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THE BOUNDEDNESS PRINCIPLE FOR LATTICE-NORMED SPACES

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Abstract: The three classical facts of the theory of normed spaces are considered: the boundedness principle, the Banach–Steinhaus theorem, and the uniform boundedness principle for a compact convex set. By means of Boolean valued analysis, the analogs of the theorems are proven in a lattice-normed space setting.

Keywords: Banach–Steinhaus theorem, Banach–Kantorovich space, cyclically compact set, Boolean valued analysis

Basing on the traditional definitions, notation, and facts of the theory of lattice-normed spaces [1] and Boolean valued analysis [2], we prove the exact analogs of the three classical theorems for arbitrary lattice-normed spaces over universally complete Riesz spaces, namely, the boundedness principle (2.4), the Banach–Steinhaus theorem (2.6), and the uniform boundedness principle for a compact convex set (3.3). The theorems (obtained by the descent method) strengthen and generalize the analogous results by I. G. Ganiev and K. K. Kudajbergenov [3] which were established for Banach–Kantorovich spaces over the lattice of measurable functions. (The proofs in [3] are based on the specific technique of the theory of measurable Banach bundles with lifting and do not use Boolean valued analysis.)

All vector spaces under consideration are assumed nonzero.

Throughout the article, E is a universally complete Riesz space over $\mathbb{R}$. Also E^{++} is the set of order unities in E; 1_E is a fixed element of E^{++}; ef is the product of $e, f \in E$ with respect to the multiplication which makes E into an ordered commutative algebra with multiplication unity 1_E and satisfies the condition $(\forall\, e, f \in E)(\, ef = 0 \Leftrightarrow e \perp f\,)$; B is the complete Boolean algebra of order projections in E; $\operatorname{Prt}(B)$ is the set of partitions of unity in B; $\mathbb{V}^{(B)}$ is the separated B-valued universe; $[\![\varphi]\!]$ is the truth value in $\mathbb{V}^{(B)}$ of a set-theoretic formula φ; $\mathbb{V}^{(B)} \vDash [\varphi]$ is a synonym for $[\![\varphi]\!] = 1_B$; $\bigsqcup_{i\in I} \pi_i x_i$ is the mixing of a family $(x_i)_{i\in I} \subset \mathbb{V}^{(B)}$ with respect to $(\pi_i)_{i\in I} \in \operatorname{Prt}(B)$; $\operatorname{cyc} X$ is the cyclic hull of a set $X \subset \mathbb{V}^{(B)}$; $\mathscr{R}$ is the ordered field of reals inside $\mathbb{V}^{(B)}$ which includes $\mathbb{R}^\wedge$ as a subfield.

The set of extensional functions from X into Y, with $X, Y \subset \mathbb{V}^{(B)}$, is denoted by $\mathscr{E}(X, Y)$. Given $\mathscr{X}, \mathscr{Y} \in \mathbb{V}^{(B)}$ and $\mathscr{T} \subset \mathscr{E}(\mathscr{X}{\downarrow}, \mathscr{Y}{\downarrow})$, we put $\mathscr{T}{\uparrow\uparrow} := \{T{\uparrow} : T \in \mathscr{T}\}{\uparrow}$. Note that $\mathbb{V}^{(B)} \vDash [\mathscr{T}{\uparrow\uparrow}$ is a set of functions from $\mathscr{X}$ into $\mathscr{Y}]$. If $\mathbb{V}^{(B)} \vDash [\mathscr{X} \neq \varnothing$ and $\mathbb{T}$ is a set of functions from $\mathscr{X}$ into $\mathscr{Y}]$ then $\mathbb{T}{\downarrow\downarrow} := \{\tau{\downarrow} : \tau \in \mathbb{T}{\downarrow}\} \subset \mathscr{E}(\mathscr{X}{\downarrow}, \mathscr{Y}{\downarrow})$.

1. Boolean Valued Representation of mix-Complete E-Normed Spaces

By a (*complete*) *E-normed space* we mean a vector space X over $\mathbb{R}$ endowed with a norm $|\!\cdot\!| : X \to E$ under which X is an (o-complete) d-decomposable lattice-normed space (see [1, 2.1.1]) subject to the condition $|X|^{\perp\perp} = E$, where $|X| := \{|x| : x \in X\}$.

Let $X := (X, |\!\cdot\!|)$ be an arbitrary E-normed space.

Say that Y is an *E-normed subspace* of X and write $Y \subset_E X$ whenever Y is a vector subspace of X and the pair $(Y, |\!\cdot\!||_Y)$ is an E-normed space. (We endow Y with the norm $|\!\cdot\!||_Y$.) As is easily seen, $Y \subset_E X$ if and only if Y is a vector subspace of X, $(\forall\, \pi \in B)(\forall\, y \in Y)(\pi_X y \in Y)$, and $|Y|^{\perp\perp} = E$. In the case $Y \subset_E X$ we also say that X is an *extension* of Y.

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†) To Academician Yu. G. Reshetnyak on his eightieth birthday.

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The formulas $x_n \to x$ and $\lim_{n\to\infty} x_n = x$ are used as synonyms for the order convergence $|x_n - x| \to 0$ in E. A subset $U \subset X$ *approximates* $x \in X$ whenever $\inf\{|u - x| : u \in U\} = 0$. Say that U is *dense* in X if U approximates each element of X.

For convenience we use the index X for the mixings evaluated in X: $\operatorname{mix}^X_{i\in I} \pi_i x_i$. Let $U \subset X$. Denote by $\operatorname{mix}_X U$ the mix-*closure* (in X) of U, the set of elements $x \in X$ representable as $x = \operatorname{mix}^X_{i\in I} \pi_i u_i$, with $(\pi_i)_{i\in I} \in \operatorname{Prt}(B)$ and $(u_i)_{i\in I} \subset U$. Say that U is mix-*closed* (in X) if $\operatorname{mix}_X U = U$. Say that U is mix-*complete* (in X) if, for all $(\pi_i)_{i\in I} \in \operatorname{Prt}(B)$ and $(u_i)_{i\in I} \subset U$, there is $u \in U$ such that $u = \operatorname{mix}^X_{i\in I} \pi_i u_i$.

Note that mix-completeness is an absolute concept in the following sense: if X and $\overline{X}$ are E-normed spaces and $U \subset X \subset_E \overline{X}$ then the mix-completeness of U in X is equivalent to that in $\overline{X}$. In case X is mix-complete in X (and hence in any extension of X) the space X is called a mix-*complete E-normed space*. As is known, each complete E-normed space is mix-complete.

By a mix-*completion* of X we mean a mix-complete E-normed space $\overline{X}$ such that $X \subset_E \overline{X}$ and $\operatorname{mix}_{\overline{X}} X = \overline{X}$. (It is clear that $X = \overline{X}$ in the case of a mix-complete space X.) Every E-normed space admits a mix-completion which is unique up to isometry (see 1.6).

The following is a consequence of [4, 1.5.2, 1.5.3]:

1.1. *If $\overline{X}$ is a mix-completion of an E-normed space X then each dense subset of X is dense in $\overline{X}$. In particular, X is dense in $\overline{X}$.*

By the Gordon theorem [2, 10.3.4, 10.4.1, 10.4.3 (2)] the descent $\mathscr{R}{\downarrow}$, endowed with the descents of the linear operations and order, is a universally complete Riesz space linearly and order isomorphic to E. For convenience we assume that $E = \mathscr{R}{\downarrow}$ and, furthermore, $\operatorname{mix}^E_{i\in I} \pi_i e_i = \bigsqcup_{i\in I} \pi_i e_i$ for all $(\pi_i)_{i\in I} \in \operatorname{Prt}(B)$ and $(e_i)_{i\in I} \subset E$. We also assume that $1_E = 1^\wedge$. (In this case $\lambda 1_E = \lambda^\wedge$ for all $\lambda \in \mathbb{R}$, and the descent of the multiplication in $\mathscr{R}$ coincides with the multiplication in E corresponding to the choice of 1_E as a multiplication unity.) Formally, we can start with considering an arbitrary complete Boolean algebra B, next introduce $E := \mathscr{R}{\downarrow}$ and $1_E := 1^\wedge$, and then make the elements $b \in B$ into the order projections in E as follows: for all $e \in E$ define $b\,e$ to be the mixing of $(e, 0)$ with respect to $(b, b^\perp)$, where $b^\perp$ is the complement of b in B.

By a *normed space over a subfield $F \subset \mathbb{R}$* we mean a vector space over F endowed with an $\mathbb{R}$-valued norm, while a *Banach space over F* is a complete normed space over F. The following is obvious:

1.2. *If $(X, +, \cdot, \|\cdot\|)$ is a Banach space over a subfield $F \subset \mathbb{R}$ then the multiplication $\cdot : F \times X \to X$ uniquely extends to $\bar{\cdot} : \mathbb{R} \times X \to X$ so that $(X, +, \bar{\cdot}, \|\cdot\|)$ is a Banach space over $\mathbb{R}$.*

By a *completion* of a normed space X over a subfield $F \subset \mathbb{R}$ we mean a Banach space over F (or over $\mathbb{R}$; see 1.2) which includes X as a dense vector subspace (over F). Note that each normed space over F admits a completion which is unique up to isometry.

1.3. If $\mathbb{V}^{(B)} \models [\mathscr{X}$ is a vector space over $\mathbb{R}^\wedge]$ then $\mathscr{X}{\downarrow}$ is a vector space over $\mathbb{R}$ with respect to the descent of the addition and the multiplication $(\lambda, x) \mapsto \lambda x$ defined by $\mathbb{V}^{(B)} \models [\lambda x = \lambda^\wedge x]$. We agree to endow $\mathscr{X}{\downarrow}$ with the above operations and thus consider $\mathscr{X}{\downarrow}$ a vector space over $\mathbb{R}$.

If $\mathbb{V}^{(B)} \models [\mathscr{X}$ is a vector space over $\mathbb{R}^\wedge]$ and X is a vector subspace of $\mathscr{X}{\downarrow}$ then $\mathbb{V}^{(B)} \models [X{\uparrow}$ is a vector subspace of $\mathscr{X}]$. In case $\mathbb{V}^{(B)} \models [\mathscr{X}$ is a normed space over $\mathbb{R}^\wedge]$ the above observation makes it possible to assume that $\mathbb{V}^{(B)} \models [X{\uparrow}$ is a normed space over $\mathbb{R}^\wedge]$.

The following is a consequence (with 1.2 taken into account) of [2, 11.3.1, 11.3.2] (also see [1, 8.3.1, 8.3.2]):

1.4. Theorem. (1) *If $\mathscr{X} \in \mathbb{V}^{(B)}$ and $\mathbb{V}^{(B)} \models [\mathscr{X}$ is a Banach space over $\mathbb{R}^\wedge]$ then $\mathscr{X}{\downarrow} := (\mathscr{X}{\downarrow}, \|\cdot\|_{\mathscr{X}}{\downarrow})$ is a complete E-normed space. Furthermore, $\operatorname{mix}^{\mathscr{X}\downarrow}_{i\in I} \pi_i x_i = \bigsqcup_{i\in I} \pi_i x_i$ for $(x_i)_{i\in I} \subset \mathscr{X}{\downarrow}$ and $(\pi_i)_{i\in I} \in \operatorname{Prt}(B)$.*

(2) *Each E-normed space X is isometric to some dense subspace $\widetilde{X} \subset_E \mathscr{X}{\downarrow}$, where $\mathbb{V}^{(B)} \models [\mathscr{X}$ is a Banach space over $\mathbb{R}^\wedge]$. This space $\mathscr{X}$ inside $\mathbb{V}^{(B)}$ is unique up to isometry. Furthermore, $\widetilde{X} = \mathscr{X}{\downarrow}$ for a complete E-normed space X.*

1.5. Theorem. (1) *If* $\mathbb{V}^{(B)} \models [\mathscr{X}$ *is a normed space over* $\mathbb{R}^\wedge]$ *then* $\mathscr{X}{\downarrow} := (\mathscr{X}{\downarrow}, \|\cdot\|_{\mathscr{X}}{\downarrow})$ *is a* mix-*complete E-normed space. If, moreover,* $X \subset_E \mathscr{X}{\downarrow}$, $(x_i)_{i\in I} \subset X$, *and* $(\pi_i)_{i\in I} \in \mathrm{Prt}(B)$ *then* $\mathrm{mix}^X_{i\in I}\, \pi_i x_i$ *exists if and only if* $\bigsqcup_{i\in I} \pi_i x_i \in X$; *in this case* $\mathrm{mix}^X_{i\in I}\, \pi_i x_i = \bigsqcup_{i\in I} \pi_i x_i$.

(2) *Each E-normed space X is isometric to some subspace* $\widetilde{X} \subset_E \mathscr{X}{\downarrow}$, *where* $\mathbb{V}^{(B)} \models [\mathscr{X}$ *is a normed space over* $\mathbb{R}^\wedge]$ *and* $\mathscr{X} = \widetilde{X}{\uparrow}$. *This space* $\mathscr{X}$ *inside* $\mathbb{V}^{(B)}$ *is unique up to isometry. Furthermore,* $\mathscr{X}{\downarrow}$ *is a* mix-*completion of* $\widetilde{X}$. *In particular, if X is a* mix-*complete E-normed space then* $\widetilde{X} = \mathscr{X}{\downarrow}$.

◁ (1): Suppose that $\mathbb{V}^{(B)} \models [\mathscr{Y}$ is a completion of $\mathscr{X}]$. In particular, $\mathbb{V}^{(B)} \models [\mathscr{X}$ is a vector subspace of the Banach space $\mathscr{Y}$ over $\mathbb{R}^\wedge]$. Then $\mathscr{X}{\downarrow}$ is a mix-complete E-normed subspace of $\mathscr{Y}{\downarrow}$. Indeed, $\mathscr{X}{\downarrow}$ is obviously closed under the linear operations, the condition $(\forall\, \pi \in B)(\forall\, x \in \mathscr{X}{\downarrow})(\pi_{\mathscr{Y}\downarrow} x \in \mathscr{X}{\downarrow})$ and the mix-completeness of $\mathscr{X}{\downarrow}$ follow from the equality $\mathrm{cyc}\, \mathscr{X}{\downarrow} = \mathscr{X}{\downarrow}$, while the relation $|\mathscr{X}{\downarrow}|^{\perp\perp} = E$ is justified by the implicit assumption $\mathbb{V}^{(B)} \models [\mathscr{X} \neq \{0\}]$. Finally, if $X \subset_E \mathscr{X}{\downarrow}$, $(x_i)_{i\in I} \subset X$, and $(\pi_i)_{i\in I} \in \mathrm{Prt}(B)$ then $x := \bigsqcup_{i\in I} \pi_i x_i = \mathrm{mix}^{\mathscr{X}\downarrow}_{i\in I} \pi_i x_i$; moreover, the existence of $\mathrm{mix}^X_{i\in I} \pi_i x_i$, as well as the containment $x \in X$, implies the equality $x = \mathrm{mix}^X_{i\in I} \pi_i x_i$.

(2): By 1.4 (2) each E-normed space X is isometric to some subspace $\widetilde{X} \subset_E \mathscr{Y}{\downarrow}$, where $\mathbb{V}^{(B)} \models [\mathscr{Y}$ is a Banach space over $\mathbb{R}^\wedge]$. Put $\mathscr{X} := \widetilde{X}{\uparrow}$. Then $\mathbb{V}^{(B)} \models [\mathscr{X}$ is a normed space over $\mathbb{R}^\wedge]$ (see 1.3). By (1) the descent $\mathscr{X}{\downarrow}$ is a mix-complete E-normed space and $\widetilde{X} \subset_E \mathscr{X}{\downarrow} = \widetilde{X}{\uparrow}{\downarrow} = \mathrm{cyc}\, \widetilde{X} = \mathrm{mix}_{\mathscr{X}\downarrow} \widetilde{X}$.

If $\mathbb{V}^{(B)} \models [\mathscr{X}$ and $\mathscr{Y}$ are normed spaces over $\mathbb{R}^\wedge]$, $X \subset_E \mathscr{X}{\downarrow}$, $Y \subset_E \mathscr{Y}{\downarrow}$, $\mathscr{X} = X{\uparrow}$, $\mathscr{Y} = Y{\uparrow}$, and f is an isometry of X onto Y then, as is easily seen, $f \in \mathscr{E}(X, Y)$ and $\mathbb{V}^{(B)} \models [f{\uparrow}$ is an isometry of $\mathscr{X}$ onto $\mathscr{Y}]$. ▷

In what follows, given a normed space $\mathscr{X}$ over $\mathbb{R}^\wedge$ inside $\mathbb{V}^{(B)}$, we endow $\mathscr{X}{\downarrow}$ with the descents of the linear operations and the norm $|\cdot| := \|\cdot\|_{\mathscr{X}}{\downarrow}$ and thus consider $\mathscr{X}{\downarrow}$ a (mix-complete) E-normed space.

1.6. Corollary. *Each E-normed space admits a* mix-*completion. If Y and Z are* mix-*completions of an E-normed space X then Y and Z are isometric; moreover, there is an isometry* $f : Y \to Z$ *such that* $f(x) = x$ *for all* $x \in X$.

◁ Existence of a mix-completion follows from 1.5 (2). Let Y and Z be mix-completions of X. For each $y \in Y$ put $f(y) := \mathrm{mix}^Z_{i\in I}\, \pi_i x_i$, where $(\pi_i)_{i\in I} \in \mathrm{Prt}(B)$ and $(x_i)_{i\in I} \subset X$ are such that $y = \mathrm{mix}^Y_{i\in I}\, \pi_i x_i$. Routine arguments show that the above definition of a function $f : Y \to Z$ is correct, which directly implies the equality $f(x) = x$ for all $x \in X$. The fact that f is an isometry of Y onto Z is also easily verified. ▷

1.7. Theorem. *Suppose that* $\mathscr{X} \in \mathbb{V}^{(B)}$, $\mathbb{V}^{(B)} \models [\mathscr{X}$ *is a normed space over* $\mathbb{R}^\wedge]$, *and let X be a dense E-normed subspace of* $\mathscr{X}{\downarrow}$. *Then the following properties of a subset* $U \subset X$ *are equivalent:*

(a) *U is dense in X;*

(b) *U is dense in* $\mathscr{X}{\downarrow}$*;*

(c) $\mathbb{V}^{(B)} \models [\, U{\uparrow}$ *is dense in* $X{\uparrow}]$*;*

(d) $\mathbb{V}^{(B)} \models [\, U{\uparrow}$ *is dense in* $\mathscr{X}\,]$.

◁ The implication (a)⇒(b) follows from [4, 1.5.2].

(b)⇒(d): From [4, 1.5.5, 1.5.6] it follows that U is dense in $\mathscr{X}{\downarrow}$ if and only if

$$(\forall\, x \in \mathscr{X}{\downarrow})(\forall\, e \in E^{++})(\exists\, u \in \mathrm{mix}_{\mathscr{X}\downarrow} U)\ |x - u| \leqslant e.$$

By 1.5 (1) the latter is equivalent to $(\forall\, x \in \mathscr{X}{\downarrow})(\forall\, e \in E^{++})(\exists\, u \in \mathrm{cyc}\, U)\ \mathbb{V}^{(B)} \models [\|x - u\| \leqslant e]$ which according to the equalities $(\mathrm{cyc}\, U){\uparrow} = (U{\uparrow}{\downarrow}){\uparrow} = U{\uparrow}$ amounts to

$$\mathbb{V}^{(B)} \models [\,(\forall\, x \in \mathscr{X})(\forall\, e \in \mathscr{R}^{++})(\exists\, u \in U{\uparrow})\ \|x - u\| \leqslant e\,].$$

The implication (d)⇒(c) is obvious.

(c)⇒(a): From (c) it follows that $\mathbb{V}^{(B)} \models [\,(\forall\, x \in X{\uparrow})(\forall\, e \in \mathscr{R}^{++})(\exists\, u \in U{\uparrow})\ \|x - u\| \leqslant e\,]$, i.e., $(\forall\, x \in X{\uparrow}{\downarrow})(\forall\, e \in E^{++})(\exists\, u \in \mathrm{cyc}\, U)\ |x - u| \leqslant e$. By [4, 1.5.5, 1.5.6] and 1.5 (1) the latter means that U is dense in $X{\uparrow}{\downarrow}$ and hence in X. ▷

The following is a consequence of [2, 10.3.8, 10.3.9]:

1.8. Theorem. *Suppose that* $\mathbb{V}^{(B)} \models [\mathscr{X}$ *is a normed space over* $\mathbb{R}^{\wedge}]$.

(1) *If* $s : \mathbb{N} \to \mathscr{X}{\downarrow}$ *and* $x \in \mathscr{X}{\downarrow}$ *then* $\mathbb{V}^{(B)} \models [s{\uparrow} : \mathbb{N}^{\wedge} \to \mathscr{X}]$ *and the convergence* $s(n) \to x$ *in* $\mathscr{X}{\downarrow}$ *is equivalent to* $\mathbb{V}^{(B)} \models [s{\uparrow}(n) \to x]$.

(2) *If* $\mathbb{V}^{(B)} \models [\,\sigma : \mathbb{N}^{\wedge} \to \mathscr{X}$ *and* $x \in \mathscr{X}\,]$ *then* $\sigma{\downarrow} : \mathbb{N} \to \mathscr{X}{\downarrow}$ *and the convergence* $\sigma{\downarrow}(n) \to x$ *in* $\mathscr{X}{\downarrow}$ *is equivalent to* $\mathbb{V}^{(B)} \models [\sigma(n) \to x]$.

2. The Boundedness Principle and the Banach–Steinhaus Theorem for E-Normed Spaces

Let X and Y be E-normed spaces. A linear operator $T : X \to Y$ is *bounded* if there is $c \in E^+$ such that $|Tx| \leqslant c\,|x|$ for all $x \in X$. The set of bounded linear operators from X into Y is denoted by $\mathscr{L}(X,Y)$. For $T \in \mathscr{L}(X,Y)$ put $|T| := \inf\{c \in E^+ : (\forall\, x \in X)\ |Tx| \leqslant c\,|x|\}$. As is easily seen, $|Tx| \leqslant |T||x|$ for all $x \in X$.

If X and Y are normed spaces over a subfield $F \subset \mathbb{R}$, the symbol $\mathscr{L}(X,Y)$ denotes the set of bounded linear (more exactly, F-linear) operators from X into Y, and the norm $\|T\| \in \mathbb{R}$ of an operator $T \in \mathscr{L}(X,Y)$ is defined by the traditional formula $\inf\{c \in \mathbb{R}^+ : (\forall\, x \in X)\ \|Tx\| \leqslant c\|x\|\}$.

If $\mathbb{V}^{(B)} \models [\mathscr{X}$ and $\mathscr{Y}$ are normed spaces over $\mathbb{R}^{\wedge}$ and $\tau \in \mathscr{L}(\mathscr{X},\mathscr{Y})]$ then the terms $\mathscr{L}(\mathscr{X},\mathscr{Y})$ and $\|\tau\|$ symbolize the elements $L \in \mathbb{V}^{(B)}$ and $e \in E$ such that $\mathbb{V}^{(B)} \models [\,L = \mathscr{L}(\mathscr{X},\mathscr{Y})\,]$ and $\mathbb{V}^{(B)} \models [\,e = \|\tau\|\,]$.

2.1. Theorem. *Suppose that* $\mathbb{V}^{(B)} \models [\mathscr{X}$ *and* $\mathscr{Y}$ *are normed spaces over* $\mathbb{R}^{\wedge}]$, $X \subset_E \mathscr{X}{\downarrow}$, *and* $Y \subset_E \mathscr{Y}{\downarrow}$. *Taking account of* 1.3, *consider* $X{\uparrow}$ *and* $Y{\uparrow}$ *as normed subspaces of* $\mathscr{X}$ *and* $\mathscr{Y}$ *over* $\mathbb{R}^{\wedge}$ *inside* $\mathbb{V}^{(B)}$.

(1) *If* $T \in \mathscr{L}(X,Y)$ *then* $T \in \mathscr{E}(X,Y)$, $\mathbb{V}^{(B)} \models [T{\uparrow} \in \mathscr{L}(X{\uparrow}, Y{\uparrow})]$, *and* $|T| = \|T{\uparrow}\|$.

(2) *If* $\mathbb{V}^{(B)} \models [\tau \in \mathscr{L}(\mathscr{X},\mathscr{Y})]$ *then* $\tau{\downarrow} \in \mathscr{L}(\mathscr{X}{\downarrow},\mathscr{Y}{\downarrow})$ *and* $|\tau{\downarrow}| = \|\tau\|$.

(3) *If* $\mathbb{V}^{(B)} \models [\tau \in \mathscr{L}(X{\uparrow}, Y{\uparrow})]$ *then* $\tau{\downarrow}|_X \in \mathscr{L}(X, Y{\uparrow}{\downarrow})$ *and* $|\tau{\downarrow}|_X| = \|\tau\|$.

(4) *For every* $T \in \mathscr{L}(X,Y)$ *there is a unique* $\tau \in \mathbb{V}^{(B)}$ *such that* $\mathbb{V}^{(B)} \models [\tau \in \mathscr{L}(X{\uparrow}, Y{\uparrow})]$ *and* $\tau{\downarrow}|_X = T$. *In this case* $\tau = T{\uparrow}$ *and* $|T| = \|\tau\|$.

◁ It suffices to employ [1, 8.3.3] and use the fact that, inside $\mathbb{V}^{(B)}$, each operator $\tau \in \mathscr{L}(\mathscr{X},\mathscr{Y})$ acting in the normed spaces $\mathscr{X}$ and $\mathscr{Y}$ over $\mathbb{R}^{\wedge}$ admits a unique extension $\bar\tau \in \mathscr{L}(\overline{\mathscr{X}},\overline{\mathscr{Y}})$, $\|\bar\tau\| = \|\tau\|$, where $\overline{\mathscr{X}}$ and $\overline{\mathscr{Y}}$ are completions of $\mathscr{X}$ and $\mathscr{Y}$. ▷

2.2. Lemma. *Let* $\overline{X}$ *and* $\overline{Y}$ *be* mix-*completions of* E-*normed spaces* X *and* Y. *Then each operator* $T \in \mathscr{L}(X,Y)$ *admits a unique extension* $\overline{T} \in \mathscr{L}(\overline{X},\overline{Y})$; *furthermore,* $|T| = |\overline{T}|$.

◁ By 1.5 (2) we may assume that $X \subset_E \overline{X} = \mathscr{X}{\downarrow}$ and $Y \subset_E \overline{Y} = \mathscr{Y}{\downarrow}$, where $\mathbb{V}^{(B)} \models [\mathscr{X}$ and $\mathscr{Y}$ are normed spaces over $\mathbb{R}^{\wedge}]$, $\mathscr{X} = X{\uparrow}$, and $\mathscr{Y} = Y{\uparrow}$. Each operator $T \in \mathscr{L}(X,Y)$ is extensional (see 2.1 (1)). Put $\overline{T} := T{\uparrow}{\downarrow} : X{\uparrow}{\downarrow} \to Y{\uparrow}{\downarrow}$. From 2.1 it follows that $\overline{T} \in \mathscr{L}(\overline{X},\overline{Y})$, $\overline{T}|_X = T$, and $|T| = |\overline{T}|$. The extension $\overline{T}$ is unique by 2.1 (4). ▷

2.3. The following slight generalization of the boundedness principle is easily derived from the classical version of the principle [5, 7.2.5]:

Let X *and* Y *be normed spaces over a subfield of* $\mathbb{R}$, *with* X *complete, and let* $\mathscr{T} \subset \mathscr{L}(X,Y)$. *If* $\sup_{T\in\mathscr{T}}\|Tx\| < \infty$ *for each* $x \in X$ *then* $\sup_{T\in\mathscr{T}}\|T\| < \infty$.

2.4. Theorem. *Let* X *and* Y *be* E-*normed spaces, with* X *complete, and let* $\mathscr{T} \subset \mathscr{L}(X,Y)$. *If* $\{|Tx| : T \in \mathscr{T}\}$ *is order bounded for all* $x \in X$ *then* $\{|T| : T \in \mathscr{T}\}$ *is order bounded.*

◁ Let $\overline{Y}$ be a mix-completion of Y (see 1.6). Then the subset $\mathscr{T} \subset \mathscr{L}(X,Y)$ is also a subset of $\mathscr{L}(X,\overline{Y})$ and the norm of any operator $T \in \mathscr{T}$ evaluated in $\mathscr{L}(X,Y)$ coincides with that in $\mathscr{L}(X,\overline{Y})$.

By 1.4 (2) and 1.5 (2) we may assume that $X = \mathscr{X}{\downarrow}$ and $\overline{Y} = \mathscr{Y}{\downarrow}$, where $\mathbb{V}^{(B)} \models [\mathscr{X}$ is a Banach space over $\mathbb{R}^\wedge$ and $\mathscr{Y}$ is a normed space over $\mathbb{R}^\wedge]$. Suppose that $(\forall\, x \in X)(\exists\, e \in E)(\forall\, T \in \mathscr{T})\ \lvert Tx \rvert \leqslant e$ (according to the hypothesis of the theorem). Since $\mathbb{V}^{(B)} \models [\,\lvert Tx\rvert = \|T{\uparrow}x\|\,]$, we have

$$(\forall\, x \in \mathscr{X}{\downarrow})(\exists\, e \in E)(\forall\, \tau \in \{T{\uparrow} : T \in \mathscr{T}\})\ \mathbb{V}^{(B)} \models [\,\|\tau x\| \leqslant e\,].$$

The equality $\mathscr{T}{\uparrow\uparrow} = \{T{\uparrow} : T \in \mathscr{T}\}{\uparrow}$ implies

$$\mathbb{V}^{(B)} \models [\,(\forall\, x \in \mathscr{X})(\exists\, e \in \mathscr{R})(\forall\, \tau \in \mathscr{T}{\uparrow\uparrow})\ \|\tau x\| \leqslant e\,].$$

Furthermore, from 2.1 it follows that $\mathbb{V}^{(B)} \models [\mathscr{T}{\uparrow\uparrow} \subset \mathscr{L}(\mathscr{X}, \mathscr{Y})]$. Applying 2.3 inside $\mathbb{V}^{(B)}$, we consequently derive

$$\mathbb{V}^{(B)} \models [\,(\exists\, e \in \mathscr{R})(\forall\, \tau \in \mathscr{T}{\uparrow\uparrow})\ \|\tau\| \leqslant e\,];$$
$$(\exists\, e \in E)(\forall\, \tau \in \{T{\uparrow} : T \in \mathscr{T}\})\ \mathbb{V}^{(B)} \models [\,\|\tau\| \leqslant e\,];$$
$$(\exists\, e \in E)(\forall\, T \in \mathscr{T})\ \|T{\uparrow}\| \leqslant e.$$

It remains to observe that $\lvert T\rvert = \|T{\uparrow}\|$ by 2.1 (1). ▹

2.5. The following generalization of the Banach–Steinhaus theorem is an easy consequence of the classical version of the theorem [5, 7.2.9]:

Suppose that X and Y are normed spaces over a subfield of $\mathbb{R}$, Y is complete, $(T_n)_{n\in\mathbb{N}} \subset \mathscr{L}(X, Y)$, U is a dense subset of X, the limit $\lim_{n\to\infty} T_n u \in Y$ exists for each $u \in U$, and there is $c \in \mathbb{R}^+$ such that $\|T_n\| \leqslant c$ for all $n \in \mathbb{N}$. Then there exists an operator $T \in \mathscr{L}(X, Y)$ such that $\|T\| \leqslant c$ and $\lim_{n\to\infty} T_n x = Tx$ for all $x \in X$.

2.6. Theorem. *Suppose that X and Y are E-normed spaces, Y is complete, $(T_n)_{n\in\mathbb{N}} \subset \mathscr{L}(X, Y)$, U is a dense subset of X, the limit $\lim_{n\to\infty} T_n u \in Y$ exists for each $u \in U$, and there is $c \in E^+$ such that $\lvert T_n\rvert \leqslant c$ for all $n \in \mathbb{N}$. Then there exists an operator $T \in \mathscr{L}(X, Y)$ such that $\lvert T\rvert \leqslant c$ and $\lim_{n\to\infty} T_n x = Tx$ for all $x \in X$.*

◃ We first prove the statement for the case in which X is mix-complete.

By 1.4 (2) and 1.5 (2) we may assume that $X = \mathscr{X}{\downarrow}$ and $Y = \mathscr{Y}{\downarrow}$, where $\mathbb{V}^{(B)} \models [\mathscr{X}$ and $\mathscr{Y}$ are normed spaces over $\mathbb{R}^\wedge$, with $\mathscr{Y}$ complete$]$.

Put $s(n) := T_n{\uparrow}$ for $n \in \mathbb{N}$. From 2.1 it follows that $s : \mathbb{N} \to \mathscr{L}(\mathscr{X}, \mathscr{Y}){\downarrow}$ and $\mathscr{L}(X,Y) = \mathscr{L}(\mathscr{X},\mathscr{Y}){\downarrow\downarrow}$, whence $\mathbb{V}^{(B)} \models [s{\uparrow} : \mathbb{N}^\wedge \to \mathscr{L}(\mathscr{X},\mathscr{Y})]$ and $\mathbb{V}^{(B)} \models [s{\uparrow}(n^\wedge) = T_n{\uparrow}]$ for all $n \in \mathbb{N}$.

According to 1.7 we have $\mathbb{V}^{(B)} \models [U{\uparrow}$ is dense in $\mathscr{X}]$. Moreover, $\|T_n{\uparrow}\| = \lvert T_n\rvert \leqslant c$ for all $n \in \mathbb{N}$ (see 2.1 (1)); therefore, $(\forall\, n \in \mathbb{N})\ \mathbb{V}^{(B)} \models [\,\|s{\uparrow}(n^\wedge)\| \leqslant c\,]$, i.e., $\mathbb{V}^{(B)} \models [\,(\forall\, n \in \mathbb{N}^\wedge)\ \|s{\uparrow}(n)\| \leqslant c\,]$. Finally, by the hypothesis of the theorem, $(\forall\, u \in U)(\exists\, y \in Y)\ s(n)u \to y$, whence by 1.8 (1) we have $(\forall\, u \in U)$ $(\exists\, y \in Y)\ \mathbb{V}^{(B)} \models [\,s{\uparrow}(n)u \to y\,]$, i.e., $\mathbb{V}^{(B)} \models [\,(\forall\, u \in U{\uparrow})(\exists\, y \in \mathscr{Y})\ s{\uparrow}(n)u \to y\,]$.

Applying 2.5 inside $\mathbb{V}^{(B)}$, we obtain $\mathbb{V}^{(B)} \models [\,(\exists\, \tau)\ \tau \in \mathscr{L}(\mathscr{X}, \mathscr{Y}),\ \|\tau\| \leqslant c,\ (\forall\, x \in \mathscr{X})\ s{\uparrow}(n)x \to \tau x\,]$ and hence there is $\tau \in \mathscr{L}(\mathscr{X}, \mathscr{Y}){\downarrow}$ such that $\|\tau\| \leqslant c$ and $(\forall\, x \in X)\ \mathbb{V}^{(B)} \models [\,s{\uparrow}(n)x \to \tau x\,]$. Put $T := \tau{\downarrow}$. Then, according to 1.8 (1) and 2.1 (2), we have $T \in \mathscr{L}(X, Y)$, $\lvert T\rvert \leqslant c$, and $(\forall\, x \in X)\ T_n x \to Tx$.

Let now X be an arbitrary (not necessarily mix-complete) E-normed space and let $\overline{X}$ be a mix-completion of X (see 1.6). Since Y is complete, it is mix-complete and therefore Y is a mix-completion of Y. By 2.2, for every $n \in \mathbb{N}$ there exists a (unique) extension $\overline{T}_n \in \mathscr{L}(\overline{X}, Y)$ of $T_n \in \mathscr{L}(X, Y)$; furthermore, $\lvert\overline{T}_n\rvert = \lvert T_n\rvert \leqslant c$. Moreover, as U is dense in X, U is dense in $\overline{X}$ (see 1.1). Finally, for each $u \in U$ we have $(\forall\, n \in \mathbb{N})\ \overline{T}_n u = T_n u$ and hence the limit $\lim_{n\to\infty} \overline{T}_n u = \lim_{n\to\infty} T_n u \in Y$ exists.

The hypotheses of the theorem are thus satisfied for $\overline{X}$, Y, $(\overline{T}_n)_{n\in\mathbb{N}}$, U, and c, while $\overline{X}$ is mix-complete. Consequently, there exists an operator $\overline{T} \in \mathscr{L}(\overline{X}, Y)$ such that $\lvert\overline{T}\,\rvert \leqslant c$ and $\lim_{n\to\infty} \overline{T}_n x = \overline{T}x$ for all $x \in \overline{X}$. By putting $T := \overline{T}\,|_X$, we obtain the required operator $T \in \mathscr{L}(X, Y)$. ▹

3. The Uniform Boundedness Principle for a mix-Compact Convex Set

Suppose that X is an E-normed space, $(x_n)_{n\in\mathbb{N}} \subset X$, and $x \in X$. Say that a *sequence* $(x_n)_{n\in\mathbb{N}}$ *approximates* x if, for each $k \in \mathbb{N}$, the set $\{x_n : n \geqslant k\}$ approximates x, i.e., $(\forall\, k \in \mathbb{N})\ \inf_{n\geqslant k} |x_n - x| = 0$.

Call a set $K \subset X$ mix-*compact* if K is mix-complete and for every sequence $(x_n)_{n\in\mathbb{N}} \subset K$ there is $x \in K$ such that $(x_n)_{n\in\mathbb{N}}$ approximates x. (It is clear that in case $E = \mathbb{R}$ mix-compactness is equivalent to compactness in the norm topology.)

3.1. As is easily seen, mix-compactness is an absolute concept in the following sense:

If X and $\overline{X}$ are E-normed spaces and $K \subset X \subset_E \overline{X}$ then the mix-*compactness of K in X is equivalent to that in $\overline{X}$.*

3.2. Theorem. *Suppose that $\mathbb{V}^{(B)} \vDash [\mathscr{X}$ is a normed space over $\mathbb{R}^\wedge]$.*

(1) *A subset $K \subset \mathscr{X}{\downarrow}$ is* mix-*compact if and only if K is* mix-*complete and $\mathbb{V}^{(B)} \vDash [K{\uparrow}$ is a compact subset of $\mathscr{X}]$.*

(2) *$\mathbb{V}^{(B)} \vDash [\mathscr{K}$ is a compact subset of $\mathscr{X}]$ if and only if $\mathscr{K}{\downarrow}$ is a* mix-*compact subset of $\mathscr{X}{\downarrow}$.*

$\lhd$ (1): The compactness of $K{\uparrow}$ inside $\mathbb{V}^{(B)}$ is equivalent to

$$\mathbb{V}^{(B)} \vDash [\,(\forall\, \sigma : \mathbb{N}^\wedge \to K{\uparrow})(\exists\, x \in K{\uparrow})(\forall\, k \in \mathbb{N}^\wedge)\ \inf\{\|\sigma(n) - x\| : n \geqslant k\} = 0\,].$$

Taking account of [2, 5.7.8] and $\operatorname{cyc} K = K$ (see 1.5 (1)), we conclude that the above relation amounts to $(\forall\, s : \mathbb{N} \to K)(\exists\, x \in K)(\forall\, k \in \mathbb{N})\ \varphi$, where $\varphi := \big(\mathbb{V}^{(B)} \vDash [\,\inf\{\|s{\uparrow}(n) - x\| : n \geqslant k^\wedge\} = 0\,]\big)$. It remains to observe that

$$\begin{aligned}\varphi \ &\Leftrightarrow\ \mathbb{V}^{(B)} \vDash \big[\,(\forall\, e \in \mathscr{R}^+)\big(\,(\forall\, n \geqslant k^\wedge)(e \leqslant \|s{\uparrow}(n) - x\|) \Rightarrow e = 0\,\big)\,\big]\\ &\Leftrightarrow\ (\forall\, e \in E^+)\big(\,(\forall\, n \geqslant k)(e \leqslant |s(n) - x|) \Rightarrow e = 0\,\big)\\ &\Leftrightarrow\ \inf\{|s(n) - x| : n \geqslant k\} = 0.\end{aligned}$$

(2): Put $K := \mathscr{K}{\downarrow}$. If $\mathbb{V}^{(B)} \vDash [\mathscr{K}$ is a compact subset of $\mathscr{X}]$ then, using the obvious mix-completeness of K and applying (1), we conclude that K is a mix-compact subset of $\mathscr{X}{\downarrow}$. Conversely, if K is a mix-compact subset of $\mathscr{X}{\downarrow}$ then, as $K{\uparrow} = \mathscr{K}$, we have $\mathbb{V}^{(B)} \vDash [\mathscr{K}$ is a compact subset of $\mathscr{X}]$ due to (1). $\rhd$

Let X be a normed space over $\mathbb{R}$ and let F be a subfield of $\mathbb{R}$. Say that a subset $U \subset X$ is *F-convex* if $(1 - \lambda)u + \lambda v \in U$ for all $u, v \in U$ and $\lambda \in [0,1]_F$, where $[0,1]_F := F \cap [0,1]$. By a convex subset of X we conventionally mean an $\mathbb{R}$-convex set. As is easily seen, if a set $U \subset X$ is closed and F-convex then U is a convex subset of X.

The following is an analog of the classical uniform boundedness principle for a compact convex set (cp. [6, 2.9]):

3.3. Theorem. *Let X and Y be E-normed spaces, let $\mathscr{T} \subset \mathscr{L}(X,Y)$, and let K be a* mix-*compact convex subset of X. If $\{|Tx| : T \in \mathscr{T}\}$ is order bounded for each $x \in K$ then $\{|Tx| : T \in \mathscr{T},\ x \in K\}$ is order bounded.*

$\lhd$ Taking account of 1.4 (2) and 1.2, we may assume that X and Y are dense E-normed subspaces of $\mathscr{X}{\downarrow}$ and $\mathscr{Y}{\downarrow}$, where $\mathbb{V}^{(B)} \vDash [\,\mathscr{X}$ and $\mathscr{Y}$ are Banach spaces over $\mathscr{R}\,]$. From 1.7 it follows that $\mathbb{V}^{(B)} \vDash [X{\uparrow}$ is dense in $\mathscr{X}]$ and $\mathbb{V}^{(B)} \vDash [Y{\uparrow}$ is dense in $\mathscr{Y}]$. Put $\mathscr{K} := K{\uparrow}$.

By 3.2 (1) we have $\mathbb{V}^{(B)} \vDash [\mathscr{K}$ is a compact subset of $\mathscr{X}]$.

By the hypothesis of the theorem $(\forall\, x, y \in K)(\forall\, \lambda \in [0,1]_{\mathbb{R}})(\exists\, z \in K)\ z = (1^\wedge - \lambda^\wedge)x + \lambda^\wedge y$. The formula $\varphi(S, 0, 1, \mathbb{R})$ which defines the equality $S = [0,1]_{\mathbb{R}}$ is bounded (see [2, 4.2.9]); therefore, $\mathbb{V}^{(B)} \vDash \big[\,([0,1]_{\mathbb{R}})^\wedge = [0^\wedge, 1^\wedge]_{\mathbb{R}^\wedge}\big]$. Consequently,

$$\mathbb{V}^{(B)} \vDash \big[\,(\forall\, x, y \in \mathscr{K})(\forall\, \lambda \in [0^\wedge, 1^\wedge]_{\mathbb{R}^\wedge})(\exists\, z \in \mathscr{K})\ z = (1^\wedge - \lambda)x + \lambda y\,\big],$$

i.e., $\mathbb{V}^{(B)} \vDash [\mathscr{K}$ is an $\mathbb{R}^\wedge$-convex subset of $\mathscr{X}]$. Since $\mathbb{V}^{(B)} \vDash [\mathscr{K}$ is closed in $\mathscr{X}]$, we conclude that $\mathbb{V}^{(B)} \vDash [\mathscr{K}$ is a convex subset of $\mathscr{X}]$.

By the hypothesis of the theorem $(\forall\, x\in K)(\exists\, e\in E)(\forall\, T\in\mathscr{T})\ |Tx|\leqslant e$ and hence

$$(\forall\, x\in K)(\exists\, e\in E)(\forall\, \tau\in\{T{\uparrow} : T\in\mathscr{T}\})\ \mathbb{V}^{(B)}\models[\,\|\tau x\|\leqslant e\,]$$

or, which is the same,

$$\mathbb{V}^{(B)}\models[\,(\forall\, x\in\mathscr{K})(\exists\, e\in\mathscr{R})(\forall\, \tau\in\mathscr{T}{\uparrow\uparrow})\ \|\tau x\|\leqslant e\,].$$

From 2.1 it follows that $\mathbb{V}^{(B)}\models[\mathscr{T}{\uparrow\uparrow}\subset\mathscr{L}(X{\uparrow},Y{\uparrow})]$. Since $\mathbb{V}^{(B)}\models[X{\uparrow}$ is dense in $\mathscr{X}]$ and $\mathbb{V}^{(B)}\models[\mathscr{Y}$ is a Banach space$]$, each operator $\tau\in\mathscr{L}(X{\uparrow},Y{\uparrow})$ inside $\mathbb{V}^{(B)}$ admits a unique extension $\bar\tau\in\mathscr{L}(\mathscr{X},\mathscr{Y})$. Denote by $\overline{\mathscr{T}}$ the element of $\mathbb{V}^{(B)}$ such that $\mathbb{V}^{(B)}\models[\overline{\mathscr{T}}=\{\bar\tau : \tau\in\mathscr{T}{\uparrow\uparrow}\}]$. Then $\mathbb{V}^{(B)}\models[\overline{\mathscr{T}}\subset\mathscr{L}(\mathscr{X},\mathscr{Y})]$ and $\mathbb{V}^{(B)}\models[(\forall\, x\in\mathscr{K})(\exists\, e\in\mathscr{R})(\forall\, \tau\in\overline{\mathscr{T}})\ \|\tau x\|\leqslant e\,]$.

Therefore, inside $\mathbb{V}^{(B)}$, the objects $\mathscr{X}$, $\mathscr{Y}$, $\overline{\mathscr{T}}$, and $\mathscr{K}$ meet the hypotheses of the uniform boundedness principle for a compact convex set (see [6, 2.9]). Applying the principle inside $\mathbb{V}^{(B)}$, we conclude that $\mathbb{V}^{(B)}\models[\,(\exists\, e\in\mathscr{R})(\forall\, \tau\in\overline{\mathscr{T}})(\forall\, x\in\mathscr{K})\ \|\tau x\|\leqslant e\,]$. Since $\mathbb{V}^{(B)}\models[\mathscr{K}\subset X{\uparrow}]$, the latter holds for $\mathscr{T}{\uparrow\uparrow}$ as well, i.e., $\mathbb{V}^{(B)}\models[\,(\exists\, e\in\mathscr{R})(\forall\, \tau\in\mathscr{T}{\uparrow\uparrow})(\forall\, x\in\mathscr{K})\ \|\tau x\|\leqslant e\,]$; hence,

$$(\exists\, e\in E)(\forall\, \tau\in\{T{\uparrow} : T\in\mathscr{T}\})(\forall\, x\in K)\ \mathbb{V}^{(B)}\models[\,\|\tau x\|\leqslant e\,]$$

or, which is the same, $(\exists\, e\in E)(\forall\, T\in\mathscr{T})(\forall\, x\in K)\ |Tx|\leqslant e$. $\triangleright$

To conclude, we will demonstrate (see 3.4) that the notion of a cyclically compact subset of a B-cyclic Banach space (see [1, 7.3.1, 7.3.3, 8.5.1]) is in a sense a particular case of the notion of a mix-compact subset of an E-normed space. For convenience, we recall the corresponding definitions.

According to [1, 8.3.5] a *B-cyclic Banach space* $(X,\|\cdot\|)$ can be thought of as an E-normed subspace $X:=\{x\in\overline{X} : (\exists\,\lambda\in\mathbb{R}^+)\ |x|\leqslant\lambda\,1_E\}$ of any complete E-normed space $\overline{X}$ that is endowed with the $\mathbb{R}$-valued norm $\|x\|:=\inf\{\lambda\in\mathbb{R}^+ : |x|\leqslant\lambda\,1_E\}$, $x\in X$.

Denote by $\mathrm{Prt}_{\mathbb{N}}(B)$ the set of sequences $\nu:\mathbb{N}\to B$ which are partitions of unity of the Boolean algebra B. For $\nu_1,\nu_2\in\mathrm{Prt}_{\mathbb{N}}(B)$, the formula $\nu_1\ll\nu_2$ abbreviates the following assertion: if $m,n\in\mathbb{N}$ and $\nu_1(m)\wedge\nu_2(n)\neq 0_B$ then $m<n$.

Let X be a B-cyclic Banach space.

Given a mix-complete subset $K\subset X$, a sequence $s:\mathbb{N}\to K$, and a partition $\nu\in\mathrm{Prt}_{\mathbb{N}}(B)$, put $s_\nu:=\mathrm{mix}^X_{n\in\mathbb{N}}\,\nu(n)s(n)$. A *cyclic subsequence* of $s:\mathbb{N}\to K$ is any sequence of the form $(s_{\nu_k})_{k\in\mathbb{N}}$, where $(\nu_k)_{k\in\mathbb{N}}\subset\mathrm{Prt}_{\mathbb{N}}(B)$ and $(\forall\, k\in\mathbb{N})\ \nu_k\ll\nu_{k+1}$.

A subset $K\subset X$ is called *cyclically compact* (see [1, 8.5.1]) if K is mix-complete and each sequence of elements in K admits a cyclic subsequence convergent to an element of K in the norm $\|\cdot\|$.

3.4. Theorem. *Let X be a B-cyclic Banach space. A subset $K\subset X$ is cyclically compact if and only if K is* mix-*compact.*

$\triangleleft$ $(\Rightarrow)$: Let K be a cyclically compact subset of X. Consider an arbitrary sequence $s:\mathbb{N}\to K$. By the definition of cyclic compactness there exist a sequence $(\nu_k)_{k\in N}\subset\mathrm{Prt}_{\mathbb{N}}(B)$ and an element $x\in K$ such that $(\forall\, k\in\mathbb{N})(\nu_k\ll\nu_{k+1})$ and $\|s_{\nu_k}-x\|\to 0$. A direct analysis of the latter relations shows that $\inf\big\{|\varkappa-x| : \varkappa\in\mathrm{mix}_X\{s(n) : n\geqslant k\}\big\}=0$ for all $k\in\mathbb{N}$ and hence the sequence s approximates $x\in K$, since for each $\varkappa=\mathrm{mix}^X_{n\geqslant k}\,\pi_n s(n)$, with $(\pi_n)_{n\geqslant k}\in\mathrm{Prt}(B)$, we have

$$\pi_m\Big(\inf_{n\geqslant k}|s(n)-x|\Big)\leqslant\pi_m|s(m)-x|=|\pi_m s(m)-\pi_m x|=\pi_m|\varkappa-x|\leqslant|\varkappa-x|$$

for all $m\geqslant k$ and, consequently,

$$\inf_{n\geqslant k}|s(n)-x|=\sup_{m\geqslant k}\pi_m\Big(\inf_{n\geqslant k}|s(n)-x|\Big)\leqslant|\varkappa-x|.$$

(⇐): Suppose now that K is a mix-compact subset of X and let $s : \mathbb{N} \to K$. According to 1.5 (2) we may assume that $X \subset_E \mathscr{X}{\downarrow}$, where $\mathbb{V}^{(B)} \models [\mathscr{X}$ is a normed space over $\mathbb{R}^\wedge]$. Put $\sigma := s{\uparrow}$. Then $\mathbb{V}^{(B)} \models [\sigma : \mathbb{N}^\wedge \to K{\uparrow}]$. Moreover, 3.1 and 3.2 (1) imply $\mathbb{V}^{(B)} \models [K{\uparrow}$ is a compact subset of $\mathscr{X}]$. Applying the classical compactness criterion inside $\mathbb{V}^{(B)}$, consider $x \in K$ and $\mathscr{N} \in \mathbb{V}^{(B)}$ such that

$$\mathbb{V}^{(B)} \models \big[\mathscr{N} : \mathbb{N}^\wedge \to \mathbb{N}^\wedge,\ \mathscr{N}(k) < \mathscr{N}(k+1),\ \|\sigma(\mathscr{N}(k)) - x)\| \leqslant \tfrac{1}{k} \text{ for each } k \in \mathbb{N}^\wedge\big].$$

Put $\nu_k(n) := [\![\mathscr{N}(k^\wedge) = n^\wedge]\!]$ for all $k, n \in \mathbb{N}$. A routine verification shows that $\nu_k \in \mathrm{Prt}_{\mathbb{N}}(B)$ and $(\forall\, k \in \mathbb{N})\ \nu_k \ll \nu_{k+1}$. Moreover, for each $k \in \mathbb{N}$ we have $\mathbb{V}^{(B)} \models [s_{\nu_k} = \sigma(\mathscr{N}(k^\wedge))]$ and, consequently, $\|s_{\nu_k} - x\| \leqslant \frac{1}{k}$. ▷

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