Formalization of inverse problems and applications to systems of equations with parameters

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We show how binary correspondences can be used for simple formalization of the notion of problem, definition of the basic components of problems, their properties, and constructions (the condition of a problem, its data and unknowns, solvability and unique solvability of a problem, inverse problem, composition and restriction of problems, etc.). As an illustration, we consider a system of differential equations which describe a process in chemical kinetics, as well as the inverse problem.

Since abstracts are restricted in length, we present only four most basic definitions. In particular, we do not mention the notion of topological problem and the related notions of stability and correctness. We also omit the definition of parametrization and the corresponding topological aspects.

Definition 1. By a *problem* we mean an arbitrary correspondence between the elements of two sets, i.e., a triple $P = (A, B, C)$, where A and B are any sets and $C \subseteq A \times B$. The sets A, B, and C are denoted by $\operatorname{Dom} P$, $\operatorname{Im} P$, and $\operatorname{Gr} P$ and called *the domain of data*, *the domain of unknowns*, and *the condition* of the problem P. The containment $(a, b) \in \operatorname{Gr} P$ is written as $P(a, b)$.

Definition 2. A *solution* to a problem P for a data instance $a \in \operatorname{Dom} P$ is an arbitrary unknown $b \in \operatorname{Im} P$ which meets the condition $P(a, b)$. The set of solutions to P for a is denoted by $P[a]$. A problem P is *solvable* for $a \in \operatorname{Dom} P$ whenever $P[a] \neq \varnothing$, and *uniquely solvable* if $P[a] = \{b\}$ for some $b \in \operatorname{Im} P$, with the corresponding solution b denoted by $P^{\mathrm{s}}(a)$.

Definition 3. The *inverse problem* to P is the inverse correspondence

$$P^{-1} := \big(\operatorname{Im} P, \operatorname{Dom} P, (\operatorname{Gr} P)^{-1}\big), \quad \text{where } (\operatorname{Gr} P)^{-1} = \{(b, a) : (a, b) \in \operatorname{Gr} P\}.$$

Definition 4. The *composition* of problems P and Q is the composition of the correspondences, which is the problem

$$Q \circ P := (\operatorname{Dom} P, \operatorname{Im} Q, \operatorname{Gr} Q \circ \operatorname{Gr} P)$$

with condition

$$\operatorname{Gr} Q \circ \operatorname{Gr} P \;=\; \big\{(a, c) \in \operatorname{Dom} P \times \operatorname{Im} Q : (\exists\, b \in \operatorname{Im} P \cap \operatorname{Dom} Q)\ P(a, b)\ \&\ Q(b, c)\big\}.$$

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As an illustration, we consider a singularly perturbed system of ordinary differential equations which arises in modeling certain processes of chemical kinetics.

Suppose that $n, m \in \mathbb{N}$, $0 < \varepsilon_0 \in \mathbb{R}$, $X := \mathbb{R}^m$, Y is a domain in $\mathbb{R}^n$, $T := \mathbb{R}$, $E := \{\varepsilon \in \mathbb{R} : 0 \leqslant \varepsilon \leqslant \varepsilon_0\}$, $F := C(X \times Y \times T \times E, \mathbb{R}^m)$, $G := C(X \times Y \times T \times E, \mathbb{R}^n)$. Consider the problem P with domain of data $\operatorname{Dom} P = F \times G \times E$, domain of unknowns $\operatorname{Im} P = C^1(T, X) \times C^1(T, Y)$, and condition

$$P\big((f, g, \varepsilon), (x, y)\big) \;\Leftrightarrow\; \begin{cases} \;\dot{x}(t) = f\big(x(t), y(t), t, \varepsilon\big), \\ \varepsilon\, \dot{y}(t) = g\big(x(t), y(t), t, \varepsilon\big) \end{cases} \text{for all } t \in T,$$

where $f \in F$, $g \in G$, $\varepsilon \in E$, $x \in C^1(T, X)$, $y \in C^1(T, Y)$.

Solution of the problem P is based on the method of integral manifolds, a convenient tool for studying multidimensional singularly perturbed systems of differential equations which makes it possible to lower the dimension of the system under study (see [1, 2, 3]). Solution of P in a sense reduces to solving the so called degenerate system which is obtained from the initial system by putting the parameter ε equal to zero. This is justified by the results of A. N. Tikhonov (see, for instance, [4]) on passing to a solution to the degenerate problem as a small parameter tends to zero.

The inverse problem to P consists in finding the unknown functions on the right-hand side of the system, given some data on the solution to the direct problem P. Relying on the close connection of the initial problem with the degenerate system, we consider the case $\varepsilon = 0$ and additionally assume that the "slow surface" defined by the equation $g\big(x, y, t, 0\big) = 0$ consists of a single sheet (with respect to the dependence of y on x). Since the right-hand parts of equations in chemical kinetics often occur polynomial, the corresponding restriction on f seems to be natural.

Therefore, for demonstration purposes, we consider the partial case of the problem P in which $m = n = 1$, $E = \{0\}$, the functions $f \in F$ are polynomials of degree one, and $g \in G$ meet the condition of the implicit function theorem, which fact allows us to replace the equation $g\big(x(t), y(t), t, 0\big) = 0$ by the equivalent equation of the form $y(t) = h\big(x(t), t\big)$.

Let $h \in C^1(\mathbb{R}^2)$. Consider the problem Q with domain of data $\operatorname{Dom} Q = \mathbb{R}^3$, domain of unknowns $\operatorname{Im} Q = C^1(\mathbb{R})^2$, and condition

$$Q\big(f, (x, y)\big) \;\Leftrightarrow\; \begin{cases} \dot{x}(t) = f_1 + f_2 x(t) + f_3 y(t), \\ y(t) = h\big(x(t), t\big) \end{cases} \text{for all } t \in \mathbb{R},$$

where $f = (f_1, f_2, f_3) \in \mathbb{R}^3$, $x, y \in C^1(\mathbb{R})$.

The formal inverse problem Q^{-1}, which has pairs of functions $(x, y) \in C^1(\mathbb{R})^2$ as data, is very simple and impractical. Finite collections of values of functions or their derivatives as data are more adequate than everywhere defined functions. The corresponding correction of the inverse problem is realized by the composition of the problem Q^{-1} and the auxiliary problem R with domain of data $\operatorname{Dom} R = (\mathbb{R}^3)^3$, domain of unknowns $\operatorname{Im} R = C^1(\mathbb{R})^2$, and condition

$$R\big((t,\alpha,\beta),(x,y)\big) \;\Leftrightarrow\; \begin{cases} x(t_1)=\alpha_1,\ x(t_2)=\alpha_2,\ x(t_3)=\alpha_3,\\ \dot x(t_1)=\beta_1,\ \dot x(t_2)=\beta_2,\ \dot x(t_3)=\beta_3, \end{cases}$$

where $t,\alpha,\beta \in \mathbb{R}^3$, $x,y \in C^1(\mathbb{R})$.

Theorem 1. *If $t,\alpha \in \mathbb{R}^3$ meet the condition*

$$\Delta := \begin{vmatrix} 1 & \alpha_1 & h(\alpha_1,t_1)\\ 1 & \alpha_2 & h(\alpha_2,t_2)\\ 1 & \alpha_3 & h(\alpha_3,t_3) \end{vmatrix} \neq 0 \tag{1}$$

then, given arbitrary $\beta \in \mathbb{R}^3$, the problem $Q^{-1}\circ R$ is uniquely solvable for the data (t,α,β), and its solution $(f_1,f_2,f_3) = (Q^{-1}\circ R)^{\mathrm{s}}(t,\alpha,\beta)$ can be calculated by Cramer's formulas $f_i = \Delta_i/\Delta$, where Δ_i is the determinant of the matrix formed from the matrix in (1) *by replacing the i th column by $\beta = (\beta_1,\beta_2,\beta_3)$.*

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Nonclosed Archimedean cones

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A *wedge* is a nonempty convex subset K of a vector space (henceforth, vector spaces are assumed to be defined over the field of reals) which meets the condition $(\forall\,\alpha \geqslant 0)(\alpha K \subseteq K)$. A wedge K is a *cone* whenever $K \cap (-K) = \{0\}$. As is known, in every (pre)ordered vector space $(X,\leqslant)$ the set $\{x \in X : x \geqslant 0\}$ is a cone (wedge). Conversely, if a cone (wedge) K is fixed in a vector space X