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SPACES OF CD_0-FUNCTIONS AND CD_0-SECTIONS OF BANACH BUNDLES

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Abstract. We first briefly expose some crucial phases in studying the space $CD_0(Q) = C(Q) + c_0(Q)$ whose elements are the sums of continuous and "discrete" functions defined on a compact Hausdorff space Q without isolated points. (In this part, special emphasis is on describing the compact space $\widetilde{Q}$ representing the Banach lattice $CD_0(Q)$ as $C(\widetilde{Q})$.) The rest of the article is dedicated to the analogous frame related to the space $CD_0(Q, \mathcal{X})$ of "continuous-discrete" sections of a Banach bundle $\mathcal{X}$ and the space of CD_0-homomorphisms of Banach bundles.

Keywords. Banach lattice, AM-space, Alexandroff duplicate, continuous Banach bundle, section of a Banach bundle, Banach $C(Q)$-module, homomorphism of Banach bundles, homomorphism of $C(Q)$-modules.

1. Introduction

1.1. A real Banach space $X = (X, +, \cdot, \|\cdot\|)$ endowed with a (partial) order $\leqslant$ is called a *Banach lattice* whenever

(1) the order $\leqslant$ makes X a lattice, i.e., for all $x, y \in X$, the supremum $x \vee y$ and infimum $x \wedge y$ exist in X (hence, the modulus $|x| := x \vee (-x)$ exists for every $x \in X$);

(2) the order $\leqslant$ agrees with the linear operations, i.e., for all $x, y, z \in X$ and $0 \leqslant \lambda \in \mathbb{R}$, the inequality $x \leqslant y$ implies $x + z \leqslant y + z$ and $\lambda x \leqslant \lambda y$;

(3) the norm $\|\cdot\|$ is monotonous with respect to the order $\leqslant$, i.e., for all $x, y \in X$, $|x| \leqslant |y|$ implies $\|x\| \leqslant \|y\|$ (hence, $\|x\| = \big\||x|\big\|$ for all $x \in X$).

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A Banach lattice X is called an *abstract M-space with unity*, or an *AM_1-space* for short, if

(4) $\|x \vee y\| = \max\{\|x\|, \|y\|\}$ for all $0 \leqslant x, y \in X$;
(5) there is an element $\mathbb{1} \in X$ such that $|x| \leqslant \mathbb{1}$ is equivalent to $\|x\| \leqslant 1$ for all $x \in X$.

The classical Banach function spaces endowed with the uniform norm and pointwise order serve as examples of AM_1-spaces:

(a) $(\mathbb{R}^n, \|\cdot\|_\infty)$, $n \in \mathbb{N}$;
(b) the space ℓ^∞ of bounded sequences;
(c) the space $L^\infty(\Omega)$ of (cosets) of essentially bounded measurable functions defined on a measure space Ω;
(d) the space $C(Q)$ of continuous functions defined on a compact Hausdorff space Q.

1.2. The theory of Banach lattices includes the following well-known fact:

Kreĭns–Kakutani Theorem. *Every AM_1-space is linearly isometric and order isomorphic to the space $C(Q)$ for a suitable compact Hausdorff space Q (moreover, such a space Q is unique up to homeomorphism).*

We can say that, in general, the space Q corresponding to a given AM_1-space according to the Kreĭns–Kakutani Theorem occurs "unobservable" ("implicit", "nonconstructive") if for no other reason than the fact that the available universal approaches to its "construction" essentially rely on the axiom of choice (or Zorn's Lemma) and employ such notions as ultrafilters, maximal ideals, etc.

Heading toward the desired compact space, some other rather bulky constructions often occur which weaken the intuitive connection with the initial AM_1-space. For instance, one of the classical ways of constructing a space Q representing a given AM_1-space X as $C(Q)$ consists in the following: first, an order completion $\overline{X}$ of X is considered; next, $\overline{X}$ is represented as the space $C(\overline{Q})$ of continuous functions defined on an extremally disconnected compact Hausdorff space $\overline{Q}$ (which occurs, for instance, as the set of all ultrafilters of the base of $\overline{X}$ endowed with a special topology); finally, the desired space Q is obtained by "gluing" together the points of $\overline{Q}$ which are not separated by the functions corresponding to the initial space X.

Another common approach to constructing a representation compact space for an AM_1-space consists in considering the second dual and employing the corresponding representation facts of the theory of commutative Banach algebras, which use such "implicit" objects as, for instance, characters of an algebra.

Perhaps, the shortest universal way to a representation compact space Q is paved in [4], where the points of Q occur as the maximal order ideals of the initial AM_1-space. (However, this construction can also hardly be called "observable" by obvious reasons.)

At the same time, it is clear that the study of the properties of a concrete AM_1-space can be considerably simplified if we manage to find an explicit and plain description of the corresponding representation compact space. As an example, consider the Banach lattice X which is the closure (with respect to the uniform norm) of the space of all functions $f\colon P \to \mathbb{R}$ that are defined on an infinite set P and are constant on P except finitely many points. (Despite of its simplicity, the space of such functions f plays an important role in some topics of the theory of regular operators in vector lattices; see, for instance, [1].) Every element of X can be described as a function $x\colon P \to \mathbb{R}$ for which there exist a number λ and a sequence of points $p_n \in P$ such that $x \equiv \lambda$ outside $\{p_n : n \in \mathbb{N}\}$ and $x(p_n) \to \lambda$ as $n \to \infty$. Since X is an AM_1-space, it is isomorphic to $C(Q)$ for some compact Hausdorff space Q. The structure of the space X becomes now absolutely clear on observing that we can take as Q the Alexandroff one-point compactification $P \cup \{\infty\}$ of the discrete topological space P. (In the compactification $P \cup \{\infty\}$, the points $p \in P$ are isolated, while the neighborhoods of ∞ are the complements of the finite subsets of P.) An isomorphism of the AM_1-space X onto $C(P \cup \{\infty\})$ is obtained by extending each function $x \in X$ to ∞ with the value $x(\infty) := \lambda$ mentioned in the above description of x.

1.3. Let now Q be an arbitrary nonempty compact Hausdorff space without isolated points and let $c_0(Q)$ be the totality of all functions $f : Q \to \mathbb{R}$ such that the set $\{q \in Q : |f(q)| > \varepsilon\}$ is finite for every number $\varepsilon > 0$. In [2] Y. A. Abramovich and A. W. Wickstead introduced the space

$$CD_0(Q) := C(Q) + c_0(Q)$$

of functions $f\colon Q \to \mathbb{R}$ each of which is representable as the sum $f = f_c + f_d$ of a continuous $f_c \in C(Q)$ and "discrete" $f_d \in c_0(Q)$ parts. First of all, it is worth noting that, since Q has no isolated points, we have the direct sum decomposition $CD_0(Q) = C(Q) \oplus c_0(Q)$ and the mappings $f \mapsto f_c$ and $f \mapsto f_d$ are the corresponding linear projections.

Y. A. Abramovich and A. W. Wickstead showed in [2] that, with respect to the uniform norm and pointwise order, the space $CD_0(Q)$ is a Banach lattice possessing certain rather exotic order-topological properties. They also observed that, despite of its "oddity," this Banach lattice is an AM_1-space and (according to the Kreĭns–Kakutani Theorem) is isomorphic to the space $C(\widetilde{Q})$ for a suitable compact Hausdorff compact space $\widetilde{Q}$. Having left aside the question of an explicit description of the corresponding compact spaces $\widetilde{Q}$, the authors of [2] nevertheless noted that, due to the unusual properties, such spaces are of interest for the general topology as well.

1.4. The spaces $CD_0(Q)$ (and other analogous spaces of "continuous-discrete" functions) became the subject of further investigations (see, for instance, [3, 5, 6]) which led to the first explicit description of the representation compact space $\widetilde{Q}$ for $CD_0(Q)$. Namely, in [8], Z. Ercan established that $\widetilde{Q}$ can be taken to be the set $Q\times\{0,1\}$ endowed with the following convergence:

$$(q_\alpha, r_\alpha) \to (q, r) \ \text{ if and only if}$$
$$f_c(q_\alpha) + r_\alpha f_d(q_\alpha) \to f_c(q) + r f_d(q) \ \text{ for every } \ f \in CD_0(Q).$$

Theorem [8]**.** *The above convergence corresponds to a compact Hausdorff topology on $Q\times\{0,1\}$. The spaces $CD_0(Q)$ and $C(Q\times\{0,1\})$ are linearly isometric and order isomorphic under the mapping that takes each element $f \in CD_0(Q)$ into the function $\widetilde{f}\colon Q\times\{0,1\} \to \mathbb{R}$ defined by the equality $\widetilde{f}(q,r) = f_c(q) + r f_d(q)$.*

The latter result has played a key role in the problem of describing the compact space $\widetilde{Q}$ which represents $CD_0(Q)$ as $C(\widetilde{Q})$.

The above approach to describing the representation compact space could be subjected to criticism by noting that the definition of its topology explicitly uses the space $CD_0(Q)$ itself, which does not allow us to reduce the study of $CD_0(Q)$ to that of $C(\widetilde{Q})$ and takes the analysis of the properties of $\widetilde{Q}$ and $C(\widetilde{Q})$ back to considering the initial space $CD_0(Q)$. Nevertheless, [8] contains an alternative description of the net convergence in $\widetilde{Q}$ which does not employ the space $CD_0(Q)$ per se and only uses the convergence in Q. Perhaps, the only remaining possible subject for criticism is introduction of a topology by means of convergence, which hinders its understanding from the traditional "neighborhood" point of view.

1.5. However that may be, the above-mentioned "demerit" was completely eliminated by V. G. Troitsky in [16]. For convenience, introduce the mappings

$$(\cdot)_c, (\cdot)_d \colon Q \to Q\times\{0,1\}$$

by putting

$$q_c := (q, 0), \quad q_d := (q, 1).$$

In addition, for every subset $P \subset Q$ put

$$P_c := \{p_c : p \in P\} = P\times\{0\}, \quad P_d := \{p_d : p \in P\} = P\times\{1\}.$$

In his "notes" [16] V. G. Troitsky described the Ercan's topology on $Q\times\{0,1\} = Q_c \cup Q_d$ as follows: the points q_d are isolated, while the base neighborhoods of each point q_c are the sets of the form $V_c \cup V_d\setminus\{q_d\}$, where V is a neighborhood of q in the initial topology of Q.

The topological space $\widetilde{Q} = Q_c \cup Q_d$ thus defined is usually called the *Alexandroff duplicate* of the compact space Q and denoted by $A(Q)$. The space $\widetilde{Q}$ indeed possesses a number of exotic properties. As is known (see [7, 3.1.G]), it is Hausdorff and compact (moreover, every subset of $\widetilde{Q}$ containing Q_c is compact), its "continuous part" Q_c is homeomorphic to Q, and the "discrete part" Q_d is open and dense in $\widetilde{Q}$. The duplicate of a circle (the so-called "Alexandroff double circle") serves as a classical example of a hereditarily normal topological space which is not perfectly normal; it is also first-countable, but not separable and thus not second-countable (see [7, 3.1.26]).

Employing the new definition of the compact space $\widetilde{Q} = Q_c \cup Q_d$, we can now easily obtain a characterization of the net convergence in $\widetilde{Q}$ (analogous to that presented in [8]): Since the points $q_d \in Q_d$ are isolated, a net in $\widetilde{Q}$ converges to q_d if and only if it stabilizes at q_d; as for the points $q_c \in Q_c$, convergence of a net (q_α, r_α) to q_c is equivalent to the following: starting at some index, the points (q_α, r_α) differ from q_d and $q_\alpha \to q$ in the initial topology of Q.

1.6. Besides a simple and explicit description of the topology of $\widetilde{Q}$ in terms of neighborhoods, V. G. Troitsky suggested the following elegant characterization of the elements of $CD_0(Q)$:

Theorem [16]. *A function $f \colon Q \to \mathbb{R}$ belongs to $CD_0(Q)$ if and only if f has a limit at every point of Q. Furthermore, the continuous part $f_c \in C(Q)$ of a function $f \in CD_0(Q)$ is calculated by the formula*

$$f_c(q) = \lim_{p\to q} f(p) \quad \text{for all } \ q \in Q.$$

This result became a very convenient tool which made it possible to considerably simplify the study of the properties of $CD_0(Q)$ and, in particular, to obtain elementary proofs of available facts on the space.

1.7. The next stage in studying the properties of CD_0-spaces is characterized by passing from real valued functions $f\colon Q \to \mathbb{R}$ to vector valued functions $f\colon Q \to X$, where X is a Banach lattice. Isomorphy of the Banach lattices $CD_0(Q, X)$ and $C(\widetilde{Q}, X)$ in the case of a compact metric space Q without isolated points is mentioned already in [8]. In a more general case the connection between the spaces of vector valued CD_0-functions and continuous functions is treated in the paper [6] by Ş. Alpay and Z. Ercan.

1.8. Further developments showed that the main facts on representation of CD_0-spaces as spaces of continuous functions remain valid after passing not only to vector valued functions, but also to sections of Banach bundles.

The space of CD_0-sections of a continuous Banach bundle $\mathcal{X}$ over Q was first considered by T. Hõim and D. A. Robbins in [13], where, in particular, a linear isometry is constructed of this space onto the Banach space $C(\widetilde{Q}, \widetilde{\mathcal{X}})$ of all continuous sections of a certain Banach bundle $\widetilde{\mathcal{X}}$ over the duplicate $\widetilde{Q}$ of Q. (The structure of the bundle $\widetilde{\mathcal{X}}$ is discussed below.) Some interconnections are also established in [13] between $C(Q)$-linear operators from $C(Q, \mathcal{X})$ into $C(Q)$ and $C(\widetilde{Q})$-linear operators from $C(\widetilde{Q}, \widetilde{\mathcal{X}})$ into $C(\widetilde{Q})$.

The present article is in essence a revised and extended compilation of [11, 12]. After outlining the basic information on continuous Banach bundles we present the main definitions and facts concerning CD_0-sections of a Banach bundle and suggest some further development of the theory, with special emphasis on the space of CD_0-homomorphisms of Banach bundles.

2. Continuous Banach bundles

2.1. Let Q be an arbitrary topological space. A *Banach bundle* (or, more precisely, a *continuous Banach bundle*) over Q is a formalization of an intuitive idea of a "continuous" function $\mathcal{X}$ which is defined on Q and maps each point $q \in Q$ into a Banach space $\mathcal{X}(q)$ (called the *stalk* of $\mathcal{X}$ at q). One of the formal approaches to defining the "continuity" of $\mathcal{X}$ (see [10, 2.1], [15, 2.4.3]) consists in indicating a so-called *continuous structure* in $\mathcal{X}$ that is a vector subspace $\mathcal{C}_{\mathcal{X}}$ of the space of sections

$$S(Q, \mathcal{X}) = \Big\{ u\colon Q \to \bigcup_{q \in Q} \mathcal{X}(q) \ :\ u(q) \in \mathcal{X}(q) \text{ for all } q \in Q \Big\}$$

(endowed with the pointwise operations, see [10, 1.7.3], [15, 2.4.3]) such that, first, the pointwise norm

$$|\!|\!| c |\!|\!| \colon Q \to \mathbb{R}, \quad |\!|\!| c |\!|\!|(q) = \|c(q)\|_{\mathcal{X}(q)} \ (q \in Q)$$

of each section $c \in \mathcal{C}_{\mathcal{X}}$ is continuous and, second, $\mathcal{C}_{\mathcal{X}}$ is stalkwise dense in $\mathcal{X}$, i.e., the set $\{c(q) : c \in \mathcal{C}_{\mathcal{X}}\}$ is dense in $\mathcal{X}(q)$ for all $q \in Q$. Given a continuous structure $\mathcal{C}_{\mathcal{X}}$, we can define the totality $C(Q, \mathcal{X})$ of *continuous sections* of $\mathcal{X}$ to be the set of all sections $u \in S(Q, \mathcal{X})$ such that $|\!|\!| u - c |\!|\!| \in C(Q)$ for $c \in \mathcal{C}_{\mathcal{X}}$.

2.2. The notion of a continuous section of a Banach bundle can be treated as a generalization of the notion of a continuous vector valued function. Indeed, if X is a Banach space, then $C(Q,X) = C(Q,\mathcal{X})$, where $\mathcal{X}$ is the constant Banach bundle whose stalks are $\mathcal{X}(q) = X$ and the continuous structure is constituted by, for instance, constant functions $c\colon Q \to X$ (see [10, 2.2.1]).

2.3. Note that there is an alternative, and in a sense equivalent, approach to introducing a continuous structure, within which continuity of sections occurs to be a purely topological notion. (An exposition of both approaches, as well as a justification of their equivalence, can be found in [9].) Denote by $Q\otimes\mathcal{X}$ the union of the pairwise disjoint copies $\{q\}\times\mathcal{X}(q)$ of the stalks $\mathcal{X}(q)$ of a Banach bundle $\mathcal{X}$ over Q:

$$Q\otimes\mathcal{X} = \big\{(q,x) : q \in Q,\ x \in \mathcal{X}(q)\big\}.$$

Given an arbitrary section $u \in S(Q,\mathcal{X})$, define the function $Q\otimes u\colon Q \to Q\otimes\mathcal{X}$ by putting $(Q\otimes u)(q) = \big(q, u(q)\big)$ for all $q \in Q$. Then the set of all "tubes"

$$\big\{(q,x) \in Q\otimes\mathcal{X} : q \in U,\ \|x - c(q)\| < \varepsilon\big\},$$

which are defined by the sections $c \in \mathcal{C}_{\mathcal{X}}$, open subsets $U \subset Q$, and numbers $\varepsilon > 0$, is a base of some open topology on $Q\otimes\mathcal{X}$ (see [9, 5.3]). Furthermore, the induced topology of the copy $\{q\}\times\mathcal{X}(q) \subset Q\otimes\mathcal{X}$ of each stalk $\mathcal{X}(q)$ coincides with the initial topology of the stalk as a Banach space, and a section $u \in S(Q,\mathcal{X})$ occurs continuous if and only if the function $Q\otimes u\colon Q \to Q\otimes\mathcal{X}$ is continuous (in the usual sense) with respect to the tubes topology (see [10, 2.1.7]).

2.4. Different continuous structures $\mathcal{C}_1$ and $\mathcal{C}_2$ in $\mathcal{X}$ may induce the same topology on $Q\otimes\mathcal{X}$. In this case, the continuous structures $\mathcal{C}_1$ and $\mathcal{C}_2$ are called *equivalent* and the Banach bundles $(\mathcal{X},\mathcal{C}_1)$ and $(\mathcal{X},\mathcal{C}_2)$ are identified. The identification is justified, in particular, by the following fact:

Theorem [10, 2.1.8]**.** *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be continuous structures in $\mathcal{X}$ and let $C(Q,\mathcal{X}\,|\,\mathcal{C}_1)$ and $C(Q,\mathcal{X}\,|\,\mathcal{C}_2)$ be the corresponding sets of continuous sections. Then the following assertions are tantamount:*

(1) *$\mathcal{C}_1$ and $\mathcal{C}_2$ are equivalent;*
(2) *$C(Q,\mathcal{X}\,|\,\mathcal{C}_1) = C(Q,\mathcal{X}\,|\,\mathcal{C}_2)$;*
(3) *$C(Q,\mathcal{X}\,|\,\mathcal{C}_1) \subset C(Q,\mathcal{X}\,|\,\mathcal{C}_2)$;*
(4) *$\mathcal{C}_1 \subset C(Q,\mathcal{X}\,|\,\mathcal{C}_2)$;*
(5) *the intersection $C(Q,\mathcal{X}\,|\,\mathcal{C}_1) \cap C(Q,\mathcal{X}\,|\,\mathcal{C}_2)$ is stalkwise dense in $\mathcal{X}$.*

2.5. It is worth taking account of the following basic properties of the set $C(Q,\mathcal{X})$ of continuous sections of a Banach bundle $\mathcal{X}$ over a compact Hausdorff space Q:

(a) *If $u \in C(Q,\mathcal{X})$ then $\|u\| \in C(Q)$.*
(b) *The set $C(Q,\mathcal{X})$ is a closed vector subspace of the Banach space $\ell^\infty(Q,\mathcal{X})$ of all bounded sections of $\mathcal{X}$ endowed with the uniform norm $\|u\| = \big\|\|u\|\big\| = \sup_{q\in Q}\|u(q)\|$.*
(c) *If $u \in C(Q,\mathcal{X})$ and $f \in C(Q)$ then $fu \in C(Q,\mathcal{X})$. In particular, $C(Q,\mathcal{X})$ is a Banach $C(Q)$-module.*
(d) *The set $C(Q,\mathcal{X})$ fills the stalks of $\mathcal{X}$. Moreover, for all $q \in Q$ and $x \in \mathcal{X}(q)$, there exists a section $u \in C(Q,\mathcal{X})$ such that $u(q) = x$ and $\|u\| \leqslant \|x\|$.*

Assertions (a)–(c) are proven, for instance, in [10, 2.3]. Assertion (d) is conventionally called the Dupré Theorem (see [9, 2.10]). Note that the statement of this theorem holds for Banach bundles over arbitrary topological spaces Q (see [14, 1.1]).

2.6. The tubes topology introduced in 2.3 makes it possible to interpret various topological notions and facts related to sections $u \in S(Q, \mathcal{X})$ in terms of the corresponding functions $Q \otimes u$. For instance (see [10, 2.3.7]), a section $u \in S(Q, \mathcal{X})$ *has limit* $x \in \mathcal{X}(q)$ at a point $q \in Q$ if and only if the limit of the function $Q \otimes u \colon Q \to Q \otimes \mathcal{X}$ at q equals (q, x):

$$\lim_{p \to q} u(p) = x \;\Leftrightarrow\; \lim_{p \to q} \big(p, u(p)\big) = (q, x) \text{ in } Q \otimes \mathcal{X}.$$

According to [10, 2.3.8] and the Dupré Theorem, the last relation is equivalent to existence of a section $v \in C(Q, \mathcal{X})$ such that $v(q) = x$ and $\lim\limits_{p \to q} \|u(p) - v(p)\| = 0$.

3. The space of CD_0-sections

Throughout the sequel, Q is a nonempty compact Hausdorff space without isolated points. All vector spaces considered in the article are assumed to be defined over the field $\mathbb{R}$ of reals.

3.1. Recall that $C(Q)$ is the set of all real valued continuous functions defined on Q; $c_0(Q)$ is the totality of all functions $f \colon Q \to \mathbb{R}$ such that the set $\{q \in Q : |f(q)| > \varepsilon\}$ is finite for every number $\varepsilon > 0$. Both $C(Q)$ and $c_0(Q)$ are Banach lattices and Banach algebras with respect to the pointwise operations, pointwise order, and uniform norm. Each of the two spaces is a Banach sublattice and subalgebra of the lattice-ordered Banach algebra $\ell^\infty(Q)$ of all bounded real valued functions defined on Q. As is easily seen, $c_0(Q)$ is the closure in $\ell^\infty(Q)$ of the space of functions with finite support and is constituted exactly by those functions $f \colon Q \to \mathbb{R}$ for which there exists a sequence of pairwise distinct points $q_n \in Q$ ($n \in \mathbb{N}$) such that $f(q_n) \to 0$ as $n \to \infty$ and $f \equiv 0$ outside $\{q_n : n \in \mathbb{N}\}$. In particular, for all $f \in \ell^\infty(Q)$ and $g \in c_0(Q)$, the inequality $|f| \leqslant |g|$ implies $f \in c_0(Q)$; therefore, $c_0(Q)$ is an order ideal of $\ell^\infty(Q)$.

3.2. The symbol $CD_0(Q)$ denotes the space of all functions $f \colon Q \to \mathbb{R}$ representable as the sums $f = g + h$ of elements $g \in C(Q)$ and $h \in c_0(Q)$:

$$CD_0(Q) \;:=\; C(Q) + c_0(Q).$$

The following statement gathers some facts on the space $CD_0(Q)$ which are established in [2, 5, 8, 16]:

(1) *Endowed with the pointwise operations, pointwise order, and uniform norm, the space $CD_0(Q)$ is an AM_1-space.*

(2) *The direct sum decomposition $CD_0(Q) = C(Q) \oplus c_0(Q)$ holds. Therefore, every function $f \in CD_0(Q)$ is uniquely representable as $f = f_c + f_d$, with $f_c \in C(Q)$ and $f_d \in c_0(Q)$.*

(3) *For each $f \in CD_0(Q)$ we have $\|f_c\| \leqslant \|f\|$.*

(4) *A function $f \colon Q \to \mathbb{R}$ belongs to $CD_0(Q)$ if and only if the limit $\lim\limits_{p \to q} f(p)$ exists for every $q \in Q$. Furthermore, $\lim\limits_{p \to q} f(p) = f_c(q)$ for all $q \in Q$. In particular, $f \in c_0(Q)$ if and only if $\lim\limits_{p \to q} f(p) = 0$ for all $q \in Q$.*

3.3. As is easily seen, $CD_0(Q)$ is a Banach algebra (with respect to the pointwise multiplication) which contains $C(Q)$ as a subalgebra and $c_0(Q)$ as an algebraic ideal. Furthermore, for all $f, g \in CD_0(Q)$ we have $(fg)_c = f_c g_c$ and $(fg)_d = f_c g_d + f_d g_d + f_d g_c$.

3.4. In what follows, $\mathcal{X}$ is an arbitrary Banach bundle over Q. Denote by $c_0(Q, \mathcal{X})$ the set of all sections of $\mathcal{X}$ whose pointwise norm belongs to $c_0(Q)$:

$$c_0(Q, \mathcal{X}) \ := \ \{u \in S(Q, \mathcal{X}) : |\!|\!|u|\!|\!| \in c_0(Q)\}.$$

Note that $c_0(Q, \mathcal{X})$ is a Banach subspace of the Banach space $\ell^\infty(Q, \mathcal{X})$ of all bounded sections of $\mathcal{X}$ (with the uniform norm) and coincides with the closure in $\ell^\infty(Q, \mathcal{X})$ of the space of sections with finite support.

The following assertion is a direct consequence of 3.2 (4):

A section $u \in S(Q, \mathcal{X})$ belongs to $c_0(Q, \mathcal{X})$ if and only if

$$\lim_{p \to q} u(p) = 0 \ \ \textit{for all } q \in Q.$$

3.5. Define $CD_0(Q, \mathcal{X})$ to be the space of all sections $u \in S(Q, \mathcal{X})$ representable as the sums $u = v + w$ of elements $v \in C(Q, \mathcal{X})$ and $w \in c_0(Q, \mathcal{X})$:

$$CD_0(Q, \mathcal{X}) \ := \ C(Q, \mathcal{X}) + c_0(Q, \mathcal{X}).$$

The following assertion is a restatement of [13, Lemma 1]:

We have the direct sum decomposition

$$CD_0(Q, \mathcal{X}) = C(Q, \mathcal{X}) \oplus c_0(Q, \mathcal{X}).$$

$\lhd$ Indeed, if $u \in C(Q, \mathcal{X}) \cap c_0(Q, \mathcal{X})$ then $|\!|\!|u|\!|\!| \in C(Q) \cap c_0(Q)$, whence due to 3.2 (2) we have $|\!|\!|u|\!|\!| = 0$ and thus $u = 0$. $\rhd$

The decomposition $CD_0(Q, \mathcal{X}) = C(Q, \mathcal{X}) \oplus c_0(Q, \mathcal{X})$ makes it possible to introduce the linear projections $(\cdot)_c$ and $(\cdot)_d$ from $CD_0(Q, \mathcal{X})$ onto the corresponding subspaces $C(Q, \mathcal{X})$ and $c_0(Q, \mathcal{X})$. Therefore, each section $u \in CD_0(Q, \mathcal{X})$ is uniquely representable as $u = u_c + u_d$, with $u_c \in C(Q, \mathcal{X})$ and $u_d \in c_0(Q, \mathcal{X})$.

3.6. *A section $u \in S(Q, \mathcal{X})$ belongs to $CD_0(Q, \mathcal{X})$ if and only if the limit $\lim\limits_{p \to q} u(p)$ exists for each $q \in Q$. Furthermore,*

$$\lim_{p \to q} u(p) = u_c(q) \ \ \textit{for all } q \in Q.$$

$\lhd$ If $u \in CD_0(Q, \mathcal{X})$ then due to 3.4 we have

$$\lim_{p \to q} \|u(p) - u_c(p)\| = \lim_{p \to q} \|u_d(p)\| = 0$$

for each $q \in Q$, whence $\lim\limits_{p \to q} u(p) = u_c(q)$ according to 2.6.

Passing to the proof of sufficiency, assume that the limit $v(q) := \lim\limits_{p \to q} u(p)$ exists for all $q \in Q$. Given $w \in C(Q, \mathcal{X})$ and $q \in Q$, we have

$$\|v(q) - w(q)\| = \Big\| \lim_{p \to q} u(p) - w(q) \Big\| = \lim_{p \to q} \|u(p) - w(p)\|;$$

therefore, $|\!|\!|v - w|\!|\!| = |\!|\!|u - w|\!|\!|_c \in C(Q)$ by 3.2 (4). Since $w \in C(Q, \mathcal{X})$ is arbitrary, we conclude that $v \in C(Q, \mathcal{X})$. On the other hand,

$$\lim_{p \to q} \|u(p) - v(p)\| = \Big\| \lim_{p \to q} u(p) - \lim_{p \to q} v(p) \Big\| = \|v(q) - v(q)\| = 0$$

for all $q \in Q$; consequently, $u - v \in c_0(Q, \mathcal{X})$ according to 3.4. $\rhd$

3.7. *If $u \in CD_0(Q, \mathcal{X})$ then $|||u||| \in CD_0(Q)$. Furthermore,*

$$|||u|||_c = |||u_c|||, \quad \big||||u|||_d\big| \leqslant |||u_d|||.$$

◁ From 3.2 (4) and 3.6 it follows that $|||u||| \in CD_0(Q)$ and $|||u|||_c = |||u_c|||$. In addition,

$$\big||||u|||_d\big| = \big||||u||| - |||u|||_c\big| = \big||||u||| - |||u_c|||\big| \leqslant |||u - u_c||| = |||u_d|||. \quad ▷$$

(Observe that the inequality $\big||||u|||_d\big| \neq |||u_d|||$ is obviously possible. For instance, if $q \in Q$, $v \in C(Q, \mathcal{X})$, $v(q) \neq 0$, and $u = v - 2\chi_{\{q\}}v$, then $|||u|||_d = 0$, while $|||u_d||| = 2\|v(q)\|\chi_{\{q\}}$.)

Therefore, the space $CD_0(Q, \mathcal{X})$ endowed with the pointwise norm $|||\cdot|||$ is a lattice-normed space over the Banach lattice $CD_0(Q)$, as well as a space with mixed norm: $\|u\| = \big\||||u|||\big\|$, $u \in CD_0(Q, \mathcal{X})$ (see [15, 7.1.1]).

3.8. *For each $u \in CD_0(Q, \mathcal{X})$ we have*

$$\|u_c\| \leqslant \|u\|, \quad \|u_d\| \leqslant 2\|u\|.$$

◁ From 3.2 (3) and 3.7 it follows that

$$\|u_c\| = \big\||||u_c|||\big\| = \big\||||u|||_c\big\| \leqslant \big\||||u|||\big\| = \|u\|.$$

It remains to observe that $\|u_d\| = \|u - u_c\| \leqslant \|u\| + \|u_c\| \leqslant 2\|u\|$. ▷

3.9. *The normed space $CD_0(Q, \mathcal{X})$ is a Banach space.*

◁ Is (u_n) is a Cauchy sequence in $CD_0(Q, \mathcal{X})$ then, according to 3.8, $(u_n)_c$ and $(u_n)_d$ are also Cauchy sequences which, due to completeness of the normed spaces $C(Q, \mathcal{X})$ and $c_0(Q, \mathcal{X})$, have the corresponding uniform limits $v \in C(Q, \mathcal{X})$ and $w \in c_0(Q, \mathcal{X})$. Therefore, the sequence (u_n) uniformly converges to the sum $v + w$ that belongs to $CD_0(Q, \mathcal{X})$. ▷

3.10. The following assertion is straightforward:

The space $CD_0(Q, \mathcal{X})$ is a Banach $CD_0(Q)$-module with respect to the pointwise multiplication. Furthermore, for all $f \in CD_0(Q)$ and $u \in CD_0(Q, \mathcal{X})$ we have

$$(fu)_c = f_c u_c, \quad (fu)_d = f_c u_d + f_d u_d + f_d u_c.$$

3.11. In what follows, we will use the notation of 1.5. Namely, we define the mappings $(\cdot)_c, (\cdot)_d \colon Q \to Q \times \{0, 1\}$ by the formulas

$$q_c := (q, 0), \quad q_d := (q, 1)$$

and, for every subset $P \subset Q$, put

$$P_c := \{p_c : p \in P\} = P \times \{0\}, \quad P_d := \{p_d : p \in P\} = P \times \{1\}.$$

As in [16], introduce a topology on $Q \times \{0, 1\} = Q_c \cup Q_d$ as follows: endow the subset $Q_d \subset Q$ with the discrete topology (i.e., declare all the points q_d isolated) and, for each $q \in Q$, declare a subset $U \subset Q \times \{0, 1\}$ to be a neighborhood of q_c whenever there exists a neighborhood $V \subset Q$ of q such that $V_c \cup V_d \backslash \{q_d\} \subset U$. The topological space thus obtained is a compact Hausdorff space which is called the *Alexandroff duplicate* of Q and denoted by $\widetilde{Q}$ (see 1.5 and the references therein).

3.12. The following result is obtained in [8] (see also [16]):

The Banach lattices $CD_0(Q)$ and $C\big(\widetilde{Q}\big)$ are linearly isometric and order isomorphic.

A linearly isometric order isomorphism between $CD_0(Q)$ and $C\big(\widetilde{Q}\big)$ is performed by the mapping $f \mapsto \widetilde{f}$ which takes each $f \in CD_0(Q)$ into the function $\widetilde{f} \colon \widetilde{Q} \to \mathbb{R}$ defined by the rule

$$\widetilde{f}(q_c) = f_c(q), \quad \widetilde{f}(q_d) = f(q) \quad \text{for all} \quad q \in Q.$$

Therefore, $\widetilde{f}(\cdot, 0) = f_c$ and $\widetilde{f}(\cdot, 1) = f$.

As is easily seen, according to 3.3, the mapping $f \mapsto \widetilde{f}$ preserves multiplication and is thus an isomorphism between the algebras $CD_0(Q)$ and $C\big(\widetilde{Q}\big)$.

3.13. The criterion presented below is immediate from 3.12:

The following properties of a function $g \colon \widetilde{Q} \to \mathbb{R}$ are equivalent:
(1) $g \in C\big(\widetilde{Q}\big)$;
(2) $g(\cdot, 0) \in C(Q)$, $g(\cdot, 1) - g(\cdot, 0) \in c_0(Q)$;
(3) $g(\cdot, 1) \in CD_0(Q)$, $g(\cdot, 0) = g(\cdot, 1)_c$.

Observe also that the images of $C(Q)$ and $c_0(Q)$ under the isometry $f \mapsto \widetilde{f}$ are described as follows:

$$\begin{aligned}
\big\{\, \widetilde{f} \;:\; f \in C(Q) \,\big\} &= \big\{\, g \in C\big(\widetilde{Q}\big) \;:\; g(\cdot, 0) = g(\cdot, 1) \,\big\} \\
&= \big\{\, g \colon \widetilde{Q} \to \mathbb{R} \;:\; g(\cdot, 0) = g(\cdot, 1) \in C(Q) \,\big\}, \\
\big\{\, \widetilde{f} \;:\; f \in c_0(Q) \,\big\} &= \big\{\, g \in C\big(\widetilde{Q}\big) \;:\; g(\cdot, 0) = 0 \,\big\} \\
&= \big\{\, g \colon \widetilde{Q} \to \mathbb{R} \;:\; g(\cdot, 0) = 0,\ g(\cdot, 1) \in c_0(Q) \,\big\}.
\end{aligned}$$

3.14. Following [13], consider the (temporarily discrete) Banach bundle $\widetilde{\mathcal{X}}$ over $\widetilde{Q} = Q_c \cup Q_d$ with stalks

$$\widetilde{\mathcal{X}}(q_c) = \widetilde{\mathcal{X}}(q_d) = \mathcal{X}(q), \quad q \in Q.$$

Given a section $u \in CD_0(Q, \mathcal{X})$, define the section $\widetilde{u} \in S\big(\widetilde{Q}, \widetilde{\mathcal{X}}\big)$ by the rule

$$\widetilde{u}(q_c) = u_c(q), \quad \widetilde{u}(q_d) = u(q) \quad \text{for all} \quad q \in Q.$$

Therefore, $\widetilde{u}(\cdot, 0) = u_c$ and $\widetilde{u}(\cdot, 1) = u$. Note that, due to 3.7, we have $|\!|\!|\widetilde{u}|\!|\!| = \widetilde{|\!|\!|u|\!|\!|}$ for all $u \in CD_0(Q, \mathcal{X})$.

Show that the set $\mathcal{C}_{\widetilde{\mathcal{X}}} := \{\widetilde{u} : u \in CD_0(Q, \mathcal{X})\}$ is a continuous structure in $\widetilde{\mathcal{X}}$. Indeed, since the mapping $u \mapsto \widetilde{u}$ is obviously linear, $\mathcal{C}_{\widetilde{\mathcal{X}}}$ is a vector subspace of $S\big(\widetilde{Q}, \widetilde{\mathcal{X}}\big)$. In addition, for each section $u \in CD_0(Q, \mathcal{X})$ we have $|\!|\!|\widetilde{u}|\!|\!| = \widetilde{|\!|\!|u|\!|\!|} \in C\big(\widetilde{Q}\big)$. Finally, $\mathcal{C}_{\widetilde{\mathcal{X}}}$ contains the set $\{\widetilde{u} : u \in C(Q, \mathcal{X})\}$ which is a stalkwise dense in $\widetilde{\mathcal{X}}$; therefore, $\mathcal{C}_{\widetilde{\mathcal{X}}}$ is itself stalkwise dense in $\widetilde{\mathcal{X}}$.

In what follows, we keep the notation $\widetilde{\mathcal{X}}$ for the continuous Banach bundle $\big(\widetilde{\mathcal{X}}, \mathcal{C}_{\widetilde{\mathcal{X}}}\big)$ over $\widetilde{Q}$.

3.15. The following result is established in [13, Proposition 6]. (For completeness of exposition, we will present the result together with our version of its proof.)

The mapping $u \mapsto \widetilde{u}$ is a linear isometry of $CD_0(Q, \mathcal{X})$ onto $C\big(\widetilde{Q}, \widetilde{\mathcal{X}}\big)$. Moreover, $\widetilde{fu} = \widetilde{f}\widetilde{u}$ for all $f \in CD_0(Q)$ and $u \in CD_0(Q, \mathcal{X})$.

◁ If $f \in CD_0(Q)$ and $u \in CD_0(Q, \mathcal{X})$ then, due to 3.10, we have

$$\widetilde{fu}\,(\cdot, 0) = (fu)_c = f_c u_c = \big(\widetilde{f}\widetilde{u}\big)(\cdot, 0), \quad \widetilde{fu}\,(\cdot, 1) = fu = \big(\widetilde{f}\widetilde{u}\big)(\cdot, 1)$$

and thereby $\widetilde{fu} = \widetilde{f}\widetilde{u}$. Consequently, $\mathcal{C}_{\widetilde{\mathcal{X}}}$ is a $C\big(\widetilde{Q}\big)$-submodule of $C\big(\widetilde{Q}, \widetilde{\mathcal{X}}\big)$. Taking account of the fact that $\mathcal{C}_{\widetilde{\mathcal{X}}}$ is stalkwise dense in $\widetilde{\mathcal{X}}$ and using the corollary [9, 4.3] to the Stone–Weierstrass Theorem for bundles, we conclude that the submodule $\mathcal{C}_{\widetilde{\mathcal{X}}}$ is uniformly dense in $C\big(\widetilde{Q}, \widetilde{\mathcal{X}}\big)$. Next, 3.8 imply that, for all $u \in CD_0(Q, \mathcal{X})$,

$$\|\widetilde{u}\| = \max\{\|\widetilde{u}(\cdot, 0)\|, \|\widetilde{u}(\cdot, 1)\|\} = \max\{\|u_c\|, \|u\|\} = \|u\|.$$

Therefore, the mapping $u \mapsto \widetilde{u}$ is a linear isometry of the Banach space $CD_0(Q, \mathcal{X})$ onto a dense subspace $\mathcal{C}_{\widetilde{\mathcal{X}}} \subset C\big(\widetilde{Q}, \widetilde{\mathcal{X}}\big)$; whence $\mathcal{C}_{\widetilde{\mathcal{X}}} = C\big(\widetilde{Q}, \widetilde{\mathcal{X}}\big)$. ▷

3.16. The criterion presented below is immediate from 3.15:

The following properties of a section $v \in S\big(\widetilde{Q}, \widetilde{\mathcal{X}}\big)$ are equivalent:

(1) $v \in C\big(\widetilde{Q}, \widetilde{\mathcal{X}}\big)$;
(2) $v(\cdot, 0) \in C(Q, \mathcal{X})$, $v(\cdot, 1) - v(\cdot, 0) \in c_0(Q, \mathcal{X})$;
(3) $v(\cdot, 1) \in CD_0(Q, \mathcal{X})$, $v(\cdot, 0) = v(\cdot, 1)_c$.

Observe also that the images of $C(Q, \mathcal{X})$ and $c_0(Q, \mathcal{X})$ under the isometry $u \mapsto \widetilde{u}$ are described as follows:

$$\begin{aligned}
\{\, \widetilde{u} \;:\; u \in C(Q, \mathcal{X}) \,\} &= \{\, v \in C\big(\widetilde{Q}, \widetilde{\mathcal{X}}\big) \;:\; v(\cdot, 0) = v(\cdot, 1) \,\} \\
&= \{\, v \in S\big(\widetilde{Q}, \widetilde{\mathcal{X}}\big) \;:\; v(\cdot, 0) = v(\cdot, 1) \in C(Q, \mathcal{X}) \,\}, \\
\{\, \widetilde{u} \;:\; u \in c_0(Q, \mathcal{X}) \,\} &= \{\, v \in C\big(\widetilde{Q}, \widetilde{\mathcal{X}}\big) \;:\; v(\cdot, 0) = 0 \,\} \\
&= \{\, v \in S\big(\widetilde{Q}, \widetilde{\mathcal{X}}\big) \;:\; v(\cdot, 0) = 0,\ v(\cdot, 1) \in c_0(Q, \mathcal{X}) \,\}.
\end{aligned}$$

4. Examples

In this section we present a series of examples which clarify the relation between $\mathcal{X}$ and $\widetilde{\mathcal{X}}$ in the case of constant bundles, as well as demonstrate that this relation meets the passage to a subbundle, a continuous change of variable, and the restriction to a topological subspace.

4.1. Consider an arbitrary Banach space X and assume that $\mathcal{X}$ is the constant Banach bundle over Q with stalk X. From the definition of $\widetilde{\mathcal{X}}$ it is clear that all its stalks coincide with X. Denote by $\operatorname{const}(Q, X)$ and $\operatorname{const}\big(\widetilde{Q}, X\big)$ the sets of all constant sections of $\mathcal{X}$ and $\widetilde{\mathcal{X}}$. As is easily seen,

$$\operatorname{const}\big(\widetilde{Q}, X\big) = \{\widetilde{c} : c \in \operatorname{const}(Q, X)\} \subset C\big(\widetilde{Q}, \widetilde{\mathcal{X}}\big).$$

Consequently, $\operatorname{const}\big(\widetilde{Q}, X\big)$ is a continuous structure in $\widetilde{\mathcal{X}}$ equivalent to $\mathcal{C}_{\widetilde{\mathcal{X}}}$; hence, $\widetilde{\mathcal{X}}$ is a constant Banach bundle over $\widetilde{Q}$ with stalk X.

Due to 2.2, the above observation allows us to translate (almost without changes) all the facts of Section 3 to the case of the spaces of vector valued functions $CD_0(Q,X) = C(Q,X) + c_0(Q,X)$ and $C\big(\widetilde{Q},X\big)$.

4.2. Let $\mathcal{X}_0$ be a subbundle of a Banach bundle $\mathcal{X}$ over Q. The latter means that $\mathcal{X}_0$ is a Banach bundle over Q such that $\mathcal{X}_0(q)$ is a Banach subspace of $\mathcal{X}(q)$ for each $q \in Q$ and, in addition, $C(Q,\mathcal{X}_0) = C(Q,\mathcal{X}) \cap S(Q,\mathcal{X}_0)$ (see [10, 2.2.2], [15, 2.4.11]). Taking account of the obvious equality $c_0(Q,\mathcal{X}_0) = c_0(Q,\mathcal{X}) \cap S(Q,\mathcal{X}_0)$, we conclude that

$$CD_0(Q,\mathcal{X}_0) \subset CD_0(Q,\mathcal{X}) \cap S(Q,\mathcal{X}_0).$$

Below we will show that the sets $CD_0(Q,\mathcal{X}_0)$ and $CD_0(Q,\mathcal{X}) \cap S(Q,\mathcal{X}_0)$ may differ or coincide; moreover, both cases are possible for a nontrivial subbundle $\mathcal{X}_0$ (i.e., for a subbundle which is neither zero, nor equal to the whole $\mathcal{X}$). However, we will first prove a simple auxiliary assertion.

4.3. *Let $\mathcal{X}_0$ be a subbundle of $\mathcal{X}$. The following properties of an arbitrary section $u \in CD_0(Q,\mathcal{X}) \cap S(Q,\mathcal{X}_0)$ are equivalent:*

(1) $u \in CD_0(Q,\mathcal{X}_0)$;

(2) $u_c \in S(Q,\mathcal{X}_0)$;

(3) $u_d \in S(Q,\mathcal{X}_0)$,

where the decomposition $u = u_c + u_d$ is taken in $CD_0(Q,\mathcal{X}) = C(Q,\mathcal{X}) \oplus c_0(Q,\mathcal{X})$.

$\triangleleft$ (1)$\Rightarrow$(2): Let $u \in CD_0(Q,\mathcal{X}_0)$. Consider the decomposition $u = u_c^\circ + u_d^\circ$ in $CD_0(Q,\mathcal{X}_0)$. Then $u_c^\circ \in C(Q,\mathcal{X}_0) \subset C(Q,\mathcal{X})$ and $u_d^\circ \in c_0(Q,\mathcal{X}_0) \subset c_0(Q,\mathcal{X})$; whence, due to the uniqueness of a decomposition $u = u_c + u_d$ in $CD_0(Q,\mathcal{X})$, we have $u_c^\circ = u_c$ and thus $u_c \in S(Q,\mathcal{X}_0)$.

The implication (2)$\Rightarrow$(3) is obvious, since $u_d = u - u_c$.

(3)$\Rightarrow$(1): Let $u_d \in S(Q,\mathcal{X}_0)$. Then $u_c = u - u_d \in S(Q,\mathcal{X}_0)$ and thus $u_c \in C(Q,\mathcal{X}) \cap S(Q,\mathcal{X}_0) = C(Q,\mathcal{X}_0)$. In addition, $u_d \in c_0(Q,\mathcal{X}) \cap S(Q,\mathcal{X}_0) = c_0(Q,\mathcal{X}_0)$. Consequently, $u = u_c + u_d \in C(Q,\mathcal{X}_0) + c_0(Q,\mathcal{X}_0) = CD_0(Q,\mathcal{X}_0)$. $\triangleright$

4.4. *Every nonzero Banach bundle $\mathcal{X}$ over Q contains nontrivial (i.e., neither zero, nor equal to the whole $\mathcal{X}$) subbundles $\mathcal{X}_1$ and $\mathcal{X}_2$ such that*

$$CD_0(Q,\mathcal{X}_1) = CD_0(Q,\mathcal{X}) \cap S(Q,\mathcal{X}_1), \tag{1}$$

$$CD_0(Q,\mathcal{X}_2) \neq CD_0(Q,\mathcal{X}) \cap S(Q,\mathcal{X}_2). \tag{2}$$

$\triangleleft$ Let $\mathcal{X}$ be an arbitrary nonzero Banach bundle over Q.

First of all, note that for each closed subset $V \subset Q$ there exists a subbundle $\mathcal{X}_V$ of $\mathcal{X}$ which has the following stalks:

$$\mathcal{X}_V(q) = \begin{cases} \{0\}, & q \in V, \\ \mathcal{X}(q), & q \in Q \backslash V. \end{cases}$$

Indeed, according to [10, 2.2.2], it suffices to show that

$$\big\{u(q) : u \in C(Q,\mathcal{X}),\ u = 0 \text{ on } V\big\} = \mathcal{X}(q) \ \text{ for all } q \in Q \backslash V.$$

Let $q \in Q \backslash V$ and $x \in \mathcal{X}(q)$. By the Dupré Theorem, there is a section $v \in C(Q,\mathcal{X})$ such that $v(q) = x$. In addition, since Q is completely regular, there exists a function $f \in C(Q)$ such that $f = 0$ on V and $f(q) = 1$. Then $fv \in C(Q,\mathcal{X})$, $fv = 0$ on V, and $(fv)(q) = x$.

As is easily seen, there are distinct points $q_1, q_2 \in Q$ at which the bundle $\mathcal{X}$ has nonzero stalks. Since Q is Hausdorff, there exists an open subset $U \subset Q$ such that $q_1 \in U$ and $q_2 \in Q \setminus \operatorname{cl} U$. In this case, the subbundle $\mathcal{X}_1 := \mathcal{X}_{\operatorname{cl} U}$ of $\mathcal{X}$ is nontrivial. Due to 4.3, for proving (1) it suffices to consider an arbitrary section $u \in CD_0(Q, \mathcal{X}) \cap S(Q, \mathcal{X}_1)$ and show that $u_c \in S(Q, \mathcal{X}_1)$. Indeed, according to 3.6, for all $q \in \operatorname{cl} U$ we have

$$u_c(q) = \lim_{p \to q} u(p) = \lim_{p \to q} u|_U(p) = 0;$$

hence, $u_c \in S(Q, \mathcal{X}_1)$.

Now take an arbitrary point $q \in Q$ at which $\mathcal{X}(q) \neq \{0\}$ and put $\mathcal{X}_2 := \mathcal{X}_{\{q\}}$. By the Dupré Theorem, there exists a section $v \in C(Q, \mathcal{X})$ such that $v(q) \neq 0$. Put $u = v - \chi_{\{q\}} v$. Then $u \in CD_0(Q, \mathcal{X}) \cap S(Q, \mathcal{X}_2)$, but $u_c = v \notin S(Q, \mathcal{X}_2)$; whence, due to 4.3, it follows that $u \notin CD_0(Q, \mathcal{X}_2)$ and thereby (2) holds. $\triangleright$

4.5. The following assertion shows that, despite the possible absence of the equality $CD_0(Q, \mathcal{X}_0) = CD_0(Q, \mathcal{X}) \cap S(Q, \mathcal{X}_0)$, the analogous equality always holds for the "continuous versions" of the spaces under consideration:

If $\mathcal{X}_0$ is a subbundle of $\mathcal{X}$ then $\widetilde{\mathcal{X}_0}$ is a subbundle of $\widetilde{\mathcal{X}}$. In particular,

$$C\big(\widetilde{Q}, \widetilde{\mathcal{X}_0}\big) = C\big(\widetilde{Q}, \widetilde{\mathcal{X}}\big) \cap S\big(\widetilde{Q}, \widetilde{\mathcal{X}_0}\big).$$

$\triangleleft$ From the definition of the bundle $\widetilde{\mathcal{X}_0}$ it is clear that each of its stalks is a Banach subspace of the corresponding stalk of $\widetilde{\mathcal{X}}$. Moreover, by 3.16 we have

$$\begin{aligned}
C\big(\widetilde{Q}, \widetilde{\mathcal{X}_0}\big) &= \big\{ v \in S\big(\widetilde{Q}, \widetilde{\mathcal{X}_0}\big) \ : \ v(\cdot, 0) \in C(Q, \mathcal{X}_0),\ v(\cdot, 1) - v(\cdot, 0) \in c_0(Q, \mathcal{X}_0) \big\} \\
&= \big\{ v \in S\big(\widetilde{Q}, \widetilde{\mathcal{X}_0}\big) \ : \ v(\cdot, 0) \in C(Q, \mathcal{X}) \cap S(Q, \mathcal{X}_0), \\
&\qquad\qquad v(\cdot, 1) - v(\cdot, 0) \in c_0(Q, \mathcal{X}) \cap S(Q, \mathcal{X}_0) \big\} \\
&= \big\{ v \in S\big(\widetilde{Q}, \widetilde{\mathcal{X}_0}\big) \ : \ v(\cdot, 0) \in C(Q, \mathcal{X}),\ v(\cdot, 1) - v(\cdot, 0) \in c_0(Q, \mathcal{X}) \big\} \\
&= \big\{ v \in S\big(\widetilde{Q}, \widetilde{\mathcal{X}_0}\big) \ : \ v \in C\big(\widetilde{Q}, \widetilde{\mathcal{X}}\big) \big\} \\
&= C\big(\widetilde{Q}, \widetilde{\mathcal{X}}\big) \cap S\big(\widetilde{Q}, \widetilde{\mathcal{X}_0}\big). \quad \triangleright
\end{aligned}$$

Returning to the example in 4.4 of a section $u \in CD_0(Q, \mathcal{X}) \cap S(Q, \mathcal{X}_2)$ which does not belong to $CD_0(Q, \mathcal{X}_2)$, note that the corresponding continuous section $\widetilde{u} \in C\big(\widetilde{Q}, \widetilde{\mathcal{X}}\big)$ goes out of $\widetilde{\mathcal{X}_2}$ at some point; namely, $\widetilde{u}(q_c) = u_c(q) \notin \widetilde{\mathcal{X}_2}(q_c)$.

4.6. Let P and Q be nonempty compact Hausdorff spaces without isolated points. Say that a function $\varphi \colon P \to Q$ is *locally unique* if, for each point $p_0 \in P$, there is a neighborhood $U \subset P$ of p_0 such that $\varphi(p) \neq \varphi(p_0)$ for all $p \in U \setminus \{p_0\}$.

As is easily seen, a function $\varphi \colon P \to Q$ is continuous and locally unique if and only if, for every point $p \in P$ and every neighborhood $V \subset Q$ of $\varphi(p)$, there is a neighborhood $U \subset P$ of p such that $\varphi(U \setminus \{p\}) \subset V \setminus \{\varphi(p)\}$.

4.7. Recall that, given a Banach bundle $\mathcal{X} = (\mathcal{X}, \mathcal{C}_{\mathcal{X}})$ over Q and a continuous function $\varphi \colon P \to Q$, the symbol $\mathcal{X} \circ \varphi$ denotes the Banach bundle over P with stalks $(\mathcal{X} \circ \varphi)(p) = \mathcal{X}\big(\varphi(p)\big)$ and continuous structure $\{c \circ \varphi : c \in \mathcal{C}_{\mathcal{X}}\}$ (see [10, 2.2.6]). As is easily seen, $u \circ \varphi \in C(P, \mathcal{X} \circ \varphi)$ for all $u \in C(Q, \mathcal{X})$.

Let P and Q be nonempty compact Hausdorff spaces without isolated points. The following properties of a continuous function $\varphi\colon P \to Q$ are equivalent:

(1) *φ is locally unique;*
(2) *the preimage $\varphi^{-1}(q)$ of every point $q \in Q$ is finite;*
(3) *if $f \in c_0(Q)$ then $f \circ \varphi \in c_0(P)$;*
(4) *if $f \in CD_0(Q)$ then $f \circ \varphi \in CD_0(P)$, $(f \circ \varphi)_c = f_c \circ \varphi$, $(f \circ \varphi)_d = f_d \circ \varphi$;*
(5) *is $\mathcal{X}$ is a Banach bundle over Q and $u \in c_0(Q, \mathcal{X})$ then $u \circ \varphi \in c_0(P, \mathcal{X} \circ \varphi)$;*
(6) *if $\mathcal{X}$ is a Banach bundle over Q and $u \in CD_0(Q, \mathcal{X})$ then $u \circ \varphi \in CD_0(P, \mathcal{X} \circ \varphi)$, $(u \circ \varphi)_c = u_c \circ \varphi$, $(u \circ \varphi)_d = u_d \circ \varphi$.*

$\triangleleft$ (6)⇒(5): If $u \in c_0(Q, \mathcal{X})$ then $u_d = u$, whence by (6) we have

$$u \circ \varphi = u_d \circ \varphi = (u \circ \varphi)_d \in c_0(P, \mathcal{X} \circ \varphi).$$

(5)⇒(4): If $f \in CD_0(Q)$ then $f \circ \varphi = f_c \circ \varphi + f_d \circ \varphi$; furthermore, $f_c \circ \varphi \in C(P)$ due to continuity of φ, and $f_d \circ \varphi \in c_0(P)$ by (5). It remains to use 3.2 (2).

(4)⇒(3): This is established in the same way as (6)⇒(5).

(3)⇒(2): Consider an arbitrary point $q \in Q$. Since $\chi_{\{q\}} \in c_0(Q)$, we have $\chi_{\varphi^{-1}(q)} = \chi_{\{q\}} \circ \varphi \in c_0(P)$ by (3). Consequently, by the definition of $c_0(P)$, the set $\varphi^{-1}(q) = \bigl\{p \in P : \chi_{\varphi^{-1}(q)}(p) > \frac{1}{2}\bigr\}$ is finite.

(2)⇒(1): This is straightforward from the fact that P is Hausdorff.

(1)⇒(6): If $u \in CD_0(Q, \mathcal{X})$ then $u \circ \varphi = u_c \circ \varphi + u_d \circ \varphi$; furthermore, $u_c \circ \varphi \in C(P, \mathcal{X} \circ \varphi)$. According to 3.5, it remains to show the inclusion $u_d \circ \varphi \in c_0(P, \mathcal{X} \circ \varphi)$. From 3.4 and (1) we have

$$\lim_{p \to p_0} u_d\bigl(\varphi(p)\bigr) = \lim_{q \to \varphi(p_0)} u_d(q) = 0$$

for all $p_0 \in P$; hence, $u_d \circ \varphi \in c_0(P, \mathcal{X} \circ \varphi)$ due to 3.4. $\triangleright$

4.8. Let P and Q be nonempty compact Hausdorff spaces without isolated points. Given a function $\varphi\colon P \to Q$, define the function $\widetilde{\varphi}\colon \widetilde{P} \to \widetilde{Q}$ by putting

$$\widetilde{\varphi}(p_c) = \varphi(p)_c, \quad \widetilde{\varphi}(p_d) = \varphi(p)_d \quad \text{for all} \quad p \in P.$$

The function $\widetilde{\varphi}\colon \widetilde{P} \to \widetilde{Q}$ is continuous if and only if $\varphi\colon P \to Q$ is continuous and locally unique.

$\triangleleft$ Since all the points of the subset $P_d \subset \widetilde{P}$ are isolated, $\widetilde{\varphi}$ is continuous on P_d regardless of the properties of φ. It remains to observe that, for every point $p \in P$, every neighborhood $U \subset P$ of p, and every neighborhood $V \subset Q$ of $\varphi(p)$, the inclusion $\varphi(U \backslash \{p\}) \subset V \backslash \{\varphi(p)\}$ is equivalent to the inclusion

$$\widetilde{\varphi}(U_c \cup U_d \backslash \{p_d\}) \subset V_c \cup V_d \backslash \{\varphi(p)_d\}. \quad \triangleright$$

4.9. *Let P and Q be nonempty compact Hausdorff spaces without isolated points and let $\varphi\colon P \to Q$ be a continuous locally unique function. Then*

(1) *$\widetilde{u \circ \varphi} = \widetilde{u} \circ \widetilde{\varphi}$ for all $u \in CD_0(Q, \mathcal{X})$;*
(2) *$\widetilde{\mathcal{X} \circ \varphi} = \widetilde{\mathcal{X}} \circ \widetilde{\varphi}$.*

◁ (1): Due to 4.7 (6), for every section $u \in CD_0(Q, \mathscr{X})$ we have

$$\widetilde{u \circ \varphi}\,(\cdot, 0) = (u \circ \varphi)_c = u_c \circ \varphi = \widetilde{u}(\cdot, 0) \circ \varphi = (\widetilde{u} \circ \widetilde{\varphi})(\cdot, 0),$$
$$\widetilde{u \circ \varphi}\,(\cdot, 1) = u \circ \varphi = \widetilde{u}(\cdot, 1) \circ \varphi = (\widetilde{u} \circ \widetilde{\varphi})(\cdot, 1).$$

(2): Obviously, $\widetilde{\mathscr{X} \circ \varphi}$ and $\widetilde{\mathscr{X}} \circ \widetilde{\varphi}$ coincide as discrete Banach bundles (i.e., they have the same stalks). According to 2.4, to prove the coincidence of $\widetilde{\mathscr{X} \circ \varphi}$ and $\widetilde{\mathscr{X}} \circ \widetilde{\varphi}$ as continuous Banach bundles, it suffices to show that the intersection $C\big(\widetilde{P}, \widetilde{\mathscr{X} \circ \varphi}\big) \cap C\big(\widetilde{P}, \widetilde{\mathscr{X}} \circ \widetilde{\varphi}\big)$ is stalkwise dense in $\widetilde{\mathscr{X}} \circ \widetilde{\varphi}$. By 3.15, 4.7 (6), and 4.9 (1) we have

$$C\big(\widetilde{P}, \widetilde{\mathscr{X} \circ \varphi}\big) \ = \ \big\{\widetilde{v} \ : \ v \in CD_0\big(P, \mathscr{X} \circ \varphi\big)\big\} \ \supset \ \big\{\widetilde{u \circ \varphi} \ : \ u \in CD_0(Q, \mathscr{X})\big\}$$
$$= \ \big\{\widetilde{u} \circ \widetilde{\varphi} \ : \ u \in CD_0(Q, \mathscr{X})\big\} \ = \ \big\{w \circ \widetilde{\varphi} \ : \ w \in C\big(\widetilde{Q}, \widetilde{\mathscr{X}}\big)\big\} \ \subset \ C\big(\widetilde{P}, \widetilde{\mathscr{X}} \circ \widetilde{\varphi}\big).$$

It remains to observe that $\big\{w \circ \widetilde{\varphi} : w \in C\big(\widetilde{Q}, \widetilde{\mathscr{X}}\big)\big\}$ is stalkwise dense in $\widetilde{\mathscr{X}} \circ \widetilde{\varphi}$. ▷

4.10. Recall that, given a Banach bundle $\mathscr{X} = (\mathscr{X}, \mathcal{C}_{\mathscr{X}})$ over Q and a topological subspace $P \subset Q$, the symbol $\mathscr{X}|_P$ denotes the Banach bundle over P with stalks $(\mathscr{X}|_P)(p) = \mathscr{X}(p)$ and continuous structure $\{c|_P : c \in \mathcal{C}_{\mathscr{X}}\}$ (see [10, 2.2.5]). Note the obvious equality $C(P, \mathscr{X}|_P) = C(P, \mathscr{X})$, where $C(P, \mathscr{X})$ is the set of all continuous sections of $\mathscr{X}$ defined on P (see [10, 2.1.2]).

Let P be a nonempty compact Hausdorff space without isolated points which is a topological subspace of Q. Then $\widetilde{P}$ is a topological subspace of $\widetilde{Q}$ and the following hold:

(1) *if $u \in CD_0(Q, \mathscr{X})$ then $u|_P \in CD_0(P, \mathscr{X}|_P)$, $(u|_P)_c = u_c|_P$, $(u|_P)_d = u_d|_P$;*
(2) *$\widetilde{u|_P} = \widetilde{u}|_{\widetilde{P}}$ for all $u \in CD_0(Q, \mathscr{X})$;*
(3) *$\widetilde{\mathscr{X}|_P} = \widetilde{\mathscr{X}}\big|_{\widetilde{P}}$; in particular, $C\big(\widetilde{P}, \widetilde{\mathscr{X}|_P}\big) = C\big(\widetilde{P}, \widetilde{\mathscr{X}}\big)$.*

◁ Let φ be the identity embedding of P into Q. Then, as is easily seen, φ is continuous and locally unique, $\widetilde{\varphi}$ is the identity embedding of $\widetilde{P}$ into $\widetilde{Q}$, $\mathscr{X}|_P = \mathscr{X} \circ \varphi$, $\widetilde{\mathscr{X}}\big|_{\widetilde{P}} = \widetilde{\mathscr{X}} \circ \widetilde{\varphi}$, $u|_P = u \circ \varphi$, $\widetilde{u}|_{\widetilde{P}} = \widetilde{u} \circ \widetilde{\varphi}$ for all $u \in CD_0(Q, \mathscr{X})$. It remains to employ 4.7 (6), 4.8, and 4.9. ▷

4.11. The definition of $\widetilde{Q}$ readily implies that the mapping $(\cdot)_c \colon q \mapsto q_c$ is a homeomorphism of Q onto the closed subset $Q_c \subset \widetilde{Q}$. This observation allows us to consider Q as a topological subspace of $\widetilde{Q}$ by taking $(\cdot)_c$ as the identification.

Under the above agreement, the following hold:

(1) *$\widetilde{u} \circ (\cdot)_c = \widetilde{u}|_Q = u$ for all $u \in C(Q, \mathscr{X})$;*
(2) *$\widetilde{\mathscr{X}} \circ (\cdot)_c = \widetilde{\mathscr{X}}\big|_Q = \mathscr{X}$.*

◁ (1): If $u \in C(Q, \mathscr{X})$ then $\widetilde{u} \circ (\cdot)_c = \widetilde{u}(\cdot, 0) = u_c = u$.

(2): Obviously, $\mathscr{X}$ and $\widetilde{\mathscr{X}} \circ (\cdot)_c$ coincide as discrete Banach bundles (i.e., they have the same stalks). In addition, by (1) and 3.15 we have

$$C(Q, \mathscr{X}) = \{\, \widetilde{u} \circ (\cdot)_c \ : \ u \in C(Q, \mathscr{X}) \,\} \subset \{\, v \circ (\cdot)_c \ : \ v \in C\big(\widetilde{Q}, \widetilde{\mathscr{X}}\big) \,\} \subset C\big(Q, \widetilde{\mathscr{X}} \circ (\cdot)_c\big).$$

According to 2.4, the inclusion $C(Q, \mathscr{X}) \subset C\big(Q, \widetilde{\mathscr{X}} \circ (\cdot)_c\big)$ implies that $\mathscr{X}$ and $\widetilde{\mathscr{X}} \circ (\cdot)_c$ coincide as continuous Banach bundles. ▷

5. The space of CD_0-homomorphisms

In what follows, $\mathcal{X}$ and $\mathcal{Y}$ are arbitrary Banach bundles over a nonempty compact Hausdorff space Q without isolated points.

5.1. For simplicity, we will introduce some abbreviating notation.

Denote by $S[\mathcal{X},\mathcal{Y}]$ the vector space of all sections of the discrete Banach bundle over Q with stalks $B\big(\mathcal{X}(q),\mathcal{Y}(q)\big)$. Therefore, $S[\mathcal{X},\mathcal{Y}]$ consists of all functions H which are defined on Q and map each point $q\in Q$ into a bounded linear operator $H(q)\colon \mathcal{X}(q)\to\mathcal{Y}(q)$.

Let $\ell^\infty[\mathcal{X},\mathcal{Y}]$ be the Banach space of all bounded sections in $S[\mathcal{X},\mathcal{Y}]$ (endowed with the uniform norm):

$$\ell^\infty[\mathcal{X},\mathcal{Y}] \;=\; \big\{\, H\in S[\mathcal{X},\mathcal{Y}] \;:\; \|H\|\in\ell^\infty(Q)\,\big\}.$$

Given $H\in S[\mathcal{X},\mathcal{Y}]$ and $u\in S(Q,\mathcal{X})$, use the symbol Hu for denoting the section of $\mathcal{Y}$ defined by the formula $(Hu)(q)=H(q)u(q)$, $q\in Q$. (In [10], the symbol $H\otimes u$ is used instead of Hu.)

Denote by $C[\mathcal{X},\mathcal{Y}]$ the Banach subspace of $\ell^\infty[\mathcal{X},\mathcal{Y}]$ constituted by all homomorphisms from $\mathcal{X}$ into $\mathcal{Y}$ (see [10, 2.4.2, 2.4.11]). $\big($In [10], the notation $\mathrm{Hom}_Q(\mathcal{X},\mathcal{Y})$ is used instead of $C[\mathcal{X},\mathcal{Y}]$.$\big)$

5.2. According to [10, 2.4.7] we have

$$C[\mathcal{X},\mathcal{Y}] \;=\; \big\{\, H\in S[\mathcal{X},\mathcal{Y}] \;:\; Hu\in C(Q,\mathcal{Y}) \text{ for all } u\in C(Q,\mathcal{X})\,\big\}.$$

$\big($In the present article, the last equality can be considered as a definition of $C[\mathcal{X},\mathcal{Y}]$, cp. [15, 2.4.9].$\big)$

5.3. *The following equality holds:*

$$\ell^\infty[\mathcal{X},\mathcal{Y}] \;=\; \big\{\, H\in S[\mathcal{X},\mathcal{Y}] \;:\; Hu\in\ell^\infty(Q,\mathcal{Y}) \textit{ for all } u\in C(Q,\mathcal{X})\,\big\}.$$

$\lhd$ We only have to prove "$\supset$." Let a section $H\in S[\mathcal{X},\mathcal{Y}]$ be such that $Hu\in\ell^\infty(Q,\mathcal{Y})$ for all $u\in C(Q,\mathcal{X})$. Given a point $q\in Q$, define the linear operator $T_q\colon C(Q,\mathcal{X})\to\mathcal{Y}(q)$ by the formula $T_q u=H(q)u(q)$. Employing the Dupré Theorem, we obtain

$$\begin{aligned}\|T_q\| \;&=\; \sup\big\{\,\|H(q)u(q)\| \;:\; u\in C(Q,\mathcal{X}),\ \|u\|\leqslant 1\,\big\}\\ &=\; \sup\big\{\,\|H(q)x\| \;:\; x\in\mathcal{X}(q),\ \|x\|\leqslant 1\,\big\} \;=\; \|H(q)\|.\end{aligned}$$

In addition, $\sup_{q\in Q}\|T_q u\| = \sup_{q\in Q}\|(Hu)(q)\|<\infty$ for all $u\in C(Q,\mathcal{X})$. Since $C(Q,\mathcal{X})$ is a Banach space, the Uniform Boundedness Principle makes it possible to conclude that $\sup_{q\in Q}\|H(q)\|=\sup_{q\in Q}\|T_q\|<\infty$. $\rhd$

5.4. Consider the set

$$c_0[\mathcal{X},\mathcal{Y}] \;=\; \big\{\, H\in S[\mathcal{X},\mathcal{Y}] \;:\; Hu\in c_0(Q,\mathcal{Y}) \text{ for all } u\in C(Q,\mathcal{X})\,\big\}$$

and call its elements c_0*-homomorphisms* from $\mathcal{X}$ into $\mathcal{Y}$.

Employing 5.3 and taking account of the fact that $c_0(Q,\mathcal{Y})$ is complete, it is easy to show that $c_0[\mathcal{X},\mathcal{Y}]$ is a closed vector subspace of $\ell^\infty[\mathcal{X},\mathcal{Y}]$. In particular, $c_0[\mathcal{X},\mathcal{Y}]$ is a Banach space with respect to the uniform norm.

5.5. *A section $H \in S[\mathcal{X}, \mathcal{Y}]$ is a c_0-homomorphism if and only if H is bounded and there exists a subset $\mathcal{U} \subset C(Q, \mathcal{X})$ such that $\mathcal{U}$ is stalkwise dense in $\mathcal{X}$ and $Hu \in c_0(Q, \mathcal{Y})$ for all $u \in \mathcal{U}$.*

$\triangleleft$ We only have to prove sufficiency. Let H and $\mathcal{U}$ satisfy the condition stated. Omitting the trivial case $H \equiv 0$, assume that $C := \sup_{q \in Q} \|H(q)\| > 0$. Consider an arbitrary section $v \in C(Q, \mathcal{X})$ and show that $Hv \in c_0(Q, \mathcal{Y})$. By 3.4 it suffices to fix $q \in Q$ and $\varepsilon > 0$ and find a neighborhood U of q such that $\|H(p)v(p)\| < \varepsilon$ for all $p \in U \backslash \{q\}$. Since $\mathcal{U}$ is stalkwise dense in $\mathcal{X}$, there exists a section $u \in \mathcal{U}$ for which $\|u(q) - v(q)\| < \frac{\varepsilon}{2C}$. Due to 3.4, we have $\lim\limits_{p \to q} H(p)u(p) = 0$; hence, there is a neighborhood U of q such that $\|u(p) - v(p)\| < \frac{\varepsilon}{2C}$ and $\|H(p)u(p)\| < \frac{\varepsilon}{2}$ for all $p \in U \backslash \{q\}$. Then for all $p \in U \backslash \{q\}$ we have

$$\|H(p)v(p)\| \leqslant \|H(p)u(p)\| + \|H(p)\| \, \|u(p) - v(p)\| < \frac{\varepsilon}{2} + C \frac{\varepsilon}{2C} = \varepsilon. \quad \triangleright$$

5.6. The following assertion shows that the boundedness of H in 5.5 is an essential condition.

For every nonempty compact Hausdorff space Q without isolated points, there exist constant Banach bundles $\mathcal{X}$ and $\mathcal{Y}$ over Q, a section $H \in S[\mathcal{X}, \mathcal{Y}]$, and a subset $\mathcal{U} \subset C(Q, \mathcal{X})$ such that $\mathcal{U}$ is stalkwise dense in $\mathcal{X}$ and $Hu \in c_0(Q, \mathcal{Y})$ for all $u \in \mathcal{U}$, but H is unbounded.

$\triangleleft$ If Q is a nonempty compact Hausdorff space without isolated points, then, obviously, Q is infinite and thus contains a sequence of pairwise distinct points $q_n \in Q$ ($n \in \mathbb{N}$). Let c_0 be the Banach space of vanishing sequences, let s_{fin} be the dense subspace of c_0 constituted by the finitely-supported sequences, and let c_0' be the dual of c_0. Define the mapping $H : Q \to c_0'$ as follows:

$$H(q_n)\, x = n \cdot x(n) \quad \text{for} \quad n \in \mathbb{N}, \ x \in c_0,$$
$$H(q) = 0 \quad \text{for all} \quad q \in Q \setminus \{q_n : n \in \mathbb{N}\}.$$

Put $\mathcal{X} = Q \times \{c_0\}$, $\mathcal{Y} = Q \times \{\mathbb{R}\}$ and denote by $\mathcal{U}$ the totality of all constant functions $u : Q \to s_{\mathrm{fin}}$. Then $\mathcal{X}$, $\mathcal{Y}$, H, and $\mathcal{U}$ possess the required properties. $\triangleright$

5.7. Every section $H \in S[\mathcal{X}, \mathcal{Y}]$ whose pointwise norm $|\!|\!| H |\!|\!|$ belongs to $c_0(Q)$ is an example of a c_0-homomorphism. However, in general, the set $c_0[\mathcal{X}, \mathcal{Y}]$ is not exhausted by the sections of the above form. Indeed, from the constructions of [13, Example 9] it follows that, given a separable compact Hausdorff space Q, there exists a c_0-homomorphism whose pointwise norm equals unity an a dense subset of Q. We can provide a stronger version of the above fact by stating that, regardless of the properties of Q, the pointwise norm of a c_0-homomorphism can be an arbitrary bounded positive function $f : Q \to \mathbb{R}$.

For every nonempty compact Hausdorff space Q without isolated points, there exist constant Banach bundles $\mathcal{X}$ and $\mathcal{Y}$ over Q such that, given an arbitrary function $0 \leqslant f \in \ell^\infty(Q)$, there is a c_0-homomorphism $H \in c_0[\mathcal{X}, \mathcal{Y}]$ with pointwise norm $|\!|\!| H |\!|\!| = f$.

$\triangleleft$ Put $\mathcal{X} = Q \times \{c_0(Q)\}$, $\mathcal{Y} = Q \times \{\mathbb{R}\}$, and define the mapping $H : Q \to c_0(Q)'$ by putting $H(q)x = f(q)x(q)$ for all $q \in Q$ and $x \in c_0(Q)$. Then, as is easily seen, $H \in S[\mathcal{X}, \mathcal{Y}]$, $|\!|\!| H |\!|\!| = f$, and $Hu \in c_0(Q, \mathcal{Y})$ for all constant functions $u : Q \to c_0(Q)$; whence, due to 5.5, we have $H \in c_0[\mathcal{X}, \mathcal{Y}]$. $\triangleright$

5.8. Consider the set

$$CD_0[\mathcal{X},\mathcal{Y}] \;=\; \bigl\{\, H \in S[\mathcal{X},\mathcal{Y}] \;:\; Hu \in CD_0(Q,\mathcal{Y}) \text{ for all } u \in C(Q,\mathcal{X}) \,\bigr\}$$

and call its elements CD_0*-homomorphisms* from $\mathcal{X}$ into $\mathcal{Y}$.

From 5.3 and the completeness of $CD_0(Q,\mathcal{Y})$ (see 3.9) it is obvious that $CD_0[\mathcal{X},\mathcal{Y}]$ is a closed vector subspace of $\ell^\infty[\mathcal{X},\mathcal{Y}]$ and is thus a Banach space with respect to the uniform norm.

5.9. *The following equality holds:*

$$CD_0[\mathcal{X},\mathcal{Y}] \;=\; \bigl\{\, H \in S[\mathcal{X},\mathcal{Y}] \;:\; Hu \in CD_0(Q,\mathcal{Y}) \text{ for all } u \in CD_0(Q,\mathcal{X}) \,\bigr\}.$$

$\lhd$ We only have to show "$\subset$." If $H\in CD_0[\mathcal{X},\mathcal{Y}]$ and $u\in CD_0(Q,\mathcal{X})$ then $|\!|\!| Hu_d |\!|\!| \leqslant \|H\| |\!|\!| u_d |\!|\!| \in c_0(Q)$ and, consequently,

$$Hu = Hu_c + Hu_d \in CD_0(Q,\mathcal{Y}) + c_0(Q,\mathcal{Y}) \subset CD_0(Q,\mathcal{Y}). \quad \rhd$$

5.10. *If $H \in \ell^\infty[\mathcal{X},\mathcal{Y}]$, $u \in C(Q,\mathcal{X})$, and $Hu \in CD_0(Q,\mathcal{Y})$ then*

$$|\!|\!|(Hu)_c|\!|\!| \leqslant \|H\|\, |\!|\!|u|\!|\!|, \quad |\!|\!|(Hu)_d|\!|\!| \leqslant 2\|H\|\, |\!|\!|u|\!|\!|.$$

$\lhd$ By 3.6, for all $q \in Q$ we have

$$\|(Hu)_c(q)\| = \lim_{p\to q} \|H(p)u(p)\| \leqslant \sup_{p\in Q} \|H(p)\| \lim_{p\to q} \|u(p)\| = \|H\|\|u(q)\|.$$

In addition, $|\!|\!|(Hu)_d|\!|\!| = |\!|\!|Hu - (Hu)_c|\!|\!| \leqslant |\!|\!|Hu|\!|\!| + |\!|\!|(Hu)_c|\!|\!| \leqslant 2\|H\|\, |\!|\!|u|\!|\!|$. $\rhd$

5.11. *A section $H \in S[\mathcal{X},\mathcal{Y}]$ is a CD_0-homomorphism if and only if H is bounded and there exists a subset $\mathcal{U} \subset C(Q,\mathcal{X})$ such that $\mathcal{U}$ is stalkwise dense in $\mathcal{X}$ and $Hu \in CD_0(Q,\mathcal{Y})$ for all $u \in \mathcal{U}$.*

$\lhd$ We only have to prove sufficiency. Let H and $\mathcal{U}$ satisfy the conditions stated. Without loss of generality, we may assume that $\mathcal{U}$ is a vector subspace of $C(Q,\mathcal{X})$.

Given a point $q \in Q$, denote by $\mathcal{U}(q)$ the dense subspace $\{u(q) : u \in \mathcal{U}\} \subset \mathcal{X}(q)$ and define the mapping $G_0(q) : \mathcal{U}(q) \to \mathcal{Y}(q)$ by putting

$$G_0(q)u(q) = (Hu)_c(q) \quad \text{for all} \quad u \in \mathcal{U}.$$

The above definition is correct, since, due to 5.10, for all sections $u_1, u_2 \in \mathcal{U}$, the equality $u_1(q) = u_2(q)$ implies

$$\|(Hu_1)_c(q) - (Hu_2)_c(q)\| \leqslant \|H\|\|u_1(q) - u_2(q)\| = 0.$$

The mapping $G_0(q)$ is obviously linear. In addition, according to 5.10, for all $u \in \mathcal{U}$ we have

$$\|G_0(q)u(q)\| = \|(Hu)_c(q)\| \leqslant \|H\|\, \|u(q)\|;$$

whence, $\|G_0(q)\| \leqslant \|H\|$ and thereby $G_0(q) \in B\bigl(\mathcal{U}(q),\mathcal{Y}(q)\bigr)$. Since $\mathcal{U}(q)$ is dense in $\mathcal{X}(q)$, and $\mathcal{Y}(q)$ is complete, $G_0(q)$ has an extension $G(q) \in B\bigl(\mathcal{X}(q),\mathcal{Y}(q)\bigr)$; moreover, $\|G(q)\| = \|G_0(q)\| \leqslant \|H\|$. Therefore, $G \in \ell^\infty[\mathcal{X},\mathcal{Y}]$ and $Gu = (Hu)_c \in C(Q,\mathcal{Y})$ for all $u \in \mathcal{U}$. By [10, 2.4.9] we have $G \in C[\mathcal{X},\mathcal{Y}]$.

Since $H - G \in \ell^\infty[\mathcal{X},\mathcal{Y}]$ and $(H - G)u = Hu - (Hu)_c \in c_0(Q,\mathcal{Y})$ for all $u \in \mathcal{U}$; therefore by 5.5 the section $H - G$ is a c_0-homomorphism. Hence, for all $v \in C(Q,\mathcal{X})$

$$Hv = Gv + (H - G)v \in C(Q,\mathcal{Y}) + c_0(Q,\mathcal{Y}) = CD_0(Q,\mathcal{Y}),$$

i.e., $H \in CD_0[\mathcal{X},\mathcal{Y}]$. $\rhd$

Note that, due to 5.6, the boundedness of H in the above assertion cannot be omitted.

5.12. It is clear that the sum of a homomorphism and a c_0-homomorphism is a CD_0-homomorphism. The following assertion shows that such sums exhaust the whole set of CD_0-homomorphisms.

The following direct sum decomposition holds:

$$CD_0[\mathcal{X},\mathcal{Y}] \; = \; C[\mathcal{X},\mathcal{Y}] \oplus c_0[\mathcal{X},\mathcal{Y}].$$

In particular, every CD_0-homomorphism $H \in CD_0[\mathcal{X},\mathcal{Y}]$ is uniquely representable as $H = H_c + H_d$, with $H_c \in C[\mathcal{X},\mathcal{Y}]$ and $H_d \in c_0[\mathcal{X},\mathcal{Y}]$.

$\triangleleft$ The only nontrivial part of the above statement is the inclusion $CD_0[\mathcal{X},\mathcal{Y}] \subset C[\mathcal{X},\mathcal{Y}] + c_0[\mathcal{X},\mathcal{Y}]$ whose justification can be easily extracted from the proof of 5.11 by taking $C(Q,\mathcal{X})$ as $\mathcal{U}$. $\triangleright$

5.13. *For all $H \in CD_0[\mathcal{X},\mathcal{Y}]$ and $u \in CD_0(Q,\mathcal{X})$, the following hold:*

$$(Hu)_c = H_c u_c, \quad (Hu)_d = H_c u_d + H_d u_d + H_d u_c.$$

In particular, if $u \in C(Q,\mathcal{X})$ then

$$(Hu)_c = H_c u, \quad (Hu)_d = H_d u.$$

$\triangleleft$ Taking account of 3.5 and 5.12, it suffices to use the equalities

$$Hu = (H_c + H_d)(u_c + u_d) = H_c u_c + (H_c u_d + H_d u_d + H_d u_c)$$

and note that $H_c u_c \in C(Q,\mathcal{Y})$ and $H_c u_d + H_d u_d + H_d u_c \in c_0(Q,\mathcal{Y})$. $\triangleright$

5.14. In a similar way, we can deduce the following assertion from 3.2 (2), 3.10, and 5.12:

The space $CD_0[\mathcal{X},\mathcal{Y}]$ is a $CD_0(Q)$-module with respect to the pointwise multiplication. Furthermore, for all $f \in CD_0(Q)$ and $H \in CD_0[\mathcal{X},\mathcal{Y}]$ we have

$$(fH)_c = f_c H_c, \quad (fH)_d = f_c H_d + f_d H_d + f_d H_c.$$

In particular, if $f \in C(Q)$ then

$$(fH)_c = f H_c, \quad (fH)_d = f H_d.$$

The obvious inequality $\|fH\| \leqslant \|f\|\|H\|$ $(f \in CD_0(Q),\ H \in CD_0[\mathcal{X},\mathcal{Y}])$ allows us to conclude that $CD_0[\mathcal{X},\mathcal{Y}]$ is a Banach $CD_0(Q)$-module.

5.15. *For every CD_0-homomorphism $H \in CD_0[\mathcal{X},\mathcal{Y}]$ we have*

$$\|H_c\| \leqslant \|H\|, \quad \|H_d\| \leqslant 2\|H\|.$$

$\triangleleft$ Let $H \in CD_0[\mathcal{X},\mathcal{Y}]$. Consider arbitrary elements $q \in Q$ and $x \in \mathcal{X}(q)$. By the Dupré Theorem, there exists a section $u \in C(Q,\mathcal{X})$ such that $u(q) = x$. According to 5.10 and 5.13 we have

$$\|H_c(q)x\| = \|H_c(q)u(q)\| = \|(H_c u)(q)\| = \|(Hu)_c(q)\| \leqslant \|H\|\,\|u(q)\| = \|H\|\,\|x\|;$$

whence $\|H_c\| \leqslant \|H\|$. Consequently,

$$\|H_d\| = \|H - H_c\| \leqslant \|H\| + \|H_c\| \leqslant 2\|H\|. \quad \triangleright$$

5.16. Let $\widetilde{\mathcal{X}}$ and $\widetilde{\mathcal{Y}}$ be Banach bundles over $\widetilde{Q}$ which are defined according to the definition 3.14. Given a CD_0-homomorphism $H \in CD_0[\mathcal{X},\mathcal{Y}]$, define the section $\widetilde{H} \in S\big[\widetilde{\mathcal{X}},\widetilde{\mathcal{Y}}\big]$ by the rule

$$\widetilde{H}(q_c) = H_c(q), \quad \widetilde{H}(q_d) = H(q) \quad \text{for all} \ \ q \in Q.$$

Therefore, $\widetilde{H}(\cdot,0) = H_c$ and $\widetilde{H}(\cdot,1) = H$.

5.17. *The mapping $H \mapsto \widetilde{H}$ is a linear isometry of the Banach space $CD_0[\mathcal{X},\mathcal{Y}]$ onto $C\big[\widetilde{\mathcal{X}},\widetilde{\mathcal{Y}}\big]$. Moreover, for all $f \in CD_0(Q)$, $u \in CD_0(Q,\mathcal{X})$, and $H \in CD_0[\mathcal{X},\mathcal{Y}]$, we have $\widetilde{Hu} = \widetilde{H}\widetilde{u}$ and $\widetilde{fH} = \widetilde{f}\widetilde{H}$.*

◁ Let $f \in CD_0(Q)$, $u \in CD_0(Q,\mathcal{X})$, and $H \in CD_0[\mathcal{X},\mathcal{Y}]$. By 5.13 we have

$$\begin{aligned} \widetilde{Hu}\,(\cdot,0) &= (Hu)_c = H_c u_c = \widetilde{H}(\cdot,0)\,\widetilde{u}(\cdot,0) = \big(\widetilde{H}\widetilde{u}\big)(\cdot,0), \\ \widetilde{Hu}\,(\cdot,1) &= Hu = \widetilde{H}(\cdot,1)\,\widetilde{u}(\cdot,1) = \big(\widetilde{H}\widetilde{u}\big)(\cdot,1) \end{aligned}$$

and thus $\widetilde{Hu} = \widetilde{H}\widetilde{u}$. Similarly, by using 5.14, we obtain $\widetilde{fH} = \widetilde{f}\widetilde{H}$.

Show that $\widetilde{H} \in C\big[\widetilde{\mathcal{X}},\widetilde{\mathcal{Y}}\big]$. Indeed, if $v \in C\big(\widetilde{Q},\widetilde{\mathcal{X}}\big)$ then $v = \widetilde{u}$ for some section $u \in CD_0(Q,\mathcal{X})$ (see 3.15); hence, $\widetilde{H}v = \widetilde{H}\widetilde{u} = \widetilde{Hu} \in C\big(\widetilde{Q},\widetilde{\mathcal{Y}}\big)$. It remains to employ 5.2.

The mapping $H \mapsto \widetilde{H}$ is obviously linear. In addition, due to 5.15, for all $H \in CD_0[\mathcal{X},\mathcal{Y}]$ we have

$$\big\|\widetilde{H}\big\| = \max\big\{\big\|\widetilde{H}(\cdot,0)\big\|, \big\|\widetilde{H}(\cdot,1)\big\|\big\} = \max\{\|H_c\|,\|H\|\} = \|H\|.$$

It remains to show that the image of the mapping $H \mapsto \widetilde{H}$ coincides with $C\big[\widetilde{\mathcal{X}},\widetilde{\mathcal{Y}}\big]$. Consider an arbitrary homomorphism $G \in C\big[\widetilde{\mathcal{X}},\widetilde{\mathcal{Y}}\big]$ and put $H := G(\cdot,1)$. As is easily seen, $H \in S[\mathcal{X},\mathcal{Y}]$. According to 3.16, for all $u \in CD_0(Q,\mathcal{X})$ we have

$$Hu = G(\cdot,1)\,\widetilde{u}(\cdot,1) = (G\widetilde{u})(\cdot,1) \in CD_0(Q,\mathcal{Y});$$

whence $H \in CD_0[\mathcal{X},\mathcal{Y}]$ (see 5.9). To prove the equality $\widetilde{H} = G$, consider arbitrary elements $(q,r) \in \widetilde{Q}$ and $x \in \widetilde{\mathcal{X}}(q,r)$ and show that

$$\widetilde{H}(q,r)\,x = G(q,r)\,x.$$

Due to 3.15 and the Dupré Theorem, there is a section $u \in CD_0(Q,\mathcal{X})$ such that $\widetilde{u}(q,r) = x$. Taking account of 3.16 and 5.13, we have

$$\widetilde{H}(\cdot,0)\,\widetilde{u}(\cdot,0) = H_c u_c = (Hu)_c = \big((G\widetilde{u})(\cdot,1)\big)_c = (G\widetilde{u})(\cdot,0) = G(\cdot,0)\,\widetilde{u}(\cdot,0).$$

In addition, $\widetilde{H}(\cdot,1)\,\widetilde{u}(\cdot,1) = H\,\widetilde{u}(\cdot,1) = G(\cdot,1)\,\widetilde{u}(\cdot,1)$. Consequently,

$$\widetilde{H}(q,r)\,x = \widetilde{H}(q,r)\,\widetilde{u}(q,r) = G(q,r)\,\widetilde{u}(q,r) = G(q,r)\,x. \quad ▷$$

5.18. The criterion presented below is immediate from 5.17. (In the case of $\mathcal{Y} = Q \times \{\mathbb{R}\}$, it is actually a restatement of [13, Proposition 8].)

The following properties of a section $G \in S\big[\widetilde{\mathcal{X}}, \widetilde{\mathcal{Y}}\big]$ are equivalent:

(1) $G \in C\big[\widetilde{\mathcal{X}}, \widetilde{\mathcal{Y}}\big]$;
(2) $G(\cdot,0) \in C[\mathcal{X},\mathcal{Y}]$, $G(\cdot,1) - G(\cdot,0) \in c_0[\mathcal{X},\mathcal{Y}]$;
(3) $G(\cdot,1) \in CD_0[\mathcal{X},\mathcal{Y}]$, $G(\cdot,0) = G(\cdot,1)_c$.

Note also that the images of the spaces $C[\mathcal{X},\mathcal{Y}]$ and $c_0[\mathcal{X},\mathcal{Y}]$ under the isometry $H \mapsto \widetilde{H}$ are described as follows:

$$\begin{aligned}
\big\{\,\widetilde{H}\ :\ H \in C[\mathcal{X},\mathcal{Y}]\,\big\} &= \big\{\,G \in C\big[\widetilde{\mathcal{X}}, \widetilde{\mathcal{Y}}\big]\ :\ G(\cdot,0) = G(\cdot,1)\,\big\}\\
&= \big\{\,G \in S\big[\widetilde{\mathcal{X}}, \widetilde{\mathcal{Y}}\big]\ :\ G(\cdot,0) = G(\cdot,1) \in C[\mathcal{X},\mathcal{Y}]\,\big\},\\
\big\{\,\widetilde{H}\ :\ H \in c_0[\mathcal{X},\mathcal{Y}]\,\big\} &= \big\{\,G \in C\big[\widetilde{\mathcal{X}}, \widetilde{\mathcal{Y}}\big]\ :\ G(\cdot,0) = 0\,\big\}\\
&= \big\{\,G \in S\big[\widetilde{\mathcal{X}}, \widetilde{\mathcal{Y}}\big]\ :\ G(\cdot,0) = 0,\ G(\cdot,1) \in c_0[\mathcal{X},\mathcal{Y}]\,\big\}.
\end{aligned}$$

5.19. In conclusion, we will describe the spaces of homomorphisms, c_0-homomorphisms, and CD_0-homomorphisms in terms of their action in Banach $C(Q)$-modules.

Let $\mathcal{U}$ and $\mathcal{V}$ be arbitrary vector subspaces of the Banach spaces $\ell^\infty(Q,\mathcal{X})$ and $\ell^\infty(Q,\mathcal{Y})$. Say that $T\colon \mathcal{U} \to \mathcal{V}$ is an *orthomorphism* (cp. [10, 6.2.11], [15, 4.1.3 (5)]) if T is a bounded linear operator and, for all $u \in \mathcal{U}$ and $q \in Q$, the equality $u(q) = 0$ implies $(Tu)(q) = 0$. Denote by $\mathrm{Orth}(\mathcal{U},\mathcal{V})$ the set of all orthomorphisms from $\mathcal{U}$ into $\mathcal{V}$. As is easily seen, $\mathrm{Orth}(\mathcal{U},\mathcal{V})$ is a closed vector subspace of the space of all bounded linear operators from $\mathcal{U}$ into $\mathcal{V}$ endowed with the operator norm.

The following properties of a function $T\colon C(Q,\mathcal{X}) \to \ell^\infty(Q,\mathcal{Y})$ are equivalent:

(1) *T is an orthomorphism;*
(2) *T is a homomorphism of Banach $C(Q)$-modules, i.e., T is a bounded linear operator and $T(fu) = fTu$ for all $f \in C(Q)$, $u \in C(Q,\mathcal{X})$;*
(3) *there exists a section $H \in \ell^\infty[\mathcal{X},\mathcal{Y}]$ such that $(Tu)(q) = H(q)u(q)$ for all $u \in C(Q,\mathcal{X})$ and $q \in Q$; furthermore, $\|T\| = \|H\|$.*

$\triangleleft$ The implication (3)⇒(2) is obvious.

(2)⇒(1): Let $u \in C(Q,\mathcal{X})$ and $q \in Q$ be such that $u(q) = 0$. According to [9, 2.11], there exist sequences of functions $f_n \in C(Q)$ and sections $u_n \in C(Q,\mathcal{X})$ such that $f_n(q) = 0$ for all $n \in \mathbb{N}$ and $\|f_n u_n - u\| \to 0$ as $n \to \infty$. Then by (2) we have

$$(Tu)(q) = \lim_{n\to\infty} \big(T(f_n u_n)\big)(q) = \lim_{n\to\infty} (f_n T u_n)(q) = \lim_{n\to\infty} f_n(q)(Tu_n)(q) = 0.$$

(1)⇒(3): Given a point $q \in Q$, define the mapping $H(q)\colon \mathcal{X}(q) \to \mathcal{Y}(q)$ by putting $H(q)u(q) = (Tu)(q)$ for all $u \in C(Q,\mathcal{X})$. The correctness of this definition is immediate from (1) and the Dupré Theorem. The mapping $H(q)$ is obviously linear. In addition, by the Dupré Theorem we have

$$\begin{aligned}
\sup_{q\in Q} \|H(q)\| &= \sup\,\big\{\,\|H(q)u(q)\|\ :\ q \in Q,\ u \in C(Q,\mathcal{X}),\ \|u\| \leqslant 1\,\big\}\\
&= \sup\,\Big\{\,\sup_{q\in Q} \|(Tu)(q)\|\ :\ u \in C(Q,\mathcal{X}),\ \|u\| \leqslant 1\Big\} = \|T\|. \quad \triangleright
\end{aligned}$$

5.20. Using 5.19, from the above-established results we can deduce a number of direct corollaries on the spaces of orthomorphisms. We will state some of them.

Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach bundles over a nonempty compact Hausdorff space Q without isolated points.

(1) *Given a section $H \in \ell^\infty[\mathcal{X},\mathcal{Y}]$, define the function $T_H : C(Q,\mathcal{X}) \to \ell^\infty(Q,\mathcal{Y})$ by putting $(T_H u)(q) = H(q)u(q)$ for all $u \in C(Q,\mathcal{X})$ and $q \in Q$. The mapping $H \mapsto T_H$ is a $C(Q)$-linear isometrical isomorphism between the following pairs of Banach $C(Q)$-modules:*

$$\begin{array}{lcl} \ell^\infty[\mathcal{X},\mathcal{Y}] & \leftrightarrow & \mathrm{Orth}\big(C(Q,\mathcal{X}),\ell^\infty(Q,\mathcal{Y})\big), \\ C[\mathcal{X},\mathcal{Y}] & \leftrightarrow & \mathrm{Orth}\big(C(Q,\mathcal{X}),C(Q,\mathcal{Y})\big), \\ c_0[\mathcal{X},\mathcal{Y}] & \leftrightarrow & \mathrm{Orth}\big(C(Q,\mathcal{X}),c_0(Q,\mathcal{Y})\big), \\ CD_0[\mathcal{X},\mathcal{Y}] & \leftrightarrow & \mathrm{Orth}\big(C(Q,\mathcal{X}),CD_0(Q,\mathcal{Y})\big). \end{array}$$

(2) *The following direct sum decomposition holds:*

$$\begin{aligned} &\mathrm{Orth}\big(C(Q,\mathcal{X}),CD_0(Q,\mathcal{Y})\big) \\ &\quad = \mathrm{Orth}\big(C(Q,\mathcal{X}),C(Q,\mathcal{Y})\big) \oplus \mathrm{Orth}\big(C(Q,\mathcal{X}),c_0(Q,\mathcal{Y})\big). \end{aligned}$$

Therefore, every orthomorphism $T\colon C(Q,\mathcal{X}) \to CD_0(Q,\mathcal{Y})$ is uniquely representable as the sum $T = T_c + T_d$ of some orthomorphisms $T_c\colon C(Q,\mathcal{X}) \to C(Q,\mathcal{Y})$ and $T_d\colon C(Q,\mathcal{X}) \to c_0(Q,\mathcal{Y})$. Moreover, $T_c = T_{H_c}$ and $T_d = T_{H_d}$, where H is the CD_0-homomorphism from $\mathcal{X}$ into $\mathcal{Y}$ determined by the equality $T = T_H$.

(3) *For every orthomorphism $T\colon C(Q,\mathcal{X}) \to CD_0(Q,\mathcal{Y})$ we have $\|T_c\| \leqslant \|T\|$, $\|T_d\| \leqslant 2\|T\|$.*

(4) *For every orthomorphism $T\colon C(Q,\mathcal{X}) \to CD_0(Q,\mathcal{Y})$ and every section $u \in C(Q,\mathcal{X})$ we have $T_c u = (Tu)_c$ and $T_d u = (Tu)_d$.*

In particular, the Banach spaces $CD_0[\mathcal{X},\mathcal{Y}]$, $C\big[\widetilde{\mathcal{X}},\widetilde{\mathcal{Y}}\big]$, $\mathrm{Orth}\big(C(Q,\mathcal{X}),CD_0(Q,\mathcal{Y})\big)$, and $\mathrm{Orth}\big(C(\widetilde{Q},\widetilde{\mathcal{X}}),C(\widetilde{Q},\widetilde{\mathcal{Y}})\big)$ are linearly isometric.

References

[1] *Abramovich Y. A., Wickstead A. W. Regular operators from and into a small Riesz space.* Indag. Math. N.S. 1991. Vol. 2, No. 3. P. 257–274.

[2] *Abramovich Y. A., Wickstead A. W. Remarkable classes of unital AM-spaces.* J. Math. Anal. Appl. 1993. Vol. 180, No. 2. P. 398–411.

[3] *Abramovich Y. A., Wickstead A. W. The regularity of order bounded operators into $C(K)$. II.* Quart. J. Math. Oxford Ser. 2. 1993. Vol. 44, No. 175. P. 257–270.

[4] *Akilov G. P., Kutateladze S. S. Ordered Vector Spaces.* Nauka, Novosibirsk. 1978. (In Russian.)

[5] *Alpay Ş., Ercan Z. $CD_0(K,E)$ and $CD_\omega(K,E)$-spaces as Banach lattices.* Positivity. 2000. Vol. 4, No. 3. P. 213–225.

[6] *Alpay Ş., Ercan Z. A remark on $CD_0(K)$-spaces.* Siberian Math. J. 2006. Vol. 47, No. 3. P. 422–424.

[7] *Engelking R. General Topology.* Heldermann, Berlin. 1989.

[8] *Ercan Z. A concrete description of $CD_0(K)$-spaces as $C(X)$-spaces and its applications.* Proc. Amer. Math. Soc. 2004. Vol. 132. P. 1761–1763.

[9] *Gierz G. Bundles of Topological Vector Spaces and Their Duality.* Springer, Berlin. 1982. (Lecture Notes in Math.; No. 955.)

[10] *Gutman A. E. Banach bundles in the theory of lattice-normed spaces.* Institute of Mathematics, Novosibirsk. 1995. (Vol. 29. Linear Operators Coordinated With Order.) P. 63–211. (In Russian.)

[11] *Gutman A. E., Koptev A. V. Spaces of CD_0-sections and CD_0-homomorphisms of Banach bundles.* Vestnik, Quart. J. of Novosibirsk State Univ., Series: Math., mech. and informatics. 2007. Vol. 7, No. 4. P. 27–48. (In Russian.)

[12] *Gutman A. E., Koptev A. V. Spaces of CD_0-functions and Alexandroff duplicate.* Vladikavkaz. Mat. Zh. 2007. Vol. 9, No. 3. P. 11–21. (In Russian.)

[13] *Hõim T., Robbins D. A. Section spaces of Banach bundles which generalize some function spaces.* Siberian Adv. Math. 2006. Vol. 16, No. 3. P. 71–81.

[14] *Koptev A. V. Some classes of Banach bundles with continuous weakly continuous sections.* Siberian Math. J. 2004. Vol. 45, No. 3. P. 495-503.

[15] *Kusraev A. G. Dominated Operators.* Nauka, Moscow. 2003. (In Russian.)

[16] *Troitsky V. G. On $CD_0(K)$-spaces.* Vladikavkaz. Mat. Zh. 2004. Vol. 6, No. 1. P. 71–73.

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ПРОСТРАНСТВА CD_0-ФУНКЦИЙ И CD_0-СЕЧЕНИЙ БАНАХОВЫХ РАССЛОЕНИЙ

А. Е. ГУТМАН, А. В. КОПТЕВ

Аннотация. В начале работы приведено краткое изложение ключевых этапов исследования пространства $CD_0(Q) = C(Q) + c_0(Q)$, элементы которого являются суммами непрерывных и «дискретных» функций на компакте Q без изолированных точек. (При этом основное внимание уделено описанию компакта $\widetilde{Q}$, реализующего банахову решетку $CD_0(Q)$ в виде $C(\widetilde{Q})$.) Оставшаяся часть статьи посвящена аналогичному кругу вопросов, связанному с пространством $CD_0(Q, \mathcal{X})$ «непрерывно-дискретных» сечений банахова расслоения $\mathcal{X}$ и с пространством CD_0-гомоморфизмов банаховых расслоений.

Ключевые слова. Банахова решетка, AM-пространство, удвоение по Александрову, непрерывное банахово расслоение, сечение банахова расслоения, банахов $C(Q)$-модуль, гомоморфизм банаховых расслоений, гомоморфизм банаховых $C(Q)$-модулей.