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THE WICKSTEAD PROBLEM

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In 1977 Anthony Wickstead raised the question of the conditions for all band preserving linear operators to be order bounded in a vector lattice. This article overviews the main ideas and results on the Wickstead problem and its variations, focusing primarily on the case of band preserving operators in a universally complete vector lattice.

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To Anthony Wickstead on his sixtieth birthday

December 13, 2007
is the date of the sixtieth birthday of
Professor Anthony Wickstead
who is associated with Queens' University at Belfast.
Professor Wickstead works in positivity,
a vast and attractive area of functional analysis
which borders many powerful theories
of modern mathematics.
His contributions to positivity
has brought him world fame and recognition.
He was the first Editor-in-Chief
of the international journal "Positivity."
His efforts and contributions in this capacity
have made this journal a natural epicenter
of research into positivity.
Russian mathematicians appreciate not only
the scientific contribution of Professor Wickstead
but also his charm and personality.
On behalf of his colleagues in Russia
and the Editorial Board
of Vladikavkaz Mathematical Journal
we heartily congratulate Professor Wickstead
and wish him many happy returns of the day.

A. Kusraev, S. Kutateladze

Introduction

WP: *When are we so happy in a vector lattice that all band preserving linear operators turn out to be order bounded?* This question was raised by Wickstead in [68]. The answer depends on the vector lattice in which the operator in question acts. There are several results that guarantee automatic order boundedness for a band preserving operator acting in concrete classes of vector lattices (cp. [2, Theorem 2], [3, Theorems 3.2 and 3.3], and [58, Corollary 2.3]). However, in this article we focus our attention on the case of universally complete vector lattices.

Abramovich, Veksler, and Koldunov were the first to announce an example of an order unbounded band preserving operator in [2, Theorem 1]. Later these authors [3, Theorem 2.1] as well as McPolin and Wickstead [58, Theorem 3.2] showed that all band preserving operators in a universally complete vector lattice E are bounded automatically if and only if E is locally one-dimensional. The Wickstead problem in the class of universally complete vector lattices was thus reduced to the characterization of locally one-dimensional vector lattices.

This led to another problem posed by Wickstead [7]: *Is the class of locally one-dimensional vector lattices coincident with the class of discrete vector lattices?* Gutman gave the negative answer in [40]: *There is a continuous (purely nonatomic) locally one-dimensional universally complete vector lattice* (cp. [39, 41]). Also, Gutman described the bases of locally one-dimensional universally complete vector lattices: these are exactly σ-distributive complete Boolean algebras.

Furthermore, it is well known in Boolean valued analysis that the condition for a universally complete vector lattice to be locally one-dimensional is related to the structure of the reals $\mathscr{R}$ inside an appropriate Boolean valued model $\mathbb{V}^{(\mathbb{B})}$. In more detail the situation is as follows (cp. [53]): By the Gordon Theorem, each universally complete vector lattice may be represented as the descent $\mathscr{R}{\downarrow}$ of the Boolean valued reals $\mathscr{R}$, while the image of the standard reals $\mathbb{R}$ (under the canonical embedding of the standard universe $\mathbb{V}$ into the Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$) is the subfield $\mathbb{R}^\wedge$ of $\mathscr{R}$ inside $\mathbb{V}^{(\mathbb{B})}$. It is easy and well-known in other terms that $\mathscr{R}{\downarrow}$ is locally one-dimensional if and only if $\mathbb{R}^\wedge = \mathscr{R}$. The same is true for Boolean valued complexes $\mathscr{C}$ and the image $\mathbb{C}^\wedge$ of the standard reals $\mathbb{C}$.

The Boolean approach to band preserving operators as developed by Kusraev in [47] reveals new interconnections. For example, the construction of an order unbounded band preserving operator can be carried out inside an appropriate Boolean valued universe by using a Hamel basis of the reals $\mathscr{R}$ considered as a vector space over its subfield $\mathbb{R}^\wedge$ (cp. [46, 54]). Of course, some important properties of $\mathscr{R}{\downarrow}$ are connected with the structure of the reals $\mathscr{R}$ as a vector space over $\mathbb{R}^\wedge$. In particular, using a Hamel basis, we can construct a discontinuous $\mathbb{R}^\wedge$-linear function in $\mathscr{R}$ which gives an order unbounded band preserving linear operator in the universally complete vector lattice $\mathscr{R}{\downarrow}$.

As was demonstrated by Kusraev in [50], similar constructions can be carried out on using a *transcendence basis* instead of a Hamel basis. This approach yielded the new characterizations of universally complete vector lattices with σ-distributive base in terms of narrower classes of band preserving linear operators, namely, of derivations and automorphisms. In particular, working with a transcendence basis, we can construct a discontinuous $\mathbb{C}^\wedge$-derivation and $\mathbb{C}^\wedge$-automorphism in $\mathscr{C}$ which gives an order unbounded band preserving derivation or automorphism in $\mathscr{C}{\downarrow}$.

Summarizing the results of [2, 40, 47, 50, 58] on the Wickstead problem, we can state the following

Theorem WP. *Assume that G is a universally complete vector lattice with a fixed order unity $\mathbb{1}$, while $G_{\mathbb{C}}$ is the complexification of G, and $\mathbb{B} := \mathfrak{E}(G) := \mathfrak{E}(\mathbb{1})$ is the Boolean algebra of all components of $\mathbb{1}$. Assume further that $\mathscr{R}$ and $\mathscr{C}$ stand for the reals and the complexes inside the Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$. Then the following are equivalent:*

WP(1) $\mathbb{B}$ *is σ-distributive;*

WP(2) $\mathscr{R} = \mathbb{R}^\wedge$ *inside* $\mathbb{V}^{(\mathbb{B})}$*;*

WP(2′) $\mathscr{C} = \mathbb{C}^\wedge$ *inside* $\mathbb{V}^{(\mathbb{B})}$*;*

WP(3) *G is locally one-dimensional;*

WP(3′) *$G_{\mathbb{C}}$ is locally one-dimensional;*

WP(4) *Every band preserving linear operator in G is order bounded;*

WP(4′) *Every band preserving linear operator in $G_{\mathbb{C}}$ is order bounded;*

WP(5) *There is no nontrivial $\mathbb{R}$-derivation in the f-algebra G;*

WP(5′) *There is no nontrivial $\mathbb{C}$-derivation in the complex f-algebra $G_{\mathbb{C}}$;*

WP(6) *Each band preserving endomorphism of the complex f-algebra $G_{\mathbb{C}}$ is a band projection;*

WP(7) *There is no band preserving automorphism of $G_{\mathbb{C}}$ other than the identity.*

The goal of this article is to examine the Wickstead problem for universally complete vector lattices and to prove the above theorem. The reader can find the necessary information on the theory of vector lattices in [10, 46, 72]; Boolean valued analysis, in [13, 53, 54]; and field theory, in [24, 67, 73]. Some aspects of the Wickstead problem are also presented in [46, Chapter 5], [54, Section 10.7], and [51].

By a vector lattice throughout the sequel we will mean a real Archimedean vector lattice, unless specified otherwise. We let := denote the assignment by definition, while $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ symbolize the naturals, the integers, the rationals, the reals, and the complexes. We denote the Boolean algebras of bands and band projections in a vector lattice E by $\mathfrak{B}(E)$ and $\mathfrak{P}(E)$; and we let $\mathfrak{E}(\mathbb{1})$ stand for the Boolean algebra of all components of $\mathbb{1}$.

PART 1. LOCALLY ONE-DIMENSIONAL VECTOR LATTICES

In this part we introduce locally one-dimensional vector lattices and σ-distributive Boolean algebras and prove that the following are equivalent for each universally complete vector lattice G with base $\mathbb{B} := \mathfrak{B}(G)$, the complete Boolean algebra of bands in G:

WP(1) $\mathbb{B}$ is σ-distributive;

WP(3) G is locally one-dimensional;

WP(4) Every band preserving linear operator in G is order bounded.

1.1. Band Preserving Operators

In this section we introduce the class of band preserving operators and briefly overview some properties of orthomorphisms.

1.1.1. Consider a vector lattice E and let D be a sublattice of E. A linear operator T from D into E is *band preserving* provided that one (and hence all) of the following holds:

(1) $e \perp f$ implies $Te \perp f \quad (e \in D,\ f \in E)$,

(2) $Te \in \{e\}^{\perp\perp} \quad (e \in D)$ (the disjoint complements are taken in E),

(3) $T(K \cap D) \subset K \quad \big(K \in \mathfrak{B}(E)\big)$.

If E is a vector lattice with the principal projection property and $D \subset E$ is an order dense ideal, then a linear operator $T : D \to E$ is band preserving if and only if T commutes with band projections; i.e.,

(4) $\pi T x = T \pi x \quad (\pi \in \mathfrak{P}(E),\ x \in D)$.

1.1.2. A band preserving operator T in E need not be order bounded (cp. Sections 1.2 and 1.3 below). However, the greatest order ideal A_T in E such that T is order bounded on A_T is a band (cp. [62]). Now, if A_T is a projection band then $A_T^{\perp}$ does not include any nonzero order ideal on which T is order bounded. Thus, if E has the projection property then to each band preserving operator T in E there is a band projection π such that πT is order bounded and $\pi^{\perp} T$ has no order bounded components; i.e., ρT is not order bounded for any nonzero $\rho \le \pi^{\perp}$.

1.1.3. An order bounded band preserving operator $\pi : D \to E$ on an order dense ideal $D \subset E$ is an *extended orthomorphism* of E (cp. [57]). Since an extended orthomorphism is disjointness preserving, it is also regular according to the Meyer Theorem [59, 33]. Let $\mathrm{Orth}(D, E)$ signify the set of all extended orthomorphisms of E that are defined on a fixed order dense ideal D. An extended orthomorphism $\alpha \in \mathrm{Orth}(E, E)$ on the whole space E is an *orthomorphism*. The collection of all orthomorphisms $\mathrm{Orth}(E)$ of E is a vector lattice under the pointwise algebraic and lattice operations. Let $\mathscr{Z}(E)$ stand for the order ideal generated by the identity operator I_E in $\mathrm{Orth}(E)$. The space $\mathscr{Z}(E)$ is often called the *ideal center* of E.

1.1.4. *Every extended orthomorphism in a vector lattice is order continuous. All extended orthomorphisms commute with one another.*

1.1.5. The space of extended orthomorphisms $\mathrm{Orth}^{\infty}(E)$ is defined as follows: Denote by $\mathfrak{M}$ the collection of all pairs (D, π), where D is an order dense ideal in E and $\pi \in \mathrm{Orth}(D, E)$. Elements (D, π) and (D', π') in $\mathfrak{M}$ are announced *equivalent* $\big($in writing $(D, \pi) \sim (D', \pi')\big)$ provided that the orthomorphisms π and π'

coincide on $D \cap D'$. The factor set $\mathfrak{M}/{\sim}$ of $\mathfrak{M}$ by $\sim$ is denoted by $\mathrm{Orth}^\infty(E)$. The set $\mathrm{Orth}^\infty(E)$ becomes a vector lattice under the pointwise addition, scalar multiplication, and lattice operations. Moreover, $\mathrm{Orth}^\infty(E)$ is an ordered algebra under composition. We will identify each orthomorphism $\pi \in \mathrm{Orth}(E)$ with the corresponding coset in $\mathrm{Orth}^\infty(E)$.

1.1.6. We now list some useful results on orthomorphisms that can be found in [10, 56, 57, 72].

(1) *The ordered algebra* $\mathrm{Orth}^\infty(E)$ *is a laterally complete semiprime f-algebra with unity I_E. Moreover,* $\mathrm{Orth}(E)$ *is an f-subalgebra of* $\mathrm{Orth}^\infty(E)$ *and $\mathscr{Z}(E)$ is an f-subalgebra of* $\mathrm{Orth}(E)$.

(2) *Every Archimedean f-algebra E with unity $\mathbb{1}$ is algebraically and latticially isomorphic to the f-algebra of orthomorphisms of E. Moreover, the ideal in E generated by $\mathbb{1}$ is mapped onto $\mathscr{Z}(E)$.*

(3) *If E is an order complete vector lattice then* $\mathrm{Orth}^\infty(E)$ *is a universally complete vector lattice and* $\mathrm{Orth}(E)$ *and $\mathscr{Z}(E)$ are order dense ideals.*

(4) *Let G be a universally complete vector lattice equipped with the f-algebra multiplication uniquely determined by a choice of an order unity in G. Also, let E and F be order dense ideals in G. Then, for every orthomorphism $\pi \in \mathrm{Orth}(E, F)$ there exists a unique $g \in G$ such that $\pi x = g \cdot x$ for all $x \in E$.*

1.1.7. An order bounded band preserving operator $\pi : D \to E$ is a *weak orthomorphism* of E provided that D is an order dense sublattice of E. In general, the weak orthomorphisms of E do not comprise a good algebraic structure, while they do in the case of semiprime f-algebra. Denote by $\mathrm{Orth}^w(A)$ the set of all weak orthomorphisms with maximal domain. The set $\mathrm{Orth}^w(A)$ endowed with pointwise operations and ordering is an f-algebra (cp. [70] for details).

Denote by $Q(A)$ the *maximal* (or *complete*) ring of quotients of an f-algebra A (cp. [55] for the definition). As was shown in [63], $\mathrm{Orth}^\infty(A)$ and $Q(A)$ are not isomorphic. Nevertheless, $\mathrm{Orth}^\infty(A)$ can be embedded in $Q(A)$ as an f-subalgebra [63, 70]. The following description of the maximal ring of quotients for an (Archimedean) semiprime f-algebra is due to Wickstead [70].

1.1.8. Theorem. *Let A be a semiprime f-algebra. Then*

(1) $\mathrm{Orth}^w(A)$ *is a von Neumann regular f-algebra with unity I_A;*
(2) $\mathrm{Orth}^\infty(A)$ *is an f-subalgebra of* $\mathrm{Orth}^w(A)$*;*
(3) *The maximal ring of quotients $Q(A)$ coincides with* $\mathrm{Orth}^w(A)$.

If, in addition, A is relatively uniformly complete then

(4) $Q(A) = \mathrm{Orth}^\infty(A) = \mathrm{Orth}^w(A)$.

1.2. A Local Hamel Basis

Following [58], we show in this section that a universally complete vector lattice is locally one-dimensional if and only if all band preserving operators in it are automatically order bounded.

1.2.1. Let G be an arbitrary universally complete vector lattice with a fixed order unity $\mathbb{1}$. We introduce some multiplication in G that makes G into a commutative ordered algebra with unity $\mathbb{1}$. A subset $\mathscr{E} \subset G$ is said to be *locally linearly independent* if whenever $e_1, \dots, e_n \in \mathscr{E}$, $\lambda_1, \dots, \lambda_n \in \mathbb{R}$, and π is a band projection in G

with $\pi(\lambda_1 e_1 + \dots + \lambda_n e_n) = 0$ and $\pi e_1, \dots, \pi e_n$ nonzero and pairwise distinct we have $\lambda_k = 0$ for all $k := 1, \dots, n$. In other words, $\mathscr{E}$ is locally linearly independent if $\pi(\mathscr{E}) \setminus \{0\}$, the set of all nonzero projections πe of the elements $e \in \mathscr{E}$, is linearly independent for each nonzero $\pi \in \mathfrak{P}(G)$. A maximal locally linearly independent set in G is a *local Hamel basis* for G.

There exists a local Hamel basis for each universally complete vector lattice.

$\vartriangleleft$ Apply the Kuratowski–Zorn Lemma to the inclusion-ordered set of all locally linearly independent sets in G. $\vartriangleright$

1.2.2. *A locally linearly independent set $\mathscr{E}$ in G is a local Hamel basis for G if and only if for every $x \in G$ there exists a partition of unity $(\pi_\xi)_{\xi \in \Xi}$ in $\mathfrak{P}(G)$ such that $\pi_\xi x$ is a finite linear combination of nonzero elements of $\pi_\xi \mathscr{E}$ for each $\xi \in \Xi$. Such representation of $\pi_\xi x$ is unique in the band $\pi_\xi(G)$.*

$\vartriangleleft$ $\leftarrow$: Assume that $\mathscr{E} \subset G$ is locally linearly independent but is not a Hamel basis. Then we may find $x \in E$ such that $\mathscr{E} \cup \{x\}$ is locally linearly independent. Therefore, there is no nonzero band projection π for which πx is a linear combination of nonzero elements from $\pi\mathscr{E}$. This contradicts the existence of a partition of unity with the mentioned properties.

$\rightarrow$: If $\mathscr{E}$ is a local Hamel basis for G then $\mathscr{E} \cup \{x\}$ is not locally linearly independent for an arbitrary $x \in G$. Thus, there exist a nonzero band projection π, reals $\lambda_0, \lambda_1, \dots, \lambda_n \in \mathbb{R}$, and elements $e_1, \dots, e_n \in \mathscr{E}$ such that $\pi(\lambda_0 x + \lambda_1 e_1 + \dots + \lambda_n e_n) = 0$, while $\pi e_1, \dots, \pi e_n$ are nonzero and pairwise distinct and not all $\lambda_0, \lambda_1, \dots, \lambda_n$ are equal to zero. Since the equality $\lambda_0 = 0$ contradicts the local linear independence of $\mathscr{E}$, it should be $\lambda_0 \neq 0$, so that πx is representable as a linear combination of $\pi e_1, \dots, \pi e_n$. Now, the existence of the required partition of unity follows from the exhaustion principle.

1.2.3. Proposition 1.2.2 admits the following reformulation: A locally linearly independent set $\mathscr{E}$ in G is a local Hamel basis if and only if for every $x \in G$ there exist a partition of unity $(\pi_\xi)_{\xi \in \Xi}$ in $\mathfrak{P}(G)$ and a family of reals $(\lambda_{\xi,e})_{\xi \in \Xi, e \in \mathscr{E}}$ such that

$$x = o\text{-}\sum_{\xi \in \Xi} \left(\sum_{e \in \mathscr{E}} \lambda_{\xi,e} \pi_\xi e \right),$$

where $\{e \in \mathscr{E} : \lambda_{\xi,e} \neq 0\}$ is finite for every $\xi \in \Xi$. Moreover, the representation is unique in the sense that if x admits one more representation

$$x = o\text{-}\sum_{\omega \in \Omega} \left(\sum_{e \in \mathscr{E}} \varkappa_{\omega,e} \rho_\omega e \right),$$

then for all $\xi \in \Xi$, $\omega \in \Omega$, and $e \in \mathscr{E}$ the relation $\pi_\xi \rho_\omega e \neq 0$ implies $\lambda_{\xi,e} = \varkappa_{\omega,e}$.

1.2.4. An element $e \in G_+$ is *locally constant* with respect to $f \in G_+$ if $e = \sup_{\xi \in \Xi} \lambda_\xi \pi_\xi f$ for some numeric family $(\lambda_\xi)_{\xi \in \Xi}$ and a family $(\pi_\xi)_{\xi \in \Xi}$ of pairwise disjoint band projections.

For each universally complete vector lattice G the following are equivalent:

(1) *All elements of G_+ are locally constant with respect to $\mathbb{1}$;*

(2) *All elements of G_+ are locally constant with respect to an arbitrary order unity $e \in G$;*

(3) *$\{\mathbb{1}\}$ is a local Hamel basis for G;*

(4) *Every local Hamel basis for G consists of pairwise disjoint members.*

$\lhd$ Obviously, (2) $\to$ (1). To prove the converse, note that, given $x \in G$, we may choose a partition of unity $(\pi_\xi)_{\xi\in\Xi}$ such that for each $\xi \in \Xi$ both $\pi_\xi x$ and $\pi_\xi e$ are multiples of $\pi_\xi \mathbb{1}$. So, $\pi_\xi x$ is a multiple of $\pi_\xi e$. A similar argument shows that $\{\mathbb{1}\}$ is a local Hamel basis if and only if so is $\{f\}$ for every order unity $f \in G$. Thus, if (4) holds and $\mathscr{E}$ is a local Hamel basis for G then $f := \sup\{e : e \in \mathscr{E}\}$ exists and $\{f\}$ is a local Hamel basis for G. It follows that (4) $\to$ (3). Clearly, (3) $\to$ (1) by 1.2.3. To complete the proof, we had to show (1) $\to$ (4). If (4) fails then we may choose a nonzero band projection π and a local Hamel basis containing two members e_1 and e_2 such that both πe_1 and πe_2 are nonzero multiples of $\pi\mathbb{1}$. Consequently, $\pi(\lambda_1 e_1 + \lambda_2 e_2) = 0$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$ and we arrive at the contradictory conclusion that $\{e_1, e_2\}$ is not locally linearly independent. $\rhd$

A universally complete vector lattice G is *locally one-dimensional* if G satisfies the equivalent conditions (1)–(4) of the above proposition.

1.2.5. Theorem. *Let G be a universally complete vector lattice. Then the following are equivalent:*

(1) *G is locally one-dimensional;*

(2) *Every band preserving operator $T : G \to G$ is order bounded.*

$\lhd$ (1) $\to$ (2): Recall that a linear operator $T : G \to G$ is band preserving if and only if $\pi T = T\pi$ for every band projection π in G (cp. 1.1.1 (4)). Assume that T is band preserving and put $\rho := T\mathbb{1}$. Since an arbitrary $e \in G_+$ can be expressed as $e = \sup_{\xi\in\Xi} \lambda_\xi \pi_\xi \mathbb{1}$, we deduce

$$\pi_\xi T e = T(\pi_\xi e) = T(\lambda_\xi \pi_\xi \mathbb{1}) = \lambda_\xi \pi_\xi T(\mathbb{1}) = \pi_\xi(e) T(\mathbb{1}) = \pi_\xi e\rho,$$

whence $Te = \rho e$. It follows that T is a multiplication operator in G which is obviously order bounded.

(2) $\to$ (1): Assume that (1) is false. According to 1.2.4 (4) there is a local Hamel basis $\mathscr{E}$ for G containing two members e_1 and e_2 that are not disjoint. Then the band projection $\pi := [e_1] \wedge [e_2]$ is nonzero. (Here and below $[e]$ is the band projection onto $\{e\}^{\perp\perp}$.) For an arbitrary $x \in G$ there exists a partition of unity $(\pi_\xi)_{\xi\in\Xi}$ such that $\pi_\xi x$ is a finite linear combination of elements of $\mathscr{E}$. Assume the elements of $\mathscr{E}$ have been labelled so that $\pi_\xi x = \lambda_1 \pi_\xi e_1 + \lambda_2 \pi_\xi e_2 + \cdots$. Define Tx to be a unique element in G with $\pi_\xi Tx := \lambda_1 \pi\pi_\xi e_2$. It is easy to check that T is a well defined linear operator from G into itself.

Take $x, y \in G$ with $x \perp y$ and let $(\pi_\xi)_{\xi\in\Xi}$ be a partition of unity such that both $\pi_\xi x$ and $\pi_\xi y$ are finite linear combination of elements from $\mathscr{E}$. Refining the partition of unity if need be, we may also require that at least one of the elements $\pi_\xi x$ and $\pi_\xi y$ equals zero for all $\xi \in \Xi$. If $\pi_\xi y \neq 0$ then $\pi_\xi x = 0$, and so the corresponding $\lambda_1 e_1$ is equal to zero. If $\pi\pi_\xi \neq 0$ then $\lambda_1 = 0$, and in any case $\pi_\xi Tx = 0$. It follows that $Tx \perp y$ and T is band preserving. If T were order bounded then T would be presentable as $Tx = ax$ $(x \in G)$ for some $a \in G$, see 1.1.6 (4). In particular, $Te_2 = ae_2$ and, since $Te_2 = 0$ by definition, we have $0 = [e_2]|a| \geq \pi|a|$. Thus $\pi e_2 = T(\pi e_1) = a\pi e_1 = 0$, contradicting the definition of π. $\rhd$

1.3. σ-Distributive Boolean Algebras

In this section we present the main result of [40]: *A universally complete vector lattice G is locally one-dimensional if and only if the base of G is σ-distributive.*

1.3.1. A σ-complete Boolean algebra $\mathbb{B}$ is said to be σ-*distributive* if $\mathbb{B}$ satisfies one of the following equivalent conditions (cp. [66, 19.1]):

(1) $\bigwedge_{n\in\mathbb{N}}\bigvee_{m\in\mathbb{N}} b^n_m = \bigvee_{m\in\mathbb{N}^{\mathbb{N}}}\bigwedge_{n\in\mathbb{N}} b^n_{m(n)}$ for all $(b^n_m)_{n,m\in\mathbb{N}}$ in $\mathbb{B}$;

(2) $\bigvee_{n\in\mathbb{N}}\bigwedge_{m\in\mathbb{N}} b^n_m = \bigwedge_{m\in\mathbb{N}^{\mathbb{N}}}\bigvee_{n\in\mathbb{N}} b^n_{m(n)}$ for all $(b^n_m)_{n,m\in\mathbb{N}}$ in $\mathbb{B}$;

(3) $\bigvee_{\varepsilon\in\{1,-1\}^{\mathbb{N}}}\bigwedge_{n\in\mathbb{N}}\varepsilon(n)b_n = \mathbb{1}$ for all $(b_n)_{n\in\mathbb{N}}$ in $\mathbb{B}$.

(Here $1b_n := b_n$ and $(-1)b_n$ is the complement of b_n.)

1.3.2. Let $\mathbb{B}$ be an arbitrary Boolean algebra. A subset of $\mathbb{B}$ with supremum unity is called a *cover* of $\mathbb{B}$. Partitions of unity in $\mathbb{B}$ are referred to as partitions of $\mathbb{B}$ for brevity. Let C be a cover of $\mathbb{B}$. A subset C_0 of $\mathbb{B}$ is said to be *refined* from C if, for each $c_0 \in C_0$, there exists $c \in C$ such that $c_0 \le c$. An element $b \in \mathbb{B}$ is *refined* from C provided that $\{b\}$ is refined from C; i.e., $b \le c$ for some $c \in C$. If $(C_n)_{n\in\mathbb{N}}$ is a sequence of covers of $\mathbb{B}$ and $b \in \mathbb{B}$ is refined from each of the covers C_n ($n \in \mathbb{N}$), then we say that b is refined from $(C_n)_{n\in\mathbb{N}}$. We also refer to a cover whose all elements are refined from $(C_n)_{n\in\mathbb{N}}$ as refined from the sequence.

1.3.3. *Let $\mathbb{B}$ be a σ-complete Boolean algebra. The following are equivalent:*

(1) $\mathbb{B}$ *is σ-distributive;*

(2) *There is a (possibly, uncountable) cover refined from each sequence of countable covers of* $\mathbb{B}$*;*

(3) *There is a (possibly, infinite) cover refined from each sequence of finite covers of* $\mathbb{B}$*;*

(4) *There is a cover refined from each sequence of two-element partitions of B.*

$\lhd$ A proof of (1)$\leftrightarrow$(2) can be found in [66, 19.3]). Item (4) is a paraphrase of 1.3.1 (3) in the definition of σ-distributivity. The implications (2)$\to$(3)$\to$(4) are obvious. $\rhd$

1.3.4. *Let $\mathbb{B}$ be a complete Boolean algebra. The following are equivalent:*

(1) $\mathbb{B}$ *is σ-distributive;*

(2) *There is a (possibly, uncountable) partition refined from each sequence of countable partitions of* $\mathbb{B}$*;*

(3) *There is a (possibly, infinite) partition refined from each sequence of finite partitions of* $\mathbb{B}$*;*

(4) *There is a partition refined from each sequence of two-element partitions of* $\mathbb{B}$.

$\lhd$ The claim follows from 1.3.3 in view of the exhaustion principle. $\rhd$

1.3.5. Let Q stand for the Stone space of $\mathbb{B}$ and denote by $\mathrm{Clop}(Q)$ the Boolean algebra of all clopen sets in Q. We say that a function $g \in C_\infty(Q)$ is *refined from a cover* C of the Boolean algebra $\mathrm{Clop}(Q)$ if, for every two points $q', q'' \in Q$ satisfying the equality $g(q') = g(q'')$, there exists an element $U \in C$ such that $q', q'' \in U$. If $(C_n)_{n\in\mathbb{N}}$ is a sequence of covers of $\mathrm{Clop}(Q)$ and a function g is refined from each of the covers C_n ($n \in \mathbb{N}$), then we say that g is *refined from* $(C_n)_{n\in\mathbb{N}}$.

1.3.6. Lemma. *For every sequence of finite covers of* $\mathrm{Clop}(Q)$, *there is a function of $C(Q)$ refined from the sequence.*

$\lhd$ Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of finite covers of $\mathrm{Clop}(Q)$. By induction, it is easy to construct a sequence of partitions $P_m = \{U^m_1, U^m_2, \dots, U^m_{2^m}\}$ of $\mathrm{Clop}(Q)$ with the following properties:

(1) for every $n \in \mathbb{N}$, there is $m \in \mathbb{N}$ such that the partition P_m is refined from C_n;
(2) $U^m_j = U^{m+1}_{2j-1} \vee U^{m+1}_{2j}$ for all $m \in \mathbb{N}$ and $j \in \{1, 2, \dots, 2^m\}$.

Given $m \in \mathbb{N}$, define the two valued function $\chi_m \in C(Q)$ as follows:

$$\chi_m := \sum_{i=1}^{2^{m-1}} \chi(U_{2i}^m),$$

where $\chi(U)$ is the characteristic function of $U \subset Q$. Since the series $\sum_{m=1}^{\infty} \frac{1}{3^m} \chi_m$ is uniformly convergent, its sum g belongs to $C(Q)$. We will show that g is refined from $(C_n)_{n\in\mathbb{N}}$. By property (1) of the sequence $(P_m)_{m\in\mathbb{N}}$, it suffices to establish that g is refined from $(P_m)_{m\in\mathbb{N}}$.

Assume the contrary and consider the least $m \in \mathbb{N}$ such that g is not refined from P_m. In this case, there are two points $q', q'' \in Q$ satisfying the equality $g(q') = g(q'')$ and belonging to distinct elements of P_m. Since g is refined from P_{m-1} (for $m > 1$), from property (2) of the sequence $(P_m)_{m\in\mathbb{N}}$ it follows that q' and q'' belong to some adjacent elements of P_m; i.e., elements of the form U_j^m and U_{j+1}^m, with $j \in \{1, \dots, 2^m - 1\}$. For definiteness, suppose that q' belongs to an element with an even index and q'', to that with an odd index; i.e., $\chi_m(q') = 1$ and $\chi_m(q'') = 0$. Since $\chi_i(q') = \chi_i(q'')$ for all $i \in \{1, \dots, m-1\}$; therefore, we have:

$$g(q') - g(q'') = \frac{1}{3^m} + \sum_{i=m+1}^{\infty} \frac{1}{3^i}\big(\chi_i(q') - \chi_i(q'')\big) \geq \frac{1}{3^m} - \sum_{i=m+1}^{\infty} \frac{1}{3^i} = \frac{1}{2 \cdot 3^m} > 0,$$

which contradicts the equality $g(q') = g(q'')$. ▷

1.3.7. Theorem. *A universally complete vector lattice G is locally one-dimensional if and only if the base of G is σ-distributive.*

◁ Let Q be the Stone space of the base of G. Suppose that G is locally one-dimensional and consider an arbitrary sequence $(P_n)_{n\in\mathbb{N}}$ of finite partitions of $\operatorname{Clop}(Q)$. By 1.3.4, to prove the σ-distributivity of G, it suffices to refine a cover of $\operatorname{Clop}(Q)$ from $(P_n)_{n\in\mathbb{N}}$. By Lemma 1.3.6, we may refine $g \in C_\infty(Q)$ from the sequence $(P_n)_{n\in\mathbb{N}}$. Since G is locally one-dimensional, there exists a partition $(U_\xi)_{\xi\in\Xi}$ of $\operatorname{Clop}(Q)$ such that g is constant on each of the sets U_ξ. Show that $(U_\xi)_{\xi\in\Xi}$ is refined from $(P_n)_{n\in\mathbb{N}}$. To this end, fix arbitrary indices $\xi \in \Xi$ and $n \in \mathbb{N}$ and establish that U_ξ is refined from P_n. We may assume that $U_\xi \neq \varnothing$. Let q_0 be an element of U_ξ. Finiteness of P_n allows us to find an element U of P_n such that $q_0 \in U$. It remains to observe that $U_\xi \subset U$. Indeed, if $q \in U_\xi$ then $g(q) = g(q_0)$ and, since g is refined from P_n, the points q and q_0 belong to the same element of P_n; i.e., $q \in U$.

Assume now that the base of G is σ-distributive and consider an arbitrary $g \in C_\infty(Q)$. By the definition of a locally one-dimensional vector lattice, it suffices to construct a partition $(U_\xi)_{\xi\in\Xi}$ of $\operatorname{Clop}(Q)$ such that g is constant on each of the sets U_ξ. Given a natural n and integer m, denote by U_m^n the interior of the closure of the set of all points $q \in Q$ for which $\frac{m}{n} \leq g(q) < \frac{m+1}{n}$ and put $P_n := \{U_m^n : m \in \mathbb{Z}\}$. By 1.3.4, from the sequence $(P_n)_{n\in\mathbb{N}}$ of countable partitions of $\operatorname{Clop}(Q)$, we may refine some partition $(U_\xi)_{\xi\in\Xi}$. It is easy that this is a desired partition. ▷

1.3.8. Theorem. *There exists a purely nonatomic locally one-dimensional universally complete vector lattice.*

◁ Theorem 1.3.7 reduces the problem to the existence of a purely nonatomic σ-distributive complete Boolean algebra. An algebra of this kind is constructed below in 1.3.9–1.3.11. ▷

1.3.9. A Boolean algebra $\mathbb{B}$ is *σ-inductive* provided that each decreasing sequence of nonzero elements of $\mathbb{B}$ has a nonzero lower bound. A subalgebra $\mathbb{B}_0$ of $\mathbb{B}$ is *dense* if, for every nonzero element $b \in \mathbb{B}$, there exists a nonzero element $b_0 \in \mathbb{B}_0$ such that $b_0 \leq b$.

Lemma. *If a σ-complete Boolean algebra $\mathbb{B}$ has a σ-inductive dense subalgebra then $\mathbb{B}$ is σ-distributive.*

$\lhd$ Let $\mathbb{B}_0$ be a σ-inductive dense subalgebra of $\mathbb{B}$. Consider an arbitrary sequence $(C_n)_{n\in\mathbb{N}}$ of countable covers of $\mathbb{B}$, denote by C the set of all elements in $\mathbb{B}$ that are refined from $(C_n)_{n\in\mathbb{N}}$, and assume by way of contradiction that C is not a cover of $\mathbb{B}$. Then there is a nonzero element $b \in \mathbb{B}$ disjoint from all elements of C.

By induction, we construct the sequences $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ as follows: Let c_1 be an element of C_1 such that $b \wedge c_1 \neq 0$. Since $\mathbb{B}_0$ is dense, there is an element $b_1 \in \mathbb{B}_0$ such that $0 < b_1 \leq b \wedge c_1$. Suppose that b_n and c_n are already constructed. Let c_{n+1} be an element of C_{n+1} such that $b_n \wedge c_{n+1} \neq 0$. As b_{n+1} we take an arbitrary element of $\mathbb{B}_0$ such that $0 < b_{n+1} \leq b_n \wedge c_{n+1}$.

Thus, we have constructed sequences $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ such that $b_n \in \mathbb{B}_0$, $b_n \leq c_n \in C_n$ and $0 < b_{n+1} \leq b_n \leq b$ for all $n \in \mathbb{N}$. Since $\mathbb{B}_0$ is σ-inductive, $\mathbb{B}_0$ contains a nonzero element b_0 that satisfies $b_0 \leq b_n$ for all $n \in \mathbb{N}$. By the inequalities $b_0 \leq c_n$, we see that b_0 is refined from $(C_n)_{n\in\mathbb{N}}$; i.e., b_0 belongs to C. On the other hand, $b_0 \leq b$, which contradicts the disjointness of b from all elements of C. $\rhd$

1.3.10. As is well known, to every Boolean algebra $\mathbb{B}$ there is a complete Boolean algebra $\overline{\mathbb{B}}$ including $\mathbb{B}$ as a dense subalgebra (cp. [66, Section 35]). This $\overline{\mathbb{B}}$ is unique to within an isomorphism and called a *completion* of $\mathbb{B}$. Obviously, a completion of a purely nonatomic Boolean algebra is purely nonatomic. Moreover, by Lemma 1.3.9, a completion of a σ-inductive algebra is σ-distributive. Therefore, in order to prove existence of a purely nonatomic σ-distributive complete Boolean algebra, it suffices to exhibit an arbitrary purely nonatomic σ-inductive Boolean algebra. The examples of these algebras are readily available. For the sake of completeness, we present here one of the simplest constructions.

1.3.11. *Let $\mathbb{B}$ be the boolean of $\mathbb{N}$ and let I be the ideal of $\mathbb{B}$ comprising all finite subsets of $\mathbb{N}$. Then the quotient algebra $\mathbb{B}/I$ (cp. [66, Section 10]) is purely nonatomic and σ-inductive.*

$\lhd$ The pure nonatomicity of $\mathbb{B}/I$ is obvious. In order to prove that $\mathbb{B}/I$ is σ-inductive, it suffices to consider an arbitrary decreasing sequence $(b_n)_{n\in\mathbb{N}}$ of infinite subsets of $\mathbb{N}$ and construct an infinite subset $b \subset \mathbb{N}$ such that the difference $b \backslash b_n$ is finite for each $n \in \mathbb{N}$. We can easily obtain the desired set $b := \{m_n : n \in \mathbb{N}\}$ by induction, letting $m_1 := \min b_1$ and $m_{n+1} := \min\{m \in b_{n+1} : m > m_n\}$. $\rhd$

PART 2. BOOLEAN APPROACH

The purpose of the this part is to present the approach of Boolean valued analysis to the Wickstead problem and prove that if G is a universally complete vector lattice and $\mathbb{B} := \mathfrak{P}(G)$ is the base of G then the following are equivalent:

WP(1) $\mathbb{B}$ is σ-distributive;

WP(2) $\mathscr{R} = \mathbb{R}^{\wedge}$ inside $\mathbb{V}^{(\mathbb{B})}$;

WP(3) G is locally one-dimensional;

WP(4) Every band preserving linear operator in G is order bounded.

In Sections 2.1, 2.2, and 2.3 we will give purely Boolean valued proofs of the equivalences WP(2) $\leftrightarrow$ WP(4), WP(2) $\leftrightarrow$ WP(3), and WP(1) $\leftrightarrow$ WP(2), respectively. It turns out that all these equivalences reduce to some simple properties of reals and cardinals in an appropriate Boolean valued model (cp. [47]).

Throughout this part $\mathscr{R}$ denotes the Boolean valued reals and $\mathbb{R}^\wedge$ is considered as a dense subfield of $\mathscr{R}$. More precisely, $\mathscr{R} \in \mathbb{V}^{(\mathbb{B})}$ and $[\![\, \mathscr{R}$ is a field of reals$\,]\!] = \mathbb{1}$, while $[\![\, \mathbb{R}^\wedge$ is a dense subfield of $\mathscr{R}\,]\!] = \mathbb{1}$ (cp. A3.3 and A3.4). The Gordon Theorem A3.6 says that if G is a universally complete vector lattice and $\mathbb{B} := \mathfrak{P}(G)$, then $\mathscr{R}{\downarrow}$ is a universally complete vector lattice isomorphic to G.

2.1. Representation of a Band Preserving Operator

In this section we show that the equivalence WP(2) $\leftrightarrow$ WP(4) is immediate from the Boolean valued representation of band preserving operators.

2.1.1. Throughout the section we let G stand for the universally complete vector lattice $\mathscr{R}{\downarrow}$. Recall that G is a faithful f-ring with unity $\mathbb{1} := 1^\wedge$.

Let $\mathrm{End}_N(G)$ be the set of all band preserving endomorphisms of G. Clearly, $\mathrm{End}_N(G)$ is a vector space. Moreover, $\mathrm{End}_N(G)$ becomes a faithful unitary module over G on letting gT be equal to $gT : x \mapsto g \cdot Tx$ for all $x \in G$. This is immediate since the multiplication by an element of G is band preserving and the composite of band preserving operators is band preserving too. By $\mathrm{End}_{\mathbb{R}^\wedge}(\mathscr{R})$ we denote the element of $\mathbb{V}^{(\mathbb{B})}$ that represents the space of all $\mathbb{R}^\wedge$-linear operators from $\mathscr{R}$ into $\mathscr{R}$. Then $\mathrm{End}_{\mathbb{R}^\wedge}(\mathscr{R})$ is a vector space over $\mathscr{R}$ inside $\mathbb{V}^{(\mathbb{B})}$, and $\mathrm{End}_{\mathbb{R}^\wedge}(\mathscr{R}){\downarrow}$ is a faithful unitary module over G.

2.1.2. *A linear operator T on a universally complete vector lattice G is band preserving if and only if T is extensional.*

$\lhd$ By the Gordon Theorem A3.6 and A2.4 (7), $T : G \to G$ is extensional if and only if, for all $x \in G$ and $\pi \in \mathfrak{P}(G)$, from $\pi x = 0$ it follows that $\pi T x = 0$. By taking $x := \pi^\perp y$ we conclude that $\pi T \pi^\perp = 0$ or, in other words, $\pi T = \pi T \pi$. Substituting $\pi^\perp$ for π, we see that $T\pi = \pi T \pi$, and so $\pi T = T\pi$. Hence, T is band preserving by 1.1.1 (4). Conversely, for a band preserving T we see that $\pi x = 0$ implies $\pi T x = 0$ by definition. $\rhd$

2.1.3. If $\sigma \in \mathbb{V}^{(\mathbb{B})}$ and $[\![\, \sigma : \mathscr{R} \to \mathscr{R}\,]\!] = \mathbb{1}$, then there exists a unique map $S : \mathscr{R}{\downarrow} \to \mathscr{R}{\downarrow}$ such that

$$[\![S(x) = \sigma(x)]\!] = \mathbb{1} \quad (x \in \mathscr{R}{\downarrow}).$$

This map S is called the *descent* of σ and is denoted by $\sigma{\downarrow}$. It is of importance that the descent is *extensional* (cp. A2.5):

$$[\![x = y]\!] \leq [\![S(x) = S(y)]\!] \quad (x, y \in \mathscr{R}{\downarrow}).$$

It is immediate from A3.6 that S is extensional if and only if $bx = by$ implies $bS(x) = bS(y)$ for all $x, y \in \mathscr{R}{\downarrow}$ and $b \in \mathbb{B} = \mathfrak{P}(\mathscr{R}{\downarrow})$.

Conversely, given an extensional map $S : \mathscr{R}{\downarrow} \to \mathscr{R}{\downarrow}$, there exists a unique function $\sigma : \mathscr{R} \to \mathscr{R}$ inside $\mathbb{V}^{(\mathbb{B})}$, such that $S = \sigma{\downarrow}$. We say that σ is the *ascent* of S and write $\sigma = S{\uparrow}$ (cp. A2.4). Thus, the descent and ascent carry out a bijection between the sets of all extensional mappings from $\mathscr{R}{\downarrow}$ into $\mathscr{R}{\downarrow}$ and all elements

$\sigma \in \mathbb{V}^{(\mathbb{B})}$ with $[\![\sigma : \mathscr{R} \to \mathscr{R}]\!] = \mathbb{1}$ (cp. the Escher rules for arrow cancellations in A2.6). Denote the latter set by $F(\mathscr{R})\!\downarrow$.

2.1.4. Let $\mathrm{Ext}(\mathscr{R}\!\downarrow)$ be the set of all extensional mappings from $\mathscr{R}\!\downarrow$ into $\mathscr{R}\!\downarrow$. The pointwise operations make this set into a unital module over the ring $\mathscr{R}\!\downarrow$. The set $F(\mathscr{R})\!\downarrow$ can be endowed with a module structure over $\mathscr{R}\!\downarrow$ by analogy to A3.5.

The bijection in 2.1.3 is an isomorphism of the modules $\mathrm{Ext}(\mathscr{R}\!\downarrow)$ *and* $F(\mathscr{R})\!\downarrow$.

$\lhd$ This is immediate from the following identities:

$$(S+T)\!\uparrow\! x = (S+T)x = Sx + Tx = S\!\uparrow\! x \oplus T\!\uparrow\! x = (S\!\uparrow \oplus T\!\uparrow)x \quad (x \in \mathscr{R}\!\downarrow);$$
$$(\alpha \cdot S)\!\uparrow\! x = (\alpha \cdot S)x = \alpha \cdot (Sx) = \alpha \odot (S\!\uparrow\! x) = (\alpha \odot S\!\uparrow)x \quad (\alpha, x \in \mathscr{R}\!\downarrow),$$

where $\oplus$ and $\odot$ stand for the operations in $\mathscr{R}$ and $F(\mathscr{R})$, while $+$ and $\cdot$ symbolize the operations in $\mathscr{R}\!\downarrow$ and $\mathrm{Ext}(\mathscr{R}\!\downarrow)$. $\rhd$

2.1.5. *The modules* $\mathrm{End}_N(G)$ *and* $\mathrm{End}_{\mathbb{R}^\wedge}(\mathscr{R})\!\downarrow$ *are isomorphic. The isomorphy may be established by sending a band preserving operator to its ascent.*

$\lhd$ Each $T \in \mathrm{End}_N(G)$ is extensional by 2.1.2, and so T has the ascent $\tau := T\!\uparrow$ presenting the unique mapping from $\mathscr{R}$ into $\mathscr{R}$ such that $[\![\tau(x) = Tx]\!]$ for all $x \in G$ (cp. 2.1.3). Using this identity and the definition of the ring structure on $\mathscr{R}\!\downarrow$, we see

$$\tau(x \oplus y) = T(x+y) = Tx + Ty = \tau(x) \oplus \tau(y) \quad (x, y \in G),$$
$$\tau(\lambda^\wedge \odot x) = T(\lambda \cdot x) = \lambda \cdot Tx = \lambda^\wedge \odot \tau(x) \quad (x \in G,\ \lambda \in \mathbb{R}).$$

Hence, $[\![\tau : \mathscr{R} \to \mathscr{R}$ is an $\mathbb{R}^\wedge$-linear function$]\!] = \mathbb{1}$; i.e., $[\![\tau \in \mathrm{End}_{\mathbb{R}^\wedge}(\mathscr{R})]\!] = \mathbb{1}$. If $\tau \in \mathrm{End}_{\mathbb{R}^\wedge}(\mathscr{R})\!\downarrow$ then the descent $\tau\!\downarrow : G \to G$ is extensional (cp. 2.1.3). The same arguments as above convince us that if τ is $\mathbb{R}^\wedge$-linear inside $\mathbb{V}^{(\mathbb{B})}$ then $\tau\!\downarrow$ is a linear operator. By 2.1.2 $\tau\!\downarrow$ is band preserving. The claim results now from 2.1.4. $\rhd$

2.1.6. In 2.1.5 we encountered the following situation: There is some ordered subfield $\mathbb{P}$ of the reals $\mathbb{R}$ that includes $\mathbb{Q}$. Consequently, $\mathbb{R}$ is a vector space over $\mathbb{P}$ and has a Hamel basis, say $\mathscr{E}$. Denote the set of all $\mathbb{P}$-linear functions in $\mathbb{R}$ by $\mathrm{End}_{\mathbb{P}}(\mathbb{R})$. For the sake of completeness, we recall the two well-known facts:

(1) *Let* $\mathbb{P}$ *be a subfield of* $\mathbb{R}$. *The general form of a* $\mathbb{P}$-*linear function* $f : \mathbb{R} \to \mathbb{R}$ *is given as*

$$f(x) = \sum_{e \in \mathscr{E}} x_e \phi(e) \quad \text{if } \ x = \sum_{e \in \mathscr{E}} x_e e,$$

where $\phi : \mathscr{E} \to \mathbb{R}$ *is an arbitrary function and the second formula is the expansion of* $x \in \mathbb{R}$ *with respect to the Hamel basis* $\mathscr{E}$ *and the coefficients* $(x_e)_{e \in \mathscr{E}}$ *are such that* $\{e \in \mathscr{E} : x_e \neq 0\}$ *is a finite set.*

$\lhd$ This is immediate from the definition and properties of a Hamel basis. $\rhd$

(2) *An arbitrary* $\mathbb{P}$-*linear function* $f : \mathbb{R} \to \mathbb{R}$ *admits the representation* $f(x) = cx$ $(x \in \mathbb{R})$ *with some* $c \in \mathbb{R}$ *if and only if* f *is bounded above or below on some interval* $]a, b[\, \subset \mathbb{R}$, *with* $a < b$.

$\lhd$ Necessity is obvious. To prove sufficiency, assume that f is bounded above by a real M on $]a, b[$. Then the open set $\{(s, t) \in \mathbb{R}^2 : a < s < b,\ M < t\}$ is disjoint from the graph of f, and so the graph of f cannot be dense in $\mathbb{R}^2$. However, if f fails to admit the desired representation then the graph of f is dense in $\mathbb{R}^2$. This is established in much the same way as in the case of the Cauchy functional equation (cp. [8, Chapter 2, Theorem 3]). $\rhd$

2.1.7. We now exhibit the two corollaries for band preserving operators which are the Boolean valued interpretations of 2.1.6 (1), (2).

(1) *A band preserving operator* $T \in \mathrm{End}_N(G)$ *is order bounded if and only if* T *may be presented as* $Tx = g \cdot x$ $(x \in G)$ *for some fixed* $g := g_T \in G$.

$\lhd$ It suffices to observe that the ascent functor preserves the property of order boundedness in 2.1.5 and apply 2.1.6 (2) inside $\mathbb{V}^{(\mathbb{B})}$. $\rhd$

(2) *For every band preserving endomorphism of* $G := \mathscr{R}{\downarrow}$ *to be order bounded it is necessary and sufficient that* $\mathbb{V}^{(\mathbb{B})} \models \mathscr{R} = \mathbb{R}^\wedge$.

$\lhd$ $\leftarrow$: If $\mathbb{R}^\wedge$ coincides with the reals $\mathscr{R}$ inside $\mathbb{V}^{(\mathbb{B})}$ then $\mathrm{End}_{\mathbb{R}^\wedge}(\mathscr{R}){\downarrow}$ is the set of all $\mathscr{R}$-linear functions in $\mathscr{R}$. However, each $\mathscr{R}$-linear function ϕ in $\mathscr{R}$ admits the representation $\phi(x) = cx$ for all $x \in \mathscr{R}$. Hence, $\mathrm{End}_N(G)$ consists of order bounded operators by (1).

$\rightarrow$: If $\mathbb{R}^\wedge \neq \mathscr{R}$, then each Hamel basis $\mathscr{E}$ for the vector space $\mathscr{R}$ over $\mathbb{R}^\wedge$ has at least two distinct elements $e_1 \neq e_2$. Defining some function $\phi_0 : \mathscr{E} \to \mathscr{R}$ so that $\phi_0(e_1)/e_1 \neq \phi_0(e_2)/e_2$, we may extend ϕ_0 to an $\mathbb{R}^\wedge$-linear function $\phi : \mathscr{R} \to \mathscr{R}$ as in 2.1.6 (1) which cannot be bounded by 2.1.6 (2). Therefore, the descent of ϕ would be a band preserving linear operator that fails to be order bounded $\big($cp. (1)$\big)$. $\rhd$

2.2. Representation of a Locally One-Dimensional Vector Lattice

A proper delineation of the notion of local Hamel basis is simply a Hamel basis in an appropriate Boolean valued model. As an easy consequence we get WP(2) $\leftrightarrow$ WP(3).

2.2.1. *The universally complete vector lattice* $G := \mathscr{R}{\downarrow}$ *is locally one-dimensional if and only if* $\mathbb{V}^{(\mathbb{B})} \models \mathscr{R} = \mathbb{R}^\wedge$.

$\lhd$ Clearly, $[\![\mathscr{R} = \mathbb{R}^\wedge]\!] = \mathbb{1}$ amounts to $\mathscr{R}{\downarrow} = \mathbb{R}^\wedge{\downarrow}$ (cp. [53, 3.3.3]). Therefore, it suffices to check that G is locally one-dimensional if and only if $G = \mathbb{R}^\wedge{\downarrow}$. However, by [53, 3.1.1] $\mathbb{R}^\wedge{\downarrow}$ consists of all mixings of the shape $\mathrm{mix}_{t\in\mathbb{R}}(b_t t^\wedge)$, where $(b_t)_{t\in\mathbb{R}}$ is an arbitrary partition of unity in $\mathbb{B}$. Considering the properties of the universally complete vector lattice G (cp. [53, 5.2.2 and 5.2.3]), we see that $G = \mathbb{R}^\wedge{\downarrow}$ means the possibility of presenting each $x \in G$ as $o\text{-}\sum_{t\in\mathbb{R}} t\chi(b_t)\mathbb{1}$ with a suitable partition of unity $(b_t)_{t\in\mathbb{R}}$ in $\mathbb{B}$. The latter rephrases as G is locally one-dimensional, since we may put $\pi_t := \chi(b_t)$ and rewrite the above presentation as

$$x = \ o\text{-}\sum_{t\in\mathbb{R},\, t>0} t\pi_t\mathbb{1} + \ o\text{-}\sum_{t\in\mathbb{R},\, t<0} t\pi_t\mathbb{1} = \sup_{t\in\mathbb{R},\, t>0} t\pi_t\mathbb{1} - \sup_{t\in\mathbb{R},\, t<0} (-t)\pi_t\mathbb{1};$$

moreover, $x^+ = \sup\{t\pi_t\mathbb{1} : t \in \mathbb{R},\ t > 0\}$ and $x^- = \sup\{-t\pi_t\mathbb{1} : t \in \mathbb{R},\ t < 0\}$. $\rhd$

2.2.2. Thus, the universally complete vector lattice $G = \mathscr{R}{\downarrow}$ is locally one-dimensional if and only if $[\![\,\mathscr{R}$ as a vector space over $\mathbb{R}^\wedge$ has dimension $1\,]\!] = \mathbb{1}$. Consequently, it stands to reason to find out what construction in $G = \mathscr{R}{\downarrow}$ corresponds to a Hamel basis for the vector space $\mathscr{R}$ over $\mathbb{R}^\wedge$. We will presume that G is furnished with the only multiplicative structure making G into an ordered commutative algebra with ring unity $\mathbb{1} := 1^\wedge$.

We will say that $x, y \in G$ differ at $\pi \in \mathfrak{P}(G)$ provided that from $\rho x = \rho y$ it follows that $\pi\rho = 0$ for all $\rho \in \mathfrak{P}(G)$. This amounts clearly to the condition $\pi(G) \subset \{|x - y|\}^{\perp\perp}$.

A subset $\mathscr{E}$ of G is *locally linearly independent* provided that, for an arbitrary nonzero band projection π in G, each collection of elements $e_1, \dots, e_n \in \mathscr{E}$ that differ pairwise at π and reals $\lambda_1, \dots, \lambda_n \in \mathbb{R}$, the condition $\pi(\lambda_1 e_1 + \dots + \lambda_n e_n) = 0$ implies that $\lambda_k = 0$ for all $k := 1, \dots, n$. An inclusion-maximal locally linearly independent subset of G is a *local Hamel basis* for G.

Observe that this definition of a local Hamel basis differs from that given in 1.2.1. The concept of a local Hamel basis in 1.2.1 (cp. [58]) corresponds to the interpretation of the set $\mathscr{E} \cup \{0\}$, where $[\![\, \mathscr{E}$ is a Hamel basis for the vector space $\mathscr{R}$ over $\mathbb{R}^{\wedge} \,]\!] = \mathbb{1}$.

There is a local Hamel basis for an arbitrary universally complete vector lattice.

$\lhd$ It suffices to apply the Kuratowski–Zorn Lemma to the inclusion ordered set of all locally linearly independent subsets of G. $\rhd$

2.2.3. *Assume that* $G := \mathscr{R}{\downarrow}$, $\mathscr{E} \in \mathbb{V}^{(\mathbb{B})}$, *and* $[\![\, \mathscr{E} \subset \mathscr{R} \,]\!] = \mathbb{1}$. *Then* $[\![\, \mathscr{E}$ *is a linearly independent subset of the vector space* $\mathscr{R}$ *over* $\mathbb{R}^{\wedge} \,]\!] = \mathbb{1}$ *if and only if* $\mathscr{E}{\downarrow}$ *is a locally linearly independent subset of* G.

$\lhd$ $\leftarrow$: Put $\mathscr{E}' := \mathscr{E}{\downarrow}$ and assume that $\mathscr{E}'$ is locally linearly independent. Given a natural n, let the formula $\varphi(n, \tau, \sigma)$ express the following: τ and σ are maps from $n := \{0, 1 \dots, n-1\}$ into $\mathbb{R}^{\wedge}$ and $\mathscr{E}$ respectively, $\sigma(k) \neq \sigma(l)$ for different k and l in n, and $\sum_{k \in n} \tau(k)\sigma(k) = 0$. Denote the formula

$$(\forall\, \tau)(\forall\, \sigma)\big(\varphi(n, \tau, \sigma) \to (\forall\, k \in n)\, \tau(k) = 0\big)$$

by $\psi(n)$. Then the linear independence of $\mathscr{E}$ inside $\mathbb{V}^{(\mathbb{B})}$ amounts to the equality

$$\mathbb{1} = [\![\, (\forall\, n \in \mathbb{N}^{\wedge})\, \psi(n) \,]\!] = \bigwedge_{n \in \mathbb{N}} [\![\, \psi(n^{\wedge}) \,]\!].$$

Hence, we are left with proving that $[\![\, \psi(n^{\wedge}) \,]\!] = \mathbb{1}$ for all $n \in \mathbb{N}$. Calculate the truth values, using the construction of the formula ψ and the rules of Boolean valued analysis (cp. [53, 2.3.8]). The result is as follows:

$$\bigwedge \Big\{ [\![\, (\forall\, k \in n^{\wedge})\, \tau(k) = 0 \,]\!] :\ \tau, \sigma \in \mathbb{V}^{(\mathbb{B})};\ [\![\, \varphi(n^{\wedge}, \tau, \sigma) \,]\!] = \mathbb{1} \Big\}.$$

Take some $\tau, \sigma \in \mathbb{V}^{(\mathbb{B})}$ and $n \in \mathbb{N}$ such that $[\![\varphi(n^{\wedge}, \tau, \sigma)]\!] = \mathbb{1}$. Then $[\![\tau : n^{\wedge} \to \mathbb{R}^{\wedge}]\!] = \mathbb{1}$ and $[\![\, \sigma : n^{\wedge} \to \mathscr{E} \,]\!] = \mathbb{1}$. Moreover, $[\![\, \sigma(k) \neq \sigma(l)$ for distinct k and l in $n^{\wedge}$, and $\sum_{k \in n^{\wedge}} \tau(k)\sigma(k) = 0 \,]\!] = \mathbb{1}$.

Let $t : n \to \mathbb{R}^{\wedge}{\downarrow}$ and $s : n \to \mathscr{E}'$ stand for the modified descents of τ and σ (cp. [53, 3.5.5]). Then

$$\mathbb{1} = [\![(\forall\, k, l \in n^{\wedge})\big(k \neq l \to \sigma(k) \neq \sigma(l)\big)]\!] = \bigwedge_{\substack{k,l \in n \\ k \neq l}} [\![\sigma(k^{\wedge}) \neq \sigma(l^{\wedge})]\!] = \bigwedge_{\substack{k,l \in n \\ k \neq l}} [\![s(k) \neq s(l)]\!],$$

and so $s(k)$ and $s(l)$ differ at the identity projection for k and l distinct. Furthermore,

$$\left[\!\!\left[\sum_{k=0}^{n-1} t(k)s(k) = 0 \right]\!\!\right] = \left[\!\!\left[\sum_{k \in n^{\wedge}} \tau(k)\sigma(k) = 0 \right]\!\!\right] = \mathbb{1}.$$

Hence, $\sum_{k=0}^{n-1} t(k)s(k) = 0$. Since $t(k) \in \mathbb{R}^{\wedge}{\downarrow}$ for all $k \in n$, there is a partition of unity $(b_\xi)_{\xi \in \Xi}$ in $\mathbb{B}$ and, to each $k \in n$, there is a numerical family $(\lambda_{\xi,k})_{\xi \in \Xi}$ such that

$$t(k) = o\text{-}\sum_{\xi \in \Xi} \lambda_{\xi,k} \chi(b_\xi) \mathbb{1} \quad (k := 0, 1, \dots, n-1).$$

Inserting these expressions into the equality $\sum_{k=0}^{n-1} t(k)s(k) = 0$, we obtain

$$0 = \sum_{k=0}^{n-1}\Bigg(o\text{-}\sum_{\xi\in\Xi}\lambda_{\xi,k}\chi(b_\xi)\mathbb{1}\Bigg)s(k) = o\text{-}\sum_{\xi\in\Xi}\chi(b_\xi)\sum_{k=0}^{n-1}\lambda_{\xi,k}s(k).$$

Consequently, $\chi(b_\xi)\sum_{k=0}^{n-1}\lambda_{\xi,k}s(k) = 0$ and, since $s(k)$ and $s(l)$ differ at $\chi(b_\xi)$ for distinct $k, l \in n$, by the definition of local linear independence we have $\lambda_{\xi,k} = 0$ $(k = 0, 1, \dots, n-1)$. Thus $t(k) = 0$ $(k = 0, 1, \dots, n-1)$, and so

$$\mathbb{1} = \bigwedge_{k\in n}[\![t(k) = 0]\!] = \bigwedge_{k\in n}[\![\tau(k^\wedge) = 0]\!] = [\![(\forall\, k \in n^\wedge)\, \tau(k) = 0]\!],$$

which was required.

$\rightarrow$: Assume that $[\![\,\mathscr{E}$ is an $\mathbb{R}^\wedge$-linearly independent set in $\mathscr{R}\,]\!] = \mathbb{1}$. Consider arbitrary $\pi \in \mathfrak{P}(G)$, $n \in \mathbb{N}$, $t : n \to \mathbb{R}$ and $s : n \to \mathscr{E}'$ such that $\pi \neq 0$, $s(k)$ and $s(l)$ differ at π for distinct $k, l \in n$, and $\pi\sum_{k=0}^{n-1} t(k)s(k) = 0$. Our goal is now to prove that $t(k) = 0$ $(k = 0, \dots, n-1)$.

Let $\tau, \sigma \in \mathbb{V}^{(\mathbb{B})}$ be the modified ascents of t and s (cp. [53, 3.5.5]). Then, inside $\mathbb{V}^{(\mathbb{B})}$, we have $\tau : n^\wedge \to \mathbb{R}^\wedge$, $\sigma : n^\wedge \to \mathscr{E}$, and

$$\Bigg((\forall\, k, l \in n^\wedge)\,\big(k \neq l \to \sigma(k) \neq \sigma(l)\big) \wedge \sum_{k\in n^\wedge}\tau(k^\wedge)\sigma(k^\wedge) = 0\Bigg) \to (\forall\, k \in n^\wedge)\, \tau(k) = 0.$$

Calculating the truth value of the latter formula, we obtain

$$b := \bigwedge_{\substack{k,l\in n\\ k\neq l}}[\![s(k) \neq s(l)]\!] \wedge \Bigg[\!\!\Bigg[\sum_{k=0}^{n-1} t(k)s(k) = 0\Bigg]\!\!\Bigg] \leq \bigwedge_{k=0}^{n-1}[\![t(k)^\wedge = 0]\!].$$

According to the initial properties of π, s, and t, by virtue of A3.6 we have $\pi \leq \chi(b)$ implying that $\pi t(k)^\wedge = 0$ for all $k \in n$ again by A3.6. Since $\pi \neq 0$, we have $t(k) = 0$ $(k = 0, \dots, n-1)$. $\triangleright$

2.2.4. *If $\mathscr{E}_0$ is a locally linearly independent subset of G and $\mathscr{E} := \mathscr{E}_0\!\uparrow$ then $[\![\,\mathscr{E}$ is $\mathbb{R}^\wedge$-linearly independent in $\mathscr{R}\,]\!] = \mathbb{1}$.*

$\triangleleft$ By 2.2.3 it suffices to show that $\mathscr{E}_0' := \mathrm{mix}(\mathscr{E}_0) = \mathscr{E}\!\downarrow = \mathscr{E}_0\!\uparrow\downarrow$ is locally linearly independent. Take some nonzero band projection π in G, elements $e_1, \dots, e_n \in \mathscr{E}_0'$ that differ at π, and reals $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ satisfying $\pi(\lambda_1 e_1 + \dots + \lambda_n e_n) = 0$. There are a partition of unity (b_ξ) in $\mathbb{B}$ and families $(g_{\xi,k}) \subset \mathscr{E}_0$ such that $e_k = o\text{-}\sum_\xi \chi(b_\xi)g_{\xi,k}$. Clearly, $\rho := \pi\chi(b_\eta) \neq 0$ for some index η. The elements $g_{\eta,1}, \dots, g_{\eta,n}$ differ pairwise at ρ and $\rho(\lambda_1 g_{\eta,1} + \dots + \lambda_n g_{\eta,n}) = 0$. Since $\mathscr{E}_0$ is locally linearly independent, $\lambda_1 = \dots = \lambda_n = 0$. $\triangleright$

2.2.5. *Assume that $G := \mathscr{R}\!\downarrow$, $\mathscr{E} \in \mathbb{V}^{(\mathbb{B})}$, and $[\![\,\mathscr{E} \subset \mathscr{R}\,]\!] = \mathbb{1}$. Then $[\![\,\mathscr{E}$ is a Hamel basis for the vector space $\mathscr{R}$ over $\mathbb{R}^\wedge]\!] = \mathbb{1}$ if and only if $\mathscr{E}\!\downarrow$ is a local Hamel basis for G.*

$\triangleleft$ Immediate from 2.2.3 and 2.2.4. $\triangleright$

2.2.6. *A universally complete vector lattice G is locally one-dimensional if and only if $\{\mathbb{1}\}$ is a local Hamel basis for G.*

$\triangleleft$ Immediate from 2.2.1 and 2.2.5. $\triangleright$

2.3. Dedekind Cuts and Continued Fractions in a Boolean Valued Model

The behavior of Dedekind cuts and continued fractions in a Boolean valued model clarifies the equivalence WP(1) $\leftrightarrow$ WP(2).

2.3.1. *For all $a \subset \mathbb{Q}$ and $\bar{a} \subset \mathbb{Q}$, the following holds:*

$$(a, \bar{a}) \text{ is a Dedekind cut} \leftrightarrow [\![(a^{\wedge}, \bar{a}^{\wedge}) \text{ is a Dedekind cut}]\!] = \mathbb{1}.$$

$\lhd$ Indeed, the formula $\varphi(a, \bar{a}, \mathbb{Q})$ stating that $a \subset \mathbb{Q}$ and $\bar{a} \subset \mathbb{Q}$ comprise a Dedekind cut, is bounded. So we are done by restricted transfer (cp. A2.2). $\rhd$

2.3.2. *If $\mathbb{B}$ is σ-distributive then $\mathbb{V}^{(\mathbb{B})} \models \mathscr{R} \subset \mathbb{R}^{\wedge}$.*

$\lhd$ Note that the claim means precisely WP(1) $\to$ WP(2). Assume that $\mathbb{B}$ is σ-distributive. By A3.9 (3) $\mathscr{P}(\omega^{\wedge}) = \mathscr{P}(\omega)^{\wedge}$. Let $\mathscr{Q}$ denote the rationals inside $\mathbb{V}^{(\mathbb{B})}$. Since the set of rationals can be defined by a restricted set-theoretic formula, we have $\mathbb{V}^{(\mathbb{B})} \models \mathscr{Q} = \mathbb{Q}^{\wedge}$ (cp. A2.2). Thus, we also conclude that $\mathscr{P}(\mathbb{Q}^{\wedge}) = \mathscr{P}(\mathbb{Q})^{\wedge}$. To demonstrate the desired inclusion we are to show only that $[\![t \in \mathscr{R}]\!] = \mathbb{1}$ implies $[\![t \in \mathbb{R}^{\wedge}]\!] = \mathbb{1}$. Assume that $[\![t \in \mathscr{R}]\!] = \mathbb{1}$; i.e., t is a Dedekind cut inside $\mathbb{V}^{(\mathbb{B})}$. We then see inside $\mathbb{V}^{(\mathbb{B})}$ that

$$\big(\exists\, a \in \mathscr{P}(\mathbb{Q}^{\wedge})\big)\big(\exists\, \bar{a} \in \mathscr{P}(\mathbb{Q}^{\wedge})\big)\, \varphi(a, \bar{a}, \mathbb{Q}^{\wedge}) \wedge t = (a, \bar{a}),$$

where φ is the same as in 2.3.1. Calculating the truth value of the above formula and considering that $\mathscr{P}(\mathbb{Q}^{\wedge}) = \mathscr{P}(\mathbb{Q})^{\wedge}$, we infer

$$\mathbb{1} = \bigvee_{a \subset \mathbb{Q}} \bigvee_{\bar{a} \subset \mathbb{Q}} [\![\varphi(a^{\wedge}, \bar{a}^{\wedge}, \mathbb{Q}^{\wedge})]\!] \wedge [\![t = (a, \bar{a})^{\wedge}]\!].$$

Choose a partition of unity $(b_\xi) \subset \mathbb{B}$ and two families (a_ξ) and $(\bar{a}_\xi)$ in $\mathscr{P}(\mathbb{Q})$ so that

$$b_\xi \le [\![\varphi(a_\xi^{\wedge}, \bar{a}_\xi^{\wedge}, \mathbb{Q}^{\wedge})]\!] \wedge [\![t = (a_\xi, \bar{a}_\xi)^{\wedge}]\!].$$

It follows that $t = \operatorname{mix}_\xi b_\xi (a_\xi, \bar{a}_\xi)^{\wedge}$, and $b_\xi \le [\![\varphi(a_\xi^{\wedge}, \bar{a}_\xi^{\wedge}, \mathbb{Q}^{\wedge})]\!]$. If $b_\xi \ne \mathbb{0}$ then $[\![\varphi(a_\xi^{\wedge}, \bar{a}_\xi^{\wedge}, \mathbb{Q}^{\wedge})]\!] = \mathbb{1}$, since $\varphi(x_1, x_2, x_3)$ is a bounded formula and the truth value $[\![\varphi(x_1^{\wedge}, x_2^{\wedge}, x_3^{\wedge})]\!]$ of a bounded formula may be either $\mathbb{0}$ or $\mathbb{1}$ by the definitions and rules of transformation of truth values (cp. [53, 2.2.3 (2)]). By restricted transfer (cp. A2.2 and [53, 2.2.9]) $\varphi(a_\xi, \bar{a}_\xi, \mathbb{Q})$; i.e., $(a_\xi, \bar{a}_\xi)$ is a Dedekind cut. It is evident now that $b_\xi \le [\![t = (a_\xi, \bar{a}_\xi)^{\wedge} \in \mathbb{R}^{\wedge}]\!]$. Hence, $[\![t \in \mathbb{R}^{\wedge}]\!] = \mathbb{1}$. $\rhd$

2.3.3. We now prove the implication WP(2) $\to$ WP(1). To this end we use continued fractions. Put

$$\mathbb{I} := \{ t \in \mathbb{R} \ :\ 0 < t < 1,\ t \text{ is irrational} \},$$
$$\mathscr{I} := \{ t \in \mathscr{R} :\ 0 < t < 1,\ t \text{ is irrational} \} \ \big(\text{inside } \mathbb{V}^{(\mathbb{B})}\big).$$

It is well known that there is a bijection $\lambda : \mathbb{I} \to \mathbb{N}^{\mathbb{N}}$ sending a real t to the sequence $\lambda(t) = a : \mathbb{N} \to \mathbb{N}$ of partial continued fractions of the continued fraction expansion of t:

$$t = \cfrac{1}{a(1) + \frac{1}{a(2) + \frac{1}{a(3) + \dots}}}.$$

Given sequences $a : \mathbb{N} \to \mathbb{N}$ and $s : \mathbb{N} \to \mathbb{I}$, consider the bounded formula $\varphi(a, s, t, \mathbb{N})$ stating that $s(1) = t^{-1}$ and

$$a(n) = \left[\frac{1}{s(n)}\right], \quad s(n+1) = \frac{1}{s(n)} - a(n),$$

for all $n \in \mathbb{N}$, where $[\alpha]$ is the integer part of $0 < \alpha \in \mathbb{R}$ which is expressed by the bounded formula $\psi(\alpha, [\alpha], \mathbb{N})$:

$$[\alpha] \in \mathbb{N} \wedge [\alpha] \leq \alpha \wedge (\forall\, n \in \mathbb{N})(n \leq \alpha \to n \leq [\alpha]).$$

The equality $\lambda(t) = a$ means the existence of a sequence $s : \mathbb{N} \to \mathbb{I}$ such that $\varphi(a, s, t, \mathbb{N})$. Call the bijection λ the *continued fraction expansion*. By transfer (cp. A1.2), the continued fraction expansion $\tilde{\lambda} : \mathscr{I} \to (\aleph_0)^{\aleph_0} = (\mathbb{N}^\wedge)^{\mathbb{N}^\wedge}$ exists inside $\mathbb{V}^{(\mathbb{B})}$.

2.3.4. *Inside $\mathbb{V}^{(\mathbb{B})}$, the restriction of $\tilde{\lambda}$ to $\mathbb{I}^\wedge$ coincides with $\lambda^\wedge$; i.e.,*

$$\mathbb{V}^{(\mathbb{B})} \models (\forall\, t \in \mathbb{I}^\wedge)\, \tilde{\lambda}(t) = \lambda^\wedge(t).$$

$\lhd$ The desired is true only if $\tilde{\lambda}(t^\wedge) = \lambda(t)^\wedge$ for all $t \in \mathbb{I}$. By the above definition of the bijection $\tilde{\lambda}$ we have to demonstrate the validity inside $\mathbb{V}^{(\mathbb{B})}$ of the following formula: $(\exists\, s \in \mathscr{I}^{\mathbb{N}^\wedge})\, \varphi(\lambda(t)^\wedge, s, t^\wedge, \mathbb{N}^\wedge)$. By the definition of λ there is a sequence $\sigma : \mathbb{N} \to \mathbb{I}$ satisfying $\varphi(\lambda(t), \sigma, t, \mathbb{N})$. Since φ is bounded, $\mathbb{1} = [\![\varphi(\lambda(t)^\wedge, \sigma^\wedge, t^\wedge, \ \mathbb{N}^\wedge)]\!]$. Note that $\sigma^\wedge : \mathbb{N}^\wedge \to \mathbb{I}^\wedge \subset \mathscr{I}$; i.e., $[\![\sigma^\wedge \in \mathscr{I}^{\mathbb{N}^\wedge}]\!] = \mathbb{1}$. Summarizing the above, we may write $[\![\, (\exists\, s \in \mathscr{I}^{\mathbb{N}^\wedge})\, \varphi(\lambda(t)^\wedge, s, t^\wedge, \mathbb{N}^\wedge)\,]\!] \geq [\![\varphi(\lambda(t)^\wedge, \sigma^\wedge, t^\wedge, \mathbb{N}^\wedge)]\!] = \mathbb{1}$. $\rhd$

2.3.5. *If $\mathbb{V}^{(\mathbb{B})} \models \mathscr{R} = \mathbb{R}^\wedge$ then $\mathbb{B}$ is σ-distributive.*

$\lhd$ By hypothesis $\mathscr{I} = \mathbb{I}^\wedge$ inside $\mathbb{V}^{(\mathbb{B})}$. Hence, $\tilde{\lambda}$ and $\lambda^\wedge$ are bijections, $\tilde{\lambda}$ extends $\lambda^\wedge$, and their images coincide. Clearly, the domains coincide in this event too (and, moreover, $\tilde{\lambda} = \lambda^\wedge$). Therefore, $(\mathbb{N}^{\mathbb{N}})^\wedge = (\mathbb{N}^\wedge)^{\mathbb{N}^\wedge}$. By A3.9 (2) we infer that $\mathbb{B}$ is σ-distributive. $\rhd$

PART 3. AUTOMORPHISMS AND DERIVATIONS

The goal of this part is to prove that if $G_{\mathbb{C}}$ is the complexification of a universally complete vector lattice G then the following are equivalent:

WP(1) $\mathbb{B}$ is σ-distributive;

WP(2′) $\mathscr{C} = \mathbb{C}^\wedge$ inside $\mathbb{V}^{(\mathbb{B})}$;

WP(4′) Every band preserving linear operator in $G_{\mathbb{C}}$ is order bounded;

WP(5′) There is no nontrivial $\mathbb{C}$-derivation in the complex f-algebra $G_{\mathbb{C}}$;

WP(6) Each band preserving endomorphism of the complex f-algebra $G_{\mathbb{C}}$ is a band projection;

WP(7) There is no band preserving automorphism of $G_{\mathbb{C}}$ other than the identity.

3.1. Band Preserving Operators in Complex Vector Lattices

Consider some properties of band preserving operators in a complex vector lattice.

3.1.1. A vector lattice E is called *square-mean closed* if for all $x, y \in E$ the set $\{(\cos\theta)x + (\sin\theta)y : 0 \leq \theta < 2\pi\}$ has a supremum $\mathfrak{s}(x, y)$ in E. Every Banach lattice as well as every relatively uniformly complete vector lattice is square-mean closed. However, a square-mean closed Archimedean vector lattice need not be relatively uniformly complete. If E is a square-mean closed f-algebra, then $\mathfrak{s}(x,y)^2 = x^2 + y^2$ for all $x, y \in E$. (It is worth mentioning that in [12, 23] the so-called *geometric-mean closed* vector lattices were also considered: this class is defined by the property that for all $x, y \in E_+$ the set $\{\frac{1}{2}x + \frac{1}{2t}y : 0 < t < +\infty\}$ has an infimum $\mathfrak{g}(x,y)$ in E. More details on the theme see in [23, Section 3], [12, 28].)

Recall that a *complex vector lattice* is the complexification $E_{\mathbb{C}} := E \oplus iE$ of a real square-mean closed vector lattice E. Thus, each element $z \in E_{\mathbb{C}}$ in a complex vector lattice has the absolute value $|z|$ defined by the formula

$$|z| := \mathfrak{s}(x,y) \quad (z := x + iy \in E_{\mathbb{C}}).$$

As usual, the notion of disjointness of elements $z := x + iy$ and $z' := x' + iy'$ in $E_{\mathbb{C}}$ is defined by the formula $z \perp z' \leftrightarrow |z| \wedge |z'| = 0$ and is equivalent to the relation $\{x, y\} \perp \{x', y'\}$. An ideal J in $E_{\mathbb{C}}$ is defined as the linear subspace which is solid: $|x| \leq |y|$ with $x \in E_{\mathbb{C}}$ and $y \in J$ implies $x \in J$. As in the real case, a band in $E_{\mathbb{C}}$ can be defined as $\{z \in E_{\mathbb{C}} : (\forall\, v \in V)\, z \perp v\}$, where V is a nonempty subset of $E_{\mathbb{C}}$. The ideals and bands of $E_{\mathbb{C}}$ are precisely the complexifications of ideals and bands of E (cp. [65, Chapter II, § 11] and [72, Section 91] for more detail).

3.1.2. Consider real vector lattices E and F. The space $L(E_{\mathbb{C}}, F_{\mathbb{C}})$ of $\mathbb{C}$-linear operators is isomorphic to the complexification of the real space $L(E, F)$ of $\mathbb{R}$-linear operators. An operator $T \in L(E_{\mathbb{C}}, F_{\mathbb{C}})$ is uniquely representable as $T = T_1 + iT_2$, where $T_1, T_2 \in L(E, F)$, and an arbitrary operator $S \in L(E, F)$ is identified with the canonical extension $\widetilde{S} \in L(E_{\mathbb{C}}, F_{\mathbb{C}})$ of S defined by the formula $\widetilde{S}z := Sx + iSy$, $z = x + iy$. In particular, if E and F are considered as real subspaces of $E_{\mathbb{C}}$ and $F_{\mathbb{C}}$ then the space $L(E, F)$ can be considered as a real subspace of $L(E_{\mathbb{C}}, F_{\mathbb{C}})$.

An operator $T = T_1 + iT_2$ is *positive* provided that $T_1 \geq 0$ and $T_2 = 0$ and *order bounded* provided that for every $e \in E_+$ there is $f \in F_+$ satisfying $|Tx| \leq f$ whenever $|x| \leq e$. The space of all order bounded linear operators from $E_{\mathbb{C}}$ into $F_{\mathbb{C}}$ is the complexification of the space of all order bounded linear operators from E into F.

If $E_{\mathbb{C}} = J \oplus J^{\perp}$ for some ideal $J \subset E_{\mathbb{C}}$ then there is a projection $P : E_{\mathbb{C}} \to E_{\mathbb{C}}$ with kernel $J^{\perp}$ and range J. The restriction of P to E is a band projection in E; in particular, P is a positive operator. More details can be found in [65, Chapter II] and [72, Section 92].

3.1.3. Suppose that F is a sublattice of a vector lattice E. As in the real case [46, 3.3.2], a linear operator T from $F_{\mathbb{C}}$ to $E_{\mathbb{C}}$ is *band preserving* provided that

$$z \perp z' \to Tz \perp z' \quad (z \in F_{\mathbb{C}},\ z' \in E_{\mathbb{C}}),$$

where the disjointness relations are understood in $E_{\mathbb{C}}$.

A linear operator $T := T_1 + iT_2$ from $F_{\mathbb{C}}$ to $E_{\mathbb{C}}$ is band preserving if and only if such are the real linear operators $T_1, T_2 : F \to E$.

$\lhd$ Assume that T_1 and T_2 are band preserving. If $z := x + iy$ and $w := u + iv$ are disjoint then $\{x, y\} \perp \{u, v\}$. Therefore, $\{x, y\} \perp \{T_1u - T_2v, T_1v + T_2u\}$. Hence, $z \perp Tw$ since $Tw = (T_1u - T_2v) + i(T_1v + T_2u)$.

Conversely, if T is band preserving and $x \in E$ and $u \in F$ are disjoint then $x \perp Tu = T_1u + iT_2u$; hence, $x \perp \{T_1u, T_2u\}$. $\rhd$

In particular, if E is a vector lattice enjoying the principal projection property and F is an order dense ideal of E then a linear operator $T = T_1 + iT_2 : F_{\mathbb{C}} \to E_{\mathbb{C}}$ is band preserving if and only if $\pi T_k z = T_k \pi z$ ($z \in F_{\mathbb{C}}$, $k = 1, 2$) for every band projection $\pi \in \mathfrak{P}(E)$. An order bounded band preserving operator in $E_{\mathbb{C}}$ is called an *orthomorphism* and the set of all orthomorphisms in $E_{\mathbb{C}}$ is denoted by $\mathrm{Orth}(E_{\mathbb{C}})$. Clearly, $\mathrm{Orth}(E_{\mathbb{C}})$ is the complexification of $\mathrm{Orth}(E)$.

3.1.4. Henceforth $\mathbb{B}$ is a complete Boolean algebra.

By the maximum principle (cp. A1.4 and [54, Theorem 4.3.9]), there is an element $\mathscr{C} \in \mathbb{V}^{(\mathbb{B})}$ for which $[\![\, \mathscr{C}$ is the complexes$\,]\!] = \mathbb{1}$. Since the equality $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$ is expressed by a bounded set-theoretic formula, from the restricted transfer principle A2.2 (cp. [54, 4.2.9 (2)]) we obtain $[\![\, \mathbb{C}^\wedge = \mathbb{R}^\wedge \oplus i^\wedge \mathbb{R}^\wedge \,]\!] = \mathbb{1}$. Moreover, $\mathbb{R}^\wedge$ is assumed to be a dense subfield of $\mathscr{R}$; therefore, we can also assume that $\mathbb{C}^\wedge$ is a dense subfield of $\mathscr{C}$. If 1 is the unity of $\mathbb{C}$ then $1^\wedge$ is the unity of $\mathscr{C}$ inside $\mathbb{V}^{(\mathbb{B})}$. We write i instead of $i^\wedge$ and $\mathbb{1}$ instead of $1^\wedge$.

The *descent* of $\mathscr{C}$ is the set $\mathscr{C}{\downarrow} := \{x \in \mathbb{V}^{(\mathbb{B})} : [\![x \in \mathscr{C}]\!] = \mathbb{1}\}$ endowed with the structure of a commutative complex ordered ring by descending the operations (cp. A2.4 and [54, Section 5.3]). Moreover, $\mathscr{C}{\downarrow} = \mathscr{R}{\downarrow} \oplus i\mathscr{R}{\downarrow}$; consequently, by the Gordon Theorem (cp. A3.6 and [54, Theorem 10.3.4]), $\mathscr{C}{\downarrow}$ is a universally complete complex vector lattice and a complex f-algebra simultaneously; moreover, $\mathbb{1} := 1^\wedge$ is the order and ring unity in $\mathscr{C}{\downarrow}$. The space $\mathscr{C}{\downarrow}$ depends only on $\mathbb{B}$ and $\mathbb{C}$; therefore, we will also use the notation $\mathbb{B}(\mathbb{C}) := \mathscr{C}{\downarrow}$.

3.1.5. Let $\mathrm{End}_N(G_{\mathbb{C}})$ be the set of all band preserving linear operators in $G_{\mathbb{C}}$, where $G := \mathscr{R}{\downarrow}$. It is clear that $\mathrm{End}_N(G_{\mathbb{C}})$ is a complex vector space. Moreover, $\mathrm{End}_N(G_{\mathbb{C}})$ becomes a faithful unitary module over $G_{\mathbb{C}}$ if the operator gT is defined by the formula $gT : x \mapsto g \cdot Tx$ ($x \in G_{\mathbb{C}}$). This follows from the fact that multiplication by an element of $G_{\mathbb{C}}$ is a band preserving operator and the composition of band preserving operators is a band preserving operator.

Denote by $\mathrm{End}_{\mathbb{C}^\wedge}(\mathscr{C})$ the element of $\mathbb{V}^{(\mathbb{B})}$ that depicts the space of all $\mathbb{C}^\wedge$-linear mappings from $\mathscr{C}$ into $\mathscr{C}$. Then $\mathrm{End}_{\mathbb{C}^\wedge}(\mathscr{C})$ is a vector space over $\mathbb{C}^\wedge$ inside $\mathbb{V}^{(\mathbb{B})}$ and $\mathrm{End}_{\mathbb{C}^\wedge}(\mathscr{C}){\downarrow}$ is a faithful unitary module over $G_{\mathbb{C}}$.

3.1.6. As in 2.1.2, we can prove that a linear operator in a universally complete vector lattice $G_{\mathbb{C}}$ is band preserving if and only if it is extensional. Since extensional mappings admit ascent, each operator $T \in \mathrm{End}_N(G_{\mathbb{C}})$ has the ascent $\tau := T{\uparrow}$ which is the unique function from $\mathscr{C}$ into $\mathscr{C}$ (inside $\mathbb{V}^{(\mathbb{B})}$) satisfying the condition $[\![\tau(x) = Tx]\!] = \mathbb{1}$ for all $x \in G_{\mathbb{C}}$ (cp. [54, Theorem 5.5.6]).

The modules $\mathrm{End}_N(G_{\mathbb{C}})$ *and* $\mathrm{End}_{\mathbb{C}^\wedge}(\mathscr{C}){\downarrow}$ *are put into isomorphy by sending a band preserving operator to its ascent.*

$\lhd$ Repeat the arguments of 2.1.2 with 3.1.3 and 3.1.4 taken into account. $\rhd$

3.2. Automorphisms and Derivations on the Complexes

We start with introducing notions and notation needed for the current and next subsections.

3.2.1. Define *a complex f-algebra* to be the complexification $A_{\mathbb{C}}$ of a real square-mean closed f-algebra A (cp. Definition 3.1.1). The multiplication in A extends naturally to $A_{\mathbb{C}}$ by the formula

$$(x + iy)(x' + iy') = (xx' - yy') + i(xy' + x'y),$$

and so $A_{\mathbb{C}}$ becomes a commutative complex algebra. Moreover, $|z_1z_2| = |z_1||z_2|$ $(z_1, z_2 \in A_{\mathbb{C}})$. In this situation $A_{\mathbb{C}}$ is called a *complex f-algebra* (cp. [17, 72]). A complex f-algebra $A_{\mathbb{C}}$ is semiprime whenever $x \perp y$ is equivalent to $xy = 0$ for all $x, y \in A_{\mathbb{C}}$.

If G is a universally complete vector lattice with a fixed order unity $\mathbb{1} \in G$ then there is a unique multiplication in G which makes G into an f-algebra and $\mathbb{1}$ into the multiplicative unity. Thus, $G_{\mathbb{C}}$ is an example of a complex f-algebra. We will always keep this circumstance in mind while considering a universally complete vector lattice an f-algebra.

3.2.2. Given an algebra A and a subalgebra A_0 of A, we call a linear operator $D : A_0 \to A$ a *derivation* provided that

$$D(uv) = D(u)v + uD(v) \quad (u, v \in A_0).$$

The kernel of a derivation is a subalgebra. A nonzero derivation is called *nontrivial.*

An *endomorphism of an algebra* is a linear multiplicative operator in it. A bijective endomorphism is an *automorphism.* The identical automorphism is commonly referred to as the *trivial automorphism.*

If the above definitions of an automorphism and a derivation relate to an algebra over a field $\mathbb{P}$ then we also speak of $\mathbb{P}$-automorphisms and $\mathbb{P}$-derivations.

For completeness of exposition, we give some properties of the complexes which we need below. In the next section we will give the Boolean valued interpretation of these properties. As above, $\mathscr{C}$ is the complexes inside $\mathbb{V}^{(\mathbb{B})}$. Recall that $\mathscr{C}$ includes the subfield $\mathbb{C}^\wedge$ inside $\mathbb{V}^{(\mathbb{B})}$. The following was obtained in [50]:

3.2.3. Theorem. *Inside $\mathbb{V}^{(\mathbb{B})}$, the field $\mathbb{C}^\wedge$ is algebraically closed in $\mathscr{C}$. In particular, if $\mathbb{V}^{(\mathbb{B})} \models \mathbb{C}^\wedge \neq \mathscr{C}$ then $\mathbb{V}^{(\mathbb{B})} \models$ "$\mathscr{C}$ is a transcendental extension of $\mathbb{C}^\wedge$."*

Thus, under the canonical embedding of the complexes into the Boolean valued model, either $\mathbb{C}^\wedge = \mathscr{C}$ or the field of complexes is a transcendental extension of some subfield of $\mathscr{C}$. The same is true for the reals. To analyze this situation, we need the notion of an algebraic or transcendence basis of a field over some subfield.

Let $\mathbb{P}$ be a subfield of $\mathbb{C}$ such that $\mathbb{C}$ is a transcendental extension of $\mathbb{P}$. By the Steinitz Theorem [24, Chapter 5, § 5, Theorem 1], there is a transcendence basis $\mathscr{E} \subset \mathbb{C}$. This means that the set $\mathscr{E}$ is algebraically independent over $\mathbb{P}$ and $\mathbb{C}$ is an algebraic extension of the field $\mathbb{P}(\mathscr{E})$ obtained by addition of the elements of $\mathscr{E}$ to $\mathbb{P}$. The field $\mathbb{P}(\mathscr{E})$ is a *pure extension* of $\mathbb{P}$.

3.2.4. *Let $\mathbb{C}$ be a transcendental extension of a field $\mathbb{P}$. Then there is a nontrivial $\mathbb{P}$-automorphism of $\mathbb{C}$.*

$\lhd$ Let $\mathscr{E}$ be a transcendence basis for the extension $\mathbb{C}$ over $\mathbb{P}$. Since $\mathbb{C}$ is an algebraically closed extension of $\mathbb{P}(\mathscr{E})$, every $\mathbb{P}$-automorphism ϕ of the field $\mathbb{P}(\mathscr{E})$ extends to a $\mathbb{P}$-automorphism Φ of the field $\mathbb{C}$ (cp. [24, Chapter 5, § 4, the Corollary to Theorem 1]). It is clear that if ϕ is nontrivial then so is Φ.

To construct a nontrivial $\mathbb{P}$-automorphism in $\mathbb{P}(\mathscr{E})$, we firstly consider the case when $\mathscr{E}$ contains only one element e; i.e., when $\mathbb{C}$ is an algebraic extension of a simple transcendental extension $\mathbb{P}(e)$. Take $a, b, c, d \in \mathbb{P}$ such that $ad - bc \neq 0$. Then $e' = (ae+b)/(ce+d)$ is a generator of the field $\mathbb{P}(e)$ different from e. The field $\mathbb{P}(e) = \mathbb{P}(e')$ is isomorphic to the field of rational fractions in one variable t; consequently, the linear-fractional substitution $t \mapsto (at + b)/(ct + d)$ defines a $\mathbb{P}$-automorphism ϕ of the field $\mathbb{P}(e)$ which takes e into e' (cp. [67, Section 39]).

Assume now that $\mathscr{E}$ contains at least two different elements e_1 and e_2 and take an arbitrary bijective mapping $\phi_0 : \mathscr{E} \to \mathscr{E}$ for which $\phi_0(e_1) = e_2$. Again, using the circumstance that $\mathbb{C}$ is an algebraically closed extension of $\mathbb{P}(\mathscr{E})$, we can construct a $\mathbb{P}$-automorphism ϕ of $\mathbb{C}$ such that $\phi_0(e) = \phi(e)$ for all $e \in \mathscr{E}$ (cp. [24, Chapter 5, Section 6, Proposition 1]). Clearly, ϕ is nontrivial. $\triangleright$

3.2.5. *Let $\mathbb{C}$ be a transcendental extension of a field $\mathbb{P}$. Then there is a nontrivial $\mathbb{P}$-derivation on $\mathbb{C}$.*

$\triangleleft$ We again use a transcendence basis $\mathscr{E}$ for the extension $\mathbb{C}$ over $\mathbb{P}$. It is well known that every derivation of $\mathbb{P}$ extends onto a purely transcendental extension; moreover, this extension is defined uniquely by prescribing arbitrary values at the elements of a transcendence basis (cp. [24, Chapter V, Section 9, Proposition 4]). Thus, for every mapping $d : \mathscr{E} \to \mathbb{C}$, there is a unique derivation $D : \mathbb{P}(\mathscr{E}) \to \mathbb{C}$ such that $D(e) = d(e)$ for all $e \in \mathscr{E}$ and $D(x) = 0$ for $x \in \mathbb{P}$. Now, $\mathbb{C}$ is a separable algebraic extension of $\mathbb{P}(\mathscr{E})$; consequently, D admits a unique extension to some derivation $\overline{D} : \mathbb{C} \to \mathbb{C}$ (cp. [24, Chapter V, Section 9, Proposition 5]). It is obvious that the freedom in the choice of d guarantees that $\overline{D}$ is nontrivial. $\triangleright$

3.2.6. Using the same arguments as above, we can show that some analogs of 3.2.3 and 3.2.5 are valid for the reals. More precisely, the following is valid:

(1) $[\![\mathbb{R}^\wedge$ is algebraically closed in $\mathscr{R}]\!] = \mathbb{1}$;

(2) If $\mathbb{V}^{(\mathbb{B})} \models \mathbb{R}^\wedge \neq \mathscr{R}$, then $\mathbb{V}^{(\mathbb{B})} \models$ "$\mathscr{R}$ is a transcendental extension of $\mathbb{R}^\wedge$;"

(3) If $\mathbb{R}$ is a transcendental extension of a field $\mathbb{P}$ then there is a nontrivial $\mathbb{P}$-derivation on $\mathbb{R}$.

However, 3.2.4 is not valid for the reals: there is no nontrivial automorphism on $\mathbb{R}$. This is connected with the fact that $\mathbb{R}$ is not an algebraically closed field.

3.2.7. Theorem. *Let $\mathbb{C}$ be an extension of an algebraically closed subfield $\mathbb{P}$. Then the following are equivalent:*

(1) $\mathbb{P} = \mathbb{C}$*;*

(2) *Every $\mathbb{P}$-linear function in $\mathbb{C}$ is order bounded;*

(3) *There are no nontrivial $\mathbb{P}$-derivations on $\mathbb{C}$;*

(4) *Each $\mathbb{P}$-linear endomorphism of $\mathbb{C}$ is the zero or identity function;*

(5) *There is no $\mathbb{P}$-linear automorphism of $\mathbb{C}$ other than the identity.*

$\triangleleft$ The equivalence (1) $\leftrightarrow$ (2) is checked by using a Hamel basis of the vector space $\mathbb{C}$ over $\mathbb{P}$. The remaining equivalences follow on replacing a Hamel basis with a transcendence basis from 3.2.4 and 3.2.5 (for details, cp. [50]). $\triangleright$

3.3. Automorphisms and Derivations on Complex f-Algebras

Consider the question of existence of nontrivial automorphisms and derivations on a universally complete complex f-algebra. In this section G is a universally complete vector lattice with a fixed multiplicative structure, E is a subring and a sublattice in G, while $G_{\mathbb{C}} := G \oplus iG$ and $E_{\mathbb{C}} := E \oplus iE$.

3.3.1. *Let $D \in L(E_{\mathbb{C}}, G_{\mathbb{C}})$ and $D = D_1 + iD_2$. The operator D is a complex derivation if and only if D_1 and D_2 are real derivations from E into G.*

$\triangleleft$ We only have to insert $D := D_1 + iD_2$ in the equality $D(uv) = D(u)v + uD(v)$, take $u := x \in E$ and $v := y \in E$, and then equate the real and imaginary parts of the resulting relation. $\triangleright$

3.3.2. *If $E^{\perp\perp} = G$ then each derivation from $E_{\mathbb{C}}$ into $G_{\mathbb{C}}$ is a band preserving operator.*

$\lhd$ By 3.1.3 and 3.3.1, we only have to establish that every real derivation is a band preserving operator. Let $D : E \to G$ be a real derivation. Take disjoint $x, y \in E$. Since the relation $x \perp y$ in an f-algebra implies $xy = 0$, we have $0 = D(xy) = D(x)y + xD(y)$. But the elements $D(x)y$ and $xD(y)$ are disjoint as well by the definition of an f-algebra; therefore, $D(x)y = 0$ and $xD(y) = 0$. Hence, since the f-algebra E is faithful, we obtain $D(x) \perp y$ and $x \perp D(y)$. Now, consider disjoint $x \in E$ and $g \in G$. By condition, the order ideal I generated by $\{x\}^{\perp} \cup \{x\}$ is order dense in G; therefore, without loss of generality we may assume that $g \in I$. At the same time, $|g| \leq y$ for some $y \in E_{+}$; consequently, $D(x) \perp g$ by the above. $\rhd$

3.3.3. Let $\mathscr{D}(\mathscr{C}{\downarrow})$ be the set of all derivations on the f-algebra $\mathscr{C}{\downarrow}$ and let $\mathscr{M}_N(\mathscr{C}{\downarrow})$ be the set of all band preserving automorphisms of $\mathscr{C}{\downarrow}$. Let $\mathscr{D}_{\mathbb{C}^\wedge}(\mathscr{C})$ and $\mathscr{M}_{\mathbb{C}^\wedge}(\mathscr{C})$ be the elements of $\mathbb{V}^{(\mathbb{B})}$ that depict the sets of all $\mathbb{C}^\wedge$-derivations and all $\mathbb{C}^\wedge$-automorphisms in $\mathscr{C}$. Clearly, $\mathscr{D}(\mathscr{C}{\downarrow})$ is a module over $\mathscr{C}{\downarrow}$ and $[\![\mathscr{D}_{\mathbb{C}^\wedge}(\mathscr{C})$ is a complex vector space$]\!] = \mathbb{1}$.

The descent and ascent produce isomorphisms between the modules $\mathscr{D}_{\mathbb{C}^\wedge}(\mathscr{C}){\downarrow}$ and $\mathscr{D}(\mathscr{C}{\downarrow})$ as well as bijections between $\mathscr{M}_{\mathbb{C}^\wedge}(\mathscr{C}){\downarrow}$ and $\mathscr{M}_N(\mathscr{C}{\downarrow})$.

$\lhd$ The proof follows from 3.1.6. We only have to note that $T \in \mathrm{End}_N(\mathscr{C}{\downarrow})$ is a derivation (automorphism) if and only if $[\![\tau := T{\uparrow}$ is a derivation (automorphism)$]\!] = \mathbb{1}$. $\rhd$

3.3.4. *An order bounded derivation and an order bounded band preserving automorphism of a universally complete f-ring $G_{\mathbb{C}}$ are trivial.*

$\lhd$ We may assume that $G_{\mathbb{C}} = \mathscr{C}{\downarrow}$. If T is a derivation (a band preserving automorphism) of the f-ring $G_{\mathbb{C}}$ then $[\![\tau := T{\uparrow}$ is a $\mathbb{C}^\wedge$-derivation ($\mathbb{C}^\wedge$-automorphism) of $\mathscr{C}]\!] = \mathbb{1}$. Moreover, T is order bounded if and only if $[\![\tau$ is order bounded in $\mathscr{C}]\!] = \mathbb{1}$. However, every order bounded $\mathbb{C}^\wedge$-derivation on the field $\mathscr{C}$ is zero and every order bounded $\mathbb{C}^\wedge$-automorphism is the identity mapping. In the first case we have $T = 0$ and in the second, $T = I$. $\rhd$

3.3.5. *If $\mathbb{V}^{(\mathbb{B})} \models \mathbb{C}^\wedge \neq \mathscr{C}$ then there exist a nontrivial derivation and a nontrivial band preserving automorphism on the universally complete complex f-algebra $\mathbb{B}(\mathbb{C}) = \mathscr{C}{\downarrow}$.*

$\lhd$ It follows from the condition $\mathbb{C}^\wedge \neq \mathscr{C}$ that $\mathscr{C}$ is a transcendental extension of $\mathbb{C}^\wedge$ inside $\mathbb{V}^{(\mathbb{B})}$ (cp. 3.2.3). By 3.2.4 and 3.2.5, there exist a nontrivial $\mathbb{C}^\wedge$-derivation $\delta : \mathscr{C} \to \mathscr{C}$ and a nontrivial $\mathbb{C}^\wedge$-automorphism $\alpha : \mathscr{C} \to \mathscr{C}$. If $D := \delta{\downarrow}$ and $A := \alpha{\downarrow}$ then, according to 3.3.3, D is a nontrivial derivation and A is a nontrivial band preserving automorphism of the f-algebra $\mathscr{C}{\downarrow}$. $\rhd$

3.3.6. Theorem. *For an arbitrary complete Boolean algebra $\mathbb{B}$ the following are equivalent:*

(1) $\mathbb{B}$ *is σ-distributive;*

(2) $\mathbb{V}^{(\mathbb{B})} \models \mathscr{C} = \mathbb{C}^\wedge$;

(3) *All band preserving linear operators on the universally complete vector lattice $\mathbb{B}(\mathbb{C}) = \mathscr{C}{\downarrow}$ are order bounded;*

(4) *There are no nonzero derivations on the complex f-algebra $\mathbb{B}(\mathbb{C}) = \mathscr{C}{\downarrow}$;*

(5) *Each band preserving endomorphism of the complex f-algebra $\mathbb{B}(\mathbb{C}) = \mathscr{C}{\downarrow}$ is a band projection;*

(6) *In the complex f-algebra $\mathbb{B}(\mathbb{C}) = \mathscr{C}{\downarrow}$ there are no nontrivial band preserving automorphisms.*

$\lhd$ (1) $\leftrightarrow$ (2): As is known (cp. Section 2.3), a Boolean algebra $\mathbb{B}$ is σ-distributive if and only if $\mathbb{V}^{(\mathbb{B})} \models \mathbb{R}^{\wedge} = \mathscr{R}$. Hence, using the restricted transfer principle A2.2 ([54, 4.2.9 (2)]), we conclude that $\mathbb{V}^{(\mathbb{B})} \models \mathscr{C} = \mathscr{R} \oplus i\mathscr{R} = \mathbb{R}^{\wedge} \oplus i\mathbb{R}^{\wedge} = \mathbb{C}^{\wedge}$. The converse is proved similarly.

(2) $\to$ (3): If $\mathbb{V}^{(\mathbb{B})} \models \mathbb{C}^{\wedge} = \mathscr{C}$ then, inside $\mathbb{V}^{(\mathbb{B})}$, the set $\mathrm{End}_{\mathbb{C}^{\wedge}}(\mathscr{C})$ consists of the functions $\tau : \mathscr{C} \to \mathscr{C}$ of the form $\tau(z) = cz$, where $c \in \mathscr{C}$. But then the operator $T := \tau{\downarrow}$ from $\mathscr{C}{\downarrow}$ into $\mathscr{C}{\downarrow}$ has the form $T(u) = gu$ for some $g \in \mathscr{C}{\downarrow}$.

(3) $\to$ (2): It follows from (3) that all band preserving linear operators are order bounded in the universally complete vector lattice $\mathscr{R}{\downarrow}$. Thus, $\mathbb{V}^{(\mathbb{B})} \models \mathbb{R}^{\wedge} = \mathscr{R}$ (cp. 2.1.7 (2)); and so $\mathbb{V}^{(\mathbb{B})} \models \mathscr{C} = \mathbb{C}^{\wedge}$.

(3) $\to$ (4): This follows from 3.3.2 and 3.3.4.

(3) $\to$ (5): A band preserving endomorphism $T : \mathscr{C}{\downarrow} \to \mathscr{C}{\downarrow}$ admits the representation $T = T_1 + iT_2$, where T_1 and T_2 are band preserving linear operators in the universally complete vector lattice $\mathscr{R}{\downarrow}$ (cp. 3.1.3). By (3), T_1 and T_2 are order bounded; consequently, $T_l x = c_l x$ $(x \in \mathscr{R}{\downarrow})$ for some constants $c_1, c_2 \in \mathscr{R}{\downarrow}$. Hence, $Tz = c \cdot z$ $(z \in \mathscr{C}{\downarrow})$, where $c := c_1 + ic_2$. Multiplicativity of T implies $c^2 = c$; therefore, the equalities $c_1^2 - c_2^2 = c_1$ and $2c_1c_2 = c_2$ are valid. If $\pi := [c_2]$ is the projection in $\mathscr{R}{\downarrow}$ onto the band $\{c_2\}^{\perp\perp}$ then from the second equality we derive $\pi c_1 = (1/2)\pi(\mathbb{1})$, while the first equality implies $-\pi(c_2^2) = (1/4)\pi(\mathbb{1})$. The last is possible only for $\pi = 0$; hence, $c_2 = 0$ and $0 \le c_1^2 = c_1$. But we also have $0 \le (\mathbb{1} - c_1)^2 = \mathbb{1} - c_1$; consequently, $c_1 \le \mathbb{1}$. Now, we see that the operator $x \mapsto T_1 x = c_1 x$ is a band projection in $\mathscr{R}{\downarrow}$ and, in view of $T_2 = 0$, its canonical extension to $\mathscr{C}{\downarrow}$ coincides with T.

(5) $\to$ (6): This is obvious.

The implications (4) $\to$ (2) and (6) $\to$ (2) follow from 3.3.5.

(4) $\to$ (2): If the equality $\mathscr{C} = \mathbb{C}^{\wedge}$ is violated inside $\mathbb{V}^{(\mathbb{B})}$ then $b := [\![\mathscr{C} = \mathbb{C}^{\wedge}]\!] < \mathbb{1}$. But then $b^* = [\![\mathscr{C} \ne \mathbb{C}^{\wedge}]\!] \ne \mathbb{0}$. The inequality $\mathscr{C} \ne \mathbb{C}^{\wedge}$ is valid in the Boolean valued model $\mathbb{V}^{(\mathbb{B}_0)}$ over the Boolean algebra $\mathbb{B}_0 := [\mathbb{0}, b^*]$. By 3.3.5, there is a nonzero derivation D on the band $b^*\mathscr{C}{\downarrow}$. The unique extension $D \oplus 0$ of the operator D coinciding with zero on the band $b\mathscr{C}{\downarrow}$ is a nonzero derivation on $\mathscr{C}{\downarrow}$, too.

(6) $\to$ (2): Similarly, using 3.3.5, for the same $b \in \mathbb{B}$ we can find a nontrivial automorphism A^* of the band $b^*\mathscr{C}{\downarrow}$. If A is the identity mapping in the band $b\mathscr{C}{\downarrow}$ then $A^* \oplus A$ is a nontrivial automorphism of $\mathscr{C}{\downarrow}$. $\rhd$

3.3.7. Corollary. *For a universally complete real vector lattice G with a fixed structure of an f-algebra, the following are equivalent:*

(1) $\mathbb{B} := \mathfrak{P}(G)$ *is a σ-distributive Boolean algebra;*

(2) *There are no nontrivial derivations on the complex f-algebra $G_{\mathbb{C}}$;*

(3) *There are no nontrivial band preserving automorphisms of the complex f-algebra $G_{\mathbb{C}}$.*

PART 4. VARIATIONS ON THE THEME

In this part we consider briefly the band preserving phenomenon in some natural environments (the endomorphisms of lattice ordered modules, bilinear operators on vector lattices, and derivations in AW^*-algebras) and state some problems that may be viewed as versions of the Wickstead problem.

4.1. The Wickstead Problem in Lattice Ordered Modules

In this section we state a kind of the Wickstead problem for lattice ordered modules.

4.1.1. Let K be a lattice ordered ring, and let X be a lattice ordered module over K. The Wickstead problem for lattice ordered modules can be stated as follows:

WP(A): *When are all band preserving K-linear endomorphisms of a lattice ordered K-module X order bounded?*

Little is known about this problem. Boolean valued analysis provides a transfer principle which might send WP to WP(A). Below we describe the class of lattice ordered modules for which this transfer works perfectly.

4.1.2. A subset S of K is *dense* provided that $S^{\perp} = \{0\}$; i.e., the equality $k \cdot S = \{0\}$ implies $k = 0$ for all $k \in K$. A ring K is *rationally complete* whenever, to each dense ideal $J \subset K$ and each group homomorphism $h : J \to K$ such that $h(kx) = kh(x)$ for all $k \in K$ and $x \in J$, there is an element r in K satisfying $h(x) = rx$ for all $x \in J$. A ring K is rationally complete if and only if K is selfinjective (cp. [54, Theorem 8.2.7 (3)]).

4.1.3. If $\mathscr{K}$ is an ordered field inside $\mathbb{V}^{(\mathbb{B})}$ then $\mathscr{K}{\downarrow}$ is a rationally complete semiprime f-ring, and there is an isomorphism χ of $\mathbb{B}$ onto the Boolean algebra $\mathfrak{B}(\mathscr{K}{\downarrow})$ of the annihilator ideals (coinciding in the case under consideration with the Boolean algebra of all bands) of $\mathscr{K}{\downarrow}$ such that

$$b \le [\![x = 0]\!] \leftrightarrow x \in \chi(b^*) \quad (x \in K,\ b \in B)$$

(cp. [54, Theorem 8.3.1]). Conversely, assume that K is a rationally complete semiprime f-ring and $\mathbb{B}$ stands for the Boolean algebra $\mathfrak{B}(K)$ of all annihilator ideals (bands) of K. Then there is an element $\mathscr{K} \in \mathbb{V}^{(\mathbb{B})}$, called the *Boolean valued representation of* K, such that $[\![\, \mathscr{K}$ is an ordered field$\,]\!] = \mathbb{1}$ and the lattice ordered rings K and $\mathscr{K}{\downarrow}$ are isomorphic (cp. [54, Theorem 8.3.2]).

4.1.4. A K-module X is *separated* provided that for every dense ideal $J \subset K$ the identity $Jx = \{0\}$ implies $x = 0$. Recall that a K-module X is *injective* whenever, given a K-module Y, a K-submodule $Y_0 \subset Y$, and a K-homomorphism $h_0 : Y_0 \to X$, there exists a K-homomorphism $h : Y \to X$ extending h_0. The Baer criterion says that a K-module X is injective if and only if for each ideal $J \subset K$ and each K-homomorphism $h : J \to X$ there exists $x \in X$ with $h(a) = xa$ for all $a \in J$ (cp. [55]).

4.1.5. Let $\mathscr{X}$ be a vector lattice over an ordered field $\mathscr{K}$ inside $\mathbb{V}^{(\mathbb{B})}$, and let $\chi : \mathbb{B} \to \mathfrak{B}(\mathscr{K}{\downarrow})$ be a Boolean isomorphism from 4.1.3. Then $\mathscr{X}{\downarrow}$ is a separated unital injective lattice ordered module over $\mathscr{R}{\downarrow}$ satisfying

$$b \le [\![x = 0]\!] \leftrightarrow \chi(b)x = \{0\} \quad (x \in \mathscr{X}{\downarrow},\ b \in \mathbb{B}).$$

Conversely, let K be a rationally complete semiprime f-ring, $\mathbb{B} := \mathfrak{B}(K)$, and let $\mathscr{K}$ be the Boolean valued representation of K. Assume that X is a unital separated injective lattice ordered K-module. Then there exists some $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ such that $[\![\, \mathscr{X}$ is a vector lattice over the ordered field $\mathscr{K}\,]\!] = \mathbb{1}$ and there are algebraic and order isomorphisms $\jmath : K \to \mathscr{K}{\downarrow}$ and $\imath : X \to \mathscr{X}{\downarrow}$ such that

$$\imath(ax) = \jmath(a)\imath(x) \quad (a \in K,\ x \in X)$$

(cp. [54, Theorems 8.3.12 and 8.3.13]). Thus, the Boolean transfer principle is applicable to unital separated injective lattice ordered modules over rationally complete semiprime f-rings. Consider an example.

4.1.6. Let $\mathbf{B}$ be a complete Boolean algebra and let $\mathbb{B}$ be a complete subalgebra of $\mathbf{B}$. We say that $\mathbf{B}$ is $\mathbb{B}$-σ-*distributive* if for every sequence $(b_n)_{n\in\mathbb{N}}$ in $\mathbf{B}$ we have

$$\bigvee_{\varepsilon\in\mathbb{B}^{\mathbb{N}}} \bigwedge_{n\in\mathbb{N}} \varepsilon(n)b_n = \mathbb{1},$$

where $\varepsilon(n)b_n := \big(\varepsilon(n)\wedge b_n\big)\vee\big(\varepsilon(n)^*\wedge b_n^*\big)$ and b^* is the complement of $b\in\mathbf{B}$. Clearly, the $\{\mathbb{0},\mathbb{1}\}$-$\sigma$-distributivity of $\mathbf{B}$ means that $\mathbf{B}$ is σ-distributive $\big($cp. 1.3.1 (3)$\big)$.

There exists a $\mathscr{B}\in\mathbb{V}^{(\mathbb{B})}$ such that $[\![\,\mathscr{B}$ is a complete Boolean algebra$\,]\!]=\mathbb{1}$ and $\mathscr{B}{\downarrow}$ is a complete Boolean algebra isomorphic to $\mathbf{B}$ (cp. [54, Theorem 4.7.11]). Moreover, $\mathbf{B}$ is $\mathbb{B}$-σ-distributive if and only if $\mathscr{B}$ is σ-distributive inside $\mathbb{V}^{(\mathbb{B})}$. We now interpret Theorem 1.3.7 inside $\mathbb{V}^{(\mathbb{B})}$ to obtain:

4.1.7. Theorem. *Let X be a universally complete vector lattice with a fixed order unity $\mathbb{1}$ and let K be an order closed sublattice containing $\mathbb{1}$. Put $\mathbf{B} := \mathfrak{E}(X) := \mathfrak{E}(\mathbb{1}_X)$ and $\mathbb{B} := \mathfrak{E}(K) := \mathfrak{E}(\mathbb{1}_K)$. Then K is a rationally complete f-algebra, X is an injective lattice ordered K-module, and the following are equivalent:*

(1) $\mathbf{B}$ *is $\mathbb{B}$-σ-distributive;*

(2) *Every element $x\in X_+$ is locally K-constant, i.e., $x=\sup_{\xi\in\Xi}a_\xi\pi_\xi\mathbb{1}$ for some family $(a_\xi)_{\xi\in\Xi}$ of elements of K and a family $(\pi_\xi)_{\xi\in\Xi}$ of pairwise disjoint band projections in X;*

(3) *Every band preserving K-linear endomorphism of X is order bounded.*

4.2. The Wickstead Problem for Bilinear Operators

In this section we present the main results of [52].

4.2.1. Let E be a vector lattice. A bilinear operator $b: E\times E\to E$ is *separately band preserving* provided that the mappings $b(\cdot,e): x\mapsto b(x,e)$ and $b(e,\cdot): x\mapsto b(e,x)$ $(x\in E)$ are band preserving for all $e\in E$ or, which is the same, provided that $b(L\times E)\subset L$ and $b(E\times L)\subset L$ for every band L in E.

4.2.2. *Assume that E is a vector lattice and $b: E\times E\to E$ is a bilinear operator. Then the following are equivalent:*

(1) *b is separately band preserving;*

(2) *$b(x,y)\in\{x\}^{\perp\perp}\cap\{y\}^{\perp\perp}$ for all $x,y\in E$;*

(3) *$b(x,y)\perp z$ for all $z\in E$ provided that $x\perp z$ or $y\perp z$.*

If E has the principal projection property, then (1)–(3) are equivalent to:

(4) *$\pi b(x,y)=b(\pi x,\pi y)$ for every $\pi\in\mathfrak{P}(E)$ and all $x,y\in E$;*

(5) *$\pi b(x,y)=b(\pi x,y)=b(x,\pi y)$ for every $\pi\in\mathfrak{P}(E)$ and all $x,y\in E$.*

$\lhd$ We omit the routine arguments which are similar to [10, Theorem 8.2]. $\rhd$

4.2.3. Let E and F be vector lattices. A bilinear operator $b: E\times E\to F$ is *orthosymmetric* provided that $|x|\wedge|y|=0$ implies $b(x,y)=0$ for arbitrary $x,y\in E$ (cp. [29]). The difference of two positive orthosymmetric bilinear operators is *orthoregular* (cp. [27, 49]). Recall also that a bilinear operator b is *symmetric* or *antisymmetric* provided that $b(x,y)=b(y,x)$ or $b(x,y)=-b(y,x)$ for all $x,y\in E$.

The following important property of orthosymmetric bilinear operators was established in [29, Corollary 2]:

Theorem. *If E and F are vector lattices then every orthosymmetric positive bilinear operator from $E \times E$ into F is symmetric.*

4.2.4. It is evident from 4.2.2 that a separately band preserving bilinear operator is orthosymmetric. Hence, all orthoregular separately band preserving operators are symmetric by 4.2.3. At the same time an order bounded separately band preserving bilinear operator b is of the form $b = \pi\odot$ with π an orthomorphism on $E^{\odot}$ and $\odot$ is the canonical bimorphism from $E \times E$ to $E^{\odot}$ (cp. [27, Section 2] and [30]). This brings up the following question:

WP(B): *Under what conditions are all separately band preserving bilinear operators in a vector lattice symmetric? Order bounded?*

In the case of a universally complete vector lattice the answer is similar to the linear case and is presented below in 4.2.5. The general case was not yet examined.

4.2.5. Theorem. *Let G be a universally complete vector lattice and let $\mathbb{B} := \mathfrak{P}(G)$ denote the complete Boolean algebra of all bands in G. Then the following are equivalent:*

(1) $\mathbb{B}$ *is σ-distributive;*

(2) *There is no nonzero separately band preserving antisymmetric bilinear operator in G;*

(3) *All separately band preserving bilinear operators in G are symmetric;*

(4) *All separately band preserving bilinear operators in G are order bounded.*

$\lhd$ The only nontrivial implication is (2) $\to$ (1).

We may assume that $G = \mathscr{R}{\downarrow}$. Suppose that $\mathbb{B}$ is not σ-distributive. Then $\mathbb{R}^{\wedge} \neq \mathscr{R}$ by WP(1) $\leftrightarrow$ WP(2) (cp. Section 2.3) and a separately band preserving antisymmetric bilinear operator can be constructed on using the bilinear version of 2.1.6 (1). Indeed, inside $\mathbb{V}^{(\mathbb{B})}$, a Hamel bases $\mathscr{E}$ for $\mathscr{R}$ over $\mathbb{R}^{\wedge}$ contains at least two different elements $e_1 \neq e_2$. Define a function $\beta_0 : \mathscr{E} \times \mathscr{E} \to \mathscr{R}$ so that $1 = \beta_0(e_1, e_2) = -\beta_0(e_2, e_1)$, and $\beta(e_1', e_2') = 0$ for all other pairs $(e_1', e_2') \in \mathscr{E} \times \mathscr{E}$ $\big($in particular, $0 = \beta_0(e_1, e_1) = \beta_0(e_2, e_2)\big)$. Then β_0 can be extended to an $\mathbb{R}^{\wedge}$-bilinear function $\beta : \mathscr{R} \times \mathscr{R} \to \mathscr{R}$. The descent b of β is a separately band preserving bilinear operator in G by 4.2.6, the bilinear version of 2.1.5. Moreover, b is nonzero and antisymmetric, since β is nonzero and antisymmetric by construction. This contradiction proves that $\mathbb{R}^{\wedge} = \mathscr{R}$ and $\mathbb{B}$ is σ-distributive. $\rhd$

4.2.6. Let $BL_N(G)$ stand for the set of all separately band preserving bilinear operators in $G = \mathscr{R}{\downarrow}$. Clearly, $BL_N(G)$ becomes a faithful unitary module over G provided that we define gT as $gT : x \mapsto g \cdot Tx$ for all $x \in G$. Denote by $BL_{\mathbb{R}^{\wedge}}(\mathscr{R})$ the element of $\mathbb{V}^{(\mathbb{B})}$ that depicts the space of all $\mathbb{R}^{\wedge}$-bilinear mappings from $\mathscr{R} \times \mathscr{R}$ into $\mathscr{R}$. Then $BL_{\mathbb{R}^{\wedge}}(\mathscr{R})$ is a vector space over $\mathbb{R}^{\wedge}$ inside $\mathbb{V}^{(\mathbb{B})}$, and $BL_{\mathbb{R}^{\wedge}}(\mathscr{R}){\downarrow}$ is a faithful unitary module over G.

The modules $BL_N(G)$ and $BL_{\mathbb{R}^{\wedge}}(\mathscr{R}){\downarrow}$ are isomorphic by sending each band preserving bilinear operator to its ascent.

$\lhd$ See 2.1.5. $\rhd$

4.2.7. *There exists a nonatomic universally complete vector lattice in which all separately band preserving bilinear operators are symmetric and order bounded.*

$\lhd$ It follows from 4.2.5 and 1.3.8. $\rhd$

4.3. The Noncommutative Wickstead Problem

The relevant information on the theory of Baer $*$-algebras and AW^*-algebras can be found in [15, 31, 46].

4.3.1. A *Baer $*$-algebra* is a complex involutive algebra A provided that, for each nonempty $M \subset A$, there is a projection, i.e., a hermitian idempotent, p satisfying $M^\perp = pA$, where $M^\perp := \{y \in A : (\forall x \in M)\, xy = 0\}$ is the right annihilator of M. Clearly, this amounts to the condition that each left annihilator has the form ${}^\perp M = Aq$ for an appropriate projection q. To each left annihilator L in a Baer $*$-algebra there is a unique projection $q_L \in A$ such that $x = xq_L$ for all $x \in L$ and $q_L y = 0$ whenever $y \in L^\perp$. The mapping $L \mapsto q_L$ is an isomorphism between the poset of left annihilators and the poset of all projections. Thus, the poset $\mathfrak{P}(A)$ of all projections in a Baer $*$-algebra is an order complete lattice. (Clearly, the formula $q \le p \leftrightarrow q = qp = pq$, sometimes pronounced as "p contains q," specifies some order on the set of projections $\mathfrak{P}(A)$.)

An element z in A is *central* provided that z commutes with every member of A; i.e., $(\forall x \in A)\, xz = zx$. The *center* of a Baer $*$-algebra A is the set $\mathscr{Z}(A)$ comprising central elements. Clearly, $\mathscr{Z}(A)$ is a commutative Baer $*$-subalgebra of A, with $\lambda\mathbb{1} \in \mathscr{Z}(A)$ for all $\lambda \in \mathbb{C}$. A *central projection* of A is a projection belonging to $\mathscr{Z}(A)$. Put $\mathfrak{P}_c(A) := \mathfrak{P}(A) \cap \mathscr{Z}(A)$.

4.3.2. A *derivation* on a Baer $*$-algebra A is a linear operator $d : A \to A$ satisfying $d(xy) = d(x)y + xd(y)$ for all $x, y \in A$. A derivation d is *inner* provided that $d(x) = ax - xa$ $(x \in A)$ for some $a \in A$. Clearly, an inner derivation vanishes on $\mathscr{Z}(A)$ and is $\mathscr{Z}(A)$-linear, i.e., $d(ex) = ed(x)$ for all $x \in A$ and $e \in \mathscr{Z}(A)$.

Consider a derivation $d : A \to A$ on a Baer $*$-algebra A. If $p \in A$ is a central projection then $d(p) = d(p^2) = 2pd(p)$. Multiplying this identity by p we have $pd(p) = 2pd(p)$ so that $d(p) = pd(p) = 0$. Consequently, every derivation vanishes on the linear span of $\mathfrak{P}_c(A)$, the set of all central projections. In particular, $d(ex) = ed(x)$ whenever $x \in A$ and e is a linear combination of central projections. Even if the linear span of central projections is dense in a sense in $\mathscr{Z}(A)$, the derivation d may fail to be $\mathscr{Z}(A)$-linear.

This brings up the natural question: *Under what conditions is every derivation Z-linear on a Baer $*$-algebra A provided that Z is a Baer $*$-subalgebra of $\mathscr{Z}(A)$?*

4.3.3. An *AW^*-algebra* is a C^*-algebra with unity $\mathbb{1}$ which is also a Baer $*$-algebra. More explicitly, an AW^*-algebra is a C^*-algebra whose every right annihilator has the form pA, with p a projection. Clearly, $\mathscr{Z}(A)$ is a commutative AW^*-subalgebra of A. If $\mathscr{Z}(A) = \{\lambda\mathbb{1} : \lambda \in \mathbb{C}\}$ then the AW^*-algebra A is an *AW^*-factor.*

A C^*-algebra A is an AW^*-algebra if and only if the following hold:

(1) *Each orthogonal family in $\mathfrak{P}(A)$ has a supremum;*

(2) *Each maximal commutative $*$-subalgebra $A_0 \subset A$ is a Dedekind complete f-algebra (or, equivalently, coincides with the least norm closed $*$-subalgebra containing all projections of A_0).*

4.3.4. Given an AW^*-algebra A, define the two sets $C(A)$ and $S(A)$ of measurable and locally measurable operators, respectively. Both are Baer $*$-algebras, cp. [31]. Suppose that Λ is an AW^*-subalgebra in $\mathscr{Z}(A)$, and Φ is a Λ valued trace on A_+. Then we may define another Baer $*$-algebra, $L(A, \Phi)$, of Φ-measurable operators. The center $\mathscr{Z}(A)$ is a vector lattice with a strong unity, while the centers of

$C(A)$, $S(A)$, and $L(A, \Phi)$ coincide with the universal completion of $\mathscr{Z}(A)$. If d is a derivation on $C(A)$, $S(A)$, or $L(A, \Phi)$ then $d(px) = pd(x)$ $\big(p \in \mathfrak{P}_c(A)\big)$ so that d can be considered as band preserving in a sense (cp. 1.1.1 (4) and 3.1.3).

WP(C): *When are all derivations on $C(A)$, $S(A)$, or $L(A, \Phi)$ inner?*

4.3.5. The classification of AW^*-algebras into types is determined from the structure of their lattices of projections $\mathfrak{P}(A)$ [46, 64]. We recall only the definition of type I AW^*-algebra. A projection $\pi \in A$ is *abelian* if $\pi A \pi$ is a commutative algebra. An algebra A has *type* I provided that each nonzero projection in A contains a nonzero abelian projection.

A C^*-algebra A is $\mathbb{B}$-*embeddable* provided that there is a type I AW^*-algebra N and a $*$-monomorphism $\imath : A \to N$ such that $\mathbb{B} = \mathfrak{P}_c(N)$ and $\imath(A) = \imath(A)''$, where $\imath(A)''$ is the bicommutant of $\imath(A)$ in N. Note that in this event A is an AW^*-algebra and $\mathbb{B}$ is a complete subalgebra of $\mathfrak{P}_c(A)$.

4.3.6. Theorem. *Let A be a type I AW^*-algebra, let Λ be an AW^*-subalgebra of $\mathscr{Z}(A)$, and let Φ be a Λ valued faithful normal semifinite trace on A. If the complete Boolean algebra $\mathbb{B} := \mathfrak{P}(\Lambda)$ is σ-distributive and A is $\mathbb{B}$-embeddable, then every derivation on $L(A, \Phi)$ is inner.*

$\lhd$ We briefly sketch the proof. Let $\mathscr{A} \in \mathbb{V}^{(\mathbb{B})}$ be the Boolean valued representation of A. Then $\mathscr{A}$ is a von Neumann algebra inside $\mathbb{V}^{(\mathbb{B})}$. Since the Boolean valued interpretation preserves classification into types, $\mathscr{A}$ is of type I. Let φ stand for the Boolean valued representation of Φ. Then φ is a $\mathscr{C}$ valued faithful normal semifinite trace on $\mathscr{A}$ and the descent of $L(\mathscr{A}, \varphi)$ is $*$-Λ-isomorphic to $L(A, \Phi)$, cp. [45]. Suppose that d is a derivation on $L(A, \Phi)$ and δ is the Boolean valued representation of d. Then δ is a $\mathscr{C}$ valued $\mathbb{C}^{\wedge}$-linear derivation on $L(\mathscr{A}, \varphi)$. Since $\mathbb{B}$ is σ-distributive, $\mathscr{C} = \mathbb{C}^{\wedge}$ inside $\mathbb{V}^{(\mathbb{B})}$ and δ is $\mathscr{C}$-linear. But it is well known that any derivation on a type I von Neumann algebra is inner, cp. [9]. Therefore, d is also inner. $\rhd$

PART 5. COMMENTS

5.1. Comments on Part 1

5.1.1. The theory of orthomorphisms stems from Nakano [60]. Orthomorphisms have been studied by many authors under various names (cp. [10]): *dilatators* (Nakano [60]), *essentially positive operators* (Birkhoff [20]), *polar preserving endomorphisms* (Conrad and Diem [32]), *multiplication operators* (Buck [25] and Wickstead [68]), and *stabilisateurs* (Meyer [59]). The main stages of this development as well as the various aspects of the theory of orthomorphisms are reflected in the books by Bigard, Keimel, and Wolfenstein [19], Aliprantis and Burkinshaw [10], Zaanen [72, Chapter 20] etc.; also see the survey papers by Bukhvalov [26, Section 2.2] and Gutman [39, Chapter 6].

5.1.2. Order continuity of an extended orthomorphism (cp. 1.1.4) was established independently by Bigard and Keimel [18] and Conrad and Diem [32] using functional representation. A direct proof was found by Luxemburg and Schep [57]. Commutativity of every Archimedean f-algebra was proved by Birkhoff and Pierce [21]; this paper also introduced the concept of f-algebra. The lattice ordered algebras were surveyed by Boulabiar, Buskes, and Triki [22, 23]. The fact that $\mathrm{Orth}(D, E)$ is a vector lattice under the pointwise algebraic and latticial operations was also obtained in [18] and [32]. Extensive is the bibliography on the theory of orthomorphisms; and so we indicate a portion of it: [2, 7, 16, 18, 33, 40, 41, 42, 43, 56, 57, 58, 62, 63, 68, 69, 71].

5.1.3. The terms "local linear independence" and "local Hamel basis" (coined in [58]) appeared in [2] as *d-independence* and *d-basis*. Using these concepts Abramovich and Kitover [4] gave complete description for a band preserving projection P on a Dedekind complete vector lattice E. The order bounded part πP of P (cp. 1.1.2) is a band projection, whereas the unbounded part $P_0 := P|_{E_0}$, with $E_0 := \pi^{\perp}(E)$, is uniquely determined from the following conditions: (1) every principal band in E_0 is laterally complete; (2) $P_0^{-1}(0)$ is componentwise closed; i.e., $\mathfrak{C}(u) \subset P_0^{-1}(0)$ for all $0 \leq u \in P_0^{-1}(0)$; (3) $L \cap P_0^{-1}(0)$ is laterally complete for each principal band L in E_0. Cp. [5] for applications of this concept.

5.1.4. The notions of d-independence and d-basis can be introduced in an arbitrary vector lattice (cp. [6]). A collection $(x_\gamma)_{\gamma\in\Gamma}$ of elements in a vector lattice E is *d-independent* provided that for each band B in E, each finite subset $\{\gamma_1, \dots, \gamma_n\}$ of Γ, and each family of nonzero scalars $c_1, \dots, c_n$ the condition $\sum_{i=1}^{n} c_i x_{\gamma_i} \perp B$ implies that $x_{\gamma_i} \perp B$ for $i = 1, \dots, n$. A d-independent system $(x_\gamma)_{\gamma\in\Gamma}$ is a *d-basis* provided that for each $x \in E$ there is a full system $(B_\alpha)_{\alpha\in\mathrm{A}}$ of pairwise disjoint bands in E and a system of elements $(y_\alpha)_{\alpha\in\mathrm{A}}$ in E such that each y_α is a linear combination of elements in $(x_\gamma)_{\gamma\in\Gamma}$ and $(x - y_\alpha) \perp B_\alpha$ for all $\alpha \in \mathrm{A}$.

5.1.5. Theorem 1.3.7 can be considered as an exhaustive answer to the Wickstead problem about the order boundedness of all band preserving operators. However, a new notion of locally one-dimensional vector lattice crept into the answer. The novelty of this notion led to the conjecture that it coincides with that of discrete (= atomic) vector lattice. In 1981, Abramovich, Veksler, and Koldunov [3, Theorem 2.1] gave a proof for existence of an order unbounded band preserving operator in every nondiscrete universally complete vector lattice, thus seemingly corroborating the conjecture that a locally one-dimensional vector lattice is discrete (also cp. [1, Section 5]). However, the proof was erroneous. Later in 1985, McPolin and Wickstead [58, Section 3] gave an example of a nondiscrete locally one-dimensional vector lattice, confuting the conjecture. However, there was an error in the example. Finally, Wickstead [7] stated the conjecture as an open question in 1993.

5.1.6. In the case of a universally complete vector lattice, a band preserving order unbounded operator can be constructed on using $\mathbb{V}^{(\mathbb{B})}$. Moreover, inside an appropriate $\mathbb{V}^{(\mathbb{B})}$ this problem reduces to existence of a discontinuous solution $\varphi : \mathscr{R} \to \mathscr{R}$ to the Cauchy functional equation $\varphi(s + t) = \varphi(s) + \varphi(t)$ $(s, t \in \mathscr{R})$ with an additional property $\varphi(\lambda s) = \lambda\varphi(s)$ $(\lambda \in \mathbb{R}^\wedge,\ s \in \mathscr{R})$. Let E be a universally complete vector lattice such that $\mathbb{V}^{(\mathbb{B})} \models \mathbb{R}^\wedge \neq \mathscr{R}$ (cp. Section 2.3) with $\mathbb{B} := \mathfrak{B}(E)$. Then $\mathscr{R}$ is an infinite-dimensional vector space over $\mathbb{R}^\wedge$ inside $\mathbb{V}^{(\mathbb{B})}$. By the Kuratowski–Zorn Lemma, there exists an $\mathbb{R}^\wedge$-linear but not $\mathscr{R}$-linear function $\varphi : \mathscr{R} \to \mathscr{R}$ inside $\mathbb{V}^{(\mathbb{B})}$. The operator $\Phi_0 := \varphi{\downarrow} : \mathscr{R}{\downarrow} \to \mathscr{R}{\downarrow}$ is linear, band preserving, but order unbounded. If ι is an isomorphism of E onto $\mathscr{R}{\downarrow}$ then $\Phi := \iota^{-1} \circ \Phi_0 \circ \iota$ is an order unbounded band preserving operator in E.

5.2. Comments on Part 2

We see that the claim of Theorem WP reduces to simple properties of reals and cardinals. However, even the reader who mastered the technique (of ascending and descending) of Boolean valued analysis might find the above demonstration bulky as compared with the standard proof in the articles by Abramovich, Veksler, and Koldunov [3], McPolin and Wickstead [58], and Gutman [40]. However, the aim of the exposition in Part 2 was not to simplify the available proof but rather demonstrate that the Boolean approach to the problem reveals many new interconnections. A few clarifications are now in order.

5.2.1. Since the space of $\mathbb{R}^\wedge$-linear functions in $\mathscr{R}$ admits a complete description that uses a Hamel basis (cp. 2.1.7 (2)); therefore, $\mathrm{End}_N(\mathscr{R}{\downarrow})$ may be described completely by means of a (strict) local Hamel basis. However, this approach will evoke some problems of unicity.

5.2.2. The dimension $\delta(\mathscr{R})$ of the vector space $\mathscr{R}$ over $\mathbb{R}^\wedge$ is a cardinal inside $\mathbb{V}^{(\mathbb{B})}$. The object $\delta(\mathscr{R})$ carries important information on the interconnection of the Boolean algebra $\mathbb{B}$ and the reals $\mathbb{R}$. By the properties of Boolean valued ordinals, we obtain the representation $\delta(\mathscr{R}) = \mathrm{mix}_\xi\, b_\xi \alpha_\xi^\wedge$, where (b_ξ) is a partition of unity in $\mathbb{B}$ and (α_ξ) is a family of standard cardinals. This representation is an instance of a "decomposition series" of $\mathbb{B}$ such that the principal ideals $[\mathbb{0}, b_\xi]$ are "α_ξ-homogeneous" in a sense.

5.2.3. If we replace the class of band preserving linear operators with the class of band preserving additive operators then the equivalence $\mathrm{WP}(1) \leftrightarrow \mathrm{WP}(4)$ fails to hold in Theorem WP.

Moreover, in each nonzero universally complete vector lattice there exist order unbounded band preserving additive operators. This reflects the fact that there is no Boolean valued model satisfying $\mathbb{V}^{(\mathbb{B})} \models \mathscr{R} = \mathbb{Q}^{\wedge}$.

5.2.4. The property of λ in 2.3.4 is usually referred to as *absolute definability.* Gordon [38] called a continuous function absolutely definable if it possesses an analogous property. For instance, the functions e^x, $\log x$, $\sin x$, and $\cos x$ are absolutely definable. In particular, these functions reside in every Boolean valued universe, presenting the mappings from $\mathscr{R}$ to $\mathscr{R}$ that are continuations of the corresponding functions $\exp^{\wedge}(\cdot)$, $\log^{\wedge}(\cdot)$, $\sin^{\wedge}(\cdot)$, and $\cos^{\wedge}(\cdot)$ from $\mathbb{R}^{\wedge}$ into $\mathbb{R}^{\wedge}$. Practically all functions admitting a constructive definition are absolutely definable.

5.2.5. Consider a band preserving operator $S : \mathscr{R}{\downarrow} \to \mathscr{R}{\downarrow}$ satisfying the Cauchy exponential equation: $S(x + y) = S(x)S(y)$ for all $x, y \in \mathscr{R}{\downarrow}$. If, moreover, S enjoys the condition $S(\lambda x) = S(x)^{\lambda}$ for all $0 < \lambda \in \mathbb{R}$ and $x \in \mathscr{R}{\downarrow}$; then we call S an *exponential operator.* Say that S is order bounded if S takes order bounded sets into order bounded sets. If σ is the ascent of S then σ is exponential inside $\mathbb{V}^{(\mathbb{B})}$. Therefore, in the class of functions bounded above on some nondegenerate interval we see that $\sigma = 0$ or $\sigma(x) = e^{cx}$ for all $x \in \mathscr{R}$ and some $c \in \mathscr{R}$. This implies that WP(1)–WP(7) of Theorem WP amount to the following:

WP(8) *Each band preserving exponential operator S on $\mathbb{B}(\mathbb{R}) := \mathscr{R}{\downarrow}$ is order bounded (and thus, S may be presented as $S(x) = e^{cx}$ for all $x \in \mathscr{R}{\downarrow}$ and some $c \in \mathscr{R}{\downarrow}$ or S is identically zero).*

5.2.6. An analogous situation takes place if S satisfies the *Cauchy logarithmic equation* $S(xy) = S(x) + S(y)$ for all $0 \ll x, y \in \mathscr{R}{\downarrow}$ and enjoys the condition $S(x^{\lambda}) = \lambda S(x)$ for all $\lambda \in \mathbb{R}$ and $x \gg 0$. (The record $0 \ll x$ means that $0 \leq x$ and $x^{\perp\perp} = \mathscr{R}{\downarrow}$.) We call an S of this sort a *logarithmic operator.* We may now formulate another equivalent claim as follows:

WP(9) *Every band preserving logarithmic operator S on $\{x \in \mathbb{B}(\mathbb{R}) := \mathscr{R}{\downarrow} : x \gg 0\}$ is order bounded (and, consequently, S may presented as $S(x) = c \log x$ for all $0 \ll x \in \mathscr{R}{\downarrow}$ with some $c \in \mathscr{R}{\downarrow}$).*

5.2.7. Instead of using continued fraction expansions in Section 2.3 we may involve binary expansions. In this event we have to construct a bijection of $\mathscr{P}(\omega)$ onto some set of reals and apply A3.9 (3) in place of A3.9 (2).

5.3. Comments on Part 3

Part 3 may be considered as an evidence of the productivity of combining algebraic and logical methods in operator theory.

5.3.1. Using the same arguments as in 3.3.5 and 3.3.6, from 3.2.6, we can infer that if $\mathbb{R}^{\wedge} \neq \mathscr{R}$ then there are nontrivial derivations on the real f-algebra $\mathscr{R}{\downarrow}$. Thus, in the class of universally complete real vector lattices with a fixed structure of an f-algebra we have WP(1) $\leftrightarrow$ WP(5); i.e., the absence of nontrivial derivations is equivalent to the σ-distributivity of the base of the algebra under consideration. At the same time there are no nontrivial band preserving automorphisms of the f-algebra $\mathscr{R}{\downarrow}$, regardless of the properties of its base.

5.3.2. It is well known that if Q is a compact space then there are no nontrivial derivations on the algebra $C(Q, \mathbb{C})$ of complex valued continuous functions on Q; for example, see [8, Chapter 19, Theorem 21]. At the same time, we see from 3.3.6 (1), (4) that if Q is an extremally disconnected compact space and the Boolean algebra of the clopen sets of Q is not σ-distributive then there is a nontrivial derivation on $C_{\infty}(Q, \mathbb{C})$.

5.3.3. Let (Ω, Σ, μ) be a measure space with the direct sum property (cp. [46, 1.1.7 and 1.1.8]). *The Boolean algebra $\mathbb{B} := \mathbb{B}(\Omega, \Sigma, \mu)$ of measurable sets modulo negligible sets is σ-distributive if and only if $\mathbb{B}$ is atomic* (*and thus isomorphic to the boolean $\mathscr{P}(A)$ of a nonempty set A*). Indeed, suppose that $\mathbb{B}$ is not atomic. By choosing a nonzero atomless coset $b_0 \in \mathbb{B}$ of finite measure, taking an instance $B_0 \in b_0$, and replacing (Ω, Σ, μ) with $(B_0, \Sigma_0, \mu|_{\Sigma_0})$, where $\Sigma_0 = \{B \cap B_0 : B \in \Sigma\}$, we may assume that μ is finite and $\mathbb{B}$ is atomless. Define a strictly positive countably additive function $\nu : \mathbb{B} \to \mathbb{R}$ by $\nu(b) = \mu(B)$ where $b \in \mathbb{B}$ is the coset of $B \in \Sigma$. Since any finite atomless measure admits halving, by induction it is easy to construct a sequence of finite partitions $P_m := \{b_1^m, b_2^m, \dots, b_{2^m}^m\}$ of $\mathbb{1} \in \mathbb{B}$ with $\mathbb{1} = b_1^1 \vee b_2^1$, $\nu(b_1^1) = \nu(b_2^1)$, and $b_j^m = b_{2j-1}^{m+1} \vee b_{2j}^{m+1}$,

$\nu(b_{2j-1}^{m+1}) = \nu(b_{2j}^{m+1})$, for all $m \in \mathbb{N}$ and $j \in \{1, 2, \dots, 2^m\}$. Since $\nu(b_j^m) \to 0$ as $m \to \infty$ for each j, there is no partition refined from $(P_m)_{m\in\mathbb{N}}$. It remains to refer to 1.3.4 (1), (3).

5.3.4. Let $L^0_{\mathbb{C}}(\Omega, \Sigma, \mu)$ be the space of all (cosets of) measurable complex valued functions, and let $L^\infty_{\mathbb{C}}(\Omega, \Sigma, \mu)$ be the space of essentially bounded measurable complex valued functions. Then the space $L^\infty_{\mathbb{C}}(\Omega, \Sigma, \mu)$ is isomorphic to some $C(Q, \mathbb{C})$; consequently, there are no nontrivial derivations on it. If the Boolean algebra $\mathbb{B}(\Omega, \Sigma, \mu)$ of measurable sets modulo sets of measure zero is not atomic (and therefore is not σ-distributive, cp. 5.3.3); then, by 3.3.6 (4), there exist nontrivial derivations on $L^0_{\mathbb{C}}(\Omega, \Sigma, \mu)$ (cp. [14, 48, 50]). The same is true about the spaces $L^\infty(\Omega, \Sigma, \mu)$ and $L^0(\Omega, \Sigma, \mu)$ of real valued measurable functions. Moreover:

5.3.5. A derivation (an automorphism) S on G is *essentially nontrivial* provided that $\pi S = 0$ ($\pi S = \pi I_G$) implies $\pi = 0$ for every band projection $\pi \in \mathfrak{P}(G)$. If (Ω, Σ, μ) is an atomless measure space with the direct sum property then (cp. [48])

(1) There is an essentially nontrivial derivation on $L^0_{\mathbb{R}}(\Omega, \Sigma, \mu)$;
(2) There is an essentially nontrivial derivation on $L^0_{\mathbb{C}}(\Omega, \Sigma, \mu)$;
(3) The identity operator is the unique automorphism of $L^0_{\mathbb{R}}(\Omega, \Sigma, \mu)$;
(4) There is an essentially nontrivial band preserving automorphism of $L^0_{\mathbb{C}}(\Omega, \Sigma, \mu)$.

Also there exists an essentially nontrivial separately band preserving antisymmetric bilinear operator in $L^0_{\mathbb{R}}(\Omega, \Sigma, \mu)$, cp. [51].

5.3.6. Two arbitrary transcendence bases for a field over a subfield have the same cardinality called the *transcendence degree* (cp. [73, Chapter II, Theorem 25]). Let $\tau(\mathscr{C})$ be the transcendence degree of $\mathscr{C}$ over $\mathbb{C}^\wedge$ inside $\mathbb{V}^{(\mathbb{B})}$. The Boolean valued cardinal $\tau(\mathscr{C})$ carries some information on the connection between the Boolean algebra $\mathbb{B}$ and the complexes $\mathscr{C}$. Each Boolean valued cardinal is a mixing of standard cardinals; i.e., the representation $\tau(\mathscr{C}) = \mathrm{mix}_\xi\, b_\xi \alpha_\xi^\wedge$ holds, where (b_ξ) is a partition of unity of $\mathbb{B}$ and (α_ξ) is some family of cardinals (cp. A36 (3) and A3.8 (1)). Moreover, for $\mathbb{B}_\xi := [\mathbb{0}, b_\xi]$ we have $\mathbb{V}^{(\mathbb{B}_\xi)} \models \tau(\mathscr{C}) = \alpha_\xi^\wedge$. In this connection, it would be interesting to characterize the complete Boolean algebras $\mathbb{B}$ such that $\tau(\mathscr{C}) = \alpha^\wedge$ inside $\mathbb{V}^{(\mathbb{B})}$ for some cardinal α.

5.3.7. Given $\mathscr{E} \subset G$, denote by $\langle\mathscr{E}\rangle$ the set of elements of the form $e_1^{n_1} \cdot \ldots \cdot e_k^{n_k}$, where $e_1, \dots, e_k \in \mathscr{E}$ and $k, n_1, \dots, n_k \in \mathbb{N}$. A set $\mathscr{E} \subset G$ is *locally algebraically independent* provided that $\langle\mathscr{E}\rangle$ is locally linearly independent in the sense of 2.2.2. This notion, presenting the external interpretation of the internal notion of algebraic independence (or transcendence), seems to turn out useful in studying the descents of fields [54, Section 8.3] or general regular rings [34].

5.3.8. Consider a band preserving operator $S : \mathscr{C}{\downarrow} \to \mathscr{C}{\downarrow}$ satisfying the Cauchy functional equation $S(u + v) = S(u)S(v)$ for all $u, v \in \mathscr{C}{\downarrow}$. If, in addition, S satisfies the condition $S(\lambda u) = S(u)^\lambda$ for arbitrary $\lambda \in \mathbb{C}$ and $u \in \mathscr{C}{\downarrow}$ then we say that S is *exponential*. Say that S is order bounded if S takes order bounded sets into order bounded sets. If σ is the ascent of S then σ is exponential inside $\mathbb{V}^{(\mathbb{B})}$; therefore, in the class of functions bounded from above on a nonzero interval, we have either $\sigma = 0$ or $\sigma(x) = e^{cx}$ $(x \in \mathscr{C})$ for some $c \in \mathscr{C}$ [8, Chapter 5, Theorem 5]. Hence, we conclude that conditions WP(1)–WP(7) of Theorem WP are also equivalent to the following: *every band preserving exponential operator in* $\mathbb{B}(\mathbb{C}) := \mathscr{C}{\downarrow}$ *is order bounded (and consequently has the form* $S = 0$ *or* $S(x) = e^{cx}$, $x \in \mathscr{C}{\downarrow}$, *for some* $c \in \mathscr{C}{\downarrow}$*).*

Appendix. Boolean Valued Analysis

A1. Boolean Valued Universes

We start with recalling some auxiliary facts about the construction and treatment of Boolean valued models.

A1.1. Let $\mathbb{B}$ be a complete Boolean algebra. Given an ordinal α, put

$$\mathbb{V}_\alpha^{(\mathbb{B})} := \big\{ x : x \text{ is a function } \wedge\ (\exists\, \beta)\big(\beta < \alpha\ \wedge\ \mathrm{dom}(x) \subset \mathbb{V}_\beta^{(\mathbb{B})}\ \wedge\ \mathrm{im}(x) \subset \mathbb{B}\big)\big\}.$$

After this recursive definition the *Boolean valued universe* $\mathbb{V}^{(\mathbb{B})}$ or, in other words, the *class of* $\mathbb{B}$ *valued sets* is introduced by

$$\mathbb{V}^{(\mathbb{B})} := \bigcup_{\alpha\in\mathrm{On}} \mathbb{V}_\alpha^{(\mathbb{B})},$$

with On standing for the class of all ordinals.

In case of the two element Boolean algebra $2 := \{\mathbb{0}, \mathbb{1}\}$ this procedure yields a version of the classical *von Neumann universe* $\mathbb{V}$ (cp. [54, Theorem 4.2.8]).

Let φ be an arbitrary formula of ZFC, Zermelo–Fraenkel set theory with choice. The *Boolean truth value* $[\![\varphi]\!] \in \mathbb{B}$ is introduced by induction on the complexity of φ by naturally interpreting the propositional connectives and quantifiers in the Boolean algebra $\mathbb{B}$ (for instance, $[\![\varphi_1 \vee \varphi_2]\!] := [\![\varphi_1]\!] \vee [\![\varphi_2]\!]$) and taking into consideration the way in which a formula is built up from atomic formulas. The Boolean truth values of the *atomic formulas* $x \in y$ and $x = y$ (with x, y assumed to be elements of $\mathbb{V}^{(\mathbb{B})}$) are defined by means of the following recursion schema:

$$[\![x \in y]\!] = \bigvee_{t \in \operatorname{dom}(y)} \big(y(t) \wedge [\![t = x]\!]\big),$$

$$[\![x = y]\!] = \bigvee_{t \in \operatorname{dom}(x)} \big(x(t) \Rightarrow [\![t \in y]\!]\big) \wedge \bigvee_{t \in \operatorname{dom}(y)} \big(y(t) \Rightarrow [\![t \in x]\!]\big).$$

The sign $\Rightarrow$ symbolizes the implication in $\mathbb{B}$; i.e., $(a \Rightarrow b) := (a^* \vee b)$, where a^* is as usual the *complement* of a. The universe $\mathbb{V}^{(\mathbb{B})}$ with the Boolean truth value of a formula is a model of set theory in the sense that the following statement is fulfilled:

A1.2. Transfer Principle. *For every theorem φ of ZFC, we have $[\![\varphi]\!] = \mathbb{1}$ (also in ZFC); i.e., φ is true inside the Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$.*

We enter into the next agreement: If $\varphi(x)$ is a formula of ZFC then, on assuming x to be an element of $\mathbb{V}^{(\mathbb{B})}$, the phrase "$x$ satisfies φ inside $\mathbb{V}^{(\mathbb{B})}$" or, briefly, "$\varphi(x)$ is true inside $\mathbb{V}^{(\mathbb{B})}$" means that $[\![\varphi(x)]\!] = \mathbb{1}$. This is sometimes written as $\mathbb{V}^{(\mathbb{B})} \models \varphi(x)$.

Given $x \in \mathbb{V}^{(\mathbb{B})}$ and $b \in \mathbb{B}$, define the function $bx : z \mapsto b \wedge x(z)$ $\big(z \in \operatorname{dom}(x)\big)$. Here we presume that $b\varnothing := \varnothing$ for all $b \in \mathbb{B}$.

There is a natural equivalence relation $x \sim y \leftrightarrow [\![x = y]\!] = \mathbb{1}$ in the class $\mathbb{V}^{(\mathbb{B})}$. Choosing a representative of the smallest rank in each equivalence class or, more exactly, using the so-called "Frege–Russell–Scott trick," we obtain a *separated Boolean valued universe* $\overline{\mathbb{V}}^{(\mathbb{B})}$ for which $x = y \leftrightarrow [\![x = y]\!] = \mathbb{1}$.

The Boolean truth value of a formula φ remains unaltered if we replace in φ each element of $\mathbb{V}^{(\mathbb{B})}$ by one of its equivalents. In this connection from now on we take $\mathbb{V}^{(\mathbb{B})} := \overline{\mathbb{V}}^{(\mathbb{B})}$ without further specification.

Observe that in $\overline{\mathbb{V}}^{(\mathbb{B})}$ the element bx is defined correctly for $x \in \overline{\mathbb{V}}^{(\mathbb{B})}$ and $b \in \mathbb{B}$, since $[\![x_1 = x_2]\!] = \mathbb{1}$ implies $[\![bx_1 = bx_2]\!] = \mathbb{1}$.

A1.3. Mixing Principle. *Let $(b_\xi)_{\xi \in \Xi}$ be a partition of unity in $\mathbb{B}$, i.e., $\sup_{\xi \in \Xi} b_\xi = \mathbb{1}$ and $\xi \neq \eta \to b_\xi \wedge b_\eta = \mathbb{0}$. To each family $(x_\xi)_{\xi \in \Xi}$ in $\mathbb{V}^{(\mathbb{B})}$ there exists a unique element x in the separated universe such that $[\![x = x_\xi]\!] \geq b_\xi$ for all $\xi \in \Xi$.*

This element x is called the *mixing* of $(x_\xi)_{\xi \in \Xi}$ by $(b_\xi)_{\xi \in \Xi}$ and is denoted by $\operatorname{mix}_{\xi \in \Xi} b_\xi x_\xi$.

A1.4. Maximum Principle. *Let $\varphi(x)$ be a formula of ZFC. Then (in ZFC) there is a $\mathbb{B}$ valued set x_0 satisfying $[\![(\exists\, x)\varphi(x)]\!] = [\![\varphi(x_0)]\!]$.*

A2. Escher Rules

Boolean valued analysis consists primarily in comparison of the instances of a mathematical object or idea in two Boolean valued models. This is impossible to achieve without some dialog between the universes $\mathbb{V}$ and $\mathbb{V}^{(\mathbb{B})}$. In other words, we need a smooth mathematical toolkit for revealing interplay between the interpretations of one and the same fact in the two models $\mathbb{V}$ and $\mathbb{V}^{(\mathbb{B})}$. The relevant *ascending-and-descending technique* rests on the functors of canonical embedding, descent, and ascent.

A2.1. We start with the canonical embedding of the von Neumann universe $\mathbb{V}$.

Given $x \in \mathbb{V}$, we denote by $x^\wedge$ the *standard name* of x in $\mathbb{V}^{(\mathbb{B})}$; i.e., the element defined by the following recursion schema: $\varnothing^\wedge := \varnothing$, $\operatorname{dom}(x^\wedge) := \{y^\wedge : y \in x\}$, $\operatorname{im}(x^\wedge) := \{\mathbb{1}\}$. Observe some properties of the mapping $x \mapsto x^\wedge$ we need in the sequel.

(1) For an arbitrary formula $\varphi(y)$ of ZFC we have (in ZFC) for each $x \in \mathbb{V}$

$$[\![(\exists\, y \in x^\wedge)\, \varphi(y)]\!] = \bigvee_{z\in x} [\![\varphi(z^\wedge)]\!],$$
$$[\![(\forall\, y \in x^\wedge)\, \varphi(y)]\!] = \bigwedge_{z\in x} [\![\varphi(z^\wedge)]\!].$$

(2) If $x, y \in \mathbb{V}$ then, by transfinite induction, we establish $x \in y \leftrightarrow \mathbb{V}^{(\mathbb{B})} \models x^\wedge \in y^\wedge$, $x = y \leftrightarrow \mathbb{V}^{(\mathbb{B})} \models x^\wedge = y^\wedge$. In other words, the standard name can be considered as an embedding of $\mathbb{V}$ into $\mathbb{V}^{(\mathbb{B})}$. Moreover, it is beyond a doubt that the standard name sends $\mathbb{V}$ onto $\mathbb{V}^{(2)}$, which fact is demonstrated by the next proposition:

(3) The following holds: $(\forall\, u \in \mathbb{V}^{(2)})\,(\exists!\, x \in \mathbb{V})\ \mathbb{V}^{(\mathbb{B})} \models u = x^\wedge$.

A formula is *bounded* or *restricted* provided that each bound variable in it is restricted by a bounded quantifier; i.e., a quantifier ranging over a particular set. The latter means that each bound variable x is restricted by a quantifier of the form $(\forall\, x \in y)$ or $(\exists\, x \in y)$.

A2.2. Restricted Transfer Principle. *Let $\varphi(x_1, \dots, x_n)$ be a bounded formula of ZFC. Then (in ZFC) for every collection $x_1, \dots, x_n \in \mathbb{V}$ we have $\varphi(x_1, \dots, x_n) \leftrightarrow \mathbb{V}^{(\mathbb{B})} \models \varphi(x_1^\wedge, \dots, x_n^\wedge)$.*

Henceforth, working in the separated universe $\overline{\mathbb{V}}^{(\mathbb{B})}$, we agree to preserve the symbol $x^\wedge$ for the distinguished element of the class corresponding to x.

Observe for example that the restricted transfer principle yields:

$$\text{“}\Phi \text{ is a correspondence from } x \text{ into } y\text{”} \ \leftrightarrow$$
$$\mathbb{V}^{(\mathbb{B})} \models \text{“}\Phi^\wedge \text{ is a correspondence from } x^\wedge \text{ into } y^\wedge\text{”};$$
$$\text{“}f : x \to y\text{”} \ \leftrightarrow\ \mathbb{V}^{(\mathbb{B})} \models \text{“}f^\wedge : x^\wedge \to y^\wedge\text{”}$$

$\big($moreover, $f(a)^\wedge = f^\wedge(a^\wedge)$ for all $a \in x\big)$. Thus, the standard name can be considered as a covariant functor from the category of sets (or correspondences) inside $\mathbb{V}$ to an appropriate subcategory of $\mathbb{V}^{(2)}$ in the separated universe $\mathbb{V}^{(\mathbb{B})}$.

A2.3. A set X is *finite* provided that X coincides with the image of a function on a finite ordinal. In symbols, this is expressed as $\mathrm{fin}(X)$; hence,

$$\mathrm{fin}(X) := (\exists\, n)(\exists\, f)\big(n \in \omega\ \wedge\ f \text{ is a function } \wedge\ \mathrm{dom}(f) = n\ \wedge\ \mathrm{im}(f) = X\big)$$

(as usual $\omega := \{0, 1, 2, \dots\}$). Obviously, the above formula is not bounded. Nevertheless there is a simple transformation rule for the class of finite sets under the canonical embedding. Denote by $\mathscr{P}_{\mathrm{fin}}(X)$ the class of all finite subsets of X; i.e., $\mathscr{P}_{\mathrm{fin}}(X) := \{Y \in \mathscr{P}(X) : \mathrm{fin}(Y)\}$. For an arbitrary set X the following holds: $\mathbb{V}^{(\mathbb{B})} \models \mathscr{P}_{\mathrm{fin}}(X)^\wedge = \mathscr{P}_{\mathrm{fin}}(X^\wedge)$.

A2.4. Given an arbitrary element x of the (separated) Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$, we define the *descent* $x{\downarrow}$ of x as $x{\downarrow} := \{y \in \mathbb{V}^{(\mathbb{B})} : [\![y \in x]\!] = \mathbb{1}\}$. We list the simplest properties of descending:

(1) The class $x{\downarrow}$ is a set, i.e., $x{\downarrow} \in \mathbb{V}$ for all $x \in \mathbb{V}^{(\mathbb{B})}$. If $[\![x \neq \varnothing]\!] = \mathbb{1}$ then $x{\downarrow}$ is nonempty.

(2) Let $\varphi(x)$ be a formula of ZFC. Then (in ZFC) for every $z \in \mathbb{V}^{(\mathbb{B})}$ such that $[\![z \neq \varnothing]\!] = \mathbb{1}$ we have

$$[\![(\forall\, x \in z)\, \varphi(x)]\!] = \bigwedge_{x\in z{\downarrow}} [\![\varphi(x)]\!],$$
$$[\![(\exists\, x \in z)\, \varphi(x)]\!] = \bigvee_{x\in z{\downarrow}} [\![\varphi(x)]\!].$$

Moreover, there exists $x_0 \in z{\downarrow}$ such that $[\![(\exists\, x \in z)\, \varphi(x)]\!] = [\![\varphi(x_0)]\!]$.

(3) Let Φ be a correspondence from X into Y in $\mathbb{V}^{(\mathbb{B})}$. Thus, Φ, X, and Y are elements of $\mathbb{V}^{(\mathbb{B})}$ and, moreover, $[\![\Phi \subset X \times Y]\!] = \mathbb{1}$. There is a unique correspondence $\Phi{\downarrow}$ from $X{\downarrow}$ into $Y{\downarrow}$ such that $\Phi{\downarrow}(A{\downarrow}) = \Phi(A){\downarrow}$ for every nonempty subset A of X inside $\mathbb{V}^{(\mathbb{B})}$. The correspondence $\Phi{\downarrow}$ is called the *descent* of Φ.

(4) The descent of the composite of correspondences inside $\mathbb{V}^{(\mathbb{B})}$ is the composite of their descents: $(\Psi \circ \Phi){\downarrow} = \Psi{\downarrow} \circ \Phi{\downarrow}$.

(5) If Φ is a correspondence inside $\mathbb{V}^{(\mathbb{B})}$ then $(\Phi^{-1}){\downarrow} = (\Phi{\downarrow})^{-1}$.

(6) Let Id_X be the identity mapping inside $\mathbb{V}^{(\mathbb{B})}$ of a set $X \in \mathbb{V}^{(\mathbb{B})}$. Then $(\mathrm{Id}_X){\downarrow} = \mathrm{Id}_{X{\downarrow}}$.

(7) Suppose that $X, Y, f \in \mathbb{V}^{(\mathbb{B})}$ are such that $[\![f : X \to Y]\!] = \mathbb{1}$, i.e., f is a mapping from X into Y inside $\mathbb{V}^{(\mathbb{B})}$. Then $f{\downarrow}$ is a unique mapping from $X{\downarrow}$ into $Y{\downarrow}$ satisfying $[\![f{\downarrow}(x) = f(x)]\!] = \mathbb{1}$ for all $x \in X{\downarrow}$. The descent of a mapping is *extensional*: $[\![x_1 = x_2]\!] \le [\![f{\downarrow}(x_1) = f{\downarrow}(x_2)]\!]$ for all $x_1, x_2 \in X{\downarrow}$ (cp. A2.5 (4)).

By virtue of (1)–(7), we can consider the descent operation as a functor from the category of $\mathbb{B}$ valued sets and mappings (correspondences) to the category of the standard sets and mappings (correspondences) (i.e., those in the sense of $\mathbb{V}$).

(8) Given $x_1, \dots, x_n \in \mathbb{V}^{(\mathbb{B})}$, denote by $(x_1, \dots, x_n)^{\mathbb{B}}$ the corresponding ordered n-tuple inside $\mathbb{V}^{(\mathbb{B})}$. Assume that P is an n-ary relation on X inside $\mathbb{V}^{(\mathbb{B})}$; i.e., $X, P \in \mathbb{V}^{(\mathbb{B})}$ and $[\![P \subset X^n]\!] = \mathbb{1}$. Then there exists an n-ary relation P' on $X{\downarrow}$ such that $(x_1, \dots, x_n) \in P' \leftrightarrow [\![(x_1, \dots, x_n)^{\mathbb{B}} \in P]\!] = \mathbb{1}$. Slightly abusing notation, we denote P' by the occupied symbol $P{\downarrow}$ and call $P{\downarrow}$ the *descent* of P.

A2.5. Let $x \in \mathbb{V}$ and $x \subset \mathbb{V}^{(\mathbb{B})}$; i.e., let x be some set composed of $\mathbb{B}$ valued sets or, in other words, $x \in \mathscr{P}(\mathbb{V}^{(\mathbb{B})})$. Put $\varnothing{\uparrow} := \varnothing$ and $\mathrm{dom}(x{\uparrow}) := x$, $\mathrm{im}(x{\uparrow}) := \{\mathbb{1}\}$ if $x \ne \varnothing$. The element $x{\uparrow}$ (of the separated universe $\mathbb{V}^{(\mathbb{B})}$, i.e., the distinguished representative of the class $\{y \in \mathbb{V}^{(\mathbb{B})} : [\![y = x{\uparrow}]\!] = \mathbb{1}\}$) is the *ascent* of x.

(1) Let $\varphi(y)$ be a formula of ZFC. Then (in ZFC) for all $x \in \mathscr{P}(\mathbb{V}^{(\mathbb{B})})$ we have

$$[\![(\forall\, y \in x{\uparrow})\, \varphi(y)]\!] = \bigwedge_{y \in x} [\![\varphi(y)]\!],$$

$$[\![(\exists\, y \in x{\uparrow})\, \varphi(y)]\!] = \bigvee_{y \in x} [\![\varphi(y)]\!].$$

Introducing the ascent of a correspondence $\Phi \subset X \times Y$, we have to bear in mind a possible distinction between the domain of departure, X, and the domain, $\mathrm{dom}(\Phi) := \{x \in X : \Phi(x) \ne \varnothing\}$. This circumstance is immaterial for the sequel; therefore, speaking of ascents, we always imply total correspondences; i.e., $\mathrm{dom}(\Phi) = X$.

(2) Let $X, Y, \Phi \in \mathbb{V}^{(\mathbb{B})}$ and let Φ be a correspondence from X into Y. There exists a (unique) correspondence $\Phi{\uparrow}$ from $X{\uparrow}$ into $Y{\uparrow}$ inside $\mathbb{V}^{(\mathbb{B})}$, such that $\Phi{\uparrow}(A{\uparrow}) = \Phi(A){\uparrow}$ is valid for every subset A of $\mathrm{dom}(\Phi)$, if and only if Φ is *extensional*; i.e., satisfies the condition $y_1 \in \Phi(x_1) \to [\![x_1 = x_2]\!] \le \bigvee_{y_2 \in \Phi(x_2)} [\![y_1 = y_2]\!]$ for $x_1, x_2 \in \mathrm{dom}(\Phi)$. In this event, $\Phi{\uparrow} = \Phi'{\uparrow}$, where $\Phi' := \{(x, y)^{\mathbb{B}} : (x, y) \in \Phi\}$. The element $\Phi{\uparrow}$ is the *ascent* of the initial correspondence Φ.

(3) The composite of extensional correspondences is extensional. Moreover, the ascent of a composite is equal to the composite of the ascents inside $\mathbb{V}^{(\mathbb{B})}$: On assuming that $\mathrm{dom}(\Psi) \supset \mathrm{im}(\Phi)$ we have $\mathbb{V}^{(\mathbb{B})} \models (\Psi \circ \Phi){\uparrow} = \Psi{\uparrow} \circ \Phi{\uparrow}$.

Note that if Φ and Φ^{-1} are extensional then $(\Phi{\uparrow})^{-1} = (\Phi^{-1}){\uparrow}$. However, in general, the extensionality of Φ in no way guarantees the extensionality of Φ^{-1}.

(4) It is worth mentioning that if an extensional correspondence f is a function from X into Y then the ascent $f{\uparrow}$ of f is a function from $X{\uparrow}$ into $Y{\uparrow}$. Moreover, the extensionality property can be stated as follows: $[\![x_1 = x_2]\!] \le [\![f(x_1) = f(x_2)]\!]$ for all $x_1, x_2 \in X$.

A2.6. Given a set $X \subset \mathbb{V}^{(\mathbb{B})}$, we denote by $\mathrm{mix}(X)$ the set of all mixings of the form $\mathrm{mix}_\xi\, b_\xi x_\xi$, where $(x_\xi) \subset X$ and (b_ξ) is an arbitrary partition of unity. The following propositions are referred to as the *arrow cancellation rules* or *ascending-and-descending rules*. There are many good reasons to call them simply the *Escher rules*.

(1) Let X and X' be subsets of $\mathbb{V}^{(\mathbb{B})}$ and let $f : X \to X'$ be an extensional mapping. Suppose also that $Y, Y', g \in \mathbb{V}^{(\mathbb{B})}$ are such that $[\![\, Y \ne \varnothing]\!] = [\![\, g : Y \to Y']\!] = \mathbb{1}$. Then $X{\uparrow}{\downarrow} = \mathrm{mix}(X)$, $Y{\downarrow}{\uparrow} = Y$, $f{\uparrow}{\downarrow} = f$ on X, and $g{\downarrow}{\uparrow} = g$.

(2) If X is a subset of $\mathbb{V}^{(\mathbb{B})}$ then $\mathbb{V}^{(\mathbb{B})} \models \mathscr{P}_{\mathrm{fin}}(X{\uparrow}) = \{\theta{\uparrow} : \theta \in \mathscr{P}_{\mathrm{fin}}(X)\}{\uparrow}$.

Suppose that $X \in \mathbb{V}$, $X \ne \varnothing$; i.e., X is a nonempty set. Let the letter ι denote the standard name embedding $x \mapsto x^{\wedge}$ $(x \in X)$. Then $\iota(X){\uparrow} = X^{\wedge}$ and $X = \iota^{-1}(X^{\wedge}{\downarrow})$. Using the above relations, we may extend the descent and ascent operations to the case in which Φ is a correspondence from X into $Y{\downarrow}$ and $[\![\, \Psi$ is a correspondence from $X^{\wedge}$ into $Y\,]\!] = \mathbb{1}$, where $Y \in \mathbb{V}^{(\mathbb{B})}$. Namely, we put $\Phi{\upharpoonleft} := (\Phi \circ \iota^{-1}){\uparrow}$ and $\Psi{\downharpoonleft} := \Psi{\downarrow} \circ \iota$. In this case, $\Phi{\upharpoonleft}$ is the *modified ascent* of Φ and $\Psi{\downharpoonleft}$ is the *modified descent* of Ψ. (If the context excludes ambiguity then we briefly speak of ascents and descents using simple arrows.) It is easy to see that $\Phi\Uparrow$ is a unique correspondence inside $\mathbb{V}^{(\mathbb{B})}$ satisfying the relation $[\![\Phi\Uparrow(x^\wedge) = \Phi(x)\uparrow]\!] = \mathbb{1}$ $(x \in X)$. Similarly, $\Psi\Downarrow$ is a unique correspondence from X into $Y\downarrow$ satisfying the equality $\Psi\Downarrow(x) = \Psi(x^\wedge)\downarrow$ $(x \in X)$. If $\Phi := f$ and $\Psi := g$ are functions then these relations take the form $[\![f\Uparrow(x^\wedge) = f(x)]\!] = \mathbb{1}$ and $g\Downarrow(x) = g(x^\wedge)$ for all $x \in X$.

A2.7. Various function spaces reside in functional analysis, and so the problem is natural of replacing an abstract Boolean valued system by some function-space analog, a model whose elements are functions and in which the basic logical operations are calculated "pointwise." An example of such a model is given by the class $\mathbb{V}^Q$ of all functions defined on a fixed nonempty set Q and acting into $\mathbb{V}$. The truth values on $\mathbb{V}^Q$ are various subsets of Q: The truth value $[\![\varphi(x_1,\dots,x_n)]\!]$ of a formula $\varphi(x_1,\dots,x_n)$ (at functions $x_1,\dots,x_n \in \mathbb{V}^Q$) is calculated as follows:

$$[\![\varphi(x_1,\dots,x_n)]\!] = \big\{q \in Q : \varphi\big(x_1(q),\dots,x_n(q)\big)\big\}.$$

Gutman and Losenkov solved the above problem by the concept of continuous polyverse which is a continuous bundle of models of set theory. It is shown that the class of continuous sections of a continuous polyverse is a Boolean valued system satisfying all basic principles of Boolean valued analysis and, conversely, each Boolean valued algebraic system can be represented as the class of sections of a suitable continuous polyverse. More details reside in [54, Chapter 6].

A2.8. Every Boolean valued universe has the collection of mathematical objects in full supply: available in plenty are all sets with extra structure (groups, rings, algebras, normed spaces, etc.). Applying the descent functor to such *internal* algebraic systems of a Boolean valued model, we distinguish some bizarre entities or recognize old acquaintances, which leads to revealing the new facts of their life and structure.

This technique of research, known as *direct Boolean valued interpretation*, allows us to produce new theorems or, to be more exact, to extend the semantical content of the available theorems by means of slavish translation. The information we so acquire might fail to be vital, valuable, or intriguing, in which case the direct Boolean valued interpretation turns out into a leisurely game.

It thus stands to reason to raise the following questions: What structures significant for mathematical practice are obtainable by the Boolean valued interpretation of the most typical algebraic systems? What transfer principles hold true in this process? Clearly, the answers should imply specific objects whose particular features enable us to deal with their Boolean valued representation which, if understood duly, is impossible to implement for arbitrary algebraic systems.

An *abstract* $\mathbb{B}$*-set* or *set with* $\mathbb{B}$*-structure* is a pair (X, d), where $X \in \mathbb{V}$, $X \neq \varnothing$, and d is a mapping from $X \times X$ into $\mathbb{B}$ such that $d(x, y) = \mathbb{0} \leftrightarrow x = y$; $d(x, y) = d(y, x)$; $d(x, y) \leq d(x, z) \vee d(z, y)$ for all $x, y, z \in X$.

To obtain an easy example of an abstract $\mathbb{B}$-set, given $\varnothing \neq X \subset \mathbb{V}^{(\mathbb{B})}$ put

$$d(x, y) := [\![x \neq y]\!] = [\![x = y]\!]^* \quad \text{for } x,\ y \in X.$$

Another easy example is a nonempty X with the *discrete* $\mathbb{B}$*-metric* d; i.e., $d(x, y) = \mathbb{1}$ if $x \neq y$ and $d(x, y) = \mathbb{0}$ if $x = y$.

Let (X, d) be some abstract $\mathbb{B}$-set. There exist an element $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ and an injection $\iota : X \to X' := \mathscr{X}\downarrow$ such that $d(x, y) = [\![\iota x \neq \iota y]\!]$ for all $x, y \in X$ and each $x' \in X'$ admits the representation $x' = \operatorname{mix}_{\xi\in\Xi} b_\xi \iota x_\xi$, where $(x_\xi)_{\xi\in\Xi} \subset X$ and $(b_\xi)_{\xi\in\Xi}$ is a partition of unity in $\mathbb{B}$. We see that an abstract $\mathbb{B}$-set X embeds in the Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ so that the Boolean distance between the members of X becomes the Boolean truth value of the negation of their equality. The corresponding element $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ is, by definition, the *Boolean valued representation* of X.

If X is a discrete abstract $\mathbb{B}$-set then $\mathscr{X} = X^\wedge$ and $\iota x = x^\wedge$ for all $x \in X$. If $X \subset \mathbb{V}^{(\mathbb{B})}$ then $\iota\uparrow$ is an injection of $X\uparrow$ into $\mathscr{X}$ (inside $\mathbb{V}^{(\mathbb{B})}$). A mapping f from a $\mathbb{B}$-set (X, d) into a $\mathbb{B}$-set (X', d') is said to be *contractive* if $d(x, y) \geq d'\big(f(x), f(y)\big)$ for all $x, y \in X$.

In case a $\mathbb{B}$-set X has some a priori structure we may try to furnish the Boolean valued representation of X with an analogous structure, so as to apply the technique of ascending and descending to the study of the original structure of X. Consequently, the above questions may be treated as instances of the unique problem of searching a well-qualified Boolean valued representation of a $\mathbb{B}$-set with some additional structure.

We call these objects *algebraic* $\mathbb{B}$*-systems*. Located at the epicenter of Boolean valued analysis, the notion of an algebraic $\mathbb{B}$-system refers to a nonempty $\mathbb{B}$-set endowed with a few contractive operations and $\mathbb{B}$-predicates, the latter meaning $\mathbb{B}$ valued contractive mappings.

The Boolean valued representation of an algebraic $\mathbb{B}$-system appears to be a standard two valued algebraic system of the same type. This means that an appropriate completion of each algebraic $\mathbb{B}$-system coincides with the descent of some two valued algebraic system inside $\mathbb{V}^{(\mathbb{B})}$.

On the other hand, each two valued algebraic system may be transformed into an algebraic $\mathbb{B}$-system on distinguishing a complete Boolean algebra of congruences of the original system. In this event, the task is in order of finding the formulas holding true in direct or reverse transition from a $\mathbb{B}$-system to a two valued system. In other words, we have to seek and reveal here some versions of transfer in the form of identity preservation, a principle of long standing in vector lattice theory.

A3. Boolean Valued Numbers, Ordinals, and Cardinals

Boolean valued analysis stems from the fact that each internal field of reals of a Boolean valued model descends into a universally complete vector lattice. Thus, a remarkable opportunity opens up to expand and enrich the treasure-trove of mathematical knowledge by translating information about the reals to the language of other noble families of functional analysis. We will elaborate upon the matter in this section.

A3.1. Recall a few definitions. Two elements x and y of a vector lattice E are *disjoint* (in symbols $x \perp y$) provided that $|x| \wedge |y| = 0$. A *band* of E is defined as the *disjoint complement* $M^{\perp} := \{x \in E : (\forall\, y \in M)\, x \perp y\}$ of a nonempty set $M \subset E$.

The inclusion-ordered set $\mathfrak{B}(E)$ of all bands in E is a complete Boolean algebra with the Boolean operations:

$$L \wedge K = L \cap K, \quad L \vee K = (L \cup K)^{\perp\perp}, \quad L^* = L^{\perp} \quad \big(L, K \in \mathfrak{B}(E)\big).$$

The Boolean algebra $\mathfrak{B}(E)$ is often referred to as the *base* of E.

A *band projection* in E is a linear idempotent operator in $\pi : E \to E$ satisfying the inequalities $0 \le \pi x \le x$ for all $0 \le x \in E$. The set $\mathfrak{P}(E)$ of all band projections ordered by $\pi \le \rho \leftrightarrow \pi \circ \rho = \pi$ is a Boolean algebra with the Boolean operations:

$$\pi \wedge \rho = \pi \circ \rho, \quad \pi \vee \rho = \pi + \rho - \pi \circ \rho, \quad \pi^* = I_E - \pi \quad \big(\pi, \rho \in \mathfrak{P}(E)\big).$$

Let $u \in E_+$ and $e \wedge (u - e) = 0$ for some $0 \le e \in E$. Then e is a *fragment* or *component* of u. The set $\mathfrak{E}(u)$ of all fragments of u with the order induced by E is a Boolean algebra where the lattice operations are taken from E and the Boolean complement has the form $e^* := u - e$.

A3.2. A Dedekind complete vector lattice is also called a *Kantorovich space* or *K-space*, for short. A Dedekind complete vector lattice E is *universally complete* if every family of pairwise disjoint elements of E is order bounded.

(1) *Let E be an arbitrary K-space. Then the correspondence $\pi \mapsto \pi(E)$ determines an isomorphism of the Boolean algebras $\mathfrak{P}(E)$ and $\mathfrak{B}(E)$. If there is an order unity $\mathbb{1}$ in E then the mappings $\pi \mapsto \pi\mathbb{1}$ from $\mathfrak{P}(E)$ into $\mathfrak{E}(\mathbb{1})$ and $e \mapsto \{e\}^{\perp\perp}$ from $\mathfrak{E}(\mathbb{1})$ into $\mathfrak{B}(E)$ are isomorphisms of Boolean algebras too.*

(2) *Each universally complete vector lattice E with order unity $\mathbb{1}$ can be uniquely endowed with multiplication so as to make E into a faithful f-algebra and $\mathbb{1}$ into a ring unity. In this f-algebra each band projection $\pi \in \mathfrak{P}(E)$ is the operator of multiplication by $\pi(\mathbb{1})$.*

A3.3. By a *field of reals* we mean every algebraic system that satisfies the axioms of an Archimedean ordered field (with distinct zero and unity) and enjoys the axiom of completeness. The same object can be defined as a one-dimensional K-space.

Recall the well-known assertion of ZFC: *There exists a field of reals $\mathbb{R}$ that is unique up to isomorphism.*

Successively applying the transfer and maximum principles, we find an element $\mathscr{R} \in \mathbb{V}^{(\mathbb{B})}$ for which $[\![\, \mathscr{R}$ is a field of reals$\,]\!] = \mathbb{1}$. Moreover, if an arbitrary $\mathscr{R}\,' \in \mathbb{V}^{(\mathbb{B})}$ satisfies the condition $[\![\, \mathscr{R}\,'$ is a field of reals $]\!] = \mathbb{1}$ then $[\![$ the ordered fields $\mathscr{R}$ and $\mathscr{R}'$ are isomorphic $]\!] = \mathbb{1}$. In other words, there exists an internal field of reals $\mathscr{R} \in \mathbb{V}^{(\mathbb{B})}$ which is unique up to isomorphism.

By the same reasons there exists an internal field of complex numbers $\mathscr{C} \in \mathbb{V}^{(\mathbb{B})}$ which is unique up to isomorphism. Moreover, $\mathbb{V}^{(\mathbb{B})} \models \mathscr{C} = \mathscr{R} \oplus i\mathscr{R}$. We call $\mathscr{R}$ and $\mathscr{C}$ the *internal reals* and *internal complexes* in $\mathbb{V}^{(\mathbb{B})}$.

A3.4. Consider another well-known assertion of ZFC: *If $\mathbb{P}$ is an Archimedean ordered field then there is an isomorphic embedding h of the field $\mathbb{P}$ into $\mathbb{R}$ such that the image $h(\mathbb{P})$ is a subfield of $\mathbb{R}$ containing the subfield of rational numbers. In particular, $h(\mathbb{P})$ is dense in $\mathbb{R}$.*

Note also that $\varphi(x)$, presenting the conjunction of the axioms of an Archimedean ordered field x, is bounded; therefore, $[\![\, \varphi(\mathbb{R}^\wedge)\,]\!] = \mathbb{1}$, i.e., $[\![\, \mathbb{R}^\wedge$ is an Archimedean ordered field $]\!] = \mathbb{1}$. "Pulling" the above assertion through the transfer principle, we conclude that $[\![\, \mathbb{R}^\wedge$ is isomorphic to a dense subfield of $\mathscr{R}\,]\!] = \mathbb{1}$. We further assume that $\mathbb{R}^\wedge$ is a dense subfield of $\mathscr{R}$ and $\mathbb{C}^\wedge$ is a dense subfield of $\mathscr{C}$. It is easy to see that the elements $0^\wedge$ and $1^\wedge$ are the zero and unity of $\mathscr{R}$.

Observe that the equalities $\mathscr{R} = \mathbb{R}^\wedge$ and $\mathscr{C} = \mathbb{C}^\wedge$ are not valid in general. Indeed, the axiom of completeness for $\mathbb{R}$ is not a bounded formula and so it may fail for $\mathbb{R}^\wedge$ inside $\mathbb{V}^{(\mathbb{B})}$. (The corresponding example is given in Section 1.3 of this paper.)

A3.5. Look now at the descent $\mathscr{R}{\downarrow}$ of the algebraic system $\mathscr{R}$. In other words, consider the descent of the underlying set of the system $\mathscr{R}$ together with the descended operations and order. For simplicity, we denote the operations and order in $\mathscr{R}$ and $\mathscr{R}{\downarrow}$ by the same symbols $+$, $\cdot$, and $\le$. In more detail, we introduce addition, multiplication, and order in $\mathscr{R}{\downarrow}$ by the formulas

$$\begin{aligned} z = x + y &\ \leftrightarrow\ [\![\, z = x + y\,]\!] = \mathbb{1}, \\ z = x \cdot y &\ \leftrightarrow\ [\![\, z = x \cdot y\,]\!] = \mathbb{1}, \\ x \le y &\ \leftrightarrow\ [\![\, x \le y\,]\!] = \mathbb{1} \quad (x, y, z \in \mathscr{R}{\downarrow}). \end{aligned}$$

Also, we may introduce multiplication by the usual reals in $\mathscr{R}{\downarrow}$ by the rule

$$y = \lambda x\ \leftrightarrow\ [\![\, y = \lambda^\wedge x\,]\!] = \mathbb{1} \quad (\lambda \in \mathbb{R},\ x, y \in \mathscr{R}{\downarrow}).$$

The fundamental result of Boolean valued analysis is the Gordon Theorem which reads as follows: *Each universally complete vector lattice is an interpretation of the reals in an appropriate Boolean valued model.* Formally:

A3.6. Gordon Theorem. *Let $\mathscr{R}$ be the reals inside $\mathbb{V}^{(\mathbb{B})}$. Then $\mathscr{R}{\downarrow}$, with the descended operations and order, is a universally complete vector lattice with order unit $1^\wedge$. Moreover, there exists an isomorphism χ of $\mathbb{B}$ onto $\mathfrak{P}(\mathscr{R}{\downarrow})$ such that*

$$\chi(b)x = \chi(b)y\ \leftrightarrow\ b \le [\![\, x = y\,]\!], \quad \chi(b)x \le \chi(b)y\ \leftrightarrow\ b \le [\![\, x \le y\,]\!]$$

for all $x, y \in \mathscr{R}{\downarrow}$ and $b \in \mathbb{B}$.

The converse is also true: *Each Archimedean vector lattice embeds in a Boolean valued model, becoming a vector sublattice of the reals (viewed as such over some dense subfield of the reals).*

A3.7. Theorem. *Let E be an Archimedean vector lattice, let $\mathscr{R}$ be the reals inside $\mathbb{V}^{(\mathbb{B})}$, and let $\jmath$ be an isomorphism of $\mathbb{B}$ onto $\mathfrak{B}(E)$. Then there is $\mathscr{E} \in \mathbb{V}^{(\mathbb{B})}$ satisfying the following:*

(1) *$\mathscr{E}$ is a vector sublattice of $\mathscr{R}$ over $\mathbb{R}^\wedge$ inside $\mathbb{V}^{(\mathbb{B})}$;*

(2) *$E' := \mathscr{E}{\downarrow}$ is a vector sublattice of $\mathscr{R}{\downarrow}$ invariant under every band projection $\chi(b)$ $(b \in \mathbb{B})$ and such that each set of pairwise disjoint elements in E' has a supremum;*

(3) *There is an order continuous lattice isomorphism $\iota : E \to E'$ such that $\iota(E)$ is a coinitial sublattice of $\mathscr{R}{\downarrow}$;*

(4) *For every $b \in \mathbb{B}$ the band projection in $\mathscr{R}{\downarrow}$ onto $\{\iota(\jmath(b))\}^{\perp\perp}$ coincides with $\chi(b)$.*

Note also that $\mathscr{E}$ and $\mathscr{R}$ coincide if and only if E is Dedekind complete. Thus, each theorem about the reals within Zermelo–Fraenkel set theory has an analog in an arbitrary Dedekind complete vector lattice. Translation of theorems is carried out by appropriate general functors of Boolean valued analysis. In particular, the most important structural properties of vector lattices such as the functional representation, spectral theorem, etc. are the ghosts of some properties of the reals in an appropriate Boolean valued model.

A3.6. Let us dwell for a while on the properties of ordinals inside $\mathbb{V}^{(\mathbb{B})}$.

(1) Clearly, $\operatorname{Ord}(x)$ is a bounded formula. Since $\lim(\alpha) \le \alpha$ for every ordinal α, the formula $\operatorname{Ord}(x) \wedge x = \lim(x)$ may be rewritten as $\operatorname{Ord}(x) \wedge (\forall\, t \in x)(\exists\, s \in x)(t \in s)$. Hence, $\operatorname{Ord}(x) \wedge x = \lim(x)$ is a bounded formula as well. Finally, the record

$$\operatorname{Ord}(x) \wedge x = \lim(x) \wedge (\forall\, t \in x)(t = \lim(t) \to t = 0)$$

convinces us that the "least limit ordinal" is a bounded formula too. Hence α is the least limit ordinal if and only if $\mathbb{V}^{(\mathbb{B})} \models$ "$\alpha^{\wedge}$ is the least limit ordinal." Since ω is the least limit ordinal, $\mathbb{V}^{(B)} \models$ "$\omega^{\wedge}$ is the least limit ordinal."

(2) It can be demonstrated that $\mathbb{V}^{(\mathbb{B})} \models$ "$\mathrm{On}^{\wedge}$ is the unique ordinal class that is not an ordinal" (with $\mathrm{On}^{\wedge}$ defined in an appropriate way). Given $x \in \mathbb{V}^{(\mathbb{B})}$, we thus have

$$[\![\operatorname{Ord}(x)]\!] = \bigvee_{\alpha \in \mathrm{On}} [\![x = \alpha^{\wedge}]\!].$$

(3) *Each ordinal inside $\mathbb{V}^{(\mathbb{B})}$ is a mixing of some set of standard ordinals. In other words, given $x \in \mathbb{V}^{(\mathbb{B})}$, we have $\mathbb{V}^{(\mathbb{B})} \models \operatorname{Ord}(x)$ if and only if there are an ordinal $\beta \in \mathrm{On}$ and a partition of unity $(b_\alpha)_{\alpha\in\beta} \subset \mathbb{B}$ such that $x = \operatorname{mix}_{\alpha\in\beta} b_\alpha \alpha^{\wedge}$.*

(4) This yields the convenient formulas for quantification over ordinals:

$$[\![(\forall\, x)\big(\operatorname{Ord}(x) \to \psi(x)\big)]\!] = \bigwedge_{\alpha \in \mathrm{On}} [\![\psi(\alpha^{\wedge})]\!],$$

$$[\![(\exists\, x)\big(\operatorname{Ord}(x) \wedge \psi(x)\big)]\!] = \bigvee_{\alpha \in \mathrm{On}} [\![\psi(\alpha^{\wedge})]\!].$$

A3.7. By transfer every Boolean valued model enjoys the classical principle of cardinal comparability. In other words, there is a $\mathbb{V}^{(\mathbb{B})}$-class Cn whose elements are only cardinals. Let $\operatorname{Card}(\alpha)$ denote the formula that declares α a cardinal. Inside $\mathbb{V}^{(\mathbb{B})}$ we then see that $\alpha \in \mathrm{Cn} \leftrightarrow \operatorname{Card}(\alpha)$. Clearly, the class of ordinals $\mathrm{On}^{\wedge}$ is similar to the class of infinite cardinals, and we denote the similarity from $\mathrm{On}^{\wedge}$ into Cn by $\alpha \mapsto \aleph_\alpha$. In particular, to each standard ordinal $\alpha \in \mathrm{On}$ there is a unique infinite cardinal $\aleph_{\alpha^{\wedge}}$ inside $\mathbb{V}^{(\mathbb{B})}$. Indeed, $[\![\operatorname{Ord}(\alpha^{\wedge})]\!] = \mathbb{1}$.

Recall that it is customary to refer to the standard names of ordinals and cardinals as *standard ordinals* and *standard cardinals* inside $\mathbb{V}^{(\mathbb{B})}$.

(1) *The standard name of the least infinite cardinal is the least infinite cardinal:*

$$\mathbb{V}^{(\mathbb{B})} \models (\omega_0)^{\wedge} = \aleph_0.$$

Inside $\mathbb{V}^{(\mathbb{B})}$ there is a mapping $|\cdot|$ from the universal class $\mathbb{U}_{\mathbb{B}}$ into the class Cn such that x and $|x|$ are equipollent for all x.

(2) *The standard names of equipollent sets are of the same cardinality:*

$$(\forall\, x \in \mathbb{V})\,(\forall\, y \in \mathbb{V})\,\big(|x| = |y| \to [\![|x^{\wedge}| = |y^{\wedge}|]\!] = \mathbb{1}\big).$$

A3.8. (1) *If the standard name of an ordinal α is a cardinal then α is a cardinal too:*

$$(\forall\, \alpha \in \mathrm{On})\ \big(\mathbb{V}^{(\mathbb{B})} \models \operatorname{Card}(\alpha^{\wedge})\big) \to \operatorname{Card}(\alpha).$$

(2) *The standard name of a finite cardinal is a finite cardinal too:*

$$(\forall\, \alpha \in \mathrm{On})\ \big(\alpha < \omega \to \mathbb{V}^{(\mathbb{B})} \models \operatorname{Card}(\alpha^{\wedge}) \wedge \alpha^{\wedge} \in \aleph_0\big).$$

A3.9. *Given $x \in \mathbb{V}^{(\mathbb{B})}$, we have $\mathbb{V}^{(\mathbb{B})} \models \operatorname{Card}(x)$ if and only if there are nonempty set of cardinals Γ and a partition of unity $(b_\alpha)_{\alpha\in\Gamma} \subset \mathbb{B}$ such that $\mathbb{V}^{(\mathbb{B})} \models \operatorname{Card}(\gamma^{\wedge})$ for all $\gamma \in \Gamma$ and $x = \operatorname{mix}_{\gamma\in\Gamma} b_\gamma \gamma^{\wedge}$. In other words, each Boolean valued cardinal is a mixing of some set of standard cardinals.*

A3.10. It is worth noting that σ-distributive Boolean algebras are often referred to as (ω, ω)-distributive Boolean algebras. This term is related to a more general notion, (α, β)-distributivity, where α and β are arbitrary cardinals.

If $\mathbb{B}$ *is a complete Boolean algebras then the following are equivalent:*

(1) $\mathbb{B}$ *is* σ*-distributive;*

(2) $\mathbb{V}^{(\mathbb{B})} \models (\aleph_0)^{\aleph_0} = (\omega^\omega)^\wedge$;

(3) $\mathbb{V}^{(\mathbb{B})} \models \mathscr{P}(\aleph_0) = \mathscr{P}(\omega)^\wedge$.

The latter is a result by Scott on (α, β)-distributive Boolean algebras which was formulated in the case $\alpha = \beta = \omega$ (cp. [13, 2.14]).

More details and references are collected in [54]. The monographs [13] and [44] are also a very good source of facts concerning Boolean valued cardinals and, in particular, continuum.

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