba(\Sigma))$.

PROOF. The containment $T_p \in \mathscr{L}(B(X))$ follows from Lemma 4.3.

For all $E \in \Sigma$ we have $A_p\mu(E) = \langle p(\cdot,E),\mu\rangle = \langle T_p1_E, \mu\rangle = A_{T_p}\mu(E)$. Since $A_{T_p} = T_p'$ (see Lemma 4.5), Proposition 2.4 implies $A_p \in \mathscr{L}_w(ba(\Sigma))$.

Demonstrate that $v_p \in \ell_w^\infty(X, ba(\Sigma))$. For every $E \in \Sigma$ we have $\langle 1_E, v_p(\cdot)\rangle = p(\cdot,E) = T_p1_E$. Consequently, $\langle s, v_p(\cdot)\rangle = T_p s$ for all $s \in St(X,\Sigma)$. Now if $f \in B(X)$ then $\langle f, v_p(x)\rangle = \lim_{n\to\infty}\langle s_n, v_p(x)\rangle = \lim_{n\to\infty}(T_p s_n)(x) = (T_p f)(x)$ for all $x \in X$, where $(s_n)_{n\in\mathbb{N}}$ is a sequence in $St(X,\Sigma)$ uniformly convergent to f. It remains to observe that $\sup_{x\in X}\|v_p(x)\| = \sup_{x\in X}\|p(x,\cdot)\| = \|p\| < \infty$.

All other containments are either evident or easily deduced from the above on using Lemma 4.5, Proposition 2.4, Remark 3.13, and the equality $p_A(\cdot,E) = A_E$ $(E \in \Sigma)$. □

Lemma 4.7. *Let $V_1, \ldots, V_n$ be ordered normed spaces and let $\alpha_1, \ldots, \alpha_n$ be mappings satisfying the following conditions for each $i = 1, \ldots, n$:*

(a) *α_i is a linear operator from V_i into V_{i+1};*
(b) *$\|\alpha_i(v)\| \leqslant \|v\|$ for all $v \in V_i$;*
(c) *$v \geqslant 0$ implies $\alpha_i(v) \geqslant 0$ for every $v \in V_i$;*
(d) *$(\alpha_{i+n-1} \cdots \alpha_{i+1}\alpha_i)(v) = v$ for all $v \in V_i$,*

where $V_{n+1} = V_1$ and $\alpha_{n+k} = \alpha_k$ for $k = 1, \ldots, n-1$. Then each of the mappings α_i is a linear isometry and order isomorphism between V_i and V_{i+1}.

Proof. Fix an arbitrary $i \in \{1, \ldots, n\}$. The operator α_i is surjective since, according to (d), $\alpha_i((\alpha_{i+n-1} \cdots \alpha_{i+2}\alpha_{i+1})(w)) = w$ for all $w \in V_{i+1}$. For every $v \in V_i$, due to (d) and (b) we have $\|v\| \leqslant \|\alpha_{i+n-1}\| \cdots \|\alpha_{i+1}\| \, \|\alpha_i\| \, \|v\| \leqslant \|v\|$. Consequently, α_i is an isometry of V_i onto V_{i+1}. It remains to observe that, by (d) and (c), $\alpha_i(v) \geqslant 0$ implies $v = (\alpha_{i+n-1} \cdots \alpha_{i+1}\alpha_i)(v) = (\alpha_{i+n-1} \cdots \alpha_{i+1})(\alpha_i(v)) \geqslant 0$ for all $v \in V_i$. □

Theorem 4.8. *The diagram, whose vertices are the five spaces $\mathscr{P}(X,\Sigma)$, $\mathscr{L}(B(X))$, $\mathscr{L}_w(ba(\Sigma))$, $ba(\Sigma, B(X))$, $\ell^\infty_w(X, ba(\Sigma))$ and edges are the twenty mappings of Definition 4.4, is commutative. Furthermore, each of the twenty mappings is an isomorphism between the corresponding spaces, where by an isomorphism we mean a linear isometry that preserves multiplication and is an order isomorphism.*

Proof. Put $V_1 = V_6 = \mathscr{P}(X,\Sigma)$, $V_2 = \mathscr{L}(B(X))$, $V_3 = \mathscr{L}_w(ba(\Sigma))$, $V_4 = ba(\Sigma, B(X))$, $V_5 = \ell^\infty_w(X, ba(\Sigma))$ and consider mappings $\alpha_i \colon V_i \to V_{i+1}$ $(i = 1, \ldots, 5)$ defined by the formulas $\alpha_1(p) = T_p$, $\alpha_2(T) = A_T$, $\alpha_3(A) = m_A$, $\alpha_4(m) = v_m$, $\alpha_5(v) = p_v$.

Show that the mappings α_i satisfy conditions (a)–(d) of Lemma 4.7 and preserve multiplication.

Condition (b) is easily checked with the help of, for instance, Lemma 4.3 and Remark 3.9. Verification of the other conditions is also straightforward. As a demonstration we will clarify the relations $A_{T_{p_{v_{m_A}}}} = A$ and $T_{p_1 * p_2} = T_{p_1}T_{p_2}$ for all $A \in \mathscr{L}_w(ba(\Sigma))$ and $p_1, p_2 \in \mathscr{P}(X,\Sigma)$. Since, as is easy to see, $\big(A_{T_{p_{v_{m_A}}}} \delta_x\big)(E) = (A\delta_x)(E)$ for all $x \in X$ and $E \in \Sigma$, the operators $A_{T_{p_{v_{m_A}}}}$ and A coincide on the weakly* dense subset $\Delta(\Sigma) \subset ba(\Sigma)$ (see Remark 3.11) and hence coincide everywhere on $ba(\Sigma)$ due to their weak* continuity.

The relation $T_{p_1 * p_2} = T_{p_1}T_{p_2}$ follows from continuity of the operators under consideration and from the equality $(T_{p_1 * p_2} 1_E)(x) = (T_{p_1}T_{p_2} 1_E)(x)$ easily verified for all $x \in X$ and $E \in \Sigma$.

So, according to Lemma 4.7 the mappings $\alpha_1, \ldots, \alpha_5$ are isomorphisms.

The fact that the mappings $T \mapsto p_T$, $A \mapsto T_A$, $m \mapsto A_m$, $v \mapsto m_v$, and $p \mapsto v_p$ are inverse to the corresponding isomorphisms $\alpha_1, \ldots, \alpha_5$ follows from the obvious equalities $p_{T_p} = p$, $m_{A_m} = m$, $v_{m_v} = v$, $p_{v_p} = p$ and from the relation $(A_{T_A}\mu)(E) = \langle T_A 1_E, \mu\rangle = \langle 1_E, A\mu\rangle = (A\mu)(E)$ that holds for all $E \in \Sigma$ and $\mu \in ba(\Sigma)$ by Lemma 4.5.

The claim of the theorem can be now easily obtained with the help of the above-established facts, condition (d) of Lemma 4.7 for the mappings $\alpha_1, \ldots, \alpha_5$, and the easily checked equalities $(m \mapsto p_m) = \alpha_5\alpha_4$, $(A \mapsto p_A) = \alpha_5\alpha_4\alpha_3$, $(m \mapsto T_m) = \alpha_1\alpha_5\alpha_4$, $(v \mapsto T_v) = \alpha_1\alpha_5$, $(p \mapsto A_p) = \alpha_2\alpha_1$, $(v \mapsto A_v) = \alpha_2\alpha_1\alpha_5$, $(p \mapsto m_p) = \alpha_3\alpha_2\alpha_1$, $(T \mapsto m_T) = \alpha_3\alpha_2$, $(T \mapsto v_T) = \alpha_4\alpha_3\alpha_2$, $(A \mapsto v_A) = \alpha_4\alpha_3$. □

Corollary 4.9. *The spaces $\mathscr{P}(X,\Sigma)$, $\mathscr{L}(B(X))$, $\mathscr{L}_w(ba(\Sigma))$, $ba(\Sigma, B(X))$, and $\ell^\infty_w(X, ba(\Sigma))$ are ordered Banach algebras.*

Proof. It suffices to observe that $\mathscr{L}(B(X))$ is an ordered Banach algebra and use Theorem 4.8. □

The following assertion implies that the ordered vector spaces $\mathscr{P}(X,\Sigma)$, $\mathscr{L}(B(X))$, $\mathscr{L}_w(ba(\Sigma))$, $ba(\Sigma, B(X))$, and $\ell^\infty_w(X, ba(\Sigma))$ are not vector lattices in general.

Theorem 4.10. *Let $(X, \Sigma, |\cdot|)$ be a measure space possessing a lifting and assume that $\{x\} \in \Sigma$ and $|\{x\}| = 0$ for all $x \in X$ and there exists a nonmeasurable subset $G \subset X$.* (As an example of such a measure space we can take, for instance, the interval $[0,1]$ with the Lebesgue measure.) *Then the ordered vector space $\mathscr{L}(B(X))$ is not a vector lattice.*

PROOF. Define $T \in \mathscr{L}(B(X))$ by putting $Tf = 1_G(f - f_\sim)$ for all $f \in B(X)$ and demonstrate that the operator T has no positive part. (The fact that T acts into $B(X)$ and is bounded follows from coincidence of f and $f_\sim$ almost everywhere and from the relations $|1_G(f-f_\sim)| \leqslant |f|+|f_\sim| = |f|+|f|_\sim \leqslant 2\|f\|$ justified by the properties of a lifting.)

Assume to the contrary that T has a positive part, T^+. Then for all $x \in G$ we have $T^+1_X \geqslant T^+1_{\{x\}} \geqslant T1_{\{x\}} = 1_{\{x\}}$, i.e., $T^+1_X \geqslant 1$ on G. Fix an arbitrary point $x \in X\backslash G$ and define a positive operator $Z_x \in \mathscr{L}(B(X))$ by the formula $Z_x f = 1_{X\backslash\{x\}}f$, $f \in B(X)$. Then for all positive $f \in B(X)$ we have $Z_x f = 1_{X\backslash\{x\}}f \geqslant 1_G f \geqslant 1_G(f - f_\sim) = Tf$, i.e., $Z_x \geqslant T$. Consequently, $Z_x \geqslant T^+$ and, in particular, $Z_x 1_X \geqslant T^+1_X$, whence $T^+1_X = 0$ on $X\backslash G$. On the other hand, as was established above, $T^+1_X \geqslant 1$ on G and hence the function T^+1_X is not measurable. □

§ 5. Countably Additive and Purely Finitely Additive Transition Functions

In this section we introduce and study the spaces $\mathscr{P}_{ca}(X,\Sigma)$ and $\mathscr{P}_{pfa}(X,\Sigma)$ of countably additive and purely finitely additive transition functions, show that they are mutually complementary bands with respect to a natural disjointness, and examine the question of the decomposition $\mathscr{P}(X,\Sigma) = \mathscr{P}_{ca}(X,\Sigma) \oplus \mathscr{P}_{pfa}(X,\Sigma)$.

Note that analogs of the spaces $\mathscr{P}(X,\Sigma)$, $\mathscr{P}_{ca}(X,\Sigma)$, and $\mathscr{P}_{pfa}(X,\Sigma)$ for the case of finitely additive transition probabilities were considered in the article by A. I. Zhdanok [2] (in their connection with the corresponding Markov chains). In the same article, the question was first raised of splitting finitely additive transition probabilities into the sums of countably additive and purely finitely additive parts. Below we study this question (in a more general setting of arbitrary transition functions) and, in particular, show that this is impossible in general.

By the way we densely embed $\mathscr{P}(X,\Sigma)$ into a Banach K-space, $\overline{\mathscr{P}}(X,\Sigma)$, and study the order properties of this embedding.

Let (X,Σ) be an arbitrary measurable space.

DEFINITION 5.1. A transition function $p \in \mathscr{P}(X,\Sigma)$ is called *countably additive* if $p(x,\cdot) \in ca(\Sigma)$ for all $x \in X$. The set of all countably additive transition functions is denoted by $\mathscr{P}_{ca}(X,\Sigma)$.

Theorem 5.2. *The spaces $\mathscr{P}_{ca}(X,\Sigma)$, $\mathscr{L}_o(B(X))$, $\mathscr{L}_{wc}(ba(\Sigma))$, $o\text{-}ca(\Sigma, B(X))$, and $\ell^\infty_w(X, ca(\Sigma))$ are ordered Banach subalgebras of $\mathscr{P}(X,\Sigma)$, $\mathscr{L}(B(X))$, $\mathscr{L}_w(ba(\Sigma))$, $ba(\Sigma, B(X))$, and $\ell^\infty_w(X, ba(\Sigma))$. The diagram, whose vertices are these five subalgebras and edges are the restrictions of the twenty mappings in Definition* 4.4, *is commutative. Each of the restrictions is an isomorphism between the corresponding spaces, where by an isomorphism we mean a linear isometry that preserves multiplication and is an order isomorphism.*

PROOF. By Theorem 4.8 it is sufficient to observe that $\mathscr{L}_o(B(X))$ is an ordered Banach subalgebra of $\mathscr{L}(B(X))$ and show the containments $T_p \in \mathscr{L}_o(B(X))$, $A_T \in \mathscr{L}_{wc}(ba(\Sigma))$, $m_A \in o\text{-}ca(\Sigma, B(X))$, $v_m \in \ell^\infty_w(X, ca(\Sigma))$, $p_v \in \mathscr{P}_{ca}(X,\Sigma)$ for all $p \in \mathscr{P}_{ca}(X,\Sigma)$, $T \in \mathscr{L}_o(B(X))$, $A \in \mathscr{L}_{wc}(ba(\Sigma))$, $m \in o\text{-}ca(\Sigma, B(X))$, $v \in \ell^\infty_w(X, ca(\Sigma))$.

In the sequel we often use Remarks 2.3 without explicit references.

Show that $T_p \in \mathscr{L}_o(B(X))$. Let a sequence $(f_n)_{n\in\mathbb{N}}$ in $B(X)$ be o-convergent to an $f \in B(X)$. Since $p(x,\cdot) \in ca(\Sigma)$ $(x \in X)$; using the Lebesgue theorem, for all $x \in X$ we have $(T_p f_n)(x) = \langle f_n, p(x,\cdot)\rangle \to \langle f, p(x,\cdot)\rangle = (T_p f)(x)$. In addition, $\|T_p f_n\| \leqslant \|T_p\| \sup_{m\in\mathbb{N}} \|f_m\|$ for all $n \in \mathbb{N}$ and hence $T_p f_n \xrightarrow{o} T_p f$.

Establish the containment $A_T \in \mathscr{L}_{wc}(ba(\Sigma))$. Suppose that $\mu \in ca(\Sigma)$, $E_n \in \Sigma$ $(n \in \mathbb{N})$, and $E_n \downarrow \varnothing$. Sequential o-continuity of T implies the convergence $T1_{E_n} \xrightarrow{o} 0$. Thus, according to the Lebesgue theorem, $(A_T\mu)(E_n) = \langle T1_{E_n}, \mu\rangle \to 0$ as $n \to \infty$, i.e., $A_T\mu \in ca(\Sigma)$.

Show that $m_A \in o\text{-}ca(\Sigma, B(X))$. Let $E_n \in \Sigma$ $(n \in \mathbb{N})$, $E_n \downarrow \varnothing$. For all $x \in X$, due to the fact that $A\delta_x \in ca(\Sigma)$ we have $m_A(E_n)(x) = (A\delta_x)(E_n) \to 0$ as $n \to \infty$. In addition, $\|m_A(E_n)\| \leqslant \|m_A\|$ for all $n \in \mathbb{N}$ and hence $m_A(E_n) \xrightarrow{o} 0$.

The containments $v_m \in \ell^\infty_w(X, ca(\Sigma))$ and $p_v \in \mathscr{P}_{ca}(X,\Sigma)$ are obvious. □

Introduce the notation $\sim$ for the Σ-inseparability relation on X; i.e., $x \sim y$ means that the points $x, y \in X$ belong to the same lump of (X, Σ).

Denote by $\overline{\mathscr{P}}(X, \Sigma)$ the set of all functions $p\colon X \times \Sigma \to \mathbb{R}$ satisfying the following conditions for all $x, y \in X$ and $E \in \Sigma$:

(a) $p(x, \cdot) \in ba(\Sigma)$;
(b) if $x \sim y$ then $p(x, E) = p(y, E)$;
(c) the function $p(\cdot, E)$ is bounded.

Make $\overline{\mathscr{P}}(X, \Sigma)$ into an ordered normed space by endowing it with the pointwise linear operations, pointwise order, and norm $\|p\| = \sup_{x \in X} \|p(x, \cdot)\| = \sup_{x \in X} |p(x, \cdot)|(X)$ that is finite due to Theorem 4.2 and the obvious equality $\overline{\mathscr{P}}(X, \Sigma) = \mathscr{P}(X, \Sigma, \overline{\Sigma})$, where $\overline{\Sigma}$ is the σ-algebra of Σ-compatible subsets of X.

The ordered normed space $\overline{\mathscr{P}}(X, \Sigma)$ is obviously isomorphic to the Banach K-space $\ell^\infty_\Sigma(X, ba(\Sigma))$ of all bounded Σ-compatible $ba(\Sigma)$-valued functions, i.e., bounded functions $v\colon X \to ba(\Sigma)$ satisfying $v(x) = v(y)$ for $x \sim y$. Thus, $\overline{\mathscr{P}}(X, \Sigma)$ is also a Banach K-space. Furthermore, in exactly the same way as in the proof of Theorem 4.8, isomorphy can be shown between $\overline{\mathscr{P}}(X, \Sigma)$ and the Banach K-spaces $\mathscr{L}(B(X, \Sigma), B(X, \overline{\Sigma}))$, $\mathscr{L}_w(ba(\overline{\Sigma}), ba(\Sigma))$, and $ba(\Sigma, B(X, \overline{\Sigma}))$. Concrete isomorphisms are determined by the formulas of Definition 4.4.

Definition 5.3. We call transition functions $p_1, p_2 \in \mathscr{P}(X, \Sigma)$ *disjoint* and write $p_1 \perp p_2$ whenever the measures $p_1(x, \cdot)$ and $p_2(x, \cdot)$ are disjoint for each $x \in X$.

Observe that $\mathscr{P}(X, \Sigma) \subset \overline{\mathscr{P}}(X, \Sigma)$, and transition functions $p_1, p_2 \in \mathscr{P}(X, \Sigma)$ are disjoint in the sense of the above definition if and only if p_1 and p_2 are disjoint in the K-space $\overline{\mathscr{P}}(X, \Sigma)$.

Theorem 5.4. *Let (X, Σ) be an atomic measurable space.*

(1) *The set $\mathscr{P}(X, \Sigma)$ minorizes $\overline{\mathscr{P}}(X, \Sigma)$.*

(2) *The set $\mathscr{P}(X, \Sigma)$ is hereditarily embedded in $\overline{\mathscr{P}}(X, \Sigma)$.*

(3) *If a family of transition functions $(p_\xi)_{\xi \in \Xi}$ has supremum or infimum in $\mathscr{P}(X, \Sigma)$ then, respectively,*

$$(\sup_{\xi \in \Xi} p_\xi)(x, \cdot) = \sup_{\xi \in \Xi} p_\xi(x, \cdot), \quad (\inf_{\xi \in \Xi} p_\xi)(x, \cdot) = \inf_{\xi \in \Xi} p_\xi(x, \cdot)$$

for all $x \in X$, where the right-hand sides are calculated in the K*-space $ba(\Sigma)$.*

(4) *The relation $\perp$ of Definition* 5.3 *is a disjointness on the vector space $\mathscr{P}(X, \Sigma)$ in the sense of Definition* 2.1.

(5) *Every element $\overline{p} \in \overline{\mathscr{P}}(X, \Sigma)$ is represented as $\overline{p} = o\text{-}\sum_{\xi \in \Xi} p_\xi$ for some family $(p_\xi)_{\xi \in \Xi} \subset \mathscr{P}(X, \Sigma)$ of pairwise disjoint transition functions.*

Proof. Show (1). Let $0 < \overline{p} \in \overline{\mathscr{P}}(X, \Sigma)$. Take an arbitrary point $x_0 \in X$ with $\overline{p}(x_0, \cdot) > 0$, denote by A the atom of Σ that contains x_0, and put $p(x, E) = 1_A(x)\overline{p}(x, E)$ for all $x \in X$ and $E \in \Sigma$. It is obvious that $p \in \mathscr{P}(X, \Sigma)$ and $0 < p \leqslant \overline{p}$.

Assertions (2)–(4) follow from (1), Theorem 2.2, and the fact that suprema and infima in $\overline{\mathscr{P}}(X, \Sigma)$ are determined by the formulas in (3) (the latter ensues, for instance, from isomorphy of $\overline{\mathscr{P}}(X, \Sigma)$ to the K-space $\ell^\infty_\Sigma(X, ba(\Sigma))$ in which the bounds are calculated pointwise).

A family $(p_\xi)_{\xi \in \Xi}$ mentioned in (5) can be defined as $p_\xi(x, E) = 1_\xi(x)\overline{p}(x, E)$ for all $\xi \in \Xi$, $x \in X$, and $E \in \Sigma$, where Ξ is the set of all atoms of Σ. $\square$

Example 5.5. We will demonstrate that the atomicity of (X, Σ) is essential for each of the assertions of Theorem 5.4 to be valid.

Let $X = [0, 1]$. Consider a Lebesgue nonmeasurable set $G \subset X$ and put

$$\Sigma = \{E \in \mathscr{L} : \ E \supset G \text{ or } E \cap G = \varnothing\},$$

where $\mathscr{L}$ is the σ-algebra of Lebesgue measurable subsets of X. As was already noted in Remark 3.5, the set G is a nonmeasurable lump of (X, Σ); thus, the measurable space (X, Σ) is not atomic.

Let $|\cdot|$ be the restriction of the Lebesgue measure onto Σ. It is easy to see that the triple $(X, \Sigma, |\cdot|)$ is a measure space in the sense of Definition 3.1. Consider a lifting λ of the quotient algebra $\widetilde{\mathscr{L}}$ (with respect to the Lebesgue measure on X) and, for each $E \in \Sigma$, put

$$\rho(E^{\sim}) = \begin{cases} \lambda(E^{\sim}) \cup G & \text{if } E \supset G, \\ \lambda(E^{\sim}) \backslash G & \text{if } E \cap G = \varnothing. \end{cases}$$

A straightforward verification shows that the mapping ρ is defined correctly and is a lifting of the quotient algebra $\widetilde{\Sigma}$ of the measure space $(X, \Sigma, |\cdot|)$ (for instance, Proposition 3.2 can be employed).

Counterexample to (1), (5). Suppose that $\mu \in ba(\Sigma)$, $\mu > 0$. Consider an element $0 < \overline{p} \in \overline{\mathscr{P}}(X, \Sigma)$ defined by the formula

$$\overline{p}(x, E) = 1_G(x)\mu(E), \quad x \in X,\ E \in \Sigma,$$

and show that it is not minorized by positive transition functions (we thus obtain a counterexample to assertions (1) and (5) of Theorem 5.4). To this end, assume that $p \in \mathscr{P}(X, \Sigma)$, $0 \leqslant p \leqslant \overline{p}$, and establish the equality $p = 0$. Given an $E \in \Sigma$, put $Z_E = \{x \in X : p(x, E) = 0\} \in \Sigma$. The definition of $\overline{p}$ readily implies the inclusion $X \backslash G \subset Z_E$. Nonmeasurability of G and measurability of Z_E yield $Z_E \cap G \neq \varnothing$, whence $Z_E \supset G$ by the definition of Σ; therefore, $Z_E = X$.

Counterexample to (2), (3). Show that the transition function $p \in \mathscr{P}(X, \Sigma)$ defined as

$$p(x, E) = 1_E(x) - 1_{\rho(E^{\sim})}(x), \quad x \in X,\ E \in \Sigma,$$

has a positive part p^+ in the ordered vector space $\mathscr{P}(X, \Sigma)$, but $p^+(x, \cdot) \neq p(x, \cdot)^+$ for $x \in G$ (we thus demonstrate that assertions (2) and (3) of Theorem 5.4 fail for (X, Σ)).

Prove that $p^+ = \mathrm{id}$, where $\mathrm{id}(x, E) = 1_E(x) = \delta_x(E)$ for all $x \in X$ and $E \in \Sigma$. Obviously, $\mathrm{id} \geqslant p$. Consider an arbitrary positive transition function $\overline{p} \in \mathscr{P}(X, \Sigma)$ satisfying $\overline{p} \geqslant p$ and show that $\overline{p} \geqslant \mathrm{id}$, i.e., $\overline{p}(\cdot, E) \geqslant 1_E$ for all $E \in \Sigma$. Fix any $E \in \Sigma$. Assume first that $E \cap G = \varnothing$. Then for all $x \in E$ we have $\overline{p}(\cdot, E) \geqslant p(\cdot, \{x\}) = 1_{\{x\}}$, i.e., $\overline{p}(\cdot, E) \geqslant 1_E$. Assume now that $E \supset G$. Then $\overline{p}(\cdot, E) \geqslant p(\cdot, \{x\}) = 1_{\{x\}}$ for all $x \in E \backslash G$, i.e., $\overline{p}(\cdot, E) \geqslant 1_{E \backslash G}$. Put $F = \{x \in X : \overline{p}(x, E) \geqslant 1\} \in \Sigma$. It is clear that $F \supset E \backslash G$. If $F \cap G = \varnothing$ then $F \cap E = E \backslash G$, which contradicts measurability of $F \cap E$. Consequently, $F \supset G$ and hence $F \supset E$; therefore, $\overline{p}(\cdot, E) \geqslant 1_E$.

Fix an arbitrary point $x \in G$. From the definitions of the function p and lifting ρ it is clear that $p(x, \cdot) = 0$, and hence $p(x, \cdot)^+ = 0$. On the other hand, $p^+(x, \cdot) = \mathrm{id}(x, \cdot) = \delta_x \neq 0$.

Counterexample to (4). Observe first that $\delta_x \perp \delta_x \circ \rho$ for all $x \in X \backslash G$. Indeed, if $x \in X \backslash G$ and $\mu = \delta_x \wedge \delta_x \circ \rho$ then $\mu(X \backslash \{x\}) \leqslant \delta_x(X \backslash \{x\}) = 0$ and $\mu(\{x\}) \leqslant (\delta_x \circ \rho)(\{x\}) = \delta_x(\varnothing) = 0$, whence $\mu = 0$.

Prove now that, for all $p_1, p_2 \in \mathscr{P}(X, \Sigma)$, the relation $p_1 \in \{p_2\}^{\perp\perp}$ implies $p_1(x, \cdot) \in \{p_2(x, \cdot)\}^{\perp\perp}$ whenever $x \in X \backslash G$. Indeed, let $p_1 \in \{p_2\}^{\perp\perp}$ and $x \in X \backslash G$. Given any $\mu \in \{p_2(x, \cdot)\}^{\perp}$, $y \in X$, and $E \in \Sigma$, put

$$\delta_x^{\mu}(y, E) = \begin{cases} \mu(E) & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases}$$

Obviously, $\delta_x^{\mu} \in \mathscr{P}(X, \Sigma)$ (due to the fact that $\{x\} \in \Sigma$) and $\delta_x^{\mu} \perp p_2$. Hence, $\delta_x^{\mu} \perp p_1$ and, in particular, $\mu = \delta_x^{\mu}(x, \cdot) \perp p_1(x, \cdot)$. Arbitrariness of $\mu \in \{p_2(x, \cdot)\}^{\perp}$ makes it possible to conclude that $p_1(x, \cdot) \in \{p_2(x, \cdot)\}^{\perp\perp}$.

Finally, given $x \in X$ and $E \in \Sigma$, put

$$p_1(x, E) = \delta_x(E),$$
$$p_2(x, E) = \begin{cases} \delta_x(E) & \text{if } x \in G, \\ \delta_x(\rho(E^{\sim})) & \text{if } x \in X \backslash G. \end{cases}$$

The containment $p_1 \in \mathscr{P}(X, \Sigma)$ is evident. Show that $p_2 \in \mathscr{P}(X, \Sigma)$. The properties of a lifting imply $p_2(x, \cdot) \in ba(\Sigma)$ for all $x \in X$. It remains to take any $E \in \Sigma$ and demonstrate Σ-measurability of the function $p_2(\cdot, E)$, i.e., the fact that the set $F = \{x \in X : p_2(x,E) = 1\}$ belongs to Σ. The definition of p_2 readily implies the equality $F = (E \cap G) \cup (\rho(E^\sim)\backslash G)$. If $E \supset G$ then by the definition of ρ we have $\rho(E^\sim) \supset G$ and hence $F = G \cup (\rho(E^\sim)\backslash G) = \rho(E^\sim) \in \Sigma$. If $E \cap G = \varnothing$ then $\rho(E^\sim) \cap G = \varnothing$ and $F = \rho(E^\sim)\backslash G = \rho(E^\sim) \in \Sigma$.

So, $p_1, p_2 \in \mathscr{P}(X,\Sigma)$. It is clear that the transition functions p_1 and p_2 are not disjoint. Show that, nevertheless, $\{p_1\}^{\perp\perp} \cap \{p_2\}^{\perp\perp} = \{0\}$ (as a result, we demonstrate that the relation $\perp$ does not satisfy condition 2.1 (4) and thus obtain a counterexample to assertion (4) of Theorem 5.4).

Suppose that $p \in \{p_1\}^{\perp\perp} \cap \{p_2\}^{\perp\perp}$. Fix an arbitrary point $x \in X\backslash G$. By what was proven above, we have $p(x,\cdot) \in \{p_1(x,\cdot)\}^{\perp\perp}$ and $p(x,\cdot) \in \{p_2(x,\cdot)\}^{\perp\perp}$. On the other hand, $p_1(x,\cdot) = \delta_x \perp \delta_x \circ \rho = p_2(x,\cdot)$. Therefore, $p(x,\cdot) = 0$ for $x \in X\backslash G$. Then for all $E \in \Sigma$ the set $\{x \in X : p(x,E) \neq 0\}$, that belongs to Σ, is included in G and is thus empty. The latter means that $p = 0$.

Definition 5.6. A transition function $p \in \mathscr{P}(X,\Sigma)$ is said to be *purely finitely additive* if $p(x,\cdot) \in pfa(\Sigma)$ for all $x \in X$. The set of all purely finitely additive transition functions is denoted by $\mathscr{P}_{pfa}(X,\Sigma)$.

As is easy to see, $\mathscr{P}_{pfa}(X,\Sigma)$ is an ordered Banach subspace of $\mathscr{P}(X,\Sigma)$.

Theorem 5.7. *Let (X,Σ) be a measurable space.*

(1) *The sets $\mathscr{P}_{ca}(X,\Sigma)$ and $\mathscr{P}_{pfa}(X,\Sigma)$ are mutually complementary $\perp$-bands in $\mathscr{P}(X,\Sigma)$, i.e.,*

$$\mathscr{P}_{ca}(X,\Sigma)^\perp = \mathscr{P}_{pfa}(X,\Sigma), \quad \mathscr{P}_{pfa}(X,\Sigma)^\perp = \mathscr{P}_{ca}(X,\Sigma).$$

(2) *If (X,Σ) is discrete then*

$$\mathscr{P}(X,\Sigma) = \mathscr{P}_{ca}(X,\Sigma) \oplus \mathscr{P}_{pfa}(X,\Sigma).$$

(3) *Let $(X,\Sigma,|\cdot|)$ be a measure space possessing a lifting and suppose that there exists a nonmeasurable subset $G \subset X$ and the Boolean algebra $\widetilde{\Sigma}$ has no atoms.* (As an example of such a measure space we can take, for instance, the interval $[0,1]$ with the Lebesgue measure.) *Then*

$$\mathscr{P}(X,\Sigma) \neq \mathscr{P}_{ca}(X,\Sigma) + \mathscr{P}_{pfa}(X,\Sigma).$$

Proof. (1) Let $p \in \mathscr{P}_{ca}(X,\Sigma)^\perp$. Given any measure $\mu \in ca(\Sigma)$, define $\mathrm{id}_\mu \in \mathscr{P}_{ca}(X,\Sigma)$ by putting $\mathrm{id}_\mu(x,E) = \mu(E)$ for all $x \in X$ and $E \in \Sigma$. The relation $p \perp \mathrm{id}_\mu$ implies that for all $x \in X$ the measure $p(x,\cdot)$ is disjoint from $\mathrm{id}_\mu(x,\cdot) = \mu$. Arbitrariness of $\mu \in ca(\Sigma)$ makes it possible to conclude that $p(x,\cdot) \in pfa(\Sigma)$ for all $x \in X$, i.e., $p \in \mathscr{P}_{pfa}(X,\Sigma)$. Therefore, $\mathscr{P}_{ca}(X,\Sigma)^\perp \subset \mathscr{P}_{pfa}(X,\Sigma)$. The inclusion $\mathscr{P}_{pfa}(X,\Sigma)^\perp \subset \mathscr{P}_{ca}(X,\Sigma)$ is established in exactly the same way, and the reverse inclusions are obvious.

Assertion (2) is a direct consequence of Remark 3.7.

(3) Let ρ be a lifting of the quotient algebra $\widetilde{\Sigma}$ of $(X,\Sigma,|\cdot|)$. According to Theorem 3.8, there exists a set $X_0 \in \Sigma$ such that $|X\backslash X_0| = 0$ and $\delta_x \circ \rho \in pfa(\Sigma)$ for all $x \in X_0$. Denote the (nonmeasurable) intersection $G \cap X_0$ by G_0 and put

$$p(x,E) = 1_{G_0}(x)(1_E(x) - 1_{\rho(E)}(x)), \quad x \in X,\ E \in \Sigma.$$

As is easy to see, $p \in \mathscr{P}(X,\Sigma)$ (observe that, for all $E \in \Sigma$, the function $p(\cdot,E) = 1_{G_0}(1_E - 1_{\rho(E)})$ vanishes almost everywhere and is thus measurable). We will show that p is not representable as the sum of a countably additive and a purely finitely additive transition functions.

By Remark 3.7, every measure $\mu \in ba(\Sigma)$ admits a unique decomposition into the sum of elements of $ca(\Sigma)$ and $pfa(\Sigma)$. Denote such elements by μ_{ca} and μ_{pfa}.

Assume to the contrary that $p = p_{ca} + p_{pfa}$, where $p_{ca} \in \mathscr{P}_{ca}(X,\Sigma)$ and $p_{pfa} \in \mathscr{P}_{pfa}(X,\Sigma)$. By the above remark, $p_{ca}(x,\cdot) = p(x,\cdot)_{ca}$ for all $x \in X$. On the other hand, as is easily seen,

$$p(x,\cdot) = \begin{cases} \delta_x - \delta_x \circ \rho & \text{if } x \in G_0, \\ 0 & \text{if } x \notin G_0, \end{cases}$$

while $\delta_x \in ca(\Sigma)$ and $\delta_x \circ \rho \in pfa(\Sigma)$ for $x \in G_0$. Consequently,

$$p_{ca}(x,\cdot) = p(x,\cdot)_{ca} = \begin{cases} \delta_x & \text{if } x \in G_0, \\ 0 & \text{if } x \notin G_0 \end{cases}$$

and hence the function $p_{ca}(\cdot, X) = 1_{G_0}$ is not measurable. This contradiction completes the proof. □

So, despite the fact that $\mathscr{P}_{ca}(X,\Sigma)$ and $\mathscr{P}_{pfa}(X,\Sigma)$ are mutually complementary $\perp$-bands, their sum is not always the entire space $\mathscr{P}(X,\Sigma)$. In accordance with representativeness of this sum in $\mathscr{P}(X,\Sigma)$, we may mention the obvious equality $(\mathscr{P}_{ca}(X,\Sigma) + \mathscr{P}_{pfa}(X,\Sigma))^{\perp\perp} = \mathscr{P}(X,\Sigma)$.

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