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SHORT COMMUNICATIONS

Convergence-Preserving Maps and Fixed-Point Theorems

A. E. Gutman[1*] and A. V. Koptev[2]**

[1]*Sobolev Institute of Mathematics, Russian Academy of Sciences, Novosibirsk, Russia*
Novosibirsk State University, Novosibirsk, Russia

[2]*Sobolev Institute of Mathematics, Russian Academy of Sciences, Novosibirsk, Russia*

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By a *convergence* (to be more precise, by a *sequential convergence*) on a set X we mean a correspondence between sequences and elements of X, i.e., any subset of $\mathcal{S}_X \times X$, where $\mathcal{S}_X := X^{\mathbb{N}}$. A convergence on X is *(pre)topological* if it coincides with convergence in some (pre)topology on X. It is easy to see that if $\tau\colon X \to \mathcal{P}(\mathcal{P}(X)) \setminus \{\varnothing\}$ and $(\forall\, x \in X)(\forall\, U \in \tau(x))(x \in U)$, then the convergence

$$\alpha \to x \quad\Longleftrightarrow\quad (\forall\, U \in \tau(x))(\exists\, \overline{n} \in \mathbb{N})(\forall\, n \geq \overline{n})(\alpha(n) \in U)$$

is pretopological. In particular, this class contains all convergences determined by metrics, partial metrics [1], generalized metrics [2], cone metrics [3], and tvs-metrics [4].

We denote the constant sequence $\mathbb{N} \times \{x\}$ by $x^\wedge$. Given $\alpha, \beta \in \mathcal{S}_X$, by $\operatorname{mix}(\alpha, \beta)$ we denote the sequence $\gamma \in \mathcal{S}_X$ such that $\gamma(2n-1) = \alpha(n)$ and $\gamma(2n) = \beta(n)$ for all $n \in \mathbb{N}$. If β is a subsequence of α, then we write $\beta \preccurlyeq \alpha$. For any convergence $\to$ on X, we define a convergence $\overset{*}{\to}$ on X by setting

$$\alpha \overset{*}{\to} x \quad\Longleftrightarrow\quad (\forall\, \beta \preccurlyeq \alpha)(\exists\, \gamma \preccurlyeq \beta)(\gamma \to x).$$

A convergence is said to be *single-valued* if

$$(\forall\, \alpha \in \mathcal{S}_X)(\forall\, x, y \in X)(\alpha \to x \,\&\, \alpha \to y \Rightarrow x = y).$$

Consider the following properties of a convergence $\to$ on a set X:

(0) $(\forall\, \alpha \in \mathcal{S}_X)(\forall\, x \in X)((\forall\, n \in \mathbb{N})(\alpha(n)^\wedge \to x) \Rightarrow \alpha \to x)$;

(1) $(\forall\, x \in X)(x^\wedge \to x)$;

(2) $(\forall\, \alpha \in \mathcal{S}_X)(\forall\, x \in X)(\alpha \to x \Rightarrow (\forall\, \beta \preccurlyeq \alpha)\, \beta \to x)$;

(3) $(\forall\, \alpha \in \mathcal{S}_X)(\forall\, x \in X)(\alpha \overset{*}{\to} x \Rightarrow \alpha \to x)$.

Theorem 1 ([5]). (a) *A convergence is pretopological if and only if it satisfies conditions* (0)–(3).

(b) *The following three properties of a single-valued convergence are pairwise equivalent*: *the convergence is pretopological*; *the convergence is topological*; *the convergence satisfies conditions* (1)–(3).

(c) *Any topological convergence on X coincides with convergence in some sequential topology on X.*

*E-mail: gutman@math.nsc.ru
**E-mail: koptev@math.nsc.ru

Recall that a topological space X is said to be *sequential* if

$$(\forall Y \subset X)(\operatorname{cl}_\sigma Y = Y \Rightarrow \operatorname{cl} Y = Y),$$

and it is called a *Fréchet space* if

$$(\forall Y \subset X)(\operatorname{cl}_\sigma Y = \operatorname{cl} Y),$$

where $\operatorname{cl} Y$ denotes the closure of Y and $\operatorname{cl}_\sigma Y := \{x \in X : (\exists\, \alpha \in \mathcal{S}_Y)(\alpha \to x)\}$ denotes the sequential closure of Y. Since a sequential topology with given convergence is unique, assertion (c) says that the notions of topological convergence and sequential topology are equivalent.

Throughout the paper, X and Y are sequential topological spaces.

We set $\mathcal{C}_X = \{\alpha \in \mathcal{S}_X : (\exists\, x \in X)(\alpha \to x)\}$. We refer to a space with single-valued convergence as a *single-valued* space. If X is a single-valued space and $\alpha \in \mathcal{C}_X$, then, for the (unique) limit of the sequence α, we use the traditional notation $\lim \alpha$. Single-valuedness occupies an intermediate position between the classical separation axioms T_1 and T_2; the T_1 axiom is equivalent to the uniqueness of a limit for constant sequences:

$$(\forall\, x, y \in X)(x^\wedge \to y \Rightarrow x = y).$$

Let $D \subset X$, and let $f\colon D \to Y$. We say that a function f is *convergence-preserving* if

$$(\forall\, \alpha \in \mathcal{S}_D)(\alpha \in \mathcal{C}_X \Rightarrow f \circ \alpha \in \mathcal{C}_Y).$$

If the space D is sequential, then, as is known, the continuity of f is equivalent to the condition

$$(\forall\, \alpha \in \mathcal{S}_D)(\forall\, x \in D)(\alpha \to x \Rightarrow f \circ \alpha \to f(x)).$$

Lemma 1. *The following equivalence holds*:

$$(\forall\, \alpha, \beta \in \mathcal{S}_X)(\forall\, x \in X)(\operatorname{mix}(\alpha, \beta) \to x \Leftrightarrow (\alpha \to x \;\&\; \beta \to x)).$$

In particular, if X is single-valued, then $\operatorname{mix}(\alpha, \beta) \in \mathcal{C}_X$ *implies* $\alpha, \beta \in \mathcal{C}_X$ *and* $\lim \alpha = \lim \beta$.

Proof. If $\alpha \to x$ and $\beta \to x$, then

$$(\forall\, \gamma \preccurlyeq \operatorname{mix}(\alpha, \beta))(\exists\, \delta \preccurlyeq \gamma)(\delta \preccurlyeq \alpha \vee \delta \preccurlyeq \beta),$$

whence $\operatorname{mix}(\alpha, \beta) \xrightarrow{*} x$ and, therefore, $\operatorname{mix}(\alpha, \beta) \to x$. The reverse implication follows from the relations $\alpha \preccurlyeq \operatorname{mix}(\alpha, \beta)$ and $\beta \preccurlyeq \operatorname{mix}(\alpha, \beta)$. ☐

Lemma 2. *Let Y be a T_1 space. Then a function $f\colon X \to Y$ is convergence-preserving if and only if it is continuous.*

Proof. Let us clarify the *only if* part. Suppose that $\alpha \to x$. Then $\operatorname{mix}(\alpha, x^\wedge) \to x$, whence

$$\operatorname{mix}(f \circ \alpha, f(x)^\wedge) = f \circ \operatorname{mix}(\alpha, x^\wedge) \to y, \qquad \text{where} \quad y \in Y;$$

therefore, $f \circ \alpha \to y$ and $f(x)^\wedge \to y$. Since Y is T_1, it follows that $f(x)^\wedge \to y$ implies $f(x) = y$. Hence $f \circ \alpha \to f(x)$. ☐

Note that T_1 separability in Lemma 2 cannot be relaxed to T_0. Indeed, suppose that X, $\alpha \in \mathcal{S}_X$, and $x \in X$ are such that $\alpha \to x \notin \operatorname{im} \alpha$. Let Y be the set $\{0, 1\}$ with the open topology $\{\varnothing, \{1\}, \{0, 1\}\}$. Then the function $f\colon X \to Y$, where $f \equiv 0$ on $X \setminus \{x\}$ and $f(x) = 1$, is convergence-preserving and discontinuous.

Theorem 2. *Suppose that X is a Fréchet space, Y is a regular single-valued sequential space, and $D \subset X$. Then a function $f\colon D \to Y$ is convergence-preserving if and only if f can be extended to a continuous function $\overline{f}\colon \operatorname{cl} D \to Y$.*

Proof. *Sufficiency* is obvious. Let us prove *necessity*. Suppose that f preserves convergence. If $\alpha, \beta \in \mathcal{S}_D$ have a common limit, then $\operatorname{mix}(\alpha, \beta) \in \mathcal{C}_X$, whence

$$\operatorname{mix}(f \circ \alpha, f \circ \beta) = f \circ \operatorname{mix}(\alpha, \beta) \in \mathcal{C}_Y$$

and, therefore, $\lim(f \circ \alpha) = \lim(f \circ \beta)$. Hence there exists a function $\overline{f}\colon \operatorname{cl} D \to Y$ such that $\lim(f \circ \alpha) = \overline{f}(x)$ for all $x \in \operatorname{cl} D$, $\alpha \in \mathcal{S}_D$, and $\alpha \to x$. Let us prove the continuity of $\overline{f}$. Arguing by contradiction, suppose that $\beta \in \mathcal{S}_{\operatorname{cl} D}$ and $\beta \to x \in \operatorname{cl} D$ but $\overline{f} \circ \beta \nrightarrow \overline{f}(x)$. Passing to a subsequence, we can assume that $\overline{f}(x) \notin \operatorname{cl} \operatorname{im}(\overline{f} \circ \beta)$. The regularity of Y implies the existence of disjoint open sets $U, V \subset Y$ for which $\overline{f}(x) \in U$ and $\operatorname{im}(\overline{f} \circ \beta) \subset V$. For each $n \in \mathbb{N}$, choose $\alpha_n \in \mathcal{S}_D$ so that $\alpha_n \to \beta(n)$. Taking into account the inclusion $\lim(f \circ \alpha_n) = \overline{f}(\beta(n)) \in V$, we can assume that $\operatorname{im}(f \circ \alpha_n) \subset V$. Therefore, $f[A] \subset V$, where $A = \bigcup_{n \in \mathbb{N}} \operatorname{im} \alpha_n$. The inclusion $\operatorname{im} \beta \subset \operatorname{cl} A$ implies $x \in \operatorname{cl} A$. Since X is a Fréchet space, it follows that $\alpha \to x$ for some $\alpha \in \mathcal{S}_A \subset \mathcal{S}_D$. Thus, $\lim(f \circ \alpha) = \overline{f}(x) \in U$, while $\operatorname{im}(f \circ \alpha) \subset f[A] \subset V$. ☐

The following examples show that the assumptions of Theorem 2 cannot be relaxed by requiring X to be only sequential (even in the case $Y = \{0, 1\}$) or by replacing the regularity requirement on Y by the Hausdorffness requirement (even in the case where X is a metric space).

Example 1. Consider the pretopology on $\mathbb{R}^2$ in which the "cross-shaped" sets

$$([s - \varepsilon, s + \varepsilon] \times \{t\}) \cup (\{s\} \times [t - \varepsilon, t + \varepsilon]), \qquad \varepsilon > 0,$$

form a neighborhood base at each point (s, t). Since the convergence in this pretopology is single-valued, it follows by Theorem 1 that it coincides with convergence in a suitable sequential topology τ on $\mathbb{R}^2$. Let $X = (\mathbb{R}^2, \tau)$. We set

$$D_0 = \{(s, t) \in \mathbb{R}^2 : s < 0\}, \qquad D_1 = \{(s, t) \in \mathbb{R}^2 : 0 < t < s\}, \qquad D = D_0 \cup D_1 \subset X.$$

The function $f\colon D \to \{0, 1\}$ identically equal to 0 on D_0 and to 1 on D_1 is convergence-preserving but does not admit a continuous extension to $\operatorname{cl} D$. (Note that D, which is an open subset of X, is a sequential space, and $\operatorname{cl} D = \operatorname{cl}_\sigma D$.)

Example 2. Let X be the classical metric space $\mathbb{R}^2$, and let $D = \mathbb{R}^2 \setminus (\mathbb{R} \times \{0\})$. Consider the topological space Y with underlying set $\mathbb{R}^2$ in which a neighborhood base at each point $y \in D$ is formed by the ordinary open disks $B(y, \varepsilon)$, $\varepsilon > 0$, and a neighborhood base at each point $y \in \mathbb{R} \times \{0\}$ consists of all sets of the form $B(y, \varepsilon) \setminus (\mathbb{R} \times \{0\}) \cup \{y\}$, $\varepsilon > 0$. The identity embedding $f\colon D \to Y$ is convergence-preserving, but it does not admit a continuous extension to $\operatorname{cl} D = X$. (Note that Y is a first countable Hausdorff space. In particular, Y is a Fréchet space.)

The following two notions were introduced in [6] (for the case of metric spaces). A function $f\colon X \to Y$ is *sequentially convergent* if

$$(\forall\, \alpha \in \mathcal{S}_X)(f \circ \alpha \in \mathcal{C}_Y \Rightarrow \alpha \in \mathcal{C}_X).$$

A function $f\colon X \to Y$ is *subsequentially convergent* if

$$(\forall\, \alpha \in \mathcal{S}_X)(f \circ \alpha \in \mathcal{C}_Y \Rightarrow (\exists\, \beta \preccurlyeq \alpha)(\beta \in \mathcal{C}_X)).$$

Lemma 3. *If X is a T_1 space and a function $f\colon X \to Y$ sequentially convergent, then f is injective and the inverse function $f^{-1}\colon \operatorname{im} f \to X$ is continuous.*

Proof. If $x_1, x_2 \in X$ and $f(x_1) = f(x_2)$, then $f \circ \operatorname{mix}(x_1^\wedge, x_2^\wedge) = f(x_1)^\wedge \in \mathcal{C}_Y$, whence we have $\operatorname{mix}(x_1^\wedge, x_2^\wedge) \in \mathcal{C}_X$ and, therefore, $x_1 = x_2$. The continuity of f^{-1} follows by Lemma 2. ☐

Corollary 1. *Let X be a regular single-valued sequential space, and let Y be a Fréchet space. Then a function $f\colon X \to Y$ is sequentially convergent if and only if f is injective and the inverse function $f^{-1}\colon \operatorname{im} f \to X$ can be extended to a continuous function $\overline{f^{-1}}\colon \operatorname{cl} \operatorname{im} f \to X$. In particular, if the image of f is closed, then the sequential convergence of f is equivalent to the existence and continuity of f^{-1}.*

Theorem 3. *Let X be a T_1 space, and let Y be a single-valued sequential space. Then the following properties of a function $f\colon X \to Y$ are pairwise equivalent*:

(a) *f is continuous and sequentially convergent*;

(b) *f is continuous, injective, and subsequentially convergent*;

(c) *f is a homeomorphism of X onto a closed subspace* $\operatorname{im} f \subset Y$.

Proof. The implication (c) ⇒ (b) is obvious.

Let us prove (b) ⇒ (a). Suppose that $\alpha \in \mathcal{S}_X$ and $f \circ \alpha \to y \in Y$. Let us show that $\alpha \in \mathcal{C}_X$. The subsequential convergence of f implies the existence of a $\beta_0 \preccurlyeq \alpha$ and an $x \in X$ for which $\beta_0 \to x$. Since $f \circ \beta_0 \preccurlyeq f \circ \alpha \to y$, it follows that $f \circ \beta_0 \to y$; hence $y = f(x)$, because f is continuous and Y is single-valued. In order to prove the relation $\alpha \overset{*}{\to} x$, we must show that $(\exists\, \gamma \preccurlyeq \beta)(\gamma \to x)$ for $\beta \preccurlyeq \alpha$. Note that $f \circ \beta \to y$, because $f \circ \beta \preccurlyeq f \circ \alpha \to y$. Again applying the subsequential convergence of f, consider $\gamma \preccurlyeq \beta$ and $x' \in X$ for which $\gamma \to x'$. The continuity of f implies $f \circ \gamma \to f(x')$. On the other hand, $f \circ \gamma \preccurlyeq f \circ \beta \to y$ implies $f \circ \gamma \to y = f(x)$. Thanks to the single-valuedness of Y, we have $f(x') = f(x)$, whence $x' = x$ (by virtue of the injectivity of f) and, therefore, $\gamma \to x$.

We proceed to (a) ⇒ (c). Since X is T_1, due to Lemma 3, it suffices to prove that $\operatorname{im} f$ is closed. Suppose that $\beta \in \mathcal{S}_{\operatorname{im} f}$ and $\beta \to y \in Y$. We set $\alpha = f^{-1} \circ \beta$. Since $f \circ \alpha = \beta \in \mathcal{C}_Y$, it follows from the sequential convergence of f that $\alpha \to x \in X$; hence $\beta = f \circ \alpha \to f(x)$, because f is continuous. The single-valuedness of Y implies $y = f(x)$ and, therefore, $y \in \operatorname{im} f$. □

Note that the separation assumptions in Theorem 3 are essential. Indeed, consider $X = \{0\} \cup \mathbb{N}$ with the topology $\{\varnothing, \{0\}, \{0\} \cup \mathbb{N}\}$ and $Y = \mathbb{N}$ with the topology $\{\varnothing, \mathbb{N}\}$. The space Y is a closed subspace of X, and the function $f\colon X \to Y$ defined by $f(x) = x + 1$ is a sequentially convergent continuous bijection, while the inverse function $f^{-1}\colon Y \to X$ is discontinuous at the point 1.

The facts mentioned above make it possible to give a simple proof of some theorems on T-contractions and similar results. As an example, consider the following theorem proved in [6].

Theorem 4 ([6]). *Let (X, d) be a complete metric space. Suppose that a function $T\colon X \to X$ is continuous, injective, and subsequentially convergent and a continuous map $S\colon X \to X$ is a T-contraction, i.e., satisfies the condition*

$$(\exists\, C \in]0, 1[\,)(\forall\, x, y \in X)\ d(TSx, TSy) \le C\, d(Tx, Ty).$$

Then S has a unique fixed point. If, in addition, T is sequentially convergent, then, for any point $x_0 \in X$, the sequence of iterations $S^n x_0$ converges to the fixed point of S.

Proof. According to Theorem 3, the function T is a homeomorphism of X to a closed (and, therefore, complete) subspace $\operatorname{im} T \subset X$. Hence the function $d_T\colon X^2 \to \mathbb{R}$ defined by $d_T(x, y) = d(Tx, Ty)$ is a metric on X, with respect to which the map S is a contraction; moreover, the space (X, d_T) is complete, and convergence in d_T coincides with convergence in d. To complete the proof of Theorem 4, it remains to refer to Banach's contraction principle. (Note also that the continuity requirement on S in the statement of Theorem 4, as well as the additional assumption of the sequential convergence of T, can be dispensed with.) □

Similar considerations apply to the main results of [7]–[33], each of which is a generalization of some known fact obtained by replacing a distance $d(x, y)$ by $d(Tx, Ty)$, where T is a (sub)sequentially convergent injection.

REFERENCES

1. S. G. Matthews, in *Ann. New York Acad. Sci.*, Vol. 728: *Papers on General Topology and Applications* (New York Acad. Sci., New York, 1994), pp. 183–197.
2. A. Branciari, Publ. Math. Debrecen **57** (1-2), 31 (2000).
3. L.-G. Huang and X. Zhang, J. Math. Anal. Appl. **332** (2), 1468 (2007).
4. Z. Kadelburg, S. Radenović, and V. Rakočević, Fixed Point Theory Appl., Article ID 170253 (2010).
5. V. Koutník, in *Math. Res.*, Vol. 24: *Convergence Structures 1984* (Akademie-Verlag, Berlin, 1985), pp. 199–204.
6. A. Beiranvand, S. Moradi, M. Omid, and H. Pazandeh, *Two Fixed-Point Theorems for Special Mappings*, `arXiv:math. FA/0903.1504v1` (2009).
7. J. R. Morales and E. Rojas, Notas Mat. **4** (2), 66 (2008).
8. J. R. Morales Medina and E. M. Rojas, Notas Mat. **5** (1), 64 (2009).
9. S. Moradi and A. Beiranvand, *A Fixed-Point Theorem for Mapping Satisfying a General Contractive Condition of Integral Type Depended an Another Function*, `arXiv:math. FA/0903.1569v1` (2009).
10. S. Moradi, *Fixed-Point Theorem For Mappings Satisfying a General Contractive Condition Of Integral Type Depended an Another Function*, `arXiv:math. FA/0903.1574v1` (2009).
11. S. Moradi, *Kannan Fixed-Point Theorem On Complete Metric Spaces And On Generalized Metric Spaces Depended an Another Function*, `arXiv:math. FA/0903.1577v1` (2009).
12. J. R. Morales and E. Rojas, *Fixed-Point Theorems for a Class of Mappings Depending of Another Function and Defined on Cone Metric Spaces*, `arXiv:math. FA/0906.2160v1` (2009).
13. J. R. Morales and E. Rojas, *T-Zamfirescu and T-Weak Contraction Mappings on Cone Metric Spaces*, `arXiv:math. FA/0909.1255v1` (2009).
14. J. R. Morales and E. Rojas, *On the Existence of Fixed Points of Contraction Mappings Depending of Two Functions on Cone Metric Spaces*, `arXiv:math. FA/0910.4921v1` (2009).
15. J. R. Morales and E. Rojas, Int. J. Math. Anal. (Ruse) **4** (4), 175 (2010).
16. R. Sumitra, V. Rhymend Uthariaraj, and R. Hemavathy, Int. Math. Forum **5** (30), 1495 (2010).
17. S. Moradi and M. Omid, Int. J. Math. Anal. (Ruse) **4** (30), 1491 (2010).
18. S. Bhatt, A. Singh, and R. C. Dimri, Int. J. Math. Archive **2** (4), 444 (2011).
19. K. P. R. Sastry, Ch. Srinivasarao, K. Sujatha, G. Praveena, and Ch. Srinivasarao, Int. J. Comp. Sci. Math. **3** (2), 133 (2011).
20. S. Moradi and D. Alimohammadi, Int. J. Math. Anal. (Ruse) **5** (47), 2313 (2011).
21. M. Sharma, R. Shrivastava, and Z. K. Ansari, J. Contemp. Appl. Math. **1** (1), 103 (2011).
22. R. Shrivastava, Z. K. Ansari, and M. Sharma, Int. J. Phys. Math. Sci. **2** (1), 83 (2011).
23. M. Öztürk and M. Başarır, Int. J. Math. Anal. **5** (3), 119 (2011).
24. S. K. Malhotra, S. Shukla, and R. Sen, Math. Aeterna **1** (6), 353 (2011).
25. M. Abbas, H. Aydi, and S. Radenović, Int. J. Math. Math. Sci., Article ID 313675 (2012).
26. V. Parvaneh, J. Basic Appl. Sci. Res. **2** (3), 2354 (2012).
27. V. Parvaneh and H. Hosseinzadeh, J. Appl. Sci. **12** (9), 848 (2012).
28. Tran Van An, Kieu Phuong Chi, Erdal Karapınar, and Tran Duc Thanh, Int. J. Math. Math. Sci., Article ID 431872 (2012).
29. Erdal Karapınar, Kieu Phuong Chi, and Tran Duc Thanh, Abstr. Appl. Anal., Article ID 518734 (2012).
30. Kieu Phuong Chi, Erdal Karapınar, and Tran Duc Thanh, Arab J. Math. Sci. **18** (2), 141 (2012).
31. J. R. Morales and E. Rojas, Int. J. Math. Math. Sci., Article ID 213876 (2012).
32. A. Razani and V. Parvanekh, Izv. Vyssh. Uchebn. Zaved. Mat., No. 3, 47 (2013) [Russian Math. (Iz. VUZ) **57** (3), 38 (2013)].
33. A. K. Dubey, R. Shukla, and R. P. Dubey, Int. J. Appl. Math. Res. **2** (1), 151 (2013).