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The Mathematical Intelligencer Flunks the Olympics

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Abstract *The Mathematical Intelligencer* recently published a note by Y. Sergeyev that challenges both *mathematics* and *intelligence*. We examine Sergeyev's claims concerning his purported *Infinity computer*. We compare his *grossone* system with the classical Levi-Civita fields and with the hyperreal framework of A. Robinson, and analyze the related algorithmic issues inevitably arising in any genuine computer implementation. We show that Sergeyev's *grossone* system is unnecessary and vague, and that whatever consistent subsystem could be salvaged is subsumed entirely within a stronger and clearer system (IST). Lou Kauffman, who published an article on a *grossone*, places it squarely *outside* the historical panorama of ideas dealing with infinity and infinitesimals.

1 Grossone Olympics

In the summer of '15, some of us were approached by an editor of *The Mathematical Intelligencer (TMI)* with a request to respond to a piece of what they felt was *pseudo-science*, published without their knowledge in *TMI*. As noted in (Dauben et al. 2015, p. 393), I. Grattan-Guinness argued that "the demarcation between science and

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pseudo-science is not clearly drawn". While agreeing with Grattan-Guinness, in the present article we argue that in some cases the demarcation is drawn clearer than in others.

Yaroslav Sergeyev has developed a positional system for infinite numbers in numerous articles over the past decade. By 2015, *MathSciNet* listed 19 such articles, starting with (Sergeyev 2003). His "Olympic Medal" note (Sergeyev 2015a) in *TMI* purports to be an application of his *grossone* system to ranking countries lexicographically according to the number of gold, silver, and bronze medals they earned in the olympics. Sergeyev's system is closely related to the field of rational functions in one variable and to the classical Levi-Civita field, with a non-Archimedean structure provided by a suitable lexicographic ordering (a more detailed comparison with the Levi-Civita fields appears in Sect. 4.1).

Sergeyev appears to be making claims of significant progress in the field of nonstandard models. The reaction of the experts to Sergeyev's claims has been lukewarm. Joel David Hamkins, a leading authority on mathematical logic and foundations, reacted as follows to Sergeyev's claims: "It seems to me that there is very little that is new in this topic, and basically nothing to support the grand claims being made about it" (Hamkins 2015). In this text, we will analyze Sergeyev's claims in more detail.

Shamseddine's group has used Levi-Civita fields to develop computer implementations exploiting infinite numbers (see Sect. 4.2), without engaging in the sort of rhetorical *flou artistique* that envelopes a typical Sergeyev performance. Pure and applied mathematicians may sometimes use different standards of rigor but Sergeyev's case is a rather different problem.

Nonstandard models of arithmetic were developed as early as 1933 by Skolem using purely constructive methods (in particular not relying on any version of the axiom of choice); see e.g., Skolem (1933, 1934, 1955), and Kanovei et al. (2013), Sect. 3.2.

Conservative extensions of the Peano axioms (PA) were studied in Kreisel (1969) and Henson et al. (1984). Subsequently Henson and Keisler (1986) described both a family of nonstandard versions of PA itself, and n-th order PA for different values of n, that are conservative extensions of PA itself and respectively n-th order PA (see Proposition 2.3 there), and also nonstandard versions containing additional stronger saturation axioms, that are not conservative extensions (see Theorem 3.2 there). All of these theories are conservative with respect to ZFC, as is IST (see Sect. 6).

Avigad (2005) showed how to use weak theories of nonstandard arithmetic to treat fragments of calculus and analysis. If (as apparently claimed in Lolli 2015) what Sergeyev is attempting to do is develop such nonstandard models, he is certainly doing it without acknowledging prior work in the field.

2 Transfering the Sine Function

A few years ago, one of the authors asked Sergeyev through email what the sine of his *grossone* was, and he replied that it is

$$\sin(\mathit{grossone}).$$

The author in question did not have the heart to ask Sergeyev what

$$\sin^2(\mathit{grossone}) + \cos^2(\mathit{grossone})$$

is, and how exactly his "infinity computer" can know it other than being told case-by-case about every possible identity in mathematics. The point is that neither the field of rational functions nor Sergeyev's *grossone* system possesses a transfer principle (see below) or any equivalent procedure.

In his list of areas where his ideas are claimed to be potentially fruitful, Sergeyev mentions differential equations. Surely for this he will need to know that the sine function is defined on the extended system with its usual properties. This is what makes the question about $\sin(grossone)$ crucial.

The *transfer principle* is a type of theorem that, depending on the context, asserts that rules, laws or procedures valid for a certain number system, still apply (i.e., are "transfered") to an extended number system. Thus, the familiar extension $\mathbb{Q} \subseteq \mathbb{R}$ preserves the property of being an ordered field. To give a negative example, the extension

$$\mathbb{R} \subseteq \mathbb{R} \cup \{\pm\infty\}$$

of the real numbers to the so-called *extended reals* does not preserve such a property. The hyperreal extension

$$\mathbb{R} \subseteq {}^*\mathbb{R}$$

preserves *all* first-order properties, including the trigonometric identity $\sin^2 x + \cos^2 x = 1$ (valid for all hyperreal x, including infinitesimal and infinite values of $x \in {}^*\mathbb{R}$). For a more detailed discussion, see the textbook *Elementary Calculus* (Keisler 1986).

The revolutionary idea that there does exist a system, sometimes called *hyperreal numbers*, satisfying such a transfer principle is due to the combined effort of Hewitt (1948), Łoś (1955), and Robinson (1961), and has roots in Leibniz's *Law of continuity* and his distinction between *assignable* and *inassignable* numbers; see Katz and Sherry (2012, 2013). We will provide an explanation of the extension $\mathbb{R} \subseteq {}^*\mathbb{R}$ in Sect. 5.

Sergeyev sometimes grudgingly acknowledges the debt to Robinson. However, in many publications Sergeyev unfortunately presents the idea as his own, as noted by Vladik Kreinovich in his *MathSciNet* review of Sergeyev's book (Kreinovich 2003). Peter W. Day's review of Sergeyev's article at Day (2006) mentions the connection to the transfer principle, lacking in Sergeyev's system. Additional critical reviews are Zlatoš (2009) and Kutateladze (2011).

Sergeyev himself introduces his symbol for infinity in the following terms:

> A new infinite unit of measure has been introduced for this purpose as the number of elements of the set $\mathbb{N}$ of natural numbers. It is expressed by the numeral ① called *grossone*. It is necessary to note immediately that ① is neither Cantor's $\aleph_0$ nor ω. Particularly, it has both cardinal and ordinal properties as usual finite natural numbers (Iudin et al. 2012, p. 8101).

It is easy to detect serious logical problems with such a definition. Sergeyev's claim that his ① has both cardinal and ordinal properties is a purely declamative pronouncement. A reader might have expected such a claim in a refereed mathematical periodical to be justified by a clever definition, but it is not. As it stands, Sergeyev's claim is merely a thinly veiled admission of an inconsistency, couched in an attempt to dress up a bug to look like a feature. Similarly, Sergeyev's attempted definition of ① as somehow "the number of elements of the set $\mathbb{N}$" contradicts other passages where ① is *included* as a member of $\mathbb{N}$, resulting in an embarrassing circularity.[1]

[1] See further on circularity of Sergeyev's definitions in footnote 5.

The point we wish to emphasize is that the plausibility that such a scheme might actually work after being sufficiently cleaned-up of superfluous pathos[2] (including inconsistencies), is entirely due to Robinson's insights implementing Leibniz's ideas about the distinction between assignable and inassignable numbers, on the one hand, and implementing Leibniz's law of continuity as the transfer principle, on the other.

In his writings, Sergeyev introduces his *grossone*, announces that it is infinite, and blithely assumes that anything algebraic, or even from analysis, that can be done with ordinary numbers can be done when the *grossone* is adjoined. Such mathematical assertions require proof, which are lacking in the analyzed note.

3 Debt to Robinson

The tendency to give insufficient credit to Robinson is clearly on display in the "Olympic medal" as the reference to Robinson's theory is concealed in an obscure phrase in such a way that an uninformed reader will be unable to gauge its significance.

For the benefit of such a reader, we provide the following clarification. As far as providing a lexicographic ordering for the olympic medals are concerned, it would be sufficient to take the *grossone* to be equal to a number p greater than the total of all the medals attributed at the olympics, for example p equal a million, and work with number representation in base p. Then obviously p will satisfy all the usual rules governing finite numbers, because p itself is a finite number. However, Sergeyev's system is obviously not tailor-made for the games. Rather, the alleged significance of Sergeyev's system is its purported applicability to a broad range of scientific problems, without any apriori limitation on the size of the sample. For this reason he wishes to use an infinite *grossone* value for p. In fact, the ordinary rational numbers suffice for this purpose, as we explain in Sect. 7.

This is where his (pseudo)mathematical claims become questionable. His framework presupposes a number system which *properly* extends the usual one, yet obeys the usual laws, i.e., a transfer principle (see Sect. 2). But Sergeyev's system does not obey a transfer principle in any mathematically identifiable form, as Sergeyev appears to acknowledge in his sin(*grossone*) comment. The *grossone* calculator will be able to compute values necessary for scientific work only to the extent that one or another version of the transfer principle is successfully implemented. While Robinson's system does obey a transfer principle, Sergeyev is sparing in acknowledging his debt to Robinson.

Thus, in his keynote address in Las Vegas '15, Sergeyev declares that

> The new computational *methodology is not related to* the non-standard analysis and gives the possibility to execute computations of a new type simplifying fields of Mathematics where the usage of infinity and/or infinitesimals is required. (Sergeyev 2015b) (emphasis added)

This strikes us as a somewhat economical way of acknowledging intellectual indebtedness. It is as if someone proclaimed himself to be the inventor of relativity theory and declared that his "methodology is not related to" the work of Albert Einstein.

Sergeyev's infringement on Robinson's framework appears to be tolerated by the decision-makers in the mathematics community, in a way that would not be tolerated if the infringement were in a field like differential geometry or Lie theory. An infringement upon

[2] The English word *pathos* is etymologically related to $\pi\acute{\alpha}\theta o\varsigma$, passion.

Robinson's framework is tolerated at least in part because the field created by Robinson has been marginalized, not least through the (combined) efforts of Paul Halmos and Errett Bishop (see e.g., Katz and Katz 2011, 2012; Kanovei et al. 2015a), and of Connes (see Kanovei et al. 2013; Katz and Leichtnam 2013). As a result, a number of Robinson's students were unable to obtain positions at PhD-granting institutions in the 1970s. An additional factor seems to be Robinson's apparent insistence that logic has to take a more prominent place in graduate programs in mathematics, provoking animosity on the part of some mathematicians.

Robinson's framework is a fruitful modern research area that has attracted many researchers. Thus, Terry Tao developed certain arguments on approximate groups exploiting ultraproducts that would be difficult to paraphrase without them. The ultraproducts form a bridge between discrete and continuous analysis, and enable a unified framework for a treatment of both Hilbert's fifth probem and Gromov's theorem on groups of polynomial growth; see Tao (2014) for details.

4 Comparison with Work by Other Scholars

In this section we will compare Sergeyev's work with that of other scholars, in chronological order.

4.1 Levi-Civita Fields

David Tall used Levi-Civita fields under the name *superreal* to popularize teaching calculus via infinitesimals in (Tall 1979). Levi-Civita fields is a classical topic with a long history. It was studied in (Robinson and Lightstone 1975). Sergeyev exploits his *grossone* in place of the variable x in the Levi-Civita fields with the lexicographic ordering, but comments that

> Levi-Civita numbers are built using a generic infinitesimal ε ...whereas our *numerical* computations with [in]finite quantities are concrete and not generic. (Sergeyev 2015c, p. 2) (emphasis added)

Two years earlier, Sergeyev compared the concrete *grossone* numeral to Levi-Civita in the following terms (we make no attempt to correct the grammar):

> [5] At the first glance the *numerals* (7) can remind numbers from the Levi-Civita field (see [20]) that is a very interesting and important precedent of algebraic manipulations with infinities and infinitesimals. However, the two mathematical objects have several crucial differences. They have been introduced for different purposes by using two mathematical languages having different accuracies and on the basis of different methodological foundations. In fact, Levi-Civita does not discuss the distinction between numbers and *numerals*. His numbers have neither cardinal nor ordinal properties; they are build [sic] using a generic infinitesimal and only its rational powers are allowed; he uses symbol ∞ in his construction; there is no any *numeral* system that would allow one to assign *numerical* values to these numbers; it is not explained how it would be possible to pass from ...a generic infinitesimal h to a concrete one (see also the discussion above on the distinction between numbers and *numerals*). In no way the said above should be considered as a criticism with respect to results of Levi-Civita. The above discussion has been introduced in this text just to

> underline that we are in front of two different mathematical tools that should be used in different mathematical contexts. (Sergeyev 2013, p. 10671, note 5) (emphasis added)

Sergeyev's use of the terms *numeral* (both as adjective and noun) and *numerical* is vague. Certainly real numbers cannot be used in computer implementations, and one needs to work instead with a specific representation such as decimals. Shamseddine and his colleagues are surely aware of this in their work with the Levi-Civita fields (see Sect. 4.2).

Sergeyev has a talent for turning *pathos*[3] into patent. Affected pathos was also characteristic of the *superior ideology* of the former Soviet Union where he was raised. Sergeyev seems to have learned the lesson of the rhetorical effectiveness of *superior ideology*. Levi-Civita may have done the same mathematics a hundred years earlier than Sergeyev, but the former says a mere x and the latter says a superior "numeral", ergo the latter is on so much higher an ideological plane.

4.2 Shamseddine's Work on Levi-Civita Fields

A group of researchers around K. Shamseddine have been developing software based on the Levi-Civita field for handling certain calculations with infinity and infinitesimals; see e.g., the article Shamseddine (2015) and http://www.bt.pa.msu.edu/index_cosy.htm.

These scholars typically refrain from assorting their work with the kind of rhetoric that typically accompanies a Sergeyev performance, such as:

1. Sergeyev does not acknowledge properly indebtedness to Robinson, particularly in the matter of the transfer principle (see Sect. 2), painting himself as a pioneer in the area.
2. Sergeyev does not acknowledge properly that what he is working with is a version of the classical Levi-Civita fields, seeking to emphasize what he claims to be the novelty of his system.
3. Sergeyev seeks to spice up his writing with an assortment of colorful *principles* that have little bearing on an actual computer implementation, such as his stylized insistence on the part being less than the whole.

With regard to this last point, Benci and di Nasso (2003) developed a mathematical theory of *numerosities* to express this idea mathematically, but its Sergeyevan incarnation seems to have little mathematical content.

4.3 Kauffman on ①

L. Kauffman is a leading topologist today. The *Kauffman bracket* Kauffman and Lins (1994) is a staple of three-manifold invariants. His article "Infinite computations and the generic finite" Kauffman (2015a) uses Sergeyev's notation ①. Sergeyev managed to cite this recent paper of Kauffman's already in three texts. Thus, Sergeyev sends the reader to Kauffman (and other texts) "In order to see the place of the new approach in the historical panorama of ideas dealing with infinite and infinitesimal" (Sergeyev 2016, p. 24). However, Kauffman himself clearly distances himself from Sergeyev's "methodology" in the following terms:

> In my paper about the Grossone, I point out that the logic of this formalism is identical (in my version) to using $1 + x + x^2 + \ldots + x^G$ as a *finite sum* with G a

[3] See our etymological comment in footnote 2.

> generic positive integer. One can then manipulate the series and look at the limiting behaviour in many cases. There is no need to invoke any new concepts about infinity. This point of view may be at variance with the interpretations of Yaroslav [Sergeyev] for his invention, but I suggest that this is what is happening here. (Kauffman 2015b)

In no way can Kauffman's work or comments be interpreted as support for Sergeyev. Nor does Kauffman place Sergeyev "in the historical panorama" etc., contrary to Sergeyev's claim. Quite the opposite, Kauffman writes that "[t]here is no need to invoke any new concepts about infinity," thereby placing Sergeyev squarely *outside* a "historical panorama of ideas dealing with the infinite."

5 The Hyperreal Extension

In an approach to analysis within Robinson's framework, one works with the pair $\mathbb{R} \subseteq {}^*\mathbb{R}$ where $\mathbb{R}$ is the usual ordered complete *Archimedean* continuum, whereas ${}^*\mathbb{R}$ is a proper extension thereof. A proper extension of the real numbers could be called a *Bernoullian* continuum, in honor of Johann Bernoulli who was the first systematically to use an infinitesimal-enriched continuum as the foundation for analysis. For historical background see Borovik and Katz (2012), Bair et al. (2013), Bascelli et al. (2014), Kanovei et al. (2015b). The extension ${}^*\mathbb{R}$ obeys the transfer principle (see Sect. 2).

The field ${}^*\mathbb{R}$ is constructed from $\mathbb{R}$ using sequences of real numbers. The main idea is to represent an infinitesimal by a sequence tending to zero. One can get something in this direction without reliance on any nonconstructive foundational material. Namely, one takes the ring of all sequences, and quotient it by the equivalence relation that declares two sequences to be equivalent if they differ only on a finite set of indices.

The resulting object is a proper ring extension of $\mathbb{R}$, where $\mathbb{R}$ is embedded by means of the constant sequences. However, this object is not a field. For example, it has zero divisors. But if one quotients it further in such a way as to obtain a field (by extending the kernel to a *maximal* ideal), then the quotient will be a field, called a hyperreal field.

To motivate the construction further, it is helpful to analyze first the construction of $\mathbb{R}$ itself using sequences of rational numbers. Let $\mathbb{Q}_C^{\mathbb{N}}$ denote the ring of Cauchy sequences of rational numbers. Then

$$\mathbb{R} = \mathbb{Q}_C^{\mathbb{N}}/\mathrm{MAX} \tag{5.1}$$

where "MAX" is the maximal ideal in $\mathbb{Q}_C^{\mathbb{N}}$ consisting of all null sequences (i.e., sequences tending to zero).

The construction of a Bernoullian field can be viewed as refining the construction of the reals via Cauchy sequences of rationals. This can be motivated by a discussion of rates of convergence as follows. In the above construction, a real number u is represented by a Cauchy sequence $\langle u_n : n \in \mathbb{N}\rangle$ of rationals. But the passage from $\langle u_n\rangle$ to u in this construction sacrifices too much information. We seek to retain some of the information about the sequence, such as its "speed of convergence". This is what one means by "relaxing" or "refining" the equivalence relation in the construction of the reals from sequences of rationals.

When such an additional piece of information is retained, two different sequences, say $\langle u_n\rangle$ and $\langle u'_n\rangle$, may both converge to $u \in \mathbb{R}$, but at different speeds. The corresponding "numbers" will differ from u by distinct infinitesimals. If $\langle u_n\rangle$ converges to u faster than

$\langle u'_n \rangle$, then the corresponding infinitesimal will be smaller. The retaining of such additional information allows one to distinguish between the equivalence class of $\langle u_n \rangle$ and that of $\langle u'_n \rangle$ and therefore obtain distinct hyperreals infinitely close to u. For example, the sequence $\langle \frac{1}{n^2} \rangle$ generates a smaller infinitesimal than $\langle \frac{1}{n} \rangle$.

A formal implementation of the ideas outlined above is as follows. Let us present a construction of a hyperreal field ${}^*\mathbb{R}$. Let $\mathbb{R}^{\mathbb{N}}$ denote the ring of sequences of real numbers, with arithmetic operations defined termwise. Then we have

$$ {}^*\mathbb{R} = \mathbb{R}^{\mathbb{N}}/\mathrm{MAX} \tag{5.2}$$

where "MAX" is a suitable maximal ideal. What we wish to emphasize is the formal analogy between (5.1) and (5.2). In both cases, the subfield is embedded in the superfield by means of constant sequences.

We now describe a construction of such a maximal ideal exploiting a suitable finitely additive measure m. The ideal MAX consists of all "negligible" sequences $\langle u_n \rangle$, i.e., sequences which vanish for a set of indices of full measure m, namely,

$$ m\big(\{n \in \mathbb{N} : u_n = 0\}\big) = 1. $$

Here $m : \mathcal{P}(\mathbb{N}) \to \{0, 1\}$ (thus m takes only two values, 0 and 1) is a finitely additive measure taking the value 1 on each cofinite set,[4] where $\mathcal{P}(\mathbb{N})$ is the set of subsets of $\mathbb{N}$. The subset $\mathcal{F}_m \subseteq \mathcal{P}(\mathbb{N})$ consisting of sets of full measure m is called a free ultrafilter. These originate with Tarski (1930). The construction of a Bernoullian continuum outlined above was therefore not available prior to that date.

The construction outlined above is known as an ultrapower construction. The first construction of this type appeared in Hewitt (1948), as did the term *hyper-real*. The transfer principle (see Sect. 2) for this extension is an immediate consequence of the theorem of Łoś; see Łoś (1955).

6 A Detailed Technical Report on GOT

The analysis presented in this section is an extension of the report (Gutman and Kutateladze 2008). We formulate our analysis in the framework of Nelson's Internal Set Theory (IST) first presented in Nelson (1977).

The difference between Nelson's approach and Robinson's can be illustrated in the context of the underlying number system as follows. Robinson extended the real number field to a hyperreal number field with infinitesimals (for example, by the ultrapower approach of Sect. 5). In contrast with Robinson's approach, Nelson proceeded axiomatically and revealed both infinitesimals and illimited numbers within the real number field itself.[5] To this end, Nelson introduced a new one-place predicate "to be standard" together with the appropriate axioms. Both Nelson's and Robinson's theories are conservative

[4] For each pair of complementary *infinite* subsets of $\mathbb{N}$, such a measure m "decides" in a coherent way which one is "negligible" (i.e., of measure 0) and which is "dominant" (measure 1).

[5] This point seems to have escaped Sergeyev, who claims it to be an advantage of the grossone system that the infinite numbers are found *within* $\mathbb{N}$, allegedly unlike nonstandard analysis; see Calude and Dinneen (2015, p. 95, note 3). Elsewhere Sergeyev claims that, on the contrary, ① is "the number of elements in $\mathbb{N}$", leading to a circularity already mentioned in footnote 1.

extensions of the traditional foundational framework of the Zermelo–Fraenkel set theory. For further discussion see Katz and Kutateladze (2015).

6.1 Logical Status of Sergeyev's Theory

Sergeyev's reasoning is not only informal but often vague and inaccurate. The inaccuracies include his definition of the grossone as "the number of elements in set of natural numbers" (which may appeal to the uneducated but mathematically speaking is nonsensical), as well as his delphic pronouncements as to "the whole being greater than the part" and the distinction between "numbers and numerals" (see Sect. 4.1). Such superfluous PATHOS needs to be removed before a consistent theory can be identified. A reader with some mathematical culture can give formal shape to Sergeyev's postulates, as done in Gutman and Kutateladze (2008) to some extent. The result is a formal theory of signature

$$S = \{ =, \in, ① \}$$

(here $\in$ is the membership relation while ① is the grossone). We will abbreviate the theory as GOT\PATHOS. Here "GOT" stands for GrossOne Theory, while "PATHOS" alludes to the inconsistencies of Sergeyev's system and his efforts to sweep them under the rug by means of *le flou artistique* via affected pathos or passionate enthusiasm; see Sect. 4.1. Thus, GOT\PATHOS is the axiomatic formal theory in the language of signature S whose axiomatic background is given by all of Sergeyev's postulates, both explicitly stated and implicitly assumed in his papers.

Fact 6.1 *Each axiom of* GOT\PATHOS *is a trivial consequence of the axioms of any classical nonstandard set theory, provided ① is understood as the factorial of an infinitely large integer.*

This is shown in Gutman and Kutateladze (2008). In particular, the axioms of GOT\PATHOS are easily proven in Nelson's IST, with ① evaluated as the factorial of an arbitrary infinitely large natural number. Therefore,

Fact 6.2 *The theory* GOT\PATHOS *is weaker than* IST.

By definition, the theory is weaker whenever is has fewer theorems. Note that, for formal theories, *weaker* does not mean *worse*; nor does *stronger* mean *better*. For instance, a theory whose theorems are all the statements, i.e., an inconsistent theory, is the strongest one, but it is hardly the best one. Nevertheless, in certain circumstances, a weaker theory cannot be regarded as *new* as compared to a stronger theory.

Fact 6.3 GOT\PATHOS *is not a new theory.*

Indeed, GOT\PATHOS is weaker than a well-known theory, IST, and moreover, the axioms of GOT\PATHOS are easily proven in IST. Consequently, any reasoning within GOT\PATHOS can be automatically converted into the corresponding and almost identical reasoning in IST. In particular, GOT\PATHOS cannot prove any *new* result, since each result proven in GOT\PATHOS is already a result of a well-known theory. Actually, even proofs within GOT\PATHOS cannot be *new*, since every such proof is almost identical to an automatically produced proof in a well-known theory.

Fact 6.4 GOT\PATHOS *is dramatically weaker than* IST.

It suffices to note that IST features a powerful and fruitful tool known as the Transfer Principle (see Sect. 2), which is absent from the theory GOT\PATHOS. In addition, GOT\PATHOS has no analogs of Idealization and Standardization Principles, which makes it almost impossible to prove any serious assertion in GOT\PATHOS without appealing to informal or implicit assumptions.

Fact 6.5 *Consistency of* GOT\PATHOS *is not justified by its originator.*

In many of Sergeyev's papers, one cannot find a single attempt formally to justify the consistency of the grossone theory. Only due to Gutman and Kutateladze (2008) do we know that GOT\PATHOS is consistent relative to IST (see also Vakil 2012). Furthermore, employing the fact that IST is consistent relative to ZFC (see Nelson's article Nelson 1977) and that ZFC is consistent relative to ZF (a result of Goedel's; see his constructible universe Goedel 1938), we may conclude that GOT\PATHOS is consistent relative to the standard set theory (this is however not surprising, since GOT\PATHOS is weaker than a well-known relatively consistent theory).

It is good to know which facts a theory can prove, but for a theory to be useful it also very important to know which facts it cannot prove. To become a generally accepted legitimate mathematical tool, a theory should be unable to prove strange or *pathological* results. The corresponding formal property of a theory is called *conservativity*.

By definition, a theory T* of signature S* is a conservative extension of a weaker theory T with smaller signature S whenever T* has exactly the same theorems in signature S as T has. Suppose that we have a generally accepted theory T (say, ZFC) and let a new theory T* (say, IST) extend T and introduce new primary notions (in our example, the notion of standard set). The fact that T* is a conservative extension of T means the following: if T* allows us to prove some result R and R does not involve new primary notions, then R is not pathological, as it can also be proven in the generally accepted theory T. Therefore, any conservative extension of a customary theory can be (and should be) accepted as a legitimate mathematical tool. Namely, it has the same deduction strength and every *sensible* fact it can prove can be proven by usual means, without any new axioms or new notions.

IST is known to be a conservative extension of ZFC, as shown by Powell's theorem presented in Nelson (1977). This nontrivial and very important fact makes IST a generally accepted mathematical theory.

Fact 6.6 *The question of conservativity of* GOT\PATHOS *is ignored by its originator.*

Again, only due to Gutman and Kutateladze (2008) do we know that GOT\PATHOS is weaker than IST, which, in its turn, is a conservative extension of ZFC. Hence, so is GOT\PATHOS: if a set-theoretic fact can be proven in GOT\PATHOS, it can also be proven in IST and, thus, in ZFC. Without knowing this, even a consistent theory need not be accepted.

Therefore, without employing nontrivial facts from contemporary nonstandard analysis, Sergeyev's reasoning remains a powerless, informal, weak theory with doubtful consistency, which cannot be generally accepted due to its doubtful conservativity. On the other hand, if we employ the facts from nonstandard analysis, the grossone theory turns out to be merely a powerless and weak theory which cannot be regarded as new.

6.2 Algorithmic Status of Sergeyev's Theory

An algorithmic problem is the task of finding an algorithm which, given a constructive object as input, produces a constructive object as output so that the output is related to the input in a desired way, and this fact is provable within a suitable theory under consideration. Therefore, solvability and complexity of an algorithmic problem depends on the underlying theory.

A solution to an algorithmic problem is an algorithm supplied with a justification, i.e., with a proof (within a theory) of the assertion that the algorithm works correctly and actually solves the problem. On the other hand, a weaker theory has fewer proofs (which is a direct consequence of the definition) and thus fewer solvable algorithmic problems.

Fact 6.7 *Within a weaker theory, there are more unprovable and undecidable statements, more unsolvable algorithmic problems, while solutions to solvable algorithmic problems are more complex.*

Recalling that GOT\PATHOS is weaker than IST, we conclude the following.

Fact 6.8 *Each algorithmic problem unsolvable in* IST *is similarly unsolvable in* GOT\PATHOS; *if an algorithmic problem has a complex solution in* IST, *it either has an even more complex solution in the system* GOT\PATHOS *or is even unsolvable in* GOT\PATHOS.

Furthermore, being a conservative extension of ZFC, IST has exactly the same solvable set-theoretical problems as ZFC has. This circumstance allows us to derive the following fact.

Fact 6.9 *Every unsolvable set-theoretical problem is unsolvable in* GOT\PATHOS; *solvable set-theoretical problems are more complex or even unsolvable in* GOT\PATHOS.

There is a number of problems listed in Gutman and Kutateladze (2008) which encounter certain theoretical obstacles to finding an algorithmic solution. Some of the problems are IST-specific, other are purely set-theoretical or analytical. According to Facts 6.8 and 6.9 we have the following fact.

Fact 6.10 *Each of the algorithmic problems enumerated in the article* Gutman and Kutateladze (2008) *is either more complex or even unsolvable in* GOT\PATHOS.

6.3 Specific Algorithmic Problems Concerning Grossone

Within GOT\PATHOS, the main tool is the "positional system with base ①" in which the role of numerals is played by "multilevel polynomials" in a single variable denoted ①, with rational coefficients and exponents. We will refer to these polynomials as *grossnumerals*. They are multilevel in the sense that the exponents (power indices) need not be numbers and may also be (multilevel) polynomials. Every grossnumeral has finite height. Suitable formal definitions are presented in Gutman and Kutateladze (2008) (and are absent from Sergeyev's papers).

If we restrict the height of grossnumerals to 1, we obtain the usual polynomials in one variable. The algorithmic problems in the classical calculus of such polynomials are far from being new. They are all solved, long ago and completely. Anything new can occur only under consideration of numerals having arbitrary finite height.

The set of grossnumerals cannot be called a "calculus" unless it is supplied with a set of algorithms which implement such key operations as reduction to canonical form and comparison. Without such algorithms, one cannot speak of any computer realization of the calculus, either.

The important point here is that the implementation of the basic calculus operations in the set of grossnumerals encounters certain theoretical obstacles in IST and ZFC.

According to Sect. 6.2, they encounter even more serious problems in the weaker GOT\PATHOS. The issues are thoroughly described in Gutman and Kutateladze (2008), and the main problem is as follows.

Fact 6.11 *There is no known algorithm that, given grossnumerals x and y, would determine which of the following holds true: $x < y$, $x = y$, or $x > y$.*

The latter problem must be solved in order to be able to speak of a *calculus*, for otherwise we would not be able to perform such elementary procedures as reducing similar terms or listing the terms in descending order by their degree. Nevertheless, algorithmic solvability of these procedures remains unknown. The corresponding hypothesis is based on rather non-trivial facts on o-minimality and decidability of the order structure of reals with exponent (see bibliographic references [11] and [13] in Gutman and Kutateladze 2008).

Thus, currently there is no algorithm able to compare grossnumerals or, for that matter, to check the inequalities

$$1 < ①^{①^{-1}} < 2.$$

Such an algorihm could hardly appear in any of Sergeyev's papers. Indeed, he provides the following characterisation of infinite numbers: "Infinite numbers in this numeral system are expressed by numerals having at least one grosspower grater [sic] than zero". Sergeyev (2007, p. 60). But the grossexponent $①^{-1}$ is indeed greater than zero; yet the number $①^{①^{-1}}$ must be infinitely close to 1 if even a most rudimentary form of the transfer principle (see Sect. 2) is to be satisfied. Yet according to Sergeyev's characterisation, $①^{①^{-1}}$ would turn out to be "infinite". Whenever Sergeyev's assertions are specific enough to be checked, one finds errors, including freshman calculus level errors.

This particular error appeared in "Blinking fractals" Sergeyev (2007) published in *Chaos, Solitons, and Fractals*, and was subsequently criticized in Gutman and Kutateladze (2008). Sergeyev blinked and modified his text in a number of online databases, so as to remove the error, including its current ResearchGate version. As of 2015, no official correction whatsoever appeared in *Chaos, Solitons, and Fractals*.

This episode indicates how far removed the questions under consideration are from any computer implementation. The comparison problem is completely ignored in Sergeyev's papers, and this is not surprising: the problem is challenging even in IST, while in GOT\PATHOS it is much more complex due to the absence of a suitable transfer principle.

With the above taken into account, it becomes clear why all screenshots of a calculator presented in Sergeyev's papers contain only grossnumerals of height 1.

Fact 6.12 *An actual grossone calculator does not exist.*

Grossnumerals of height 1 are just ordinary polynomials of one variable, and software for the corresponding calculus is commonplace nowadays. Contemporary symbolic computation packages provide much more sophisticated machinery. The grossone theory is so poorly designed and underdeveloped that a toy calculator is the only tool which can be created on its basis.

7 Olympic Ranks Need No "Numerical Infinities"

In his note "The Olympic medals, ranks, lexicographic ordering, and numerical infinities," Sergeyev represents the basics of grossone theory (as he does in each of his numerous papers containing the symbol ①) under the pretext of applying it to a "mathematical problem" related to the lexicographic ranking method. The problem is caused by the fact that, contrary to other known ranking methods, the lexicographic method does not assign numerical ranks to various medal distributions, it only orders them, i.e., determines which distribution is higher and which is lower. Sergeyev suggests using grossnumerals as "numerical" ranks of arbitrary medal distributions and emphasizes that his suggestion solves the problem without upper bounds on the number of medals awarded by a single country as well as on the number of the medal classes (gold, silver, etc.).

We will demonstrate that the approach suggested by Sergeyev is useless and any application of a theory infinite numbers is overkill for such a trivial aim. Indeed, the lexicographic order can be made numerical in a very easy, reasonable, and practical way by means of ordinary standard rational numbers.

Suppose that there are infinitely (but countably) many medal classes. List them in descending order and associate with successive natural numbers: 1 for "gold", 2 for "silver", 3 for "bronze", 4, 5, 6, etc. for all the rest. Each competitor can win an arbitrary finite set of medals which can be encoded by a finite word with positive integers as "letters". For instance, the word $w = \langle 5, 0, 12, 1 \rangle$ encodes the fact that a competitor has won 5 medals of class 1, 0 medals of class 2, 12 medals of class 3, 1 medal of class 4, and 0 medals of any other class. The task is to invent a practical method (an algorithm) of calculating a number $R(w)$ for any word w in such a way that the equality $R(u) > R(v)$ be equivalent to $u \succ v$, where $\succ$ is the lexicographic order on words:

$$u \succ v \;\Leftrightarrow\; u_1 = v_1, \ldots, u_{n-1} = v_{n-1}, u_n > v_n \text{ for some } n.$$

(Here w_n is the nth letter of a word w, with $w_n = 0$ for n greater than the length of w.)

The method proposed by Sergeyev consists in defining the "numerical" rank $R_S(w)$ of a word $w = \langle w_1, \ldots, w_L \rangle$ of length L as the grossnumeral

$$R_S(w) \;=\; w_1①^{L-1} + w_2①^{L-2} + \cdots + w_{L-1}①^1 + w_L①^0.$$

How useful is such a solution, however? Sergeyev regards $R_S(w)$ as a "numerical" rank just because it is a "number" in the sense of his grossone theory. Both theoretically and practically, this is nothing but a mere replacement of a word $\langle w_1, \ldots, w_L \rangle$ with a more bulky expression of the form $w_1①^{L-1} + \cdots + w_L①^0$. This expression cannot be written in any other numerical form and cannot be used in any software other than the hypothetical "Infinity Calculator" based on the mythical "Infinity Computer technology".

We will now indicate a very simple and honest method of solving the above-stated "problem". Note first that, for the aim under consideration, there is no need for any artificial numbers, and the standard rational numbers with their standard order are undoubtedly sufficient. This is so because, as is well known, every countable linear order embeds into the standard ordered set of rationals, and this is true, in particular, for the lexicographically ordered set of words which represent medal distributions. So, the task is merely in choosing a specific order-preserving rational encoding of the words. The encoding can be as simple as follows. Given a word $w = \langle w_1, \ldots, w_L \rangle$, set

$$R(w) \;=\; \sum_{n=1}^{L} 2^{-(w_1+\cdots+w_{n-1}+n-1)} \sum_{m=1}^{w_n} 2^{-m}.$$

Here $R(w) \in [0, 1)$ is the rational number whose binary representation (representation in the positional numeral system with base 2) has the form

$$0.\underbrace{11\ldots1}_{w_1\text{ ones}}0\underbrace{11\ldots1}_{w_2\text{ ones}}0\ \ldots\ 0\underbrace{11\ldots1}_{w_L\text{ ones}}.$$

It is an easy exercise to show that the encoding R meets the required condition, i.e., assigns greater ranks $R(w)$ to lexicographically greater words w. Note also that medal distributions are uniquely (and easily) determined by their numerical ranks. It is also worth observing that R reflects certain emotional aspects related to medals wins: the awarding of the first medal of a given class is felt as a more exciting and significant achievement than awarding the second one, and so on. This circumstance results in the fact that the medal distributions with close numerical ranks are also "psychologically" close.

As an illustration, we present the 2014 Winter Olympics medal table of competitors (in lexicographic order) and their medal distributions supplemented with the corresponding exact binary ranks, and approximate decimal ranks.

2014 Winter Olympics medal table

Country	Medals			Binary	Decimal
Russia	13	11	9	0.111111111111101111111111101111111111	0.9999389
Norway	11	5	10	0.1111111111101111101111111111	0.9997520
Canada	10	10	5	0.111111111101111111111011111	0.9995114
United States	9	7	12	0.11111111101111111011111111111 1	0.9990196
Netherlands	8	7	9	0.11111111011111110111111111	0.9980392
Germany	8	6	5	0.1111111101111110111111	0.9980311
Switzerland	6	3	2	0.1111110111011	0.9915771
Belarus	5	0	1	0.11111001	0.9726562
Austria	4	8	5	0.111101111111101111 1	0.9686870
France	4	4	7	0.1111011110111111 1	0.9677658
Poland	4	1	1	0.11110101	0.9570312
China	3	4	2	0.11101111011	0.9350585
South Korea	3	3	2	0.1110111011	0.9326171
Sweden	2	7	6	0.1101111111011111 1	0.8745040
Czech Republic	2	4	2	0.1101111011	0.8701171
Slovenia	2	2	4	0.1101101111	0.8583984
Japan	1	4	3	0.1011110111	0.7412109
Finland	1	3	1	0.1011101	0.7265625
Great Britain	1	1	2	0.101011	0.6718750
Ukraine	1	0	1	0.1001	0.5625000
Slovakia	1	0	0	0.1	0.5000000
Italy	0	2	6	0.0110111111	0.4365234
Latvia	0	2	2	0.011011	0.4218750
Australia	0	2	1	0.01101	0.4062500
Croatia	0	1	0	0.01	0.2500000
Kazakhstan	0	0	1	0.001	0.1250000

8 Publication Venue

This rebuttal did not appear in the journal *The Mathematical Intelligencer* where Sergeyev's note originally appeared because five successive versions of our rebuttal were rejected by that journal, in spite of at least one favorable referee report.

9 Conclusion

The Olympic medals ranking was considered in Sergeyev's note in *The Mathematical Intelligencer* without any serious mathematical treatment. The note's shortcomings include serious issues of attribution of prior work.

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