The technique of definable terms in Boolean valued analysis

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Let Φ be a set of first-order formulas of set-theoretic signature. A formula φ is said to be *of class* Φ ("φ is Φ" for short) whenever $\mathrm{ZFC} \vdash [\varphi \Leftrightarrow \varphi']$ for some φ' in Φ. Let $\tau(\bar{x})$ be any term introduced in (a conservative extension of) ZFC by means of a definition of the form $\tau(\bar{x}) = y \Leftrightarrow \varphi(\bar{x}, y)$. Say that τ is *of class* Φ ("τ is Φ") whenever φ is of class Φ. Say that τ is Φ-*definable* via a term σ ("τ is $\Phi(\sigma)$") whenever there is a formula $\varphi(\bar{x}, y, z)$ of class Φ such that $\mathrm{ZFC} \vdash \big[\tau(\bar{x}) = y \Leftrightarrow \varphi\big(\bar{x}, y, \sigma(\bar{x})\big)\big]$.

In what follows, we denote formulas and terms by φ and τ, σ, ρ with possible indices; Δ_0 is the smallest set containing the formulas $x \in y$ and closed under the connectives $\vee$, $\neg$, $(\exists\, x \in y)$; Σ_1 is constituted by the formulas $(\exists\, x)\, \varphi$, with φ in Δ_0. A formula φ is *of class* Δ_1 ("φ is Δ_1") whenever φ and $\neg\varphi$ are Σ_1.

Lemma. (1) *If* φ, τ, $\tau_1, \dots, \tau_n$ *are* Σ_1 *then so are* $\varphi(\tau_1, \dots, \tau_n)$ *and* $\tau(\tau_1, \dots, \tau_n)$.

(2) *If* $\tau_1, \dots, \tau_n$ *are* Σ_1 *and* φ *is* Δ_1 *then* $\varphi(\tau_1, \dots, \tau_n)$ *is* Δ_1.

(3) *If* τ *is* Σ_1 *and* φ *is* Δ_1 *then* $\{\tau(\bar{x}) : \bar{x} \in y,\ \varphi(\bar{x}, y)\}$ *is* Σ_1.

(4) *If* τ *is* $\Sigma_1(\sigma)$ *and* ρ *is* $\Sigma_1(\tau)$ *then* ρ *is* $\Sigma_1(\sigma)$.

(5) *If* τ, $\tau_1, \dots, \tau_n$ *are* $\Sigma_1(\sigma)$ *then so is* $\tau(\tau_1, \dots, \tau_n)$.

(6) *If* τ *is* Σ_1 *then* $\tau(\sigma)$ *is* $\Sigma_1(\sigma)$.

(7) *If* τ *is* Σ_1 *and* φ *is* Δ_1 *then* $\{\tau(\bar{x}) : \bar{x} \in \sigma,\ \varphi(\bar{x}, \sigma)\}$ *is* $\Sigma_1(\sigma)$.

(8) *If* τ *is* Σ_1 *and* φ *is* Δ_1 *then* $\{\tau(\bar{x}) : \bar{x} \in \sigma,\ \varphi(\bar{x}, \sigma)\}^{\mathbb{N}}$ *is* $\Sigma_1(\sigma^{\mathbb{N}})$.

The following example shows that statements (3) and (7) do not extend to the case in which φ is Σ_1.

Example. *Assume that ZFC is consistent and put* $\varphi(x) := (\exists\, z)(z \subseteq \mathbb{N} \wedge z \notin x)$. *Then* φ *is* Σ_1, φ *is not* Δ_1, *and* $\{x \in y : \varphi(x)\}$ *is not* Σ_1.

In what follows, $(\cdot)^\wedge$ stands for the canonical embedding of $\mathbb{V}$ into the Boolean valued universe $\mathbb{V}^{(B)}$.

Theorem. *If* ρ *is* Σ_1, τ *is* $\Sigma_1(\sigma)$, *and all the parameters of* ρ, σ, τ *are in* $\bar{x}$ *then the following is provable in ZFC: for every complete Boolean algebra* B *and all* $\bar{x}$

(1) $\mathbb{V}^{(B)} \models \big[\, \rho(\bar{x})^\wedge = \rho(\bar{x}^\wedge) \,\big]$;

(2) $\mathbb{V}^{(B)} \models \big[\, \sigma(\bar{x})^\wedge = \sigma(\bar{x}^\wedge) \,\big] \ \Rightarrow\ \mathbb{V}^{(B)} \models \big[\, \tau(\bar{x})^\wedge = \tau(\bar{x}^\wedge) \,\big]$.

Let $\mathbb{R}_{\mathrm{D}}$ and $\mathbb{R}_{\mathrm{C}}$ stand for the set of reals defined as Dedekind cuts and, respectively, classes of Cauchy sequences in $\mathbb{Q}$.

Corollary (ZFC). *Let* B *be a complete Boolean algebra.*

(1) $\mathbb{V}^{(B)} \models \big[\mathbb{R}_{\mathrm{D}}^\wedge \subseteq \mathbb{R}_{\mathrm{D}}\big]$; $\mathbb{V}^{(B)} \models \big[\mathcal{P}_{\mathrm{fin}}(X)^\wedge = \mathcal{P}_{\mathrm{fin}}(X^\wedge)\big]$ *for all* X.

(2) *The following properties of* B *are pairwise equivalent:* B *is* σ-*distributive*; $\mathbb{V}^{(B)} \models \big[\mathcal{P}(\mathbb{N})^\wedge = \mathcal{P}(\mathbb{N})\big]$; $\mathbb{V}^{(B)} \models \big[(\mathbb{N}^{\mathbb{N}})^\wedge = \mathbb{N}^{\mathbb{N}}\big]$; $\mathbb{V}^{(B)} \models \big[\mathbb{R}_{\mathrm{D}}^\wedge = \mathbb{R}_{\mathrm{D}}\big]$; $\mathbb{V}^{(B)} \models \big[\mathbb{R}_{\mathrm{C}}^\wedge \subseteq \mathbb{R}_{\mathrm{C}}\big]$; $\mathbb{V}^{(B)} \models [\mathbb{R}^\wedge$ *and* $\mathbb{R}$ *are isomorphic ordered fields*$]$.

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