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FINITE DIMENSIONALITY AND SEPARABILITY OF THE STALKS OF BANACH BUNDLES

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Abstract: Topological characteristics are studied of the set of points at which the stalks of an ample Banach bundle over an extremally disconnected compact space are finite-dimensional or separable. Some connection is established between finite dimensionality or separability of the stalks of a bundle and the analogous properties of the stalks of the ample hull of the bundle. We obtain a new criterion for existence of a dual bundle.

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As is known (see [1]), each Banach–Kantorovich space $\mathscr{U}$ is isomorphic to an order dense ideal of the space $C_\infty(Q,\mathscr{X})$ of extended continuous sections of a suitable ample Banach bundle $\mathscr{X}$ over an extremally disconnected compact space Q. Furthermore, the properties of the bundle $\mathscr{X}$ or those of its individual stalks affect the analogous global or local properties of the space $\mathscr{U}$. In particular, local finite dimensionality and order separability of $\mathscr{U}$ are closely related to finite dimensionality and separability of the stalks of $\mathscr{X}$.

In the paper, we study topological characteristics of the set of points at which the stalks of an ample Banach bundle are finite-dimensional or separable, establish a connection between finite dimensionality or separability of the stalks of a bundle and the analogous properties of the stalks of the ample hull of the bundle, and obtain a new criterion for existence of a dual bundle in the separable case.

In what follows, $\mathscr{X}$ is an arbitrary continuous Banach bundle over an extremally disconnected compact space Q, $\overline{\mathscr{X}}$ is the ample hull of $\mathscr{X}$, $\mathbb{N}=\{1,2,\dots\}$, $\omega=\{0,1,2,\dots\}$, and $\overline{\omega}=\omega\cup\{\infty\}$. We use the terminology and notation of [1, 2].

1. Preliminary Information

1.1. The *dimension* of $\mathscr{X}$ is the function $\dim\mathscr{X}:Q\to\overline{\omega}$ that maps each point $q\in Q$ into the dimension $\dim\mathscr{X}(q)\in\omega$ of the stalk $\mathscr{X}(q)$ in case the latter is finite-dimensional, and takes the value $(\dim\mathscr{X})(q)=\infty$ otherwise. The dimension of $\mathscr{X}$ is *bounded* (on $P\subset Q$) whenever $\dim\mathscr{X}\le n$ (on P) for some $n\in\omega$. The dimension of $\mathscr{X}$ is *locally bounded* on $P\subset Q$ if it is bounded on a neighborhood of each point $p\in P$.

Given $F,G:Q\to\overline{\omega}$ and $d\in\overline{\omega}$, put $\{F\le d\}:=\{q\in Q:F(q)\le d\}$; the notations $\{F<d\}$, $\{F=d\}$, $\{F=G<d\}$, etc. are introduced along the same lines.

1.2. (1) *Let* $u_1,\dots,u_n\in C(Q,\mathscr{X})$ *and* $q\in Q$. *If* $u_1(q),\dots,u_n(q)$ *are linearly independent then* $u_1,\dots,u_n$ *are pointwise linearly independent on a neighborhood of* q.

(2) *For every* $n\in\omega$, *the sets* $\{\dim\mathscr{X}<n\}$ *and* $\{\dim\mathscr{X}\le n\}$ *are closed, while* $\{\dim\mathscr{X}>n\}$ *and* $\{\dim\mathscr{X}\ge n\}$ *are open.*

$\triangleleft$ (1): Follows from [3, 18.1].

(2): It suffices to show that $\{\dim\mathscr{X}\ge n\}$ is open. The case $n=0$ is trivial; assume that $n>0$. For every $q\in\{\dim\mathscr{X}\ge n\}$ the stalk $\mathscr{X}(q)$ contains n linearly independent elements which, due to

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Dupré's Theorem [4, 1.1], are values of n sections in $C(Q, \mathscr{X})$. Then, according to (1), the point q is in $\{\dim \mathscr{X} \geq n\}$ together with a neighborhood of q. ▹

Remark. Assertion 1.2 holds true for an arbitrary topological space Q.

1.3. *Given $n \in \omega$, consider the five conditions:*

$$\begin{array}{ll} \{\dim \mathscr{X} = n\} \text{ is open;} & \\ \{\dim \mathscr{X} < n\} \text{ is open;} & \{\dim \mathscr{X} \leq n\} \text{ is open;} \\ \{\dim \mathscr{X} > n\} \text{ is closed;} & \{\dim \mathscr{X} \geq n\} \text{ is closed.} \end{array} \qquad (*)$$

If all stalks of $\mathscr{X}$ are finite-dimensional then the following are equivalent:

(1) *one of the conditions $(*)$ holds for all $n \in \omega$;*

(2) *each of the conditions $(*)$ holds for all $n \in \omega$;*

(3) *all sets in $(*)$ are clopen for all $n \in \omega$;*

(4) *$\{\dim \mathscr{X} = n\}$ is closed for all $n \in \omega$, and the dimension of $\mathscr{X}$ is bounded;*

(5) *there is a finite partition of Q into clopen subsets such that the dimension of $\mathscr{X}$ is constant on each element of the partition.*

◃ Assertions (1)–(3) are pairwise equivalent due to [2, 3.2.8]. Condition (3) implies (5), since the sets $\{\dim \mathscr{X} = n\}$, $n \in \omega$, constitute a partition and thus a cover of the compact space Q. The implications (5)⇒(4)⇒(1) are obvious. ▹

Remark. Assertion 1.3 holds true for an arbitrary compact space Q.

1.4. *For every clopen subset $U \subset Q$, the restriction of the ample hull $\overline{\mathscr{X}}|_U$ is the ample hull of the restriction $\mathscr{X}|_U$.*

2. Finite Dimensionality of the Stalks of Banach Bundles

This section is devoted to studying topological characteristics of the set of points at which the stalks of a Banach bundle are finite-dimensional, as well as to establishing an interrelation between finite dimensionality of the stalks of the bundle $\mathscr{X}$ and those of its ample hull $\overline{\mathscr{X}}$.

2.1. Theorem. *If $\mathscr{X}$ is ample then $\{\dim \mathscr{X} = n\}$ is clopen for all $n \in \omega$.*

◃ It suffices, given a number $n \in \mathbb{N}$, to show that $P := \{\dim \mathscr{X} \geq n\}$ is clopen.

The case $P = \varnothing$ is trivial. Suppose that $P \neq \varnothing$. First, associate with each point $p \in P$ some sections $u_1^p, \dots, u_n^p \in C(Q, \mathscr{X})$ and a clopen neighborhood U_p of p.

Now let $p \in P$. Since $\dim \mathscr{X}(p) \geq n$, employing the ε-Perpendicular Lemma [5, 8.4.1], we can choose $x_1, \dots, x_n$ in $\mathscr{X}(p)$ which meet the following conditions: $\|x_1\| = \dots = \|x_n\| = 1$ and, in case $n \neq 1$, the inequality

$$\|x_k - x\| \geq \tfrac{1}{2}$$

holds for all $k \in \{2, \dots, n\}$ and $x \in \operatorname{lin}\{x_1, \dots, x_{k-1}\}$. By Dupré's Theorem there are sections $u_1^p, \dots, u_n^p \in C(Q, \mathscr{X})$ with values $u_1^p(p) = x_1, \dots, u_n^p(p) = x_n$. According to the lemma of [6], there exists a neighborhood U_p of p such that

$$\tfrac{1}{2}|\!|\!| u |\!|\!|(p) \leq |\!|\!| u |\!|\!| \leq \tfrac{3}{2}|\!|\!| u |\!|\!|(p) \text{ on } U_p \text{ for all } u \in \operatorname{lin}\{u_1^p, \dots, u_n^p\}.$$

We may assume U_p to be clopen. Next,

$$|\!|\!| u_1^p |\!|\!| \leq \tfrac{3}{2}, \ \dots, \ |\!|\!| u_n^p |\!|\!| \leq \tfrac{3}{2} \text{ and } |\!|\!| u_1^p |\!|\!| \geq \tfrac{1}{2} \text{ on } U_p$$

and, if $n \neq 1$,

$$|\!|\!| u_k^p - u |\!|\!| |_{U_p} \geq \tfrac{1}{2} |\!|\!| u_k^p - u |\!|\!|(p) = \tfrac{1}{2}\|x_k - u(p)\| \geq \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{4}$$

for all $k \in \{2, \dots, n\}$ and $u \in \operatorname{lin}\{u_1^p, \dots, u_{k-1}^p\}$.

By the Exhaustion Principle (see, e.g., [1, 1.2.1 and 1.2.8]) there is a family $(V_p)_{p\in P}$ of pairwise disjoint clopen subsets of Q such that $V_p \subset U_p$ for all $p \in P$ and $\operatorname{cl}\bigcup_{p\in P} V_p = \operatorname{cl}\bigcup_{p\in P} U_p$. Given $k \in \{1,\dots,n\}$, define a section u_k over $\bigcup_{p\in P} V_p$ by putting $u_k = u_k^p$ on V_p for all $p \in P$. In this case we have

$$\|u_1\| \le \tfrac{3}{2},\ \dots,\ \|u_n\| \le \tfrac{3}{2} \ \text{ and } \ \|u_1\| \ge \tfrac{1}{2}$$

and, if $n \neq 1$,

$$\|u_k - u\| \ge 1/4 \ \text{ for all } k \in \{2,\dots,n\}, \quad u \in \operatorname{lin}\{u_1,\dots,u_{k-1}\}. \tag{$**$}$$

Since $\mathscr{X}$ is ample, the sections $u_1,\dots,u_n$ extend to the continuous sections $\bar u_1,\dots,\bar u_n$ over $\operatorname{cl}\bigcup_{p\in P} V_p$. Furthermore, $\|\bar u_1\| \ge 1/2$ and, if $n \neq 1$, we easily see that $\bar u_1,\dots,\bar u_n$ meet the analog of $(**)$. Whence $\bar u_1,\dots,\bar u_n$ are pointwise linearly independent. In particular, $\operatorname{cl}\bigcup_{p\in P} V_p \subset P$. On the other hand,

$$P \subset \operatorname{cl} P \subset \operatorname{cl}\bigcup_{p\in P} U_p \subset \operatorname{cl}\bigcup_{p\in P} V_p.$$

Therefore, $P = \operatorname{cl}\bigcup_{p\in P} V_p$ is clopen. ▷

Remark. From the above it is clear that if $\mathscr{X}$ is ample then for every $n \ge 1$ there are sections $u_1,\dots,u_n \in C(Q,\mathscr{X})$ pointwise linearly independent on $\{\dim \mathscr{X} \ge n\}$.

2.2. Corollary. *Suppose that $\mathscr{X}$ is ample.*

(1) *If all stalks of $\mathscr{X}$ are finite-dimensional then there exists a finite partition of Q into clopen subsets such that the dimension of $\mathscr{X}$ is constant on each element of the partition.*

(2) *The set $\{\dim \mathscr{X} < \infty\}$ is open and σ-closed, while $\{\dim \mathscr{X} = \infty\}$ is closed and σ-open.*

(3) *The following are pairwise equivalent:*

(a) $\{\dim \mathscr{X} < \infty\}$ *is clopen;*

(b) $\{\dim \mathscr{X} = \infty\}$ *is clopen;*

(c) *the set of values of* $\dim \mathscr{X}$ *is finite.*

(4) *If the stalk $\mathscr{X}(q)$ at a σ-isolated point $q \in Q$ is infinite-dimensional then the stalks of $\mathscr{X}$ are infinite-dimensional on a neighborhood of q.*

◁ Assertion (1) follows from 2.1 and 1.3 (see (1)⇒(4)); (2) is an immediate consequence of 2.1; in (3), the equivalence (a)⇔(b) is obvious, the implication (a)⇒(c) follows from (1) and 1.4, while 2.1 justifies the implication (c)⇒(a). For proving (4) note that if a σ-isolated point q does not belong to an open σ-closed set (and $\{\dim \mathscr{X} < \infty\}$ is such a set by (2)) then q cannot be a limit point of the set (see, e.g., [1, 1.1.8]). ▷

2.3. *All stalks of $\overline{\mathscr{X}}$ are finite-dimensional if and only if the dimension of $\mathscr{X}$ is bounded. In this case the dimension of $\overline{\mathscr{X}}$ is also bounded and* $\max\dim \overline{\mathscr{X}} = \max\dim \mathscr{X}$.

◁ If all stalks of $\overline{\mathscr{X}}$ are finite-dimensional then, due to 2.2 (1), the dimension of $\overline{\mathscr{X}}$ is bounded and so is the dimension of $\mathscr{X}$.

Let the dimension of $\mathscr{X}$ be bounded now and $\max\dim \mathscr{X} = n > 0$ (the case $n = 0$ is trivial). Suppose that the stalk of $\overline{\mathscr{X}}$ at $q \in Q$ contains k linearly independent elements. By Dupré's Theorem these elements are values of some sections $u_1,\dots,u_k \in C(Q,\overline{\mathscr{X}})$. According to 1.2 (1) the sections $u_1,\dots,u_k$ are pointwise linearly independent on a neighborhood U of q. Furthermore, since $\mathscr{X}$ is a dense subbundle of $\overline{\mathscr{X}}$, for each $j \in \{1,\dots,k\}$ the set $D_j := \{p \in Q : u_j(p) \in \mathscr{X}(p)\}$ is comeagre in Q (see [1, 2.5.8]) and so is the intersection $\bigcap_{j=1}^k D_j$. Then $\bigcap_{j=1}^k D_j$ is dense in Q and thus contains a point $p \in U$. Therefore, $u_1(p),\dots,u_k(p)$ are linearly independent and belong to $\mathscr{X}(p)$. We thus have $k \le n$. Consequently, the dimension of every stalk of $\overline{\mathscr{X}}$ is at most n. ▷

2.4. Corollary. *If $\mathscr{X}$ has constant finite dimension then $\mathscr{X}$ is an ample bundle.*

◁ According to 2.3 we have $\dim \overline{\mathscr{X}} \equiv \dim \mathscr{X} < \infty$ and hence $\overline{\mathscr{X}} = \mathscr{X}$. ▷

2.5. Theorem. *If all stalks of $\mathscr{X}$ are finite-dimensional then the following are pairwise equivalent:*

(1) *$\mathscr{X}$ is ample;*

(2) *$\{\dim \mathscr{X} = n\}$ is clopen for all $n \in \omega$;*

(3) *there exists a finite partition of Q into clopen subsets such that the dimension of $\mathscr{X}$ is constant on each element of the partition.*

(See also the equivalent conditions in 1.3.)

◁ Condition (1) implies (3) by 2.2 (1); the implication (3)⇒(2) is obvious (see also 1.3); for justifying (2)⇒(1) it suffices to observe that, in case (2), assertions 2.4 and 1.4 imply coincidence of $\mathscr{X}$ with $\overline{\mathscr{X}}$ on the sets $\{\dim \mathscr{X} = n\}$. ▷

2.6. (1) *The following hold:*

$$\{\mathscr{X} = \overline{\mathscr{X}}\} \cap \{\dim \mathscr{X} < \infty\} = \{\mathscr{X} = \overline{\mathscr{X}}\} \cap \{\dim \overline{\mathscr{X}} < \infty\}$$
$$= \{\dim \mathscr{X} = \dim \overline{\mathscr{X}} < \infty\} = \bigcup_{n\in\omega} \{\dim \mathscr{X} = \dim \overline{\mathscr{X}} = n\} = \bigcup_{n\in\omega} \operatorname{int}\{\dim \mathscr{X} = n\}.$$

(2) *Suppose that $q \in Q$ and $\dim \mathscr{X}(q) = n < \infty$. The stalks $\mathscr{X}(q)$ and $\overline{\mathscr{X}}(q)$ coincide if and only if $\dim \mathscr{X} = n$ on a neighborhood of q.*

(3) *The following hold:*

$$\{\dim \overline{\mathscr{X}} = 0\} = \operatorname{int}\{\dim \mathscr{X} = 0\};$$
$$\{\dim \overline{\mathscr{X}} = n\} = \operatorname{cl}\operatorname{int}\{\dim \mathscr{X} = n\} \text{ for all } n \in \omega;$$
$$\{\dim \overline{\mathscr{X}} < \infty\} = \bigcup_{n\in\omega} \operatorname{cl}\operatorname{int}\{\dim \mathscr{X} = n\}$$
$$= \operatorname{int}\{\dim \mathscr{X} = 0\} \cup \bigcup_{n\in\mathbb{N}} \operatorname{cl}\operatorname{int}\{\dim \mathscr{X} = n\}.$$

◁ (1): The first three equalities are rather obvious. We will prove the last equality.

"$\subset$": Suppose that $\dim \overline{\mathscr{X}}(q) = \dim \mathscr{X}(q) = n < \infty$ for some point $q \in Q$. It is clear that $q \in \{\dim \overline{\mathscr{X}} = n\} \cap \{\dim \mathscr{X} \geq n\} \subset \{\dim \mathscr{X} = n\}$. Furthermore, the sets $\{\dim \overline{\mathscr{X}} = n\}$ and $\{\dim \mathscr{X} \geq n\}$ are open due to 2.1 and 1.2 (2). Therefore, $q \in \operatorname{int}\{\dim \mathscr{X} = n\}$.

"$\supset$": Suppose that $n \in \omega$ and $\operatorname{int}\{\dim \mathscr{X} = n\} \neq \varnothing$. Consider an arbitrary clopen set $U \subset \operatorname{int}\{\dim \mathscr{X} = n\}$. From 1.4 and 2.4 we have $\overline{\mathscr{X}}|_U = \mathscr{X}|_U$. Since the clopen sets form a base for the topology of Q, we conclude that $\overline{\mathscr{X}} = \mathscr{X}$ on $\operatorname{int}\{\dim \mathscr{X} = n\}$, i.e., $\operatorname{int}\{\dim \mathscr{X} = n\} \subset \{\overline{\mathscr{X}} = \mathscr{X}\}$.

(2) is an immediate consequence of (1).

(3): Let $n \in \omega$. By (1) we have $\operatorname{int}\{\dim \mathscr{X} = n\} \subset \{\dim \overline{\mathscr{X}} = n\}$ and, since $\{\dim \overline{\mathscr{X}} = n\}$ is closed (see 2.1), $\operatorname{cl}\operatorname{int}\{\dim \mathscr{X} = n\} \subset \{\dim \overline{\mathscr{X}} = n\}$. The reverse inclusion is obvious in the case $\{\dim \overline{\mathscr{X}} = n\} = \varnothing$. Now let $q \in \{\dim \overline{\mathscr{X}} = n\}$ and let U be an arbitrary neighborhood of q. Taking account of 2.1 we may assume that $U \subset \{\dim \overline{\mathscr{X}} = n\}$ and U is clopen. From 1.4 and 2.3 we immediately obtain the equality $\dim \mathscr{X}(p) = \dim\big(\mathscr{X}|_U\big)(p) = n$ for some point $p \in U$. Then $p \in \operatorname{int}\{\dim \mathscr{X} = n\}$ due to (1). Consequently, $U \cap \operatorname{int}\{\dim \mathscr{X} = n\} \neq \varnothing$, whence $q \in \operatorname{cl}\operatorname{int}\{\dim \mathscr{X} = n\}$. Therefore, $\{\dim \overline{\mathscr{X}} = n\} \subset \operatorname{cl}\operatorname{int}\{\dim \mathscr{X} = n\}$.

It remains to observe that $\{\dim \mathscr{X} = 0\}$ is closed (see 1.2 (2)), and the interiors of closed subsets of an extremally disconnected space are closed. ▷

2.7. Corollary. *The stalk $\overline{\mathscr{X}}(q)$ at $q \in Q$ is finite-dimensional if and only if the dimension of $\mathscr{X}$ is bounded on some neighborhood of q. In this case $\dim \mathscr{X} \leq \dim \overline{\mathscr{X}}(q)$ on a neighborhood of q.*

◁ If $\dim \overline{\mathscr{X}}(q) = n < \infty$ then, due to 2.1, $U := \{\dim \overline{\mathscr{X}} = n\}$ is a (clopen) neighborhood of q and the following hold on U: $\dim \mathscr{X} \leq \dim \overline{\mathscr{X}} = n = \dim \overline{\mathscr{X}}(q)$. Conversely, if $\dim \mathscr{X} \leq n < \infty$ on a clopen neighborhood U of q then, taking account of 1.4 and 2.3, we have $\dim \overline{\mathscr{X}} < \infty$ on U and, in particular, $\dim \overline{\mathscr{X}}(q) < \infty$. ▷

2.8. Corollary. *Suppose that all stalks of $\mathscr{X}$ are finite-dimensional.*

(1) *The stalks of $\mathscr{X}$ and $\overline{\mathscr{X}}$ coincide on an open dense set.*

(2) *The set $\{\dim \overline{\mathscr{X}} < \infty\}$ is open, σ-closed, and dense in Q. The equality $\{\dim \overline{\mathscr{X}} < \infty\} = Q$ is equivalent to boundedness of $\dim \mathscr{X}$.*

$\triangleleft$ By [2, 3.2.9 (1)], assertion (1) follows from 2.6 (1), while (2) is a consequence of 2.6 (3) and 2.3. $\triangleright$

3. Separability of the Stalks of Banach Bundles

In this section we present certain conditions for the stalks of an ample bundle to be separable and establish a separability criterion for the ample hull $\overline{\mathscr{X}}$ of $\mathscr{X}$.

3.1. Theorem. *Suppose that $\mathscr{X}$ is ample and $q \in Q$ is not σ-isolated. The stalk $\mathscr{X}(q)$ is separable if and only if it is finite-dimensional.*

$\triangleleft$ Let $\mathscr{X}(q)$ be infinite-dimensional. Show that none of its countable subsets $\{x_k : k \in \mathbb{N}\}$ can be dense.

Denote by B the open unit ball of $\mathscr{X}(q)$. Since a ball in an infinite-dimensional Banach space is not precompact, $\mathscr{X}(q)$ has no finite ε-net for B with some $0 < \varepsilon < 1$. Therefore, for every n we can choose an element $y_n \in B$ whose distance to $\{x_1, \dots, x_n\}$ is greater than ε. By Dupré's Theorem, for each $n \in \mathbb{N}$ there are sections $u_n, v_n \in C(Q, \mathscr{X})$ with values $u_n(q) = x_n$ and $v_n(q) = y_n$. Given n, choose a clopen neighborhood U_n of q so that

$$\| v_n - u_1 \| \geq \varepsilon, \ \dots, \ \| v_n - u_n \| \geq \varepsilon \ \text{ and } \ \| v_n \| \leq 1 \ \text{ on } U_n.$$

According to assertion 12 of [6], there exists a sequence $(D_n)_{n\in\mathbb{N}}$ of pairwise disjoint clopen subsets of Q which meets the following conditions:

$$D_k \subset U_k, \quad q \in \operatorname{cl} \bigcup_{n\geq k} D_n \ \text{ for all } k \in \mathbb{N}, \quad q \notin \bigcup_{n\in\mathbb{N}} D_n.$$

Consider the open set $D_0 := Q \setminus \operatorname{cl} \bigcup_{n\in\mathbb{N}} D_n$ and define a section v on the open dense subset $D_0 \cup \bigcup_{n\in\mathbb{N}} D_n$ of Q by putting $v = 0$ on D_0 and $v = v_n$ on D_n for each $n \in \mathbb{N}$. As is easily seen, the section v is continuous and bounded and, since $\mathscr{X}$ is ample, v extends to a global continuous section $\bar{v}$. For every $k \in \mathbb{N}$ we have

$$\| \bar{v} - u_k \| |_{D_n} = \| v_n - u_k \| |_{D_n} \geq \varepsilon \ \text{ for all } n \geq k,$$

whence $\lim_{p\to q} \| \bar{v} - u_k \| (p) \geq \varepsilon$. Consequently,

$$\| \bar{v}(q) - x_k \| = \| \bar{v} - u_k \| (q) = \lim_{p\to q} \| \bar{v} - u_k \| (p) \geq \varepsilon \ \text{ for all } k \in \mathbb{N}.$$

Therefore, the set $\{x_k : k \in \mathbb{N}\}$ is not dense in $\mathscr{X}(q)$. $\triangleright$

3.2. Lemma. (1) *Every infinite extremally disconnected compact space* (*and, in particular, every infinite clopen subset of such a space*) *contains a point which is not σ-isolated.*

(2) *Every finite σ-open subset of an extremally disconnected compact space consists of isolated points.*

(3) *If a closed σ-open subset of an extremally disconnected compact space is infinite then it is uncountable.*

$\triangleleft$ (1): If q is a limit point of an infinite countable set $P \subset Q$ then the intersection of the neighborhoods $Q \setminus \{p\}$ ($p \in P \setminus \{q\}$) of q does not contain elements of P different from q and therefore is not a neighborhood of q.

(2): It suffices to show that the set $\{q\}$ can be σ-open only if $q \in Q$ is isolated. As is easily seen, if $\{q\}$ is σ-open, there exists a family $(U_n)_{n\in\mathbb{Z}}$ of pairwise disjoint nonempty clopen subsets of Q such that $\bigcup_{n\in\mathbb{Z}} U_n = Q \setminus \{q\}$. If q is not isolated then $q \in \operatorname{cl} Q \setminus \{q\}$. Without loss of generality, we may assume that $q \in \operatorname{cl} \bigcup_{n<0} U_n$. Then the open sets $\operatorname{cl} \bigcup_{n<0} U_n, U_0, U_1, \dots$ form a cover of the compact space Q without a finite subcover.

(3): Assume that a closed set $\{q_0, q_1, \dots\} \subset Q$ of pairwise distinct points q_n coincides with the intersection $\bigcap_{n\in\mathbb{N}} U_n$ of some open subsets $U_n \subset Q$. Since an infinite compact set cannot consist of isolated points, we may assume that q_0 is not isolated. On the other hand, $\bigcap_{n\in\mathbb{N}} U_n \setminus \{q_n\} = \{q_0\}$, which contradicts (2). $\triangleright$

3.3. Corollary. *The following properties of an ample bundle $\mathscr{X}$ are pairwise equivalent:*

(1) *the stalks of $\mathscr{X}$ are separable at all non-σ-isolated points;*

(2) *the stalks of $\mathscr{X}$ are finite-dimensional at all nonisolated points;*

(3) *the set $\{\dim \mathscr{X} = \infty\}$ is countable;*

(4) *the set $\{\dim \mathscr{X} = \infty\}$ is finite;*

(5) *there exists a partition of Q into clopen sets $Q_0, Q_1, \dots, Q_n$ ($n \in \omega$) such that the dimension of $\mathscr{X}$ is constant and finite on each of the sets $Q_1, \dots, Q_n$, while Q_0 is a finite set of isolated points at which the stalks of $\mathscr{X}$ are infinite-dimensional.*

$\triangleleft$ According to 2.2 (2), $Q_0 := \{\dim \mathscr{X} = \infty\}$ is closed and σ-open.

(1)$\Rightarrow$(2): From 3.1 it follows that, in case (1), the set Q_0 consists of σ-isolated points and, due to 2.2 (4), is open. Taking account of 3.2 (1) we conclude that the clopen set Q_0 is finite, and all its points are therefore isolated.

(2)$\Rightarrow$(4): The set Q_0, is closed in Q, and so Q_0 is compact, while every compact set of isolated points is finite.

(4)$\Rightarrow$(5): If Q_0 is finite then, according to 3.2 (2), it consists of isolated points. Then the sets Q_0 and $Q \backslash Q_0$ are clopen and, in case $Q_0 \neq Q$, assertion 2.2 (1) is applicable to the ample bundle $\mathscr{X}|_{Q \backslash Q_0}$.

The implication (5)$\Rightarrow$(1) is obvious. The equivalence of (3) and (4) follows from 3.2 (3). $\triangleright$

Recall that a bundle $\mathscr{X}$ is *separable* whenever there exists a countable subset of $C(Q, \mathscr{X})$ which is stalkwise dense in $\mathscr{X}$.

3.4. Theorem. *The following properties of an ample bundle $\mathscr{X}$ are pairwise equivalent:*

(1) *the bundle $\mathscr{X}$ is separable;*

(2) *all stalks of $\mathscr{X}$ are separable;*

(3) *the stalks of $\mathscr{X}$ are separable at all isolated points and all non-σ-isolated points;*

(4) *the stalks of $\mathscr{X}$ are finite-dimensional everywhere except a finite set of isolated points at which the stalks of $\mathscr{X}$ are separable;*

(5) *there exists a partition of Q into clopen sets $Q_0, Q_1, \dots, Q_n$ ($n \in \omega$) such that the dimension of $\mathscr{X}$ is constant and finite on each of the sets $Q_1, \dots, Q_n$, while Q_0 is a finite set of isolated points at which the stalks of $\mathscr{X}$ are separable.*

$\triangleleft$ The implications (1)$\Rightarrow$(2) and (2)$\Rightarrow$(3) are obvious. The implications (3)$\Rightarrow$(4) and (4)$\Rightarrow$(5) follow respectively from 3.3 ((1)$\Rightarrow$(5)) and 3.3 ((4)$\Rightarrow$(5)). It remains to show that (5)$\Rightarrow$(1).

Let $Q_0, Q_1, \dots, Q_n$ be a partition mentioned in (5). It suffices to associate with each Q_i a countable subset of $C(Q, \mathscr{X})$ which is stalkwise dense in $\mathscr{X}$ on Q_i. We will assume that Q_i are nonempty.

A countable set of continuous sections, which is stalkwise dense in $\mathscr{X}$ on Q_0, can be obtained by choosing a countable dense subset in each stalk of $\mathscr{X}$ over Q_0 and then applying Dupré's Theorem.

Now take $i \in \{1, \dots, n\}$ and suppose that $\dim \mathscr{X} \equiv m$ on Q_i. The case $m = 0$ is trivial. Assume that $m > 0$. According to Remark 2.1 there exist sections $u_1, \dots, u_m \in C(Q, \mathscr{X})$ which are pointwise linearly independent on Q_i. It is clear that the set of all linear combinations $\sum_{j=1}^{m} \lambda_j u_j$ with rational coefficients λ_j is stalkwise dense in $\mathscr{X}$ on Q_i. $\triangleright$

Corollary. *Every ample bundle with finite-dimensional stalks is separable.*

3.5. Theorem. *The following are pairwise equivalent:*

(1) *the bundle $\overline{\mathscr{X}}$ is separable;*

(2) *the stalks of $\mathscr{X}$ are separable on a finite set $Q_0 \subset Q$ of isolated points, and the dimension of $\mathscr{X}$ is bounded on $Q \backslash Q_0$;*

(3) *the stalks of $\mathscr{X}$ are separable on a finite set $S \subset Q$, and the dimension of $\mathscr{X}$ is locally bounded on $Q \backslash S$.*

$\triangleleft$ The implication (1)$\Rightarrow$(2) follows from 3.4 ((1)$\Rightarrow$(5)), since $\mathscr{X}(q) \subset \overline{\mathscr{X}}(q)$; (2)$\Rightarrow$(3) is obvious.

We now proceed with (3)⇒(1). Consider a finite subset $S \subset Q$ satisfying (3). By 2.7, from boundedness of $\dim \mathscr{X}$ on a neighborhood of each point of $Q\backslash S$ it follows that $\dim \overline{\mathscr{X}} < \infty$ on $Q\backslash S$, and the stalks of $\overline{\mathscr{X}}$ are therefore separable on $Q\backslash S$. Moreover, according to 3.3, finiteness of the set $\{\dim \overline{\mathscr{X}} = \infty\} \subset S$ ensures that the stalks of $\overline{\mathscr{X}}$ are finite-dimensional and, in particular, separable at all nonisolated points, and if a point $q \in S$ is isolated then the stalk $\overline{\mathscr{X}}(q) = \mathscr{X}(q)$ is separable by (3). So, the bundle $\overline{\mathscr{X}}$ satisfies 3.4 (2) and is thus separable. ▷

4. Existence of a Dual Bundle

In this section, after proving several auxiliary assertions on homomorphisms of Banach bundles, we establish a connection between finite dimensionality and separability of the stalks of $\mathscr{X}$ and existence of a dual bundle $\mathscr{X}'$.

In what follows, $\mathscr{R}$ is the constant bundle over Q with stalk $\mathbb{R}$.

4.1. Theorem. *Given a Banach subbundle $\mathscr{X}_0$ of $\mathscr{X}$ and a homomorphism $H_0 \in \mathrm{Hom}(\mathscr{X}_0, \mathscr{R})$, there is a homomorphism $H \in \mathrm{Hom}(\mathscr{X}, \mathscr{R})$ such that $H(q)\big|_{\mathscr{X}_0(q)} = H_0(q)$ for all $q \in Q$ and $\|H\| = \|H_0\|$.*

◁ Let $H_0 \in \mathrm{Hom}(\mathscr{X}_0, \mathscr{R})$. It suffices to consider the case $\|H_0\| = 1$.

It is clear that the linear operator $T_0 : C(Q, \mathscr{X}_0) \to C(Q)$ defined by the formula $T_0 u = \langle u \mid H_0 \rangle$ meets the inequality

$$T_0 u \leq |\!|\!| u |\!|\!| \quad \text{for all } u \in C(Q, \mathscr{X}_0).$$

According to the Hahn–Banach–Kantorovich Theorem (see, e.g., [7, 1.4.13 (1)]) T_0 extends to a linear operator $T : C(Q, \mathscr{X}) \to C(Q)$ such that

$$Tu \leq |\!|\!| u |\!|\!| \quad \text{for all } u \in C(Q, \mathscr{X}).$$

Since $\{u(q) : u \in C(Q, \mathscr{X})\} = \mathscr{X}(q)$ by Dupré's Theorem, for each point $q \in Q$ the formula

$$\langle u(q) \mid H(q) \rangle = (Tu)(q), \quad u \in C(Q, \mathscr{X}),$$

correctly defines some functional $H(q) \in \mathscr{X}(q)'$. According to [1, Theorem 2.4.7] the mapping $q \mapsto H(q)$ belongs to $\mathrm{Hom}(\mathscr{X}, \mathscr{R})$ and is the desired homomorphism. ▷

4.2. Corollary. *For every point $q \in Q$, every separable Banach subspace $X \subset \mathscr{X}(q)$, and every functional $x' \in X'$, there is a homomorphism $H \in \mathrm{Hom}(\mathscr{X}, \mathscr{R})$ such that $H(q)\big|_X = x'$ and $\|H(q)\| = \|x'\|$.*

◁ Let $X \subset \mathscr{X}(q)$ be a separable Banach subspace and $x' \in X'$. By Dupré's Theorem there exists a countable set $\mathscr{U} \subset C(Q, \mathscr{X})$ such that $\{u(q) : u \in \mathscr{U}\}$ is a dense subset of X. The linear hull of $\mathscr{U}$ induces a subbundle $\mathscr{X}_0$ of $\mathscr{X}$ (see [1, 2.2.2]). Furthermore, $\mathscr{X}_0(q) = X$. It is clear that the subset of $C(Q, \mathscr{X}_0)$ consisting of all rational linear combinations of elements of $\mathscr{U}$ is countable and stalkwise dense in $\mathscr{X}_0$. Therefore, [3, Corollary 19.16] implies existence of a homomorphism $H_0 \in \mathrm{Hom}(\mathscr{X}_0, \mathscr{R})$ satisfying $H_0(q) = x'$ and $\|H_0\| = \|x'\|$. Application of 4.1 yields a homomorphism $H \in \mathrm{Hom}(\mathscr{X}, \mathscr{R})$ such that

$$H(q)\big|_X = H(q)\big|_{\mathscr{X}_0(q)} = H_0(q) = x';$$
$$\|H(q)\| \leq \|H\| = \|H_0\| = \|x'\|;$$
$$\|H(q)\| \geq \left\|H(q)\big|_{\mathscr{X}_0(q)}\right\| = \|H_0(q)\| = \|x'\|. \ ▷$$

4.3. Corollary. *If $\mathscr{X}$ is ample then, for every $q \in Q$, every separable Banach subspace $X \subset \mathscr{X}(q)$, and every functional $x' \in X'$, there exists an element $\bar{x}' \in \mathscr{X}'(q)$ such that $\bar{x}'\big|_X = x'$ and $\|\bar{x}'\| = \|x'\|$.*

4.4. Theorem. *If the dual bundle $\mathscr{X}'$ exists and the stalk $\mathscr{X}(q)$ at a non-σ-isolated point $q \in Q$ is separable, then $\mathscr{X}(q)$ is finite-dimensional and coincides with $\overline{\mathscr{X}}(q)$.*

◁ Assume to the contrary that a separable stalk $\mathscr{X}(q)$ is infinite-dimensional. By 3.1 the stalk $\overline{\mathscr{X}}(q)$ is not separable. Consequently, there exists a separable Banach subspace $X \subset \overline{\mathscr{X}}(q)$ such that

$\mathscr{X}(q) \subset X$ and $\mathscr{X}(q) \neq X$. Employing the Separation Theorem, consider a nonzero functional $x' \in X'$ which vanishes on $\mathscr{X}(q)$. According to 4.3 there is an element $\bar{x}' \in \overline{\mathscr{X}}'(q)$ such that $\bar{x}'|_X = x'$ and $\|\bar{x}'\| = \|x'\|$. Furthermore,

$$\|\bar{x}'|_{\mathscr{X}(q)}\| = \|x'|_{\mathscr{X}(q)}\| = 0 < \|x'\| = \|\bar{x}'\|.$$

Therefore, by [2, Theorem 3.3.5] the bundle $\mathscr{X}$ has no dual bundle.

So, within the hypotheses of the theorem under proof, the space $\mathscr{X}(q)$ is finite-dimensional. It remains to refer to [2, Proposition 3.4.11]. ▷

4.5. Theorem. *If the stalks of $\mathscr{X}$ at all non-σ-isolated points are separable then the following are pairwise equivalent:*

(1) *the dual bundle $\mathscr{X}'$ exists;*

(2) *$\mathscr{X}$ is ample;*

(3) *there is a partition of Q into clopen sets $Q_0, Q_1, \dots, Q_n$ $(n \in \omega)$ such that the dimension of $\mathscr{X}$ is constant and finite on each of the sets $Q_1, \dots, Q_n$, while Q_0 is a finite set of isolated points.*

In each of the cases (1)–(3), *the equality $\mathscr{X}'(q) = \mathscr{X}(q)'$ holds for all $q \in Q$.*

◁ (1)⇒(2): From (1) and 4.4 it follows directly that the stalks of $\overline{\mathscr{X}}$ are finite-dimensional at all non-σ-isolated points. According to 3.3 the stalks of $\overline{\mathscr{X}}$ are finite-dimensional on the set of all nonisolated points. Then, on the latter set, we have $\mathscr{X} = \overline{\mathscr{X}}$ due to [2, 3.4.11]. In addition, the stalks of $\mathscr{X}$ and $\overline{\mathscr{X}}$ coincide at isolated points. Therefore, $\mathscr{X} = \overline{\mathscr{X}}$.

The implication (2)⇒(3) follows from 3.3 ((1)⇒(5)).

(3)⇒(1): Let $Q_0, Q_1, \dots, Q_n$ satisfy (3). For each $i \in \{0, \dots, n\}$ the totality $\{u|_{Q_i} : u \in C(Q, \mathscr{X})\}$ is a subset of $C(Q_i, \mathscr{X}|_{Q_i})$ which is stalkwise dense in $\mathscr{X}|_{Q_i}$. Then, according to [2, 3.2.12], every homomorphism $H \in \mathrm{Hom}(\mathscr{X}, \mathscr{R})$ meets the condition $|\!|\!| H|_{Q_i} |\!|\!| \in C(Q_i)$ and, consequently, $|\!|\!| H |\!|\!| \in C(Q)$. By [2, Theorem 3.3.2] the dual bundle $\mathscr{X}'$ exists.

The equality $\mathscr{X}'(q) = \mathscr{X}(q)'$ at nonisolated points $q \in Q$ follows from [2, 3.4.10 (2) and 3.4.4 (8)]. ▷

4.6. Corollary. *Suppose that all stalks of $\mathscr{X}$ are separable. The dual bundle $\mathscr{X}'$ exists if and only if $\mathscr{X}$ is ample.*

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