Siberian Mathematical Journal, Vol. 49, *No.* 5, *pp.* 835–841, 2008

ON THE THEORY OF GROSSONE

A. E. Gutman and S. S. Kutateladze UDC 517.13+510.635

Abstract: A trivial formalization is given for the informal reasonings of a series of papers by Ya. D. Sergeyev on a positional numeral system with an infinitely large base, grossone; the system which is groundlessly opposed by its originator to the classical nonstandard analysis.

Keywords: nonstandard analysis, infinitesimal analysis, positional numeral system

In recent years Sergeyev has published a series of papers [1–5] in which a positional numeral system is advanced related to the notion of grossone.‡) Sergeyev opposes his system to nonstandard analysis and regards the former as resting on different mathematical, philosophical, etc. doctrines. The aim of the present note is to properly position the papers by Sergeyev on developing numeral systems. It turns out that a model of Sergeyev's system is provided by the initial segment $\{1, 2, \dots, \nu!\}$ of the nonstandard natural scale up to the factorial $\nu!$ of an arbitrary actual infinitely large natural ν. Such a factorial serves as a model of Sergeyev's grossone, thus demonstrating the place occupying by the numeral system he proposed.

As the main source we have chosen [4], the latest available paper by Sergeyev, which contains a detailed description of his basic ideas.

> [4]: ... the approach used in this paper is different also with respect to the nonstandard analysis ... and built using Cantor's ideas.

We are about to show that, contrary to what is expected by the author of [4], his indistinct definitions of grossone and the concomitant notions admit an extremely accurate and trivial formalization within the classical nonstandard analysis.

> [4]: The infinite radix of the new system is introduced as the number of elements of the set $\mathbb{N}$ of natural numbers expressed by the numeral ① called *grossone*.

Use the formalism of IST, the internal set theory by E. Nelson [6] or any of the classical external set theories, for instance, EXT by K. Hrbaček [7] or NST by T. Kawai [8] (also see the monographs [9, 10]). As usual, ${}^\circ X$ denotes the standard core of a set X, i.e., the totality of all standard elements of X. In particular, ${}^\circ\mathbb{N}$ is the totality of all finite (standard) naturals. Fix an arbitrary infinitely large natural ν and denote its factorial by ①:

$$ ① = \nu!\,, \quad \text{where } \nu \in \mathbb{N},\ \nu \approx \infty. $$

Show that ① possesses all properties of "grossone" (postulated as well as implicitly presumed in [4]).

A possible approach to an adequate formalization (in the sense of [4]) of the notion of size or "the number of elements" of an arbitrary set A of standard naturals (i.e., of an external subset $A \subset {}^\circ\mathbb{N}$) consists in assigning the natural $\|A\| = |{}^*A \cap \{1, 2, \dots, ①\}|$ to each A, where ${}^*A$ is the standardization of A and $|X|$ is the size (in the usual sense) of a finite internal set X. In this case it is clear that $\|{}^\circ\mathbb{N}\| = ①$, which agrees with the fore-quoted "definition" of grossone. Note also that, due to the external induction, the function $A \mapsto \|A\|$ possesses the additivity property (presumed in [4]): $\big\|\bigcup_{k=1}^{n} A_k\big\| = \sum_{k=1}^{n} \|A_k\|$ for every family of pairwise disjoint sets $A_1, \dots, A_n \subset {}^\circ\mathbb{N}$, $n \in {}^\circ\mathbb{N}$.

†) On the centenary of the birth of S. L. Sobolev.

‡) The term "grossone" belongs to Sergeyev, has no relevance to the usual meaning of the noun "gross" in English, and stems most likely from "groß" in German or "grosso" in Italian. (Editor of the Translation.)

Novosibirsk. Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 49, No. 5, pp. 1054–1063, September–October, 2008. Original article submitted June 18, 2008.

Another approach (which is more trivial and much closer to that of [4]) to defining the number of elements consists in “replacing” the set $^\circ\mathbb{N}$ with the initial segment $\mathscr{N} = \{1, 2, \dots, ①\}$ of the natural scale and considering the usual size $|A| \in \mathbb{N}$ of each internal set $A \subset \mathscr{N}$. In this case, again, $|\mathscr{N}| = ①$; and the additivity of the counting measure $A \mapsto |A|$ needs no argument.

[4]: The new numeral ① allows us to write down the set, $\mathbb{N}$, of natural numbers in the form

$$\mathbb{N} = \{1,\ 2,\ 3,\ \dots,\ ① - 2,\ ① - 1,\ ①\}$$

because *grossone has been introduced as the number of elements of the set of natural numbers* (similarly, the number 3 is the number of elements of the set $\{1, 2, 3\}$). Thus, grossone is the biggest natural number . . .

While crediting the author of [4] for the audacious extrapolation of the properties of the number 3, we nevertheless cannot accept the fore-quoted agreement if for no other reason than the fact that the set $\mathbb{N}$ of naturals (in the popular sense of this fundamental notion) has no greatest element (with respect to the classical order). To keep the traditional sense for the symbol $\mathbb{N}$ (and being governed by “Postulate 3. *The part is less than the whole*” of [4]), instead of reusing this symbol for the proper subset $\{1, 2, \dots, ①\} \subset \mathbb{N}$ we decided to give the latter a less radical notation, $\mathscr{N}$.

[4]: *The Infinite Unit Axiom* consists of the following three statements:

Infinity. For any finite natural number n it follows $n < ①$.

Identity. The following relations link ① to identity elements 0 and 1

$$0 \cdot ① = ① \cdot 0 = 0, \quad ① - ① = 0, \quad \frac{①}{①} = 1, \quad ①^0 = 1, \quad 1^① = 1, \quad 0^① = 0.$$

Divisibility. For any finite natural number n sets $\mathbb{N}_{k,n}$, $1 \leqslant k \leqslant n$, being the nth parts of the set, $\mathbb{N}$, of natural numbers have the same number of elements indicated by the numeral $\frac{①}{n}$, where

$$\mathbb{N}_{k,n} = \{k,\ k+n,\ k+2n,\ k+3n,\ \dots\}, \quad 1 \leqslant k \leqslant n, \quad \bigcup_{k=1}^{n} \mathbb{N}_{k,n} = \mathbb{N}.$$

Since $① = \nu!$ is an infinitely large number, it satisfies *Infinity*. Every natural meets *Identity*, and so does ①. Presenting the factorial of an infinitely large number, ① is divisible by every standard natural. Moreover, if $n \in {}^\circ\mathbb{N}$, $1 \leqslant k \leqslant n$, and

$$\begin{aligned} {}^\circ\mathbb{N}_{k,n} &= \{k + (m-1)n \,:\, m \in {}^\circ\mathbb{N}\}, \\ \mathscr{N}_{k,n} &= \mathscr{N} \cap \{k + (m-1)n \,:\, m \in \mathbb{N}\}; \end{aligned}$$

then $\|{}^\circ\mathbb{N}_{k,n}\| = |\mathscr{N}_{k,n}| = \frac{①}{n}$. Hence, ① meets *Divisibility.*

[4]: It is worthy to emphasize that, since the numbers $\frac{①}{n}$ have been introduced as numbers of elements of sets $\mathbb{N}_{k,n}$, they are integer.

If a number is declared natural, it naturally cannot occur unnatural. To remove all doubts, we suggest a rigorous and detailed justification for satisfiability of the above postulate: for every $n \in {}^\circ\mathbb{N}$ we have $n < \nu$ and thus

$$\text{the number} \quad \frac{①}{n} = \frac{\nu!}{n} = \frac{1 \cdot 2 \cdot \ldots \cdot n \cdot \ldots \cdot \nu}{n} \quad \text{is integer.}$$

[4]: The introduction of grossone allows us to obtain the following interesting result: the set $\mathbb{N}$ is not a monoid under addition. In fact, the operation $①+1$ gives us as the result a number grater than ①. Thus, by definition of grossone, $① + 1$ does not belong to $\mathbb{N}$ and, therefore, $\mathbb{N}$ is not closed under addition and is not a monoid.

Indeed, $① \in \{1, 2, \dots, ①\} = \mathscr{N}$, but $① + 1 \notin \{1, 2, \dots, ①\} = \mathscr{N}$. (However, taking it into account that $\mathscr{N}$ is not the set of all naturals, the above trivial observation is unlikely “interesting.”)

[4]: . . . adding the Infinite Unit Axiom to the axioms of natural numbers defines the set of extended natural numbers indicated as $\widehat{\mathbb{N}}$ and including $\mathbb{N}$ as a proper subset

$$\widehat{\mathbb{N}} = \{1,\ 2,\ \dots,\ ① - 1,\ ①,\ ① + 1,\ \dots,\ ①^2 - 1,\ ①^2,\ ①^2 + 1,\ \dots\}.$$

In fact, ${}^\circ\mathbb{N}$ and $\mathscr{N}$ are both proper subsets of the set $\mathbb{N}$ of *all* naturals. (As is known, the radical formalism of IST saves us from considering "extended numbers.")

We permit ourselves to pass over other numerous descriptions of the properties of grossone and the accompanying notions in [4], since the corresponding analysis is quite analogous to that above (and equally trivial). However, we cannot help commenting the declared elimination of Hilbert's paradox of the Grand Hotel:

> [4]: . . . it is well known that Cantor's approach leads to some "paradoxes" . . . Hilbert's Grand Hotel has an infinite number of rooms . . . If a new guest arrives at the Hotel where every room is occupied, it is, nevertheless, possible to find a room for him. To do so, it is necessary to move the guest occupying room 1 to room 2, the guest occupying room 2 to room 3, etc. In such a way room 1 will be available for the newcomer . . .
>
> . . . In the paradox, the number of the rooms in the Hotel is countable. In our terminology this means that it has ① rooms . . . Under the Infinite Unit Axiom this procedure is not possible because the guest from room ① should be moved to room ① + 1 and the Hotel has only ① rooms. Thus, when the Hotel is full, no more new guests can be accommodated — the result corresponding perfectly to Postulate 3 and the situation taking place in normal hotels with a finite number of rooms.

The following unpretentious "paradox of the Gross Hotel" is brought to the audience's attention: Even though all grossrooms $1, 2, \dots, ①$ are occupied, it is easy to accommodate one more client in the Gross Hotel. To this end it suffices to move the guest occupying room n to room $n+1$ for each *finite* n. Since $n+1 < ①$ for all finite n, all former guests get their rooms in the Gross Hotel, while room 1 becomes free for a newcomer.

Besides a babbling theorization around grossone, [4] includes an "applied" part dedicated to a new positional numeral system with base ①. (The system is meant for becoming a foundation for "Infinity Computer" [5] which is able to operate infinitely large and infinitesimal numbers.) Unfortunately, the corresponding exposition remains highly informal, and even crucial definitions are substituted with allusions and illustrating examples.

> [4]: In order to construct a number C in the new numeral positional system with base ① we subdivide C into groups corresponding to powers of ①:
>
> $$C = c_{p_m} ①^{p_m} + \cdots + c_{p_1} ①^{p_1} + c_{p_0} ①^{p_0} + c_{p_{-1}} ①^{p_{-1}} + \cdots + c_{p_{-k}} ①^{p_{-k}}.$$
>
> . . . Finite numbers c_i are called *infinite grossdigits* and can be both positive and negative; numbers p_i are called *grosspowers* and can be finite, infinite, and infinitesimal (the introduction of infinitesimal numbers will be given soon). The numbers p_i are such that $p_i > 0$, $p_0 = 0$, $p_{-i} < 0$ and
>
> $$p_m > p_{m-1} > \cdots > p_2 > p_1 > p_{-1} > p_{-2} > \cdots > p_{-(k-1)} > p_{-k}.$$
>
> . . . Finite numbers in this new numeral system are represented by numerals having only one grosspower equal to zero . . .
>
> . . . all grossdigits c_i, $-k \leqslant i \leqslant m$, can be integer or fractional . . . Infinite numbers in this numeral system are expressed by numerals having at least one grosspower grater than zero . . . Numerals having only negative grosspowers represent infinitesimal numbers.

In the fore-quoted definitions, the combinations of the terms "finite," "infinite," and "number" seem to be used quite vaguely. For instance, it is unclear from the text whether a numeral is assumed infinite (and in what sense) if it is not finite (in some sense). Following the definitions of [4] literally, a grosspower can be finite, infinite, and (or?) infinitesimal, while "finite" means $c①^0$ (a grossdigit c, a rational numeral), "infinite" is expressed by a numeral having al least one strictly positive grosspower, and "infinitesimal" is a numeral whose grosspowers are all strictly negative. Seemingly, this implies that a grosspower cannot be equal to, say, $①^0 + ①^{-1}$, but the subsequent examples of [4] show that this is not so, and arbitrary numerals can serve as grosspowers. In addition, the reason is completely unclear for choosing the terms "infinite" and "infinitesimal" exactly for the classes of numerals mentioned in the quote. For instance, the numeral $a = ①^{①^{-1}}$ (with grosspower $①^{-1} > 0$) is "infinite" by definition, while, obviously, $1 < a < 2$. On the other hand, the numeral $b = ①^{①^{-1}} - 1$ is also considered "infinite" and not "infinitesimal," while, as is easily seen, b is infinitely close to zero in the sense that $-c < b < c$ for every finite $c > 0$.

Regardless of terminological discipline, the fore-quoted definition of numerals C cannot be considered formal if for no other reason than the participating notion of ("infinite" and "infinitesimal") grosspowers depends on the initial notion of numeral, thus leading to a vicious circle. In addition, from the illustrations of [4] it is clear that the positional system proposed admit syntactically different numerals with coincident values: for instance, $0① ^0 \equiv 0①^1$, $1①^0 \equiv 1①^{0①^0}$. (The notion of the value of a term and the equivalence relation $\equiv$ are clarified below in this note.) At the same time, [4] misses not only the corresponding stipulations (easy to guess though) but also any attempts of justifying the unambiguity of the positional system, even under implicit stipulations. Observe also that the description of [4] for the algorithms of calculating the sum and product of numerals (i.e., of finding the corresponding equivalent numeral) is very superficial, since it does not touch upon the problem of recognizing equivalent numerals (which is necessary for collecting similar terms) and that of comparing them (which is necessary for collating the summands in order of their "grosspowers"). It is thus not surprising that the patent application [5] reports on the development of "Infinity Calculator" that is able only to deal with the numerals admitting "finite exponents."

To provide some justification, in the remaining part of the note we briefly describe one of the possible approaches to formalization of the notion of numeral as well as the corresponding algorithmic procedures. (In this way, some statements reflecting the implicit assumptions of [4] will be formulated as hypotheses.)

Let $\mathscr{Q}$ be any traditional "constructive" model of the ordered field of rationals (for instance, the one constituted by finite and periodic decimal fractions or irreducible vulgar fractions). The elements of $\mathscr{Q}$ are called *rational numerals*. Denote by 0 and 1 the rational numerals corresponding to zero and unity.

Enrich the language of the set theory ZFC by the classically definable constants $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ as well as by the terms $x+y$, $x-y$, $x\cdot y$, x^y and the formulas $x<y$. (All these and other similar signature enlargements are Beth-definable via the corresponding axiomatic extensions.) Observe that we rely on the traditional assumption that ZFC is consistent (hence so are any conservative extensions of the theory).

Given a rational numeral $q \in \mathscr{Q}$, let $\varphi_q(x)$ be the formula which defines the corresponding rational number in ZFC. In particular, for every $q \in \mathscr{Q}$ we have $\mathrm{ZFC} \vdash \big((\exists!\, x)\, \varphi_q(x) \,\&\, (\exists\, x \in \mathbb{Q})\, \varphi_q(x)\big)$. Enrich the language of ZFC by the corresponding definable constants $q \in \mathscr{Q}$. More precisely, (conservatively) extend ZFC to a theory $\mathrm{ZFC}_{\mathscr{Q}}$ by enlarging the signature with the set of constants $\mathscr{Q}$ and extending the axiomatics by the family of formulas $\varphi_q(q)$, $q \in \mathscr{Q}$. We thus have $\mathrm{ZFC}_{\mathscr{Q}} \vdash (q \in \mathbb{Q})$ for all $q \in \mathscr{Q}$, $\mathrm{ZFC}_{\mathscr{Q}} \vdash (q_1 \neq q_2)$ for $q_1 \neq q_2$, and $\mathrm{ZFC}_{\mathscr{Q}} \vdash (0 = 0_{\mathbb{Q}})$, $\mathrm{ZFC}_{\mathscr{Q}} \vdash (1 = 1_{\mathbb{Q}})$.

"Constructivity" of the model $\mathscr{Q}$ allows us to assume that the addition, subtraction, multiplication, and comparison of rational numerals are decidable in $\mathrm{ZFC}_{\mathscr{Q}}$, i.e., for all $q, q_1, q_2 \in \mathscr{Q}$, the following sentences are decidable in $\mathrm{ZFC}_{\mathscr{Q}}$: $(q_1 + q_2 = q)$, $(q_1 - q_2 = q)$, $(q_1 \cdot q_2 = q)$, $(q_1 < q_2)$. (Recall that a sentence φ is *decidable* in a theory T whenever $T \vdash \varphi$ or $T \vdash \neg\varphi$.) In this case $\mathscr{Q}$ is endowed with computable operations $\oplus$, $\ominus$, $\odot$ and a decidable linear strong order relation $\prec$ such that for all $q_1, q_2 \in \mathscr{Q}$ the sentences $(q_1 + q_2 = q_1 \oplus q_2)$, $(q_1 - q_2 = q_1 \ominus q_2)$, $(q_1 \cdot q_2 = q_1 \odot q_2)$ are provable in $\mathrm{ZFC}_{\mathscr{Q}}$ and the inequality $q_1 \prec q_2$ is equivalent to $\mathrm{ZFC}_{\mathscr{Q}} \vdash (q_1 < q_2)$.

Let $\mathscr{C}$ be the set of all terms of signature $\mathscr{Q} \cup \{+, -, \cdot\}$. The elements of $\mathscr{C}$ are called *rational terms*. It is clear that $\mathrm{ZFC}_{\mathscr{Q}} \vdash (c \in \mathbb{Q})$ for all $c \in \mathscr{C}$. By means of the available addition, subtraction, and multiplication in $\mathscr{Q}$, define the (computable) function $\mathrm{val} : \mathscr{C} \to \mathscr{Q}$ by recursion on the complexity of rational terms: $\mathrm{val}(q) := q$ for $q \in \mathscr{Q}$, $\mathrm{val}(c_1 + c_2) := \mathrm{val}(c_1) \oplus \mathrm{val}(c_2)$, $\mathrm{val}(c_1 - c_2) := \mathrm{val}(c_1) \ominus \mathrm{val}(c_2)$, $\mathrm{val}(c_1 \cdot c_2) := \mathrm{val}(c_1) \odot \mathrm{val}(c_2)$. Then, as is easily seen, $\mathrm{ZFC}_{\mathscr{Q}} \vdash \big(\mathrm{val}(c) = c\big)$ for all $c \in \mathscr{C}$. Introduce the relations $\equiv$ and $\prec$ on $\mathscr{C}$ as follows:

$$c_1 \equiv c_2 \;\Leftrightarrow\; \mathrm{ZFC}_{\mathscr{Q}} \vdash (c_1 = c_2) \;\Leftrightarrow\; \mathrm{ZFC}_{\mathscr{Q}} \vdash \big(\mathrm{val}(c_1) = \mathrm{val}(c_2)\big) \;\Leftrightarrow\; \mathrm{val}(c_1) = \mathrm{val}(c_2);$$

$$c_1 \prec c_2 \;\Leftrightarrow\; \mathrm{ZFC}_{\mathscr{Q}} \vdash (c_1 < c_2) \;\Leftrightarrow\; \mathrm{ZFC}_{\mathscr{Q}} \vdash \big(\mathrm{val}(c_1) < \mathrm{val}(c_2)\big) \;\Leftrightarrow\; \mathrm{val}(c_1) \prec \mathrm{val}(c_2).$$

Since the function $\mathrm{val} : \mathscr{C} \to \mathscr{Q}$ is computable and the relation $\prec$ is decidable in $\mathscr{Q}^2$, it is clear that $\equiv$ and $\prec$ are decidable in $\mathscr{C}^2$.

Let x be a variable. Denote by $\mathscr{T}(x)$ the set of expressions obtained by applying the following rules finitely many times: if $q \in \mathscr{Q}$ then $q \in \mathscr{T}(x)$; if $t_1, t_2 \in \mathscr{T}(x)$ then $(t_1 + t_2), (t_1 - t_2), (t_1 \cdot t_2) \in \mathscr{T}(x)$;

if $t \in \mathscr{T}(x)$ then $x^t \in \mathscr{T}(x)$. Introducing the notation $\chi(t) := x^t$ and considering χ a unary functional symbol, we can formally define $\mathscr{T}(x)$ as the set of all terms of signature $\mathscr{Q} \cup \{+, -, \cdot, \chi\}$. Call the elements of $\mathscr{T}(x)$ *x-terms*. We will sometimes write down x-terms t as $t(x)$ for the sake of visualization. It is clear that $\mathrm{ZFC}_{\mathscr{Q}} \vdash (\forall\, x \in \mathbb{R}_+)\big(t(x) \in \mathbb{R}\big)$ for all $t \in \mathscr{T}(x)$.

Given a nonempty finite family $(t_i)_{i \in I}$ of x-terms and any natural numeration $I = \{i_1, \dots, i_n\}$, denote the x-term $\big(\cdots\big((t_{i_1} + t_{i_2}) + t_{i_3}\big) + \cdots + t_{i_n}\big)$ by $\sum_{k=1}^{n} t_{i_k}$ or $\sum_{i \in I} t_i$. Next, given $c \in \mathscr{C}$ and $t \in \mathscr{T}(x)$, define the x-term cx^t by letting $cx^0 := c$ and $cx^t := c \cdot x^t$ for $t \neq 0$.

Put $\mathscr{P}_1(x) := \mathscr{C} = \{cx^0 : c \in \mathscr{C}\}$ and recursively define

$$\mathscr{P}_{k+1}(x) := \big\{\textstyle\sum_{i=1}^{n} c_i x^{p_i} \ : \ n \in \mathbb{N},\ c_i \in \mathscr{C},\ p_i \in \mathscr{P}_k(x)\big\}.$$

Clearly, $\mathscr{P}_k(x) \subset \mathscr{P}_{k+1}(x)$ for all $k \in \mathbb{N}$. Denote the union $\bigcup_{k=1}^{\infty} \mathscr{P}_k(x)$ by $\mathscr{P}(x)$ and call the elements of $\mathscr{P}(x)$ *generalized x-polynomials* or simply *x-polynomials* for brevity. Define the function $h : \mathscr{P}(x) \to \mathbb{N}$ by putting $h(p) := \min\{k \in \mathbb{N} : p \in \mathscr{P}_k(x)\}$ and call $h(p)$ the *height* of an x-polynomial p.

Say that x-terms t_1, t_2 are *equivalent* and write $t_1 \equiv t_2$ if $\mathrm{ZFC}_{\mathscr{Q}} \vdash (\forall\, x \in \mathbb{R}_+)\big(t_1(x) = t_2(x)\big)$. It is clear that $\equiv$ is an equivalence relation on $\mathscr{T}(x)$ which extends the relation of the same designation which was previously defined on the set of rational terms.

Given arbitrary $p, \tilde{p} \in \mathscr{P}(x)$, define x-polynomials $p \boxplus \tilde{p}$, $p \boxminus \tilde{p}$, $p \boxdot \tilde{p}$ as follows: If $p = \sum_{i \in I} c_i x^{p_i}$ and $\tilde{p} = \sum_{j \in J} c_j x^{p_j}$, where $I = \{1, \dots, m\}$ and $J = \{m+1, \dots, n\}$, then $p \boxplus \tilde{p} := \sum_{k \in I \cup J} c_k x^{p_k}$, $p \boxminus \tilde{p} := p \boxplus \sum_{j \in J} (0 - c_j) x^{p_j}$, $p \boxdot \tilde{p} := \sum_{(i,j) \in I \times J} (c_i \cdot c_j) x^{p_i \boxplus p_j}$.

Define the function $\pi : \mathscr{T}(x) \to \mathscr{P}(x)$ by recursion on the complexity of x-terms: given $q \in \mathscr{Q}$, put $\pi(q) := q$; given $t_1, t_2 \in \mathscr{T}(x)$, put $\pi(t_1 + t_2) := \pi(t_1) \boxplus \pi(t_2)$, $\pi(t_1 - t_2) := \pi(t_1) \boxminus \pi(t_2)$, $\pi(t_1 \cdot t_2) := \pi(t_1) \boxdot \pi(t_2)$; and, given $t \in \mathscr{T}(x)$, put $\pi(x^t) := 1x^{\pi(t)}$. Obviously, $\pi : \mathscr{T}(x) \to \mathscr{P}(x)$ is a computable function and $\pi(t) \equiv t$ for all $t \in \mathscr{T}(x)$.

Introduce the relation $\prec$ on $\mathscr{P}(x)$ by letting $p_1 \prec p_2$ if $\mathrm{ZFC}_{\mathscr{Q}} \vdash (\exists\, y \in \mathbb{R}_+)(\forall\, x \geqslant y)\big(p_1(x) < p_2(x)\big)$. Clearly, $\prec$ is a strong (nonlinear) order which extends the order previously introduced on the set of rational terms.

In [11] it is proven that the structure $\mathscr{R} = (\mathbb{R}, 0, 1, +, -, \cdot, <, \exp)$ is o-minimal. The latter means that the truth domain of every unary predicate definable by a first-order formula of the structure is a union of finitely many points and finitely many bounded or unbounded intervals. (The monograph [12] is also a good source of information concerning o-minimal structures.) Owing to the identity $a^b = \exp\big(b \exp^{-1}(a)\big)$, given an x-polynomial p, the function $\big(x \mapsto p(x)\big) : \mathbb{R}_+ \to \mathbb{R}$ is definable in the signature of the structure $\mathscr{R}$ and hence, for all $p_1, p_2 \in \mathscr{P}(x)$, the unary predicates $p_1(x) < p_2(x)$ and $p_1(x) = p_2(x)$ are definable in $\mathscr{R}$. Since $\mathscr{R}$ is o-minimal, it is immediate that, for all $p_1, p_2 \in \mathscr{P}(x)$, one and only one of the following sentences is valid in $\mathscr{R}$:

$$\begin{gathered}
(\exists\, y \in \mathbb{R}_+)(\forall\, x \geqslant y)\big(p_1(x) < p_2(x)\big), \\
(\exists\, y \in \mathbb{R}_+)(\forall\, x \geqslant y)\big(p_1(x) > p_2(x)\big), \\
(\exists\, y \in \mathbb{R}_+)(\forall\, x \geqslant y)\big(p_1(x) = p_2(x)\big).
\end{gathered}$$

Moreover, the last sentence is equivalent to $(\forall\, x \in \mathbb{R}_+)\big(p_1(x) = p_2(x)\big)$. Indeed, since the function $f : x \mapsto p_1(x) - p_2(x)$ is analytic, in the case of $f \not\equiv 0$ the set $\{x \in \mathbb{R}_+ : f(x) = 0\}$ cannot contain a limit point and is thus finite due to o-minimality of $\mathscr{R}$. This argument justifies the assumption that, for each pair $p_1, p_2 \in \mathscr{P}(x)$, exactly one of the statements $p_1 \prec p_2$, $p_1 \equiv p_2$, $p_1 \succ p_2$ holds, and the relations $\equiv$ and $\prec$ on $\mathscr{P}(x)$ are decidable. (By accepting this hypothesis, observe that its version for the x-polynomials of height 1 is obvious, while the study of the general case can be probably simplified with the help of the results and ideas of the paper [13] devoted to the decidability of the structure $\mathscr{R}$.)

Put $\mathscr{A}_1(x) := \mathscr{Q} = \{qx^0 : q \in \mathscr{Q}\}$ and recursively define

$$\begin{aligned}
\mathscr{A}_{k+1}(x) \ := \ \big\{ \textstyle\sum_{i=1}^{n} q_i x^{a_i} \ :\ & n \in \mathbb{N},\ q_i \in \mathscr{Q},\ a_i \in \mathscr{A}_k(x), \\
& a_1 \succ a_2 \succ \cdots \succ a_n,\ q_i \neq 0 \text{ whenever } a_i \neq 0 \big\}.
\end{aligned}$$

As is easily seen, $\mathscr{A}_k(x) \subset \mathscr{P}_k(x)$ and $\mathscr{A}_k(x) \subset \mathscr{A}_{k+1}(x)$ for all $k \in \mathbb{N}$. Put $\mathscr{A}(x) = \bigcup_{k=1}^{\infty} \mathscr{A}_k(x)$ and call the elements of $\mathscr{A}(x)$ *x-numerals*. As soon as the order $\prec$ on $\mathscr{P}(x)$ is decidable, $\mathscr{A}(x)$ is a decidable subset of $\mathscr{P}(x)$ $\big($as well as of $\mathscr{T}(x)\big)$.

Define the function $\alpha : \mathscr{P}(x) \to \mathscr{A}(x)$ by recursion on the height of x-polynomials. In the case of $h(p) = 1$, i.e., $p \in \mathscr{P}_1(x) = \mathscr{C}$, put $\alpha(p) := \mathrm{val}(p) \in \mathscr{Q} \subset \mathscr{A}(x)$. On assuming that the function α is already defined on $\mathscr{P}_k(x)$, consider an arbitrary x-polynomial $p = \sum_{i=1}^{n} c_i x^{p_i}$ of height $k+1$. Since $h(p_1), \dots, h(p_n) < h(p)$, the x-numerals $\alpha(p_1), \dots, \alpha(p_n) \in \mathscr{A}(x)$ are available which can be ordered and grouped by means of the decidable comparison of x-polynomials: $\{\alpha(p_1), \dots, \alpha(p_n)\} = \{a_1, \dots, a_m\}$, where $a_1 \succ a_2 \succ \dots \succ a_m$. Let $I = \{1, \dots, n\}$, $J = \{1, \dots, m\}$ and put $I(j) := \{i \in I : \alpha(p_i) = a_j\}$ and $q_j := \mathrm{val}\big(\sum_{i \in I(j)} c_i\big) \in \mathscr{Q}$ for each $j \in J$. It is clear that $p \equiv \sum_{j \in J} q_j x^{a_j}$. Consequently, we can take as $\alpha(p)$ the sum $\sum_{j \in J} q_j x^{a_j}$ except for the summands $q_j x^{a_j}$ of the form $0x^a$, with $a \neq 0$, and put $\alpha(p) := 0$ if all summands $q_j x^{a_j}$ have the form mentioned. In result, we obtain a computable function $\alpha : \mathscr{P}(x) \to \mathscr{A}(x)$ such that $\alpha(p) \equiv p$ for all $p \in \mathscr{P}(x)$.

Consider the composite $\mathrm{val} : \mathscr{T}(x) \to \mathscr{A}(x)$ of $\pi : \mathscr{T}(x) \to \mathscr{P}(x)$ and $\alpha : \mathscr{P}(x) \to \mathscr{A}(x)$. It is clear that val is a computable function which extends the already-introduced function $\mathrm{val} : \mathscr{C} \to \mathscr{Q}$ of the same designation and meets the condition $\mathrm{val}(t) \equiv t$ for all $t \in \mathscr{T}(x)$. Having accepted the hypothesis that $a_1 \equiv a_2$ implies $a_1 = a_2$ for all $a_1, a_2 \in \mathscr{A}(x)$, we can call the x-numeral $\mathrm{val}(t) \in \mathscr{A}(x)$ the *value* of an x-term $t \in \mathscr{T}(x)$. Therefore, x-terms are equivalent whenever they have equal values. By means of the function val we can effectively extend the available computable addition, subtraction, and multiplication from $\mathscr{Q}$ onto $\mathscr{A}(x)$ by letting $a_1 \oplus a_2 := \mathrm{val}(a_1 + a_2)$, $a_1 \ominus a_2 := \mathrm{val}(a_1 - a_2)$, $a_1 \odot a_2 := \mathrm{val}(a_1 \cdot a_2)$, and add the computable x-power operation: $x{\uparrow}a := \mathrm{val}(x^a)$. In result, we can realistically call $\mathscr{A}(x)$ the "calculus of x-numerals."

Let IST be the internal set theory by E. Nelson [6]. (However, instead of IST, we may employ any of the classical nonstandard set theories which conservatively extends ZFC.) Enrich the language of IST by the constants $q \in \mathscr{Q}$ and introduce the new constant $\mathbb{O}$ axiomatizable as the factorial of an infinitely large natural. More precisely, (conservatively) extend IST + $\mathrm{ZFC}_{\mathscr{Q}}$ to a theory $\mathrm{IST}_{\mathbb{O}}$ by enlarging the signature with the constant $\mathbb{O}$ and adding the axiom $(\exists\, \nu \in \mathbb{N} \setminus {}^{\circ}\mathbb{N})(\mathbb{O} = \nu!)$.

By transfer, $\mathrm{IST}_{\mathbb{O}} \vdash (c \in {}^{\circ}\mathbb{Q})$ for all $c \in \mathscr{C}$ and, moreover, for each x-numeral $a \in \mathscr{A}(x)$, the following sentence is provable in $\mathrm{IST}_{\mathbb{O}}$: $\big(x \mapsto a(x)\big)$ is a standard continuous function from $\mathbb{R}_+$ into $\mathbb{R}$ and if $a(x) \not\equiv 0$ then there is a number $y \in \mathbb{R}_+$ such that $a(x) \neq 0$ for $x \geqslant y$. Consequently, for all $a_1, a_2 \in \mathscr{A}(x)$ we have

$$a_1 = a_2 \;\Leftrightarrow\; \mathrm{ZFC}_{\mathscr{Q}} \vdash (\exists\, y \in \mathbb{R}_+)(\forall\, x \geqslant y)\big(a_1(x) = a_2(x)\big) \;\Leftrightarrow\; \mathrm{IST}_{\mathbb{O}} \vdash \big(a_1(\mathbb{O}) = a_2(\mathbb{O})\big);$$
$$a_1 \prec a_2 \;\Leftrightarrow\; \mathrm{ZFC}_{\mathscr{Q}} \vdash (\exists\, y \in \mathbb{R}_+)(\forall\, x \geqslant y)\big(a_1(x) < a_2(x)\big) \;\Leftrightarrow\; \mathrm{IST}_{\mathbb{O}} \vdash \big(a_1(\mathbb{O}) < a_2(\mathbb{O})\big).$$

Call the elements of $\mathscr{G} := \{a|_{\mathbb{O}}^{x} : a \in \mathscr{A}(x)\}$ *grossnumerals*. Since the substitutions $a \mapsto a|_{\mathbb{O}}^{x}$ and $g \mapsto g|_{x}^{\mathbb{O}}$ are mutually inverse computable bijections between $\mathscr{A}(x)$ and $\mathscr{G}$, the set of grossnumerals is endowed with the translated decidable order $\prec$ and computable addition $\oplus$, subtraction $\ominus$, multiplication $\odot$, and grosspower $\mathbb{O}{\uparrow}$ which make $\mathscr{G}$ the "calculus of grossnumerals." For all $g, g_1, g_2 \in \mathscr{G}$ we have

$$\mathrm{IST}_{\mathbb{O}} \vdash (g \in \mathbb{R});$$
$$g_1 = g_2 \;\Leftrightarrow\; \mathrm{IST}_{\mathbb{O}} \vdash (g_1 = g_2); \quad g_1 \prec g_2 \;\Leftrightarrow\; \mathrm{IST}_{\mathbb{O}} \vdash (g_1 < g_2);$$
$$\mathrm{IST}_{\mathbb{O}} \vdash (g_1 \oplus g_2 = g_1 + g_2); \quad \mathrm{IST}_{\mathbb{O}} \vdash (g_1 \ominus g_2 = g_1 - g_2);$$
$$\mathrm{IST}_{\mathbb{O}} \vdash (g_1 \odot g_2 = g_1 \cdot g_2); \quad \mathrm{IST}_{\mathbb{O}} \vdash (\mathbb{O}{\uparrow}g = \mathbb{O}^{g}).$$

Of course, the numbers expressed by grossnumerals are not representative in $\mathbb{R}$. More precisely, every number of this kind belongs in $\mathrm{IST}_{\mathbb{O}}$ to the external set $\{f(\mathbb{O}) : f \in {}^{\circ}\mathscr{F}\}$, where $\mathscr{F}$ is the standard countable set defined as the smallest ring of functions $f : \mathbb{R}_+ \to \mathbb{R}$ which contains the rational constants and the functions $x \mapsto x^{f(x)}$ for each $f \in \mathscr{F}$. Moreover, the hypothesis is plausible that, for each $g \in \mathscr{G}$, the formula $(|g| \approx \infty) \vee (\exists\, r \in {}^{\circ}\mathbb{Q})(g \approx r)$ is provable in $\mathrm{IST}_{\mathbb{O}}$, which, in particular, implies that no grossnumeral can express a standard irrational number (even to within an infinitesimal).

References

1. *Sergeyev Ya. D.*, Arithmetic of Infinity, Edizioni Orizzonti Meridionali, Cosenza (2003).
2. *Sergeyev Ya. D.*, "A few remarks on philosophical foundations of a new applied approach to Infinity," Scheria, **26–27**, 63–72 (2005).
3. *Sergeyev Ya. D.*, "Misuriamo l'infinito: Un semplice modo per insegnare i concetti delle grandezze infinite," Period. Mat., **6(2)**, 11–26 (2006).
4. *Sergeyev Ya. D.*, "Blinking fractals and their quantitative analysis using infinite and infinitesimal numbers," Chaos Solitons Fractals, **33(1)**, 50–75 (2007).
5. *Sergeyev Ya. D.*, Infinity Computer. Computer System for Storing Infinite, Infinitesimal, and Finite Quantities and Executing Arithmetical Operations with Them, International Patent Application, Submitted 08.03.04.
6. *Nelson E.*, "Internal set theory. A new approach to nonstandard analysis," Bull. Amer. Math. Soc., **83**, No. 6, 1165–1198 (1977).
7. *Hrbaček K.*, "Axiomatic foundations for nonstandard analysis," Fund. Math., **98**, No. 1, 1–24 (1978).
8. *Kawai T.*, "Axiom systems of nonstandard set theory," in: Logic Symposia, Proc. Conf. Hakone 1979, 1980, Springer-Verlag, Berlin etc., 1981, pp. 57–65.
9. *Kanovei V. and Reeken M.*, Nonstandard Analysis, Axiomatically, Springer-Verlag, Berlin (2004) (Springer Monogr. in Math.).
10. *Gordon E. I., Kusraev A. G., and Kutateladze S. S.*, Infinitesimal Analysis. Selected Topics [in Russian], Nauka, Moscow (2008).
11. *Wilkie A. J.*, "Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function," J. Amer. Math. Soc., **9**, No. 4, 1051–1094 (1996).
12. *van den Dries L.*, Tame Topology and o-Minimal Structures, Cambridge University Press, Cambridge (1998) (London Math. Soc. Lecture Notes Series; 248).
13. *Macintyre A. and Wilkie A. J.*, "On the decidability of the real exponential field," in: Kreiseliana: About and Around Georg Kreisel, A K Peters, Wellesley, MA, 1996, pp. 441–467.

A. E. Gutman; S. S. Kutateladze
Sobolev Institute of Mathematics, Novosibirsk, Russia
E-mail address: gutman@math.nsc.ru; sskut@math.nsc.ru