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Этот выпуск содержит тезисы некоторых докладов, представленных на Международной конференции «Геометрический анализ и теория управления» (8–12 декабря 2016 года). Темы докладов относятся к современным направлениям в геометрии, теории управления и анализе, а также к приложениям методов метрической геометрии и анализа к смежным областям математики и прикладным задачам.

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The issue contains abstracts of some of the talks presented on the International Conference "Geometric Analysis and Control Theory" (8–12 of December, 2016). Topics of talks concern modern trends in geometry, control theory and analysis, as well as applications of the methods of the metric geometry and analysis to related fields of mathematics and applied problems.

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Dedicated to Sergey Konstantinovich Vodopyanov
in the honor of his 70th birthday

Contents

On integrability of geodesic flows on the two-torus

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Some questions related to the well-known conjecture given by V. V. Kozlov about integrability of geodesic flows on the 2-torus will be discussed. It is known that its integrability on all energy levels is equivalent to existence of smooth periodic solutions of a quasi-linear system of PDEs on metric and coefficients of the first integral. This system has many beautiful properties, e. g. it is a semi-Hamiltonian system. The only known solutions of this system correspond to the existence of the first or of the second degree polynomial integral.

The integrability of geodesic flow on the 2-torus in a non-zero magnetic field on all energy levels is a very restrictive requirement. In the only known example there is an additional linear in momenta first integral. Thus it is natural to consider the question of integrability of the magnetic geodesic flow only on a fixed energy level.

Finally a connection between geodesic flows and natural mechanical systems will be discussed.

Volume geodesic distortion and Ricci curvature for extremals flows

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We study the variation of a smooth volume form along extremals of a variational problem with nonholonomic constraints and an action-like Lagrangian. We introduce a new invariant describing the interaction of the volume with the dynamics and we study its basic properties.

We then show how this invariant, together with the generalized Ricci curvature appear in the expansion of the volume at regular points of the exponential map. This generalizes the well-known expansion of the Riemannian volume in terms of Ricci curvature to a wide class of geometric structures, including all sub-Riemannian manifolds. This is a joint work with D. Barilari and E. Paoli.

Isometric approximation in bounded sets and its applications

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We give a short survey of results related to the isometric approximation problem in bounded sets of $\mathbb{R}^n$, starting from [1]. After this, we present some applications in the extension problems for bilipschitz and quasisymmetric maps with small constants. In particular, we showed in [2] that a $(1+\varepsilon)$-bilipschitz map $f : A \to \mathbb{R}^n$ can be extended to a $(1 + C\varepsilon)$-bilipschitz map $F : \mathbb{R}^n \to \mathbb{R}^n$ if A statisfies a geometrical thickness condition. Furthermore, similar extension results were obtained in [3] for $(1+\varepsilon)$-power-quasisymmetric maps.

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Nilpotent approximation of a wheel robot with a trailer

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Consider a wheel robot with a trailer on the plane. Possible positions of such robot can be expressed by vector $q = (x, y, \theta, \varphi)$, where $(x, y) \in \mathbb{R}^2$ is a midpoint of the robot and $\theta, \varphi \in S^1$ are the angles of robot and trailer orientation. Corresponding control system can be expressed in the following way [1]:

$$\dot{q} = u_1 X_1(q) + u_2 X_2(q), \tag{1}$$

$$q = (x, y, \theta, \varphi)^T \in M = \mathbb{R}^2 \times (S^1)^2, \quad (u_1, u_2) \in \mathbb{R}^2, \tag{2}$$

$$X_1(q) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \\ -\frac{\sin\varphi}{l_t} \end{pmatrix}, \qquad X_2(q) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -\frac{l_r \cos\varphi}{l_t} - 1 \end{pmatrix}, \tag{3}$$

The problem is to move a robot with a trailer from one configuration to another, i.e., to find a path $q(t)$, s.t.

$$q(0) = q_0 = (x_0, y_0, \theta_0, \varphi_0), \qquad q(t_1) = q_1 = (x_1, y_1, \theta_1, \varphi_1),$$

where $q_0, q_1 \in M$.

The optimality criterion has form:

$$\int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt.$$

The method of nipotent approximation [2] is applied to the control problem (1)–(3) with arbitrary values of l_t, l_r. The corresponding nipotent problem is sub- Riemannian problem on Engel group which has been studied quite recently in a series of works [3, 4, 5]. The problem of finding optimal synthesis in the general case is reduced to a system of three algebraic equations in elliptic functions and elliptic integrals [6]. It seems impossible to analytically solve such equations, therefore, a software for computing optimal trajectories for the nilpotent sub-Riemannian problem on the Engel group is being developed in Wolfram Mathematica [7]. This software has already been used to devise several algorithms for computing approximate paths close to optimal in terms of (4) for a mobile robot with a trailer. Those algorithms will be applied for controlling a real mobile robot with a trailer.

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Infinite dimensional sub-Riemannian geometry

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In this talk, I will start by defining weak and strong sub-Riemannian geometry on Banach manifolds, and I will compare these manifolds to the finite dimensional case. In particular, we will see that there is no Pontryagin maximum principle in infinite dimensions, because a third type of geodesic called "elusive geodesics" can appear. They correspond to curves for which the range of the differential of the endpoint mapping is dense and not closed. However, we will see that we still have the existence of a Hamiltonian geodesic flow for normal geodesics, at least in the strong case. I will also give a variant of Chow-Rashevski's theorem, and a variant of Sussmann's orbit theorem.

Covering and packing of the plane with two kinds of circles

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The classical results about packing and covering of the plane with equal circles are presented in [1, 2]. It was proved that the maximum packing density of the plane equals $D_1 = \frac{\pi}{\sqrt{12}} \approx 0.9069$, and the minimum covering density of the plane equals $d_1 = \frac{2\pi}{\sqrt{27}} \approx 1.2092$. Basic research methods on this subject were studied in [3, 4]. The problem of sharp density bounds for packing and covering of the plane with two or more kinds of circles is very important. It is known the value $D_n = 1 - \left(1 - \frac{\pi}{\sqrt{12}}\right)^n$ bounds from above the density of any packing of the plane with n different circles [5]. The problem of determining the sharp lower bound d_n for a coverage density with two or more circles is much more difficult.

A publication [5] determines lowest bound $d_2 = 1.018955\ldots$ for a covering density of the plane with two kinds of circles. With an algorithm proposed in above-mentioned publication this value can be calculated with any desired accuracy. In the present paper we prove there is the rigorous analytical formula for calculating d_2. We have obtained the following result.

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Theorem. *The lowest bound for a density of any cover of the plane with two kinds of circles is defined by the formula:*

$$d_2 = \frac{2\pi}{\sqrt{27}}\left(1 - \sqrt{3}\cdot\tan\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)\right) \approx 1.018955.$$

The min-density covers of the flat areas with disks have many applications. For example, it can be used to design energy-efficient wireless sensor networks. Designers try to reduce density of the covers because this reduces energy consumption of sensor networks. It is important that the number of different types of circles is small and the cover has a simple structure. Therefore, it is necessary to set the task of finding the min-density covering with low complexity, as this provides significant advantages in the design and construction of sensor networks. The concept of the covering (packing) complexity with a repetitive structure is associated with a number of different kinds of circles involved in covering (packing) of the minima typical fragment. The precise definitions of the complexity of a covering and of a packing are given in this article.

Optimal covering and packing with first-level complexity ($L1$) correspond to the models studied in [1, 2]. In this paper we have presented examples of covering and packing models with complexity $L2$. Among them we choose the best models which are expected to optimal in this class. Their densities are equal $D_2^* = \frac{5\pi}{\sqrt{12}}\left(59 - 24\sqrt{6}\right) \approx 0.0962$ and $d_2^* = \frac{3\pi}{\sqrt{75}} \approx 1.0882$.

It is worth noting that the covering proposed in [5] has unlimited complexity. This is due to the fact that the minimal fragment asymptotically filled with an unlimited number of infinitely small identical circles which have specific locations.

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Geometry of phase portrait of one cell complex

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We study one model of initial stage of interaction of 3 adjacent cells K_1, K_2, K_3 in the proneulal cluster of *Drosophila melanogaster*. Here, the main components of the processes simulated in the model are: the complex (AS-C) of proteins Achaete and Scute, and the proteins Notch and Delta, denoted here by (N) and (Dl), which control intercellular interaction.

Let x_j, y_j, z_j be, respectively, the concentrations of (AS-C), (Dl), (N) in the cells K_j, $j = 1, 2, 3$, as in the model of two cells interaction, see [1].

Then the model of concentrations kinetics in this three-cells complex can be represented in the form of non-linear dynamical system in the positive octant $\mathbb{R}^9_+$:

$$\begin{cases} \dfrac{dx_1}{dt} = f_1(z_1) - x_1; \ \dfrac{dy_1}{dt} = \sigma_1(x_1) - y_1; \ \dfrac{dz_1}{dt} = \zeta_{2,1}(y_2) + \zeta_{3,1}(y_3) - z_1; \\ \dfrac{dx_2}{dt} = f_2(z_2) - x_2; \ \dfrac{dy_2}{dt} = \sigma_2(x_2) - y_2; \ \dfrac{dz_2}{dt} = \zeta_{1,2}(y_1) + \zeta_{3,2}(y_3) - z_2; \\ \dfrac{dx_3}{dt} = f_3(z_3) - x_3; \ \dfrac{dy_3}{dt} = \sigma_3(x_3) - y_3; \ \dfrac{dz_3}{dt} = \zeta_{1,3}(y_1) + \zeta_{2,3}(y_2) - z_3. \end{cases} \quad (1)$$

The functions f_j are positive and monotonically decreasing, they describe negative feedbacks N$\cdots\cdots\blacktriangleleft$(AS-C) in the cells K_j. The functions $\zeta_{k,j}$ and σ_j $k, j = 1, 2, 3$, $k \neq j$, are positive and monotonically increasing, they describe positive feedbacks in the intercellular interaction $(\mathrm{Dl})_k \to (\mathrm{N})_j$, and $(\mathrm{AS-C})_j \to (\mathrm{Dl})_j$ in the cell K_j. The positive summands in the equations describe synthesis of the proteins, and the negative summands correspond to their natural degradation.

The cells K_j are assumed to be identical at the initial stage of the processes, so, following [2], we consider the case

$$f_1 = f_2 = f_3 = f(z) = \frac{A}{a + z^2}; \quad \sigma_1 = \sigma_2 = \sigma_3 = \sigma = \frac{Bx^2}{b + x^2};$$
$$\zeta_{i,j} = \zeta(y) = \frac{Cy}{c + y}; \quad i, j = 1, 2, 3, \ i \neq j.$$

All the parameters are positive.

The equilibrium points of the system (1) are determined by the equations

$$z_1 = g(z_2) + g(z_3); \quad z_2 = g(z_1) + g(z_3); \quad z_3 = g(z_1) + g(z_2), \quad (2)$$

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where $g(z) = \dfrac{\alpha}{(a + z^2)^2 + \gamma}$. As above, all the parameters are positive.

I. For all $\alpha > 0$, $a > 0$, $\gamma > 0$ the system (2) has exactly one symmetric solution: $z_1 = z_2 = z_3 = z_0$, determined by the equation $z_0 = 2g(z_0)$, so the system (1) has one symmetric equilibrium point S_0 with coordinates

$$x_0 = x_1 = x_2 = x_3 = f(z_0); \quad y_0 = y_2 = y_3 = y_1 = \sigma(x_0).$$

II. We assume from now on that the parameter $\alpha > 0$ is *sufficiently large*, namely, let $\min\limits_{z>0} \frac{dg}{dz} < -1$. If this condition is not satisfied, then the system (1) has a unique equilibrium point S_0, and this point is stable. In the opposite case, which is more complicated from the geometric view point, the system (2) has six partially symmetric solutions

$$\begin{array}{lll} z_1 = z_2 > z_3; & z_1 = z_3 > z_2; & z_2 = z_3 > z_1; \\ z_1 = z_2 < z_3; & z_1 = z_3 < z_2; & z_2 = z_3 < z_1. \end{array}$$

All these six solutions appear from the equations $z_j = g(z_j) + g(2g(z_j))$, $j = 1, 2, 3$, and determines the equilibrium point S_3^+, S_2^+, S_1^+ and S_3^-, S_2^-, S_1^-, respectively.

Theorem 1. *All the equilibrium points of the system* (1) *are listed in the paragraphs* I, II *above. That is, each equilibrium point of* (1) *is either symmetric, or partially symmetric.*

Linearization of the system (1) at its equilibrium points is described by 9×9 block matrix M of the form

$$M = \begin{pmatrix} A_1 & B_2 & B_3 \\ B_1 & A_2 & B_3 \\ B_1 & B_2 & A_3 \end{pmatrix}, \text{ where } A_j = \begin{pmatrix} -1 & 0 & -p_j \\ q_j & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}; \; B_j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & r_j & 0 \end{pmatrix}$$

and

$-p_j = \dfrac{df(z_j)}{dz_j}$; $q_j = \dfrac{d\sigma(x_j)}{dx_j}$; $r_j = \dfrac{d\zeta(y_j)}{dy_j}$. Direct calculations imply the following

Theorem 2. *For sufficiently large values of the parameter α, the equilibrium points S_1^+, S_2^+, S_3^+ are stable, and S_1^-, S_2^-, S_3^-, S_0 are unstable.*

The biological interpretation of this phase portrait geometry is the following:

If a trajectory of the system (1) is attracted by the point S_1^+, then the cell K_j becomes the parental cell in the considered proneural cluster.

Three-dimensional plane $P^3 \subset \mathbb{R}^9$, defined by the equations $x_1 = x_2 = x_3$, $y_1 = y_2 = y_3$, $z_1 = z_2 = z_3$, is an invariant manifold of the system (1), and contains the equilibrium point S_0.

Theorem 3. *For sufficiently large values of the parameters, the plane P^3 contains an unstable cycle of the system* (1)*.*

S. Smale has constructed one very hypothetical model of interaction of two cells, see [3]. It was not related to any biological process, and was represented in the form of 4-dimensional nonlinear dynamical system. For appropriate choice of parameters of this system, its phase portrait contains a stable cycle. In absence of interaction there are no periodic trajectories in this two-cells complex dynamics.

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Invariants of Euler equations with commuting operators

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The scalar Euler operator

$$L_n = D^n + \sum_{k=1}^{n} C_n^k \frac{a_k}{x^k} D^{n-k} = D^n + n\frac{a_1}{x} D^{n-1} + \cdots + n\frac{a_{n-1}}{x^{n-1}} D + \frac{a_n}{x^n}, \qquad (1)$$
$$a_k = \text{const}, \qquad D = d/dx,$$

is a particular case of a linear differential operator. In general, studying the commuting linear operators is a complicated problem [1, 2]. The case of the Euler operators is more simple. Commuting operators L_m and L_n have common eigenfunction $\psi(x)$ which is a common solution of the pair of the equations

$$L_m\psi = \lambda^m \psi, \qquad L_n\psi = \lambda^n \psi, \qquad \lambda = \text{const} \neq 0. \qquad (2)$$

Number l of linearly independent solutions of system (2) is called the rank of the pair of operators L_m, L_n. The rank coincides with the highest common factor of the numbers m and n. Denote by $L_n = \Lambda_n\{q_1, \ldots, q_n\}$ the Euler operator (1) factorizable in the form

$$\begin{gathered} L_n = x^{-n}(xD + a_1 + q_1)(xD + a_1 + q_2) \cdots (xD + a_1 + q_n), \\ q_1, \ldots, q_n = \text{const}, \quad q_1 + \cdots + q_n = n(1-n)/2. \end{gathered} \tag{3}$$

If q_j are integers different by modulo n, then eigenfunction $\psi(x)$ of operator (3) is expressible in terms of elementary functons. In [3] we have constructed the common eigenfunctions for the rank two commuting Euler operators L_4, L_6 and L_{10}, which satisfy the foregoing conditions. It is not a difficult problem in the case of arbitrary constant q_j, too. For the rank l commuting Euler operators equations (2) are factorizable in the form

$$(L_m - \lambda^m)\psi = N_{m-l}M_l\psi = 0, \qquad (L_n - \lambda^n)\psi = N_{n-l}M_l\psi = 0$$

with some linear operators M_l, N_{m-l}, N_{n-l}. Thier common eigenfunction solves the linear equation of order l

$$M_l\psi = 0. \tag{4}$$

For the rank two Euler operators a differential substitution exists that reduces equation (4) to the Bessel equation. For example, for the commuting operators

$$\begin{aligned} L_4 &= \Lambda_4\{K, K-2, 1-K, -K-5\}, \qquad K = \text{const}, \\ L_6 &= \Lambda_6\{K, K-2, K-4, 1-K, -K-3, -K-7\}, \end{aligned}$$

the operator M_2 is given by

$$\begin{aligned} M_2 = D^2 + \frac{1}{x}\Big(2a_1 + \frac{4(2K+1)}{2(2K+1) - \lambda^2x^2}\Big)D + \\ + \frac{1}{x^2}(a_1^2 - a_1 - K^2 - K - 4) + \frac{4(2K+1)(a_1+K+2)}{x^2(2(2K+1) - \lambda^2x^2)} - \lambda^2. \end{aligned}$$

Equation (4) with this operator becomes the Bessel equation

$$\phi'' + \frac{2a_1}{x}\phi' + (a_1 - K)(a_1 + K - 1)\frac{\phi}{x^2} = \lambda^2\phi$$

via the substitution $\psi = \phi' + (a_1 - K - 2)\phi/x$. In the case of the rank three Euler equations a differential substitution exists that reduces equation (4) to an equation of the form $L_3\phi = \lambda^3\phi$, where L_3 is an Euler operator of the third order, and so on.

Invariants of the point equivalence transformations $\tilde{x} = \xi(x)$, $\tilde{y} = \eta(x)y + \zeta(x)$ of the Euler equation with operator (1)

$$L_ny \equiv y^{(n)} + \frac{na_1}{x}y^{(n-1)} + \cdots + \frac{na_{n-1}}{x^{n-1}}y' + \frac{a_n}{x^n}y = 0 \tag{5}$$

are the functions of x, y, y', …, $y^{(n-1)}$, a_1, …, a_n. Relative invariants of equations (5) of different orders are defined by the same formulas. And $n-1$ of that invariants, $j_1, \ldots, j_{n-1}$, depend on the constant parameters of operator (1) only. Namely, when $n=2$ equation (5) has a single constant invariant

$$j_1 = a_2 - a_1(a_1 - 1),$$

when $n=3$ there are two invariants: j_1 and

$$j_2 = a_3 + 3(1-a_1)j_1 - a_1(a_1-1)(a_1-2).$$

When $n=4$ equation (5) has three invariants: the foregoing j_1, j_2, and

$$j_3 = a_4 + 2(3-2a_1)j_2 - 3j_1^2 - (6a_1^2 - 18a_1 + 11)j_1 - a_1(a_1-1)(a_1-2)(a_1-3),$$

when $n=5$ there are four invariants j_1, j_2, j_3 and

$$\begin{aligned} j_4 = a_5 &+ 5(2-a_1)(j_3 + 3j_1^2) - 10j_1j_2 - 5(2a_1^2 - 8a_1 + 7)j_2 + \\ &+ 5(2-a_1)(2a_1^2 - 8a_1 + 5)j_1 - a_1(a_1-1)(a_1-2)(a_1-3)(a_1-4), \end{aligned}$$

and so on. Absolute invarianta of equation (5) are found fron the invariance condition $XI = 0$, where

$$\begin{aligned} X = \sum_{k=1}^{[(n-1)/2]} (2k+1)j_{2k}\partial_{j_{2k}} &+ \Big(2j_1 - \frac{n+1}{6}\Big)\partial_{j_1} + \\ +\Big(4j_3 - \frac{n+1}{30}\Big)\partial_{j_3} &+ \Big(6j_5 - \frac{n+1}{42}\Big)\partial_{j_5} + \cdots \end{aligned}$$

It turns out that the Euler equations of different orders possess similar invariants, when the corresponding operators (1) commute. Namely, if we introduce the modified relative invariants

$$\begin{aligned} &J_1 = (n-1)j_1, \\ &J_2 = (n-1)(n-2)j_2, \\ &J_3 = (n-1)[(n-2)(n-3)j_3 + 6nj_1^2 - n(n+1)j_1], \\ &J_4 = (n-1)(n-2)[(n-3)(n-4)j_4 + 60nj_1j_2 - 5n(n+1)j_2], \ \ldots, \end{aligned} \tag{6}$$

then they coincide for the commuting Euler operators.

As an example we consider the following commuting Euler operators and the corresponding invariants (6):

$L_4 = \Lambda_4\{3, -2, -3, -4\}$, $j_1 = -2$, $j_2 = 6$, $j_3 = -26$,
$J_1 = -6$, $J_2 = 36$, $J_3 = 252$,

$L_5 = \Lambda_5\{3, -1, -3, -4, -5\}$, $j_1 = -3/2$, $j_2 = 3$, $j_3 = -33/4$, $j_4 = 30$,
$J_1 = -6$, $J_2 = 36$, $J_3 = 252$, $J_4 = -20880$.

The same invariants J_1, J_2, J_3, J_4 correspond to the operators

$$
\begin{aligned}
&L_6 = \Lambda_6\{3, -1, -2, -4, -5, -6\},\\
&L_7 = \Lambda_7\{3, -1, -2, -3, -5, -6, -7\}, \quad L_8 = L_4^2,\\
&L_9 = \Lambda_9\{3, -1, -2, -3, -4, -5, -7, -8, -9\}, \quad L_{10} = L_5^2,\\
&L_{11} = \Lambda_{11}\{3, -1, -2, -3, -4, -5, -6, -7, -9, -10, -11\}, \quad L_{12} = L_4^3, \ \ldots
\end{aligned}
$$

commuting with L_4, L_5. Their common eigenfunction is given by

$$\psi = \frac{\mathrm{e}^{\lambda x}}{x^{3+a_1}}(\lambda^3 x^3 - 3\lambda^2 x^2 + 6\lambda x - 6).$$

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Privileged coordinates for nonsmooth Carnot–Carathéodory spaces

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Following the works [1, 2] we define the *equiregular Carnot–Carathéodory space of class C^k* as C^∞-smooth manifold $\mathbb{M}$ with a fixed family of C^k-smooth subbundles of the tangent bundle

$$H_1 \subsetneq H_2 \subsetneq \ldots \subsetneq H_r = T\mathbb{M}$$

such that $[H_i, H_j] \subset H_{i+j}$. Note that the classical equiregular sub-Riemannian space with horizontal distribution $H \in C^{r+k-1}$ is a C^k-smooth Carnot–Carathéodory space.

Let $X_1, \ldots, X_n$ be a basis of vector fields such that $H_k = \operatorname{span}\{X_1, \ldots, X_{\dim H_k}\}$ and assign to each field the weight $d_j = \min\{m : X_j \in H_m\}$. Using the canonical coordinates of the 1st kind

$$\theta_q(u_1, \ldots, u_n) = \exp(u_1 X_1 + \ldots + u_n X_n)(q)$$

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we define quasimetric d_∞ on $\mathbb{M}$ by

$$d_\infty(x, y) = \max_{j=1,\dots,n} |u_j|^{\frac{1}{d_j}} \quad \text{if } y = \theta_x(u_1, \dots, u_n).$$

In [1–3] it is shown that if in a neighborhood of $p \in \mathbb{M}$ we consider a family of dilations

$$\Delta_\varepsilon : \theta_p(x_1, \dots, x_n) \mapsto \theta_p(\varepsilon^{d_1} x_1, \dots, \varepsilon^{d_n} x_n)$$

then there are homogeneous limits

$$\widehat{d_\infty}(x, y) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} d_\infty(\delta_\varepsilon x, \delta_\varepsilon y), \quad \widehat{X}_j = \lim_{\varepsilon \to 0} (\delta_\varepsilon^{-1})_* \varepsilon^{d_j} X_j(\delta_\varepsilon x) \tag{1}$$

which define a structure of a nilpotent graded Lie group (quasimetric on it and a basis of its Lie algebra correspondingly). These results allow to generalize notion of a nilpotent tangent cone to a sub-Riemannian space to the class of aforementioned equiregular spaces.

For smooth sub-Riemannian spaces it is known [4] that in (smooth) local coordinates ϕ_p limits (1) exist if and only if

$$d(p, \phi_p(u_1, \dots, u_n)) \sim |u_1|^{\frac{1}{d_1}} + \dots + |u_n|^{\frac{1}{d_n}}, \tag{2}$$

where d can be either Carnot–Carathéodory metric or d_∞. Such coordinate systems are called *privileged*.

Since for C^k-smooth spaces most of canonical coordinate systems are also just C^k-smooth, we study the class of nonsmooth coordinate systems that allow limits (1). It turns out that the notion of privileged coordinates becomes more blurred in this case.

Theorem 1. *The condition* (2) *is necessary for existence of limits* (1) *but is not enough in general.*

Since existence of limits (1) is known for canonical coordinates of the 1st kind θ_p we obtain further results in terms of transition map $\Phi = \theta_p^{-1} \circ \phi_p$ where ϕ_p is the new coordinate system.

Theorem 2. *If ϕ_p is a homeomorphism, there is an uniform limit*

$$L(x) := \lim_{\varepsilon \to 0} \delta_\varepsilon^{-1} \circ \Phi \circ \delta_\varepsilon(x), \tag{3}$$

and L is also a homeomorphism then homogeneous limit of quasimetrics exists in coordinates ϕ_p and is a quasimetric isometric to $\widehat{d_\infty}$, L being the isometry.

We stress out that condition (3) is sufficient for limit of quasimetrics but not neccessary. Moreover it is not sufficient for the limit of vector fields even if ϕ_p is a diffeomorphism.

Theorem 3. *If ϕ_p is a diffeomorphism, there is an uniform limit*

$$\lambda(x) := \lim_{\varepsilon\to 0} D\delta_{\varepsilon}^{-1} \circ D\Phi \circ D\delta_{\varepsilon}, \tag{4}$$

and $\det \lambda(0) \neq 0$ *then there are homogeneous limits of vector fields in coordinates ϕ_p and λ is an isomorphism of homogeneous Lie algebras.*

Finally, we obtain regularity requirement for these conditions to fall into classical one.

Theorem 4. *If Φ is a C^r-diffeomorphism where r is the rank of Carnot–Carathéodory space, then conditions* (2)*,* (3) *and* (4) *are equivalent.*

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Local isometric coverings of the Lie group $SO_0(2,1)$ with a sub-Riemannian metric

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Below G denotes a connected Lie group with covering epimorphism $m : G \to SO_0(2,1)$ of Lie groups; $K = m^{-1}(SO(2))$. Then the Lie algebra $\mathfrak{g}$ of the Lie group G is isomorphic to the Lie algebra $\mathfrak{so}(2,1)$ of the Lie group $SO_0(2,1)$.

Theorem 1. *Let e be the unit of G, $\{a,b,c\}$ a basis of the Lie algebra $\mathfrak{g} = \mathfrak{so}(2,1)$: $[a,b] = -c$, $[b,c] = a$, $[c,a] = b$, $K = \exp(sc)$, $s \in \mathbb{R}$; $\Delta(e)$ a vector subspace in $\mathfrak{g}$ with the basis $\{a,b\}$, $\langle\cdot,\cdot\rangle$ a scalar product on $\Delta(e)$ with the orthonormal* *basis* $\{a, b\}$. *Then the left-invariant distribution* Δ *on the Lie group* G *with given* $\Delta(e)$ *is completely nonholonomic and the pair* $(\Delta(e), \langle\cdot,\cdot\rangle)$ *defines a left-invariant and invariant relative to right shifts by elements of the subgroup* $K \subset G$ *sub-Riemannian metric* δ *on* G. *Additionally, every parameterized by arclength geodesic* $\gamma = \gamma(t)$, $t \in \mathbb{R}$, *in* (G, δ) *with condition* $\gamma(0) = e$ *is a product of two* 1*–parameter subgroups:*

$$\gamma(t) = \gamma(\beta, \phi; t) = \exp(t(\cos\phi a + \sin\phi b - \beta c)) \exp(t\beta c), \tag{1}$$

where ϕ, β *are arbitrary real constants.*

Theorem 2. *Every element* $g \in G$ *has a unique representation in the form* $g = sk$ *or* $g = k_2 s_2$, *where* $k, k_2 \in K$, $s, s_2 \in \Sigma(e) := \exp(\Delta(e))$. *Moreover,* $k = k_2$ *and* $m(sk) = m(s)m(k)$, $m(k_2 s_2) = m(k_2)m(s_2)$.

Theorem 3. *Let* $g \in G$ *and* $g = sk$, *where* $s \in \Sigma(e) := \exp(\Delta(e))$, $k \in K$, $\gamma(t) = \gamma(\beta, \phi; t)$, $0 \leq t \leq t_1 = \delta(e, g)$, *be a parameterized by arclength shortest arc in* (G, δ), *joining points* e *and* g. *Then the distance* $\delta(e, g)$ *is equal to arc length of the projection* $x(t) = pr(\gamma(t)) = p(m(\gamma(t)))$, $0 \leq t \leq t_1$, *(the curve of constant geodesic curvature* β *in* L^2, *joining points* O *and* $pr(g) = pr(s)$*) and bounding together with the rectilinear segment* $[O, pr(g)]$ *of length* $r(g)$ *in* L^2 *a digon in* L^2 *with area* $S(g)$.

Theorem 4. *Let* $g = sk$, *where* $s \in \Sigma(e) := \exp(\Delta(e))$, $k \in K$. *Then the following statements hold.*

0. *If* $k = e$ *then* $\delta(e, g) = r(g) = |\exp^{-1}(g)|$.

I. *If* $s = e$ *and* $m : G \to SO_0(2, 1)$ *is a (finite)* l*–sheeted covering, then*

$$|\beta| \geq \frac{l+2}{\sqrt{l(l+4)}}, \quad \delta(e, g) = \frac{2\pi}{\sqrt{\beta^2 - 1}} \tag{2}$$

and $|\beta|$ *is a unique number defined by the equality* $S(g) = 2\pi\Big(\frac{|\beta|}{\sqrt{\beta^2-1}} - 1\Big)$. *If* $s = e$ *and* $m : G \to SO_0(2, 1)$ *is a universal covering, then the inequality in* (2) *is replaced by the inequality* $|\beta| > 1$.

If $s \neq e$, $k \neq e$, *then the following statements are valid.*

II. *If* $S(g) = \pi\big(\operatorname{ch}\frac{r(g)}{2} - 1\big) := \theta$ *then* $|\beta| = \operatorname{cth}\frac{r(g)}{2}$, $\delta(e, g) = \pi \operatorname{sh}\frac{r(g)}{2}$. *Furthermore, if* m *is a* l*–sheeted covering, then* $|\beta| = \frac{l+1}{\sqrt{l(l+2)}}$ *only in the case* $S(g) = l\pi$; *otherwise we have* $|\beta| > \frac{l+1}{\sqrt{l(l+2)}}$.

III. *If* $S(g) > \theta$ *then* $1 < |\beta| < \operatorname{cth}\frac{r(g)}{2}$. *Furthermore, if* m *is a* l*–sheeted covering, then* $|\beta| > \frac{l+1}{\sqrt{l(l+2)}}$, *and if* $S(g) = l\pi$, *then* $|\beta| < \frac{l+2}{\sqrt{l(l+4)}}$. *Generally,*

$$\delta(e, g) = (2\pi - \gamma)/\sqrt{\beta^2 - 1},$$

where

$$S(g) = (2\pi - \gamma)\frac{|\beta|}{\sqrt{\beta^2 - 1}} - 2\arccos(-|\mu|);$$

$$\cos\gamma = \beta^2 - (\beta^2 - 1)\operatorname{ch} r(g), \quad 0 \le \gamma < \pi. \tag{3}$$

IV. *If $S(g) < \theta$ then the following statements are valid.*

a) If $S(g) = \sqrt{2(\operatorname{ch} r(g) - 1)} - 2\arccos\sqrt{\frac{2}{1+\operatorname{ch} r(g)}} := \alpha$ then

$$|\beta| = 1, \quad \delta(e, g) = \sqrt{2(\operatorname{ch} r(g) - 1)}.$$

b) If $S(g) < \alpha$ then $0 < |\beta| < 1$, $|\beta|$ is an implicit solution of equation

$$S(g) = 2\Big(\frac{|\beta|}{\sqrt{1-\beta^2}}\operatorname{arch}\Big(|\mu|\operatorname{ch}\frac{r(g)}{2}\Big) - \arccos|\mu|\Big),$$

$$\delta(e, g) = \frac{2}{\sqrt{1-\beta^2}}\operatorname{arch}\Big(|\mu|\operatorname{ch}\frac{r(g)}{2}\Big).$$

c) If $S(g) > \alpha$ then $1 < |\beta| < \operatorname{cth}\frac{r(g)}{2}$. If m is a l–sheeted covering and $S(g) = l\pi$ then $|\beta| < \frac{l+1}{\sqrt{l(l+2)}}$. Generally (3)*, $\delta(e, g) = \gamma/\sqrt{\beta^2 - 1}$, where*

$$S(g) = \gamma\frac{|\beta|}{\sqrt{\beta^2 - 1}} - 2\arccos|\mu|, \quad 0 < \gamma < \pi.$$

V. *The above-mentioned conditions uniquely define $\delta(e, g)$.*

Theorem 5. 1) *Let $g \in (G, \delta)$, $C(g)$ be the cut locus for g. Then $C(g) = gC(e)$.*

2) *If (G, δ) is a simply connected Lie group, then $C(e) = K - \{e\}$. Otherwise, $C(e) = K - \{e\} \cup k\Sigma(e)$, where $k \in K: \ k^2 = e, \ k \neq e$.*

Let $\gamma = \gamma(\beta, \psi; t)$, $0 \le t \le T$, be a noncontinuable shortest arc.

Theorem 6. I. *If $\beta = 0$ then every segment of the geodesic* (1) *is a shortest arc.*

II. *Let $m : G \to SO_0(2, 1)$ be a universal covering. Then*

1) *If $|\beta| > 1$ then $T = \frac{2\pi}{\sqrt{\beta^2-1}}$.*

2) *If $|\beta| \le 1$ then every segment of the geodesic* (1) *is a shortest arc.*

III. *Let $\beta \neq 0$ and $m : G \to SO_0(2, 1)$ be a (finite) l–sheeted covering. Then*

1) *If $|\beta| \ge \frac{l+2}{\sqrt{l(l+4)}}$ then $T = \frac{2\pi}{\sqrt{\beta^2-1}}$.*

2) *If $0 < |\beta| < 1$ then $T \in \big(\frac{l\pi}{|\beta|}, \frac{(l+1)\pi}{|\beta|}\big)$, moreover*

$$\operatorname{tg}\Big(\frac{|\beta|T}{2}\Big) = \begin{cases} \frac{|\beta|}{\sqrt{1-\beta^2}}\operatorname{th}\Big(\frac{T\sqrt{1-\beta^2}}{2}\Big), & \text{if } l \text{ is even,} \\ -\frac{\sqrt{1-\beta^2}}{|\beta|}\operatorname{cth}\Big(\frac{T\sqrt{1-\beta^2}}{2}\Big), & \text{if } l \text{ is odd.} \end{cases}$$

3) *If* $|\beta| = 1$ *then* $T \in (l\pi, (l+1)\pi)$, *moreover*

$$\operatorname{tg}\frac{T}{2} = \begin{cases} \frac{T}{2}, & \text{if } l \text{ is even}, \\ -\frac{2}{T}, & \text{if } l \text{ is odd}. \end{cases}$$

4) *If* $1 < |\beta| < \frac{l+1}{\sqrt{l(l+2)}}$ *then* $T \in \left(\frac{l\pi}{|\beta|}, \frac{(l+1)\pi}{|\beta|}\right)$, *moreover*

$$\operatorname{tg}\left(\frac{|\beta|T}{2}\right) = \begin{cases} \frac{|\beta|}{\sqrt{\beta^2-1}} \operatorname{tg}\left(\frac{T\sqrt{\beta^2-1}}{2}\right), & \text{if } l \text{ is even}, \\ -\frac{\sqrt{\beta^2-1}}{|\beta|} \operatorname{ctg}\left(\frac{T\sqrt{\beta^2-1}}{2}\right), & \text{if } l \text{ is odd}. \end{cases} \quad (4)$$

5) *If* $\frac{l+1}{\sqrt{l(l+2)}} < |\beta| < \frac{l+2}{\sqrt{l(l+4)}}$ *then* $T \in \left(\frac{(l+1)\pi}{|\beta|}, \frac{(l+2)\pi}{|\beta|}\right)$ *and* (4) *is true.*

6) *If* $|\beta| = \frac{l+1}{\sqrt{l(l+2)}}$ *then* $T = \pi\sqrt{l(l+2)}$.

Two-dimensional hypergeometric series providing solutions to algebraic functions of two variables

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In the paper we study regions of convergence of hypergeometric series providing solutions to the general tetranomial algebraic equation

$$a_0 + a_l y^l + a_m y^m + a_n y^n = 0,$$

where l, m, n are coprime integers and $l < m < n$.

By simple monoidal transformations the equation can be reduced to given kind by fixing any pair of coefficients a_p, a_q. The solutions to equations are represented as double power series in this case. We will give description of the convergence regions D_{pq} of these series as explicit functional inequalities on discriminant Δ_{pq} of the equations.

Combinatorial description of such convergence regions is done in work by Passare and Tsikh [1].

For the reduced equation denote the auxilliary pair of coefficients as a_t, a_s.

Theorem 1. *For any pair* $p, q \in \{0, l, m, n\}$ *the convergence region* D_{pq} *of series providing solution to the reduced tetranomial equation is given either by inequality*

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of one of the following types

$$\varepsilon^{\mp(t-p)(s-p)}\Delta_{pq}\Big(\pm\varepsilon^{t-p}\,|a_t|\,,\pm\varepsilon^{s-p}\,|a_s|\Big)\lessgtr 0,$$

$$\Delta_{pq}\Big(-\,|a_t|\,,-\,|a_s|\Big)>0,$$

or by two inequalities of one of the following types

$$\Delta_{pq}\Big(\varepsilon^{t-p}\,|a_t|\,,\varepsilon^{s-p}\,|a_s|\Big)\lessgtr 0,\quad \Delta_{pq}\Big(-\,|a_t|\,,-\,|a_s|\Big)\lessgtr 0,$$

$$\Delta_{pq}\Big(-\varepsilon^{t-p}\,|a_t|\,,-\varepsilon^{s-p}\,|a_s|\Big)\lessgtr 0,\quad \Delta_{pq}\Big(\varepsilon^{p-t}\,|a_t|\,,\varepsilon^{p-s}\,|a_s|\Big)\lessgtr 0,$$

where ε *is the antiderivative root of* -1 *by the power* $q-p$*.*

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Asymptotic ensemble stabilizability of the Bloch equation

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Let $I\subset\mathbb{R}$ be some interval, and consider a subset $\mathcal{E}\subset I$. For every $e\in\mathcal{E}$ we consider the dynamical system

$$\begin{pmatrix}\dot{x}_e\\ \dot{y}_e\\ \dot{z}_e\end{pmatrix}=\begin{pmatrix}0 & -e & u_2\\ e & 0 & u_1\\ -u_2 & -u_1 & 0\end{pmatrix}\begin{pmatrix}x_e\\ y_e\\ z_e\end{pmatrix}. \tag{1}$$

We are concerned with the following control problem:

(P)*Design a control function* $\mathbf{u}:[0,+\infty)\to\mathbb{R}^2$ *such that for every* $e\in\mathcal{E}$ *the solution of equation* (1) *is (asymptotically) driven to* $X_e=(0,0,-1)^T$*.*

The problem can be easily solved in the case where $\mathcal{E}$ is a finite set. For a countable set $\mathcal{E}$ of energies, a global control providing asymptotic pointwise convergence can be found. The problem is still open for a continuous set $\mathcal{E}$.

Elementary differentials of integer order on variable torus

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Let F_0 be a fixed compact Riemann surface of genus $g = 1$. The Teichmueller space $\mathbb{T}_1 = \mathbb{T}_1(F_0)$ consists of classes $[F_\mu, \{a_1^\mu, b_1^\mu\}]$ of conformally equivalent marked compact Riemann surfaces of genus one, which are parameterized by points of $H = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Here $F_\mu = \mathbb{C}/\Gamma_\mu$, where Γ_μ is generated by generators $A_{\mu 1}(z) = z + 1$, $B_{\mu 1}(z) = z + \mu$.

Definition. By q-differential ϕ with respect to group Γ on $\mathbb{C}$ call a differential $\phi(z)dz^q$ such that $\phi(Tz)(T'z)^q = \phi(z)$, $z \in \mathbb{C}$, $T \in \Gamma$.

Theorem. *On variable torus F_μ for every natural number $m > 1$, $q \in \mathbb{Z}$ there exists an elementary q-differential $\tau_{q;Q}^{(m)}$ with pole in every point $Q = Q(\mu) \in F_\mu$ exactly of order m, locally holomorphic w.r.t. μ, which has general view divisor $(\tau_{q;Q}^{(m)}) = \frac{R_1 \cdot \ldots \cdot R_m}{Q^m}$, where $\varphi(R_1) = \varphi(Q^m) - \varphi(R_2 \cdot \ldots \cdot R_m)$ and φ is Jacobi mapping from F_μ to Jacobi manifold $J(F_\mu)$. Here divisors $R_2, \ldots, R_m$ and $Q = Q(\mu)$ can be choosed as locally holomorphic sections bundle integer divisors of degrees $m - 1$ and 1 respectively over $\mathbb{T}_1$ for μ from every sufficiently small neighborhood $U(\mu_0) \subset \mathbb{T}_1$.*

Several precise solutions for a certain nonlinear equation of third order

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In the paper of B. K. Beloshapka [1] there is a nonlinear equation of the third order on $u = u(x,t)$

$$u_x \cdot u_t(u_{xxt} \cdot u_t - u_{xtt} \cdot u_x) + u_{xt}((u_x)^2 \cdot u_{tt} - (u_t)^2 \cdot u_{xx}) = 0. \tag{1}$$

Problem 1. 1) Consider equation (1) in a bar $\Pi = \{(x,t) \in \mathbb{R}^2 : x + t \in (0, y_0)\}$, $e^{-y_0} + \sqrt{2}\sin(y_0 - \frac{\pi}{4}) = 0$, $y_0 \in (\pi + \frac{\pi}{4}, \pi + \frac{\pi}{2})$. Let function vanish on a boundary of a bar

$$u|_{x+t=0} = 0, \quad u_x|_{x+t=0} = 0, \quad u_t|_{x+t=0} = 0, \quad u|_{x+t=y_0} = 0. \tag{2}$$

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Nonvanishing solution to (1),(2) is $u(x,t) = e^{-(x+t)} + \sqrt{2}\sin((x+t) - \frac{\pi}{4})$ (see Fig 1.)

2) Consider equation (1) in a square $x \in (0,\pi)$, $t \in (0,\pi)$ with vanishing boundary conditions

$$u|_{x=0} = 0, \quad u|_{x=\pi} = 0, \quad u|_{t=0} = 0, \quad u|_{t=\pi} = 0. \tag{3}$$

Nonvanishing solution to (1),(3) is the function $u(x,t) = \sin x \cdot \sin t$.

Problem 2. Consider equation (1) in $K = \{(x,t) \in \mathbb{R}^2 : x \in (0,\pi), t \in (0,1)\}$ with boundary conditions

$$u(0,t) = u(\pi,t) = u(x,0) = u(x,1) = 0. \tag{4}$$

The solution to (1),(4) is a function $u(x,t) = \sin x \cdot t \cdot \ln t$ (Fig 2.) The derivative $u_t = \sin x \cdot (\ln t + 1)$ is unbounded in K.

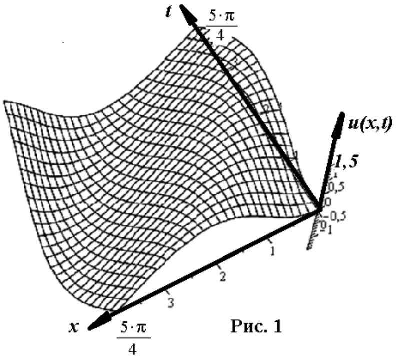

Рис. 1

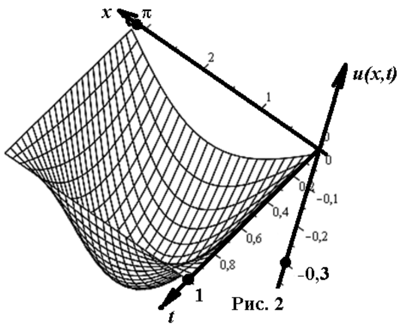

Рис. 2

Problem 3. Consider equation (1) in a half-bar $x \in (0,\pi)$, $t \in (0,\infty)$ with boundary conditions

$$u|_{x=0} = 0, \quad u|_{x=\pi} = 0, \quad u|_{t=0} = \frac{\sin(nx)}{n^5}, \quad u_t|_{t=0} = \frac{\sin(nx)}{n^4}. \tag{4}$$

When n is a natural number the solution $u(x,t) = \frac{e^{nt}\sin(nx)}{n^5}$ to (1),(4) is unstable (when $n = 2$ – Fig. 3, when $n = 20$ – Fig. 4).

Remark 1. Solutions to (1) are functions $u(x,t) = c + \tanh(d + ax + bt)$, e^{x+y}; $\ln(x+y)$; $\sin x \cdot \ln y$; $\sin x \cdot e^y$.

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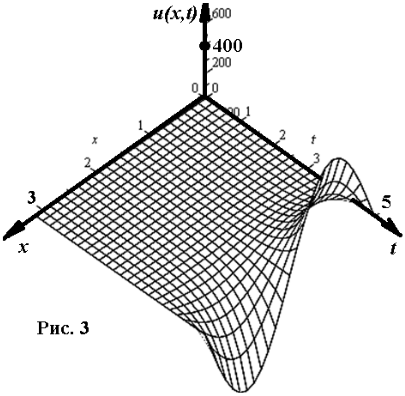

Рис. 3

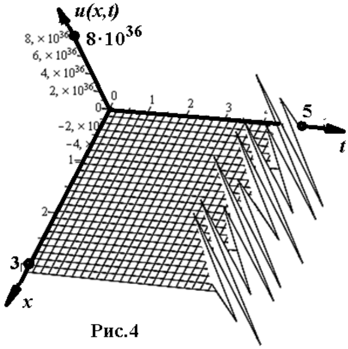

Рис.4

Canonical domains for constructing almost orthogonal quasi-isometric grids

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Let $\mathcal{D}$ be a curvilinear quadrangle with the angles $\varphi_1, \dots, \varphi_4$ $(0 < \varphi_i < \pi)$ and the conformal modulus $\mathcal{M}$. For constructing a quasi-isometric grid in $\mathcal{D}$, in [1] we use a one-parameter family $\{D_\delta\}$ of *canonical* domains with the same angles φ_i, where δ is a real-valued parameter with range from $-\infty$ to $+\infty$, and the conformal modulus $\mathcal{M}(D_\delta)$ that varies from 0 to $+\infty$ for $-\infty < \delta < +\infty$. We suppose that the boundaries of every member of $\{D_\delta\}$ are some Lyapunov manifolds. To construct a quasi-isometric grid in $\mathcal{D}$ we find a quasi-isometric mapping between $\mathcal{D}$ and an appropriate canonical domain D_{δ^*} from the one-parameter family. In [1] we prove that there exists a certain value δ^* for which the mapping from D_{δ^*} onto $\mathcal{D}$ is conformal, the vertices are mapped into the corresponding vertices, and the derivative of the mapping is bounded everywhere.

In [2] as the family $\{D_\delta\}$ we use a particular combination of geodesic quadrangles on the surfaces of constant curvature. A canonical domain D_δ is constructed of a regular rectangle by replacing the corners with four geodesic quadrangles from some particular surfaces of constant curvature. Each of these four geodesic quadrangles has three right angles, and the fourth angle is chosen to match the angle of the physical domain $\mathcal{D}$. This procedure allows us to construct such a canonical domain for every angle defect $\gamma = \sum_{i=1}^{4} \varphi_i - 2\pi$ of the domain $\mathcal{D}$ due to use either spherical or Lobachevsky geometries (for positive and negative angle defect respectively).

In this article we present the family $\{D_\delta\}$ with monotonic function $\mathcal{M}(D_\delta)$ of δ and prove that for every $\mathcal{D}$ there exists a unique δ^* such that the mapping from D_{δ^*} onto $\mathcal{D}$ is conformal and has bounded derivative. Note that this uniqueness theorem is of fundamental importance for applications.

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The module of a family of vector measures on a Riemann surface

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On a Riemann surface we consider the capacity of a condenser and the module of a family of measures associated with the class of curves connecting the condenser plates. It is proved that this quantities are equal. Earlier similar results were obtained by M. Ohtsuka and H. Aikawa [1] in $\mathbb{R}^n$.

Let $\mathcal{R}$ be a Riemann surface glued from finitely or countably many domains in the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ so that the following conditions are satisfied: each point in $\mathcal{R}$ projects onto a point $w = x + iy = \operatorname{pr} W$ in one in the glued domains, each point in $\mathcal{R}$ has a neighbourhood which is a univalent disk or a multivalent point at the centre of the disk.

The operation of projection $W \to \operatorname{pr} W = w = x + iy$ induces the Lebesgue 2-dimensional measure σ, the Hausdorff 1-dimensional measure $\mathcal{H}^1$, the Euclidean metric ds on $\mathcal{R}$.

Let G be an open set in $\mathcal{R}$ such that the closure $\overline{G}$ is a compact set in $\mathcal{R}$, $a \cdot b$ be an inner product of two vectors.

Define the p-module of a measure family Σ in G by the equality

$$M_p(\Sigma) = \inf \int_G \rho^p \, d\sigma,$$

where the infimum is taken over all nonnegative Borel functions ρ such that $\int\limits_G \rho\, d\mu \geq 1$ for all measures $\mu \in \Sigma$, $p > 1$.

If Γ is a family of locally rectifiable curves then with every curve $\gamma \in \Gamma$ one may associate a measure $\mathcal{H}^1_\gamma$ in $\mathcal{R}$ by setting $\mathcal{H}^1_\gamma(A) = \mathcal{H}^1(\gamma \cap A)$ for an arbitrary Borel set $A \subset \mathcal{R}$.

The p-module of the curve family Γ is defined as the p-module of the family of measures $\{\mathcal{H}^1_\gamma : \gamma \in \Gamma\}$. Define a vector measure $\nu = (\nu_1, \nu_2)$ where every component ν_i is a signed measure. The total variation $|\nu|$ of the vector measure ν is defined as follows. For any borel set E

$$|\nu|(E) = \sup \sum_j \Big(\sum_{k=1}^{2} \nu_k(E_j)^2 \Big)^{\frac{1}{2}},$$

where the supremum is taken over all finite partitions $\{E_j\}$ of E into Borel sets.

Let $\mathcal{F}_0$ be a set of vector measures ν. We put $|\mathcal{F}_0| = \{|\nu| : \nu \in \mathcal{F}_0\}$.

If $M_p(|\mathcal{F}_0|) = 0$ then we say that $\mathcal{F}_0$ is p-exceptional (abbreviated p-exc.). If a statement concerning vector measures ν fails to hold on by for p-exc. system $\mathcal{F}_0$ then we say that it holds p-a. e.

Let $\zeta = (\zeta_1, \zeta_2)$ be a vector-valued function and let $\nu = (\nu_1, \nu_2)$ be a vector measure defined on G. We write $\zeta \wedge \nu$ if $\int_G \zeta \cdot d\nu \geq 1$. Define the module of family of vector measures $\mathcal{F}$ by

$$M_p(\mathcal{F}) = \int_G |\zeta|^p \, d\sigma,$$

where infimum is taken over all vector-valued functions ζ such that $\zeta \wedge \nu$ for p-a. e. $\nu \in \mathcal{F}$.

A condenser in G is an ordered triple of sets (F_0, F_1, G), where F_0, F_1 are disjoint compact sets in $\overline{G}$.

Let $\Gamma = \Gamma(F_0, F_1, G)$ be a family of curves γ which are locally rectifiable and connecting F_0 and F_1 in G.

Thus for $\gamma \in \Gamma$ we have a vector measure $dw|_\gamma = (dx, dy)|_\gamma$ and a measure $ds_\gamma = |dw||_\gamma$. We write $d\Gamma = \{dw|_\gamma : \gamma \in \Gamma\}$, $|d\Gamma| = \{ds_\gamma : \gamma \in \Gamma\}$ and define the generalized capacity of condenser.

Let $D(F_0, F_1, G)$ be the family of all p-precise functions u on G such that u tends to j as $W \to F_j$ along p-a. e. curves $\gamma \in \Gamma$, $j = 0, 1$ [1]. We define

$$C_p(F_0, F_1, G) = \inf_{u \in D(F_0, F_1, G)} \int_G |\nabla u|^p \, d\sigma.$$

The following assertion holds

Theorem. $C_p(F_0, F_1, G) = M_p(d\Gamma) = M_p(|d\Gamma|)$.

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The weak limit of a sequence of mappings and the biting convergence

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The first results for mappings with finite distortion were obtained by S. K. Vodop'yanov and V. M. Gol'dshtein [1]. In [2] F. W. Gehring and T. Iwaniec proved the theorem on closure of the class of mappings with finite distortion with respect to the weak convergence in the Sobolev space $W^{1,n}_{\mathrm{loc}}(V;\mathbb{R}^n)$ (see also [3, 4]). This theorem is a refinement of Reshetnyak's convergence theorem [5, Lemma 7] concerning mappings of bounded distortion (see also [6, Theorem 9.2] and [7, Ch. 4, Lemma 1.16]). In the proving of results on stability of classes of conformal mappings Reshetnyak's convergence theorem plays the important role (see, for example, [4–8]). We consider the classes of mappings which are analogues of mappings of finite distortion. These classes are defined in the following way. Let $m, n \in \mathbb{N}$, $2 \le k \le \min\{m, n\}$. We assume that continuous functions $F : \mathbb{R}^{m\times n} \to \mathbb{R}$ and $G : \mathbb{R}^{m\times n} \to \mathbb{R}$ satisfy the following conditions [9]:

(H1) F is a quasiconvex function [10], i.e. $F(\zeta) \le \int_{(0,1)^n} F(\zeta + \varphi'(x))\,dx$ for $\varphi \in C_0^\infty((0,1)^n;\mathbb{R}^m)$ and $\zeta \in \mathbb{R}^{m\times n}$;

(H2) G is a null Lagrangian [11], i.e. G and $-G$ are quasiconvex;

(H3) F and G are positively homogeneous of degree k, i.e. $F(t\zeta) = t^k F(\zeta)$, $G(t\zeta) = t^k G(\zeta)$ for $t > 0$ and $\zeta \in \mathbb{R}^{m\times n}$;

(H4) $\sup\{K \ge 0 : F(\zeta) \ge K G(\zeta),\ \zeta \in \mathbb{R}^{m\times n}\} = 1$;

(H5) $c_F = \inf\{F(\zeta) : |\zeta| = 1\} > 0$.

Let $\mathfrak{F} = \mathfrak{F}_{F,G}$ be the class of mappings $v \in W^{1,k}_{\mathrm{loc}}(V;\mathbb{R}^m)$ (defined on domains $V \subset \mathbb{R}^n$) for which there exists a measurable function $K(x) \ge 1$, finite almost everywhere, such that $F(v'(x)) \le K(x)G(v'(x))$ a.e. in V. For $v \in \mathfrak{F}$ we define

$$K(x, v) = K_{F,G}(x, v) = \begin{cases} \frac{F(v'(x))}{G(v'(x))} & \text{if } G(v'(x)) > 0; \\ 1 & \text{otherwise.} \end{cases}$$

In [12] we proved the theorem on closure of the class $\mathfrak{F}$ with respect to the weak convergence in the Sobolev space $W^{1,k}_{\mathrm{loc}}(V;\mathbb{R}^m)$.

Theorem 1 ([12]). *Let $F : \mathbb{R}^{m\times n} \to \mathbb{R}$ and $G : \mathbb{R}^{m\times n} \to \mathbb{R}$ satisfy* (H1)–(H5) *and V be a domain in $\mathbb{R}^n$. Suppose that $(v_j : V \to \mathbb{R}^m)_{j\in\mathbb{N}}$ is a sequence of mappings of the class $\mathfrak{F} = \mathfrak{F}_{F,G}$ which converge weakly in $W^{1,k}_{\mathrm{loc}}(V;\mathbb{R}^m)$ to $v : V \to$*

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$\mathbb{R}^m$ *and suppose that* $K(x,v_j) \le M(x) < \infty$ *a.e. in* V *for* $j \in \mathbb{N}$. *Then* $v \in \mathfrak{F}$. *Moreover, for any subsequence* $(v_{j_\mu})_{\mu\in\mathbb{N}}$, *one has* $K(x,v) \le \limsup_{\mu\to\infty} K(x,v_{j_\mu})$ *a.e. in* V.

We improve the last inequality in the case F satisfies the following condition:
(H1′) $F(\zeta) = \sum_{t=1}^{s} \alpha_t |\zeta_{J_t I_t}|^k$ for some $\alpha_t > 0$, $J_t \in \Gamma_m$, $I_t \in \Gamma_n$, $t = 1,\dots,s$.

Here for $l \in \mathbb{N}$ the set $\Gamma_l := \{I = (i_1,\dots,i_r) : 1 \le i_1 < \ldots < i_r \le l,\ i_\lambda \in \{1,\dots,l\},\ \lambda = 1,\dots,r,\ r \in \{1,\dots,l\}\}$ consists of all r-tuples of ordered indices $1 \le i_1 < \ldots < i_r \le l$, $r \in \{1,\dots,l\}$, and for $J = (j_1,\dots,j_q) \in \Gamma_m$ and $I = (i_1,\dots,i_p) \in \Gamma_n$ $\zeta_{JI} := (\zeta_{j_\mu i_\nu})_{\mu=1,\dots,q;\,\nu=1,\dots,p} \in \mathbb{R}^{q\times p}$ is the submatrix of $\zeta = (\zeta_{ji})_{j=1,\dots,m;\,i=1,\dots,n} \in \mathbb{R}^{m\times n}$. Observe that (H1′) implies the convexity of F. Let h_j, $j \in \mathbb{N}$, and h be Lebesgue measurable functions defined on a measurable set $E \subset \mathbb{R}^n$. Following J. K. Brooks and R. V. Chacon [13], we say that the sequence $(h_j)_{j\in\mathbb{N}}$ converge to h in a *biting sence* if there is a sequence of measurable sets $(E_\nu)_{\nu\in\mathbb{N}}$ whose union is E, $E = \bigcup_{\nu=1}^{\infty}$, such that $h, h_j \in L^1(E_\nu)$, $j,\nu \in \mathbb{N}$, and $\lim_{j\to\infty} \int_{E_\nu} \varphi h_j = \int_{E_\nu} \varphi h$ for $\varphi \in L^\infty(E_\nu)$, $\nu \in \mathbb{N}$. Then we write $h = b * \lim_{j\to\infty} h_j$.

Theorem 2. *Let* $F : \mathbb{R}^{m\times n} \to \mathbb{R}$ *and* $G : \mathbb{R}^{m\times n} \to \mathbb{R}$ *satisfy* (H1′) *and* (H2)–(H5), *and let* $v, v_j : V \to \mathbb{R}^m$, $j \in \mathbb{N}$, *be from Theorem* 1. *Then there is a subsequence* $(v_{j_\mu})_{\mu\in\mathbb{N}}$ *such that* $K(x,v) \le b * \lim_{\mu\to\infty} K(x,v_{j_\mu})$ *a.e. in* V.

In the case $k = n = m$, $F(\zeta) = |\zeta|^n$, $G(\zeta) = \det\zeta$, the class $\mathfrak{F}_{F,G}$ cosnsists of mappings with finite distortion, and $K_{F,G}(x,v) = K_O(x,v)$ is the outer dilatation function of a mapping v. Theorems 1 and 2 for this class were obtained in [2] (see also [3, 4]). Theorems 1 and 2 are also refinements of the convergence results for some classes from [8, 9, 14–16].

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Ricci solitons on K–symmetric Lorentzian manifolds

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There are a number of generalizations of Einstein’s equation on a (pseudo) Riemannian manifold (M, g). The equation of the Ricci soliton is one of them (see reviews [1, 2]). The Ricci solitons were considered at first by R. S. Hamilton in [3] and defined by the equation $r = \Lambda g + L_X g$, where r is the Ricci tensor, Λ is a real constant and $L_X g$ is the Lie derivative of the metric g in the direction of the complete differentiable vector field X.

In general the problem of finding of the Ricci solitons is quite difficult, therefore one can consider some restrictions either on a structure of the manifold M or on the parameters of the Ricci soliton equation.

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In this paper we investigate the Ricci soliton equation on the Lorentzian k-symmetric spaces, which were studied by D. V. Alekseevsky and A. S. Galaev earlier [4, 5]. In the case of two– and three–symmetric spaces we have obtained a classification of Ricci solitons on such manifolds.

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Composition operator on Lorentz spaces

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We provide the necessary and sufficient conditions under which a measurable mapping induces a bounded composition operator on Lorentz spaces. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measurable spaces. A measurable mapping $\varphi : X \to Y$ induces a *bounded composition operator* on Lorentz spaces $C_\varphi : L_{r,s}(Y) \to L_{p,q}(X)$ by the rule $C_\varphi f = f \circ \varphi$ whenever $f \circ \varphi \in L_{p,q}(X)$ and $\|C_\varphi f\|_{L_{p,q}(X)} \leq K\|f\|_{L_{r,s}(Y)}$ for every function $f \in L_{r,s}(Y)$.

Remind the notion of Lorentz space. For a measurable function $f : X \to \mathbb{C}$ define the distribution function as $\mu_f(s) = \{x \in X : |f(x)| > s\}$. The non-increasing rearrangement of f is a function $f^*(t) = \inf\{s > 0 : \mu_f(s) \leq t\}$. The Lorentz space $L_{p,q}(X)$ is a set of all measurable functions for which

$$\|f\|_{L_{p,q}(X)} = \begin{cases} \left(\frac{q}{p}\int\limits_0^\infty (t^{\frac{1}{p}} f^*(t))^q \, \frac{dt}{t}\right)^{\frac{1}{q}}, & \text{if } 1 < p < \infty,\ 1 \leq q < \infty, \\ \sup\limits_{t>0} t^{\frac{1}{p}} f^*(t), & \text{if } 1 < p < \infty,\ q = \infty. \end{cases}$$

Theorem. *A measurable mapping $\varphi : X \to Y$ satisfying $\mathcal{N}^{-1}$-property induces a bounded composition operator $C_\varphi : L_{r,s}(Y) \to L_{p,s}(X)$, $p \leq r$ if and only if $(\mu(\varphi^{-1}(A)))^{\frac{1}{p}} \leq K(\nu(A))^{\frac{1}{r}}$ for any $A \in \mathcal{B}$.*

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Composition operators on L_p have been investigated intensively (see [1] and references herein). In the case of Lorentz spaces most of the research has been concerned with a composition operator from $L_{p,q}$ to $L_{p,q}$ (domain and image spaces have the same parameters [2]). Here we initiate the studies of a composition operator from $L_{r,s}$ to $L_{p,q}$ (with different parameters).

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Differentiation of a convolution on the roto-translation group

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The roto-translation group $\mathcal{RT}$ is the set of points in $\mathbb{R}^3$ with the group operation

$$(x_1,y_1,\theta_1)\cdot(x_2,y_2,\theta_2) = (x_1+x_2\cos\theta_1-y_2\sin\theta_1, y_1+x_2\sin\theta_1+y_2\cos\theta_1, \theta_1+\theta_2).$$

The basis left-invariant vector fields at a point $g=(x,y,\theta)\in\mathcal{RT}$ have the form

$$X_1(g) = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}, \quad X_2(g) = \frac{\partial}{\partial\theta}, \quad X_3(g) = -\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y}.$$

The following commutation relations hold:

$$[X_1,X_2] = -X_3, \quad [X_3,X_2] = X_1, \quad [X_1,X_3] = 0.$$

These equalities mean that the Lie algebra $\mathfrak{g}$ of the left-invariant vector fields of the roto-translation group is representable as $\mathfrak{g} = \mathfrak{g}_1\oplus\mathfrak{g}_2$, where $\mathfrak{g}_1 = \operatorname{span}\{X_1,X_2\}$, $\mathfrak{g}_2 = \operatorname{span}\{X_3\}$. The fields X_1, X_2 are called *horizontal*.

The group finds application, for example, in problems of the modeling of the image recognition of the human vision (see [1, 2, 3]).

Let u and v be measurable functions on $\mathcal{RT}$. Their *convolution* is defined as

$$(u*v)(g) = \int\limits_{\mathcal{RT}} u(h)v(h^{-1}g)\,d\mu(h) = \int\limits_{\mathcal{RT}} u(gh^{-1})v(h)\,d\mu(h),$$

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where $d\mu$ is a left-invariant Haar measure on $\mathcal{RT}$ (it is easy to check that we may take the usual Lebesgue measure in $\mathbb{R}^3$).

Now we are ready to state our main result.

Theorem ([4]). *Suppose that $f\colon \mathcal{RT} \to \mathbb{R}$ is a function of class C^1, and let $\psi\colon \mathcal{RT} \to \mathbb{R}$ be an infinitely differentiable function with compact support. Then there exist differential operators D_{1i}, D_{2i} such that*

$$X_i(f * \psi) = X_1 f * D_{1i}\psi + X_2 f * D_{2i}\psi, \quad i = 1, 2, 3.$$

Sometimes the explicit form of the operators D_{1i}, D_{2i} is unimportant. However, it follows from the proof of the theorem that

$$D_{11} = \cos\theta \,\mathrm{Id} - \sin\theta \tfrac{\partial}{\partial x},$$

$$D_{21} = \cos\theta \,\mathrm{Id} + (y\sin\theta\cos\theta - x\sin^2\theta)\tfrac{\partial}{\partial x} - (x\sin\theta\cos\theta + y\sin^2\theta)\tfrac{\partial}{\partial y} + \sin\theta\tfrac{\partial}{\partial\theta},$$

$$D_{12} = (x\sin 2\theta - y\cos 2\theta + 2x\sin\theta - 2y\cos\theta)\,\mathrm{Id} - \left(\tfrac{1}{2}(y^2 - x^2)\sin 2\theta + xy\cos 2\theta\right)\tfrac{\partial}{\partial x} + (x\cos\theta + y\sin\theta)^2\tfrac{\partial}{\partial y},$$

$$D_{22} = (1 + \cos\theta)\,\mathrm{Id} + (x\cos\theta + y\sin\theta)\tfrac{\partial}{\partial x},$$

$$D_{13} = -2\sin\theta\,\mathrm{Id} + (y\cos^2\theta - x\sin\theta\cos\theta)\tfrac{\partial}{\partial x} - (x\cos^2\theta + y\sin\theta\cos\theta)\tfrac{\partial}{\partial y} + \cos\theta\tfrac{\partial}{\partial\theta},$$

$$D_{23} = -\cos\theta\tfrac{\partial}{\partial x},$$

where Id is the identity operator.

The theorem allows us to express the derivatives of a convolution in terms of only the horizontal derivatives. For the class of homogeneous nilpotent Lie groups, similar relations were established by D. Jerison [5]. His principal idea is to find, for a vector field X, a vector field Y so that $Xu * v = u * Yv$. Inspired by this idea, we formulate Lemma 1 below. The roto-translation group does not belong to the class mentioned above, and therefore we cannot carry over Jerison's reasoning, which rests largely on homogeneity and nilpotency, to the group $\mathcal{RT}$. We perform straightforward calculations that turn out to be successful because of the specific form of the fields X_1, X_2, X_3.

The following lemma plays a crucial role in the proof of Theorem 1.

Lemma ([4]). *Suppose that $f\colon \mathcal{RT} \to \mathbb{R}$ is a function of class C^1, and let $\psi\colon \mathcal{RT} \to \mathbb{R}$ be an infinitely differentiable function with compact support. Then $X_i f * \psi = f * \widetilde{X}_i\psi$, $i = 1, 2, 3$, where*

$$\widetilde{X}_1 = \frac{\partial}{\partial x}, \quad \widetilde{X}_2 = (y\cos\theta - x\sin\theta)\frac{\partial}{\partial x} - (x\cos\theta + y\sin\theta)\frac{\partial}{\partial y} + \frac{\partial}{\partial\theta}, \quad \widetilde{X}_3 = \frac{\partial}{\partial y}.$$

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On existence and stability of a cycle of one piecewise linear dynamical system

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Let the step functions L, Γ, Π be defined as follows: $\mathrm{L}(u) = A_1 > 2$ for $0 \le u \le 1$, $\mathrm{L}(u) = 0$ for $1 < u$; such functions describe negative feedbacks in mathematical models of gene network functioning. Positive feedbacks in these models are described by increasing function $\Gamma(u) = 0$ for $0 \le u \le 1$, and $\Gamma(u) = A_2 > 2$ if $1 < u$.

The step-function $\Pi(u) = 0$ for $0 \le u < 1$ and for $1 < \beta < u$; $\Pi(u) = A_3 > 2$ for $1 \le u \le \beta$, describes a variable feedback in the hypothetical gene network model studied below.

The right hand sides of the following dissipative dynamical system compose a piecewise linear vector field.

$$\frac{dx_1}{dt} = \mathrm{L}(x_3) - k_1 x_1; \quad \frac{dx_2}{dt} = \Gamma(x_1) - k_2 x_2; \quad \frac{dx_3}{dt} = \Pi(x_2) - k_3 x_3. \qquad (\mathrm{L}\Gamma\Pi)$$

Geometry of its phase portrait is the main object of our considerations. Here, the variables $x_j \ge 0$ denote concentrations of interacting components in the model, the step functions L, Γ, Π describe velocities of synthesis of these components, and the coefficients $k_j > 0$ correspond to velocities of their natural

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degradation. Just for simplicity of exposition, we consider here the case $k_j = 1$ for all $j = 1, 2, 3$.

We show that the parallelepiped $Q = [0, A_1] \times [0, A_2] \times [0, A_3]$ is an invariant domain of the system (LΓΠ). The planes $x_1 = 1$; $x_2 = 1$; $x_2 = \beta$; $x_3 = 1$ divide this domain Q to 12 smaller parallelepipeds (blocks) which we denote by indices $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$:

$\varepsilon_1 = 0$ for $x_1 < 1$ and $\varepsilon_1 = 1$ for $x_1 \geq 1$;

$\varepsilon_3 = 0$ for $x_3 < 1$ and $\varepsilon_3 = 1$ for $x_3 \geq 1$;

$\varepsilon_2 = 0$ for $x_2 < 1$, $\varepsilon_2 = 1$ for $\beta \geq x_2 \geq 1$, $\varepsilon_2 = 2$ for $\beta < x_2$.

Lemma 1. *In each block $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ trajectories of the system* (LΓΠ) *are straight line segments, and all straight lines containing these trajectories intersect in one point $P(\varepsilon_1, \varepsilon_2, \varepsilon_3)$.*

In each block $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ except $\{1, 2, 0\}$, we have $P(\varepsilon_1, \varepsilon_2, \varepsilon_3) \notin \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$. The block $\{1, 2, 0\}$ contains the attractor of the system (L, Γ, Π), this is the point with coordinates $(A_1, A_2, 0) = P(1, 2, 0)$, and trajectories of all other points of $\{1, 2, 0\}$ remain in this block for all $t > 0$.

Behavior of most trajectories of the system (LΓΠ), which are not attracted to $P(1, 2, 0)$ is represented by the diagram

$$\begin{array}{ccccccc}
\{1,0,0\} & \xrightarrow{\varphi_3} & \{1,1,0\} & \xrightarrow{\varphi_4} & \{1,1,1\} & \xrightarrow{\psi_1} & \{1,2,1\} \\
\uparrow \varphi_2 & & & & \downarrow \varphi_5 & & \downarrow \psi_2 \\
\{0,0,0\} & \xleftarrow{\varphi_1} & \{0,0,1\} & \xleftarrow{\varphi_6} & \{0,1,1\} & \xleftarrow{\psi_3} & \{0,2,1\}
\end{array} \qquad \text{(D)}$$

Let U_6 be the union of six blocks listed in the left side of this diagram (D), and U_8 be the union of all blocks of this diagram. Denote by Φ_8 the composition $\psi_3\psi_2\psi_1\varphi_4\varphi_3\varphi_2\varphi_1\varphi_6$. We have constructed a convex pentagon $H \subset F = \{0, 0, 1\} \cap \{0, 1, 1\}$ such that $\Phi_8(H) \subset H$, this is a projective transformation of H into itself, and it can be expressed explicitly.

In order to give qualitative description of the phase portrait of the system (LΓΠ), we consider below and in our numerical experiments the case $A_1 = A_2 = A_3 = 3$; $\beta = 1,8$.

Lemma 2. *The dynamical system* (LΓΠ) *does not have cycles in the domain U_6.*

Lemma 3. *In the domain U_8, the dynamical system* (LΓΠ) *does not have cycles which enclose after m, $m \geq 2$, rounds according to the diagram* (D).

Theorem 1. *The system* (LΓΠ) *has in the domain U_8 a unique stable cycle $\mathcal{C}$ which encloses after one round according to* (D).

Linearization matrix of the Poincaré map $\Phi_8 : H \to H$ at the intersection point $H \cap \mathcal{C}$ has complex eigenvalues with negative real parts.

Starting from the polygons F and H, we have constructed in the domain U_8 an invariant piecewise linear surface of the system (LΓΠ). This surface is composed by six triangles with common vertex $(1, 1, 1)$ located in the domain U_6, and infinite sequence of "narrow" quadrangles q_n which tend to the edges of the piecewise linear cycle $\mathcal{C}$ as $n \to \infty$. Each of these triangles and quadrangles is contained in corresponding block of the diagram (D).

Similar results for analogous, more simple dynamical systems of the type (LLL) and (LΓΓ), including the cases of smooth analogues of the step functions considered here, were obtained earlier in [1, 3, 4]. Biological interpretations of these results concerning negative, positive and variable feedbacks are described in [2, 3], see also references therein.

Numerous numerical experiments realized in the frame of this work used methods implemented in the package *deSolve*[*] of the programming language R[†].

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Patalogical solutions of the Euler–Lagrange equation and existence/regularity of minimizers in one-dimensional variational problems[‡]

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Let us consider classical one-dimensional problems

[*] https://cran.r-project.org/web/packages/deSolve/index.html

[†] https://www.r-project.org

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$$J(u) = \int_a^b L(x, u(x), \dot{u}(x))dx \to \min, \quad u(a) = A, \ u(b) = B. \tag{1}$$

We assume that $L(x, u, v) : \mathbb{R}^3 \to \mathbb{R}$ is of class C^3 and $L_{vv}(x, u, v) > 0$ everywhere. These assumptions on the integrand L will be regarded as basic throughout.

In the case when the solution $u : [a, b] \to \mathbb{R}$ is Lipschitz and L satisfies the basic assumptions we have that $u \in C^3[a, b]$ and satisfies the Euler equation (see [1])

$$\ddot{u} = \frac{L_u - L_{xv} - L_{uv}\dot{u}}{L_{vv}}. \tag{2}$$

The function $u : [a, b] \to \mathbb{R}$ is called a pathological solution of the Euler equation (2) on the interval $[a, b]$ if there exists the interval $[c, d) \subset [a, b]$, with possibility $d < c$, such that $u \in C^3[c, d)$ and satisfies the equation (2) in $[c, d)$, $\dot{u}(x) \to \infty$ as $x \to d$, $\|u(x)\|_{C[c,d]} < \infty$.

The following theorems are valid.

Theorem 1. *Let L satisfy the basic assumptions and*

$$L(x, u, v) \geq \alpha|v| + \beta, \quad \alpha > 0, \quad \beta \in \mathbb{R}.$$

Assume also that there are no pathological solutions of the Euler equation (2) *on the interval* $[a, b]$ *which satisfy the condition*

$$\dot{u}(c) = (B - A)/(b - a).$$

Then each problem (1), (2) *admits a solution in $W^{1,1}$-class and all such solutions are equi-bounded in C^3-norm.*

Theorem 2. *Let L satisfy the basic assumptions. Assume also that there are no pathological solutions of the Euler equation* (2) *on the interval* $[a, b]$ *which satisfy the condition*

$$\dot{u}(c) = (B - A)/(b - a).$$

Then there are no Lavrentiev phenomena in the problem (1), (2), *i.e.*

$$I_1 = I_\infty,$$

where

$$I_1 = inf\{J(u) \,:\, u(a) = A, u(b) = B, u \in W^{1,1}[a, b]\},$$
$$I_\infty = inf\{J(u) \,:\, u(a) = A, u(b) = B, u \in W^{1,\infty}[a, b]\}.$$

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Some properties of (q_1, q_2)-quasimetric spaces

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A pair (X, d), where X is a set, $d : X \times X \to \mathbb{R}^+ \cup \{0\}$, is called (q_1, q_2)-quasimetric space, and d is called quasimetric, if

– d is a distance function, i. e. $d(x, y) = 0 \Leftrightarrow x = y$,

– $d(x, z) \le q_1 d(x, y) + q_2 d(y, z)$ $\forall x, y, z \in X$ for some constants $q_1, q_2 > 0$ (generalized (q_1, q_2)-triangle inequality).

If (q_1, q_2)-quasimetric d satisfies q_0-symmetry condition, i. e. $d(x, y) \le q_0 d(y, x)$ $\forall x, y \in X$ for some constant $q_0 > 0$ then d is called q_0-symmetric (q_1, q_2)-quasimetric.

Let $B(x, r) = \{y \in X \mid d(x, y) < r\}$. Define open set in (X, d) as follows. A set $Y \subset X$ is called open if for every $y \in Y$ there is a number $\varepsilon > 0$ such that $B(y, \varepsilon) \subset Y$. Open sets define a topology τ in (X, d).

1-symmetric distance function d^s is called symmetrization of d if d^s defines the same topology τ in (X, d).

Properties of topology τ and symmetrizations were studied in [1] in more general situations recently.

In our talk we discuss some properties of (q_1, q_2)-quasimetric spaces, in particular, symmetrizations of (q_1, q_2)-quasimetric spaces.

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Classification of biconservative hypersurfaces

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The classification of constant mean curvature (CMC) hypersurfaces are associated with the problem of eigenvalues of the shape operator or differential equations arises from Laplacian operator. We obtain some classification of biconservative hypersurfaces in non-flat Lorentz space forms.

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Formalization of inverse problems and applications to systems of equations with parameters

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We show how binary correspondences can be used for simple formalization of the notion of problem, definition of the basic components of problems, their properties, and constructions (the condition of a problem, its data and unknowns, solvability and unique solvability of a problem, inverse problem, composition and restriction of problems, etc.). As an illustration, we consider a system of differential equations which describe a process in chemical kinetics, as well as the inverse problem.

Since abstracts are restricted in length, we present only four most basic definitions. In particular, we do not mention the notion of topological problem and the related notions of stability and correctness. We also omit the definition of parametrization and the corresponding topological aspects.

Definition 1. By a *problem* we mean an arbitrary correspondence between the elements of two sets, i.e., a triple $P = (A, B, C)$, where A and B are any sets and $C \subseteq A \times B$. The sets A, B, and C are denoted by $\operatorname{Dom} P$, $\operatorname{Im} P$, and $\operatorname{Gr} P$ and called *the domain of data*, *the domain of unknowns*, and *the condition* of the problem P. The containment $(a, b) \in \operatorname{Gr} P$ is written as $P(a, b)$.

Definition 2. A *solution* to a problem P for a data instance $a \in \operatorname{Dom} P$ is an arbitrary unknown $b \in \operatorname{Im} P$ which meets the condition $P(a, b)$. The set of solutions to P for a is denoted by $P[a]$. A problem P is *solvable* for $a \in \operatorname{Dom} P$ whenever $P[a] \neq \varnothing$, and *uniquely solvable* if $P[a] = \{b\}$ for some $b \in \operatorname{Im} P$, with the corresponding solution b denoted by $P^{\mathrm{s}}(a)$.

Definition 3. The *inverse problem* to P is the inverse correspondence

$$P^{-1} := \big(\operatorname{Im} P, \operatorname{Dom} P, (\operatorname{Gr} P)^{-1}\big), \quad \text{where } (\operatorname{Gr} P)^{-1} = \{(b, a) : (a, b) \in \operatorname{Gr} P\}.$$

Definition 4. The *composition* of problems P and Q is the composition of the correspondences, which is the problem

$$Q \circ P := (\operatorname{Dom} P, \operatorname{Im} Q, \operatorname{Gr} Q \circ \operatorname{Gr} P)$$

with condition

$$\operatorname{Gr} Q \circ \operatorname{Gr} P = \big\{(a, c) \in \operatorname{Dom} P \times \operatorname{Im} Q : (\exists\, b \in \operatorname{Im} P \cap \operatorname{Dom} Q)\ P(a, b)\ \&\ Q(b, c)\big\}.$$

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As an illustration, we consider a singularly perturbed system of ordinary differential equations which arises in modeling certain processes of chemical kinetics.

Suppose that $n, m \in \mathbb{N}$, $0 < \varepsilon_0 \in \mathbb{R}$, $X := \mathbb{R}^m$, Y is a domain in $\mathbb{R}^n$, $T := \mathbb{R}$, $E := \{\varepsilon \in \mathbb{R} : 0 \leqslant \varepsilon \leqslant \varepsilon_0\}$, $F := C(X \times Y \times T \times E, \mathbb{R}^m)$, $G := C(X \times Y \times T \times E, \mathbb{R}^n)$. Consider the problem P with domain of data $\operatorname{Dom} P = F \times G \times E$, domain of unknowns $\operatorname{Im} P = C^1(T, X) \times C^1(T, Y)$, and condition

$$P\big((f, g, \varepsilon), (x, y)\big) \;\Leftrightarrow\; \begin{cases} \;\dot{x}(t) = f\big(x(t), y(t), t, \varepsilon\big), \\ \varepsilon\, \dot{y}(t) = g\big(x(t), y(t), t, \varepsilon\big) \end{cases} \text{for all } t \in T,$$

where $f \in F$, $g \in G$, $\varepsilon \in E$, $x \in C^1(T, X)$, $y \in C^1(T, Y)$.

Solution of the problem P is based on the method of integral manifolds, a convenient tool for studying multidimensional singularly perturbed systems of differential equations which makes it possible to lower the dimension of the system under study (see [1, 2, 3]). Solution of P in a sense reduces to solving the so called degenerate system which is obtained from the initial system by putting the parameter ε equal to zero. This is justified by the results of A. N. Tikhonov (see, for instance, [4]) on passing to a solution to the degenerate problem as a small parameter tends to zero.

The inverse problem to P consists in finding the unknown functions on the right-hand side of the system, given some data on the solution to the direct problem P. Relying on the close connection of the initial problem with the degenerate system, we consider the case $\varepsilon = 0$ and additionally assume that the "slow surface" defined by the equation $g\big(x, y, t, 0\big) = 0$ consists of a single sheet (with respect to the dependence of y on x). Since the right-hand parts of equations in chemical kinetics often occur polynomial, the corresponding restriction on f seems to be natural.

Therefore, for demonstration purposes, we consider the partial case of the problem P in which $m = n = 1$, $E = \{0\}$, the functions $f \in F$ are polynomials of degree one, and $g \in G$ meet the condition of the implicit function theorem, which fact allows us to replace the equation $g\big(x(t), y(t), t, 0\big) = 0$ by the equivalent equation of the form $y(t) = h\big(x(t), t\big)$.

Let $h \in C^1(\mathbb{R}^2)$. Consider the problem Q with domain of data $\operatorname{Dom} Q = \mathbb{R}^3$, domain of unknowns $\operatorname{Im} Q = C^1(\mathbb{R})^2$, and condition

$$Q\big(f, (x, y)\big) \;\Leftrightarrow\; \begin{cases} \dot{x}(t) = f_1 + f_2 x(t) + f_3 y(t), \\ y(t) = h\big(x(t), t\big) \end{cases} \text{for all } t \in \mathbb{R},$$

where $f = (f_1, f_2, f_3) \in \mathbb{R}^3$, $x, y \in C^1(\mathbb{R})$.

The formal inverse problem Q^{-1}, which has pairs of functions $(x, y) \in C^1(\mathbb{R})^2$ as data, is very simple and impractical. Finite collections of values of functions or their derivatives as data are more adequate than everywhere defined functions. The corresponding correction of the inverse problem is realized by the composition of the problem Q^{-1} and the auxiliary problem R with domain of data $\operatorname{Dom} R = (\mathbb{R}^3)^3$, domain of unknowns $\operatorname{Im} R = C^1(\mathbb{R})^2$, and condition

$$R\big((t,\alpha,\beta),(x,y)\big) \;\Leftrightarrow\; \begin{cases} x(t_1) = \alpha_1,\ x(t_2) = \alpha_2,\ x(t_3) = \alpha_3, \\ \dot{x}(t_1) = \beta_1,\ \dot{x}(t_2) = \beta_2,\ \dot{x}(t_3) = \beta_3, \end{cases}$$

where $t, \alpha, \beta \in \mathbb{R}^3$, $x, y \in C^1(\mathbb{R})$.

Theorem 1. *If $t, \alpha \in \mathbb{R}^3$ meet the condition*

$$\Delta := \begin{vmatrix} 1 & \alpha_1 & h(\alpha_1, t_1) \\ 1 & \alpha_2 & h(\alpha_2, t_2) \\ 1 & \alpha_3 & h(\alpha_3, t_3) \end{vmatrix} \neq 0 \tag{1}$$

then, given arbitrary $\beta \in \mathbb{R}^3$, the problem $Q^{-1} \circ R$ is uniquely solvable for the data (t, α, β), and its solution $(f_1, f_2, f_3) = (Q^{-1} \circ R)^{\mathrm{s}}(t, \alpha, \beta)$ can be calculated by Cramer's formulas $f_i = \Delta_i/\Delta$, where Δ_i is the determinant of the matrix formed from the matrix in (1) by replacing the i th column by $\beta = (\beta_1, \beta_2, \beta_3)$.

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Nonclosed Archimedean cones

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A *wedge* is a nonempty convex subset K of a vector space (henceforth, vector spaces are assumed to be defined over the field of reals) which meets the condition $(\forall\, \alpha \geqslant 0)(\alpha K \subseteq K)$. A wedge K is a *cone* whenever $K \cap (-K) = \{0\}$. As is known, in every (pre)ordered vector space $(X, \leqslant)$ the set $\{x \in X : x \geqslant 0\}$ is a cone (wedge). Conversely, if a cone (wedge) K is fixed in a vector space X

then the order $\leqslant_K$ defined by the rule $x \leqslant_K y \Leftrightarrow y - x \in K$ makes X into a (pre)ordered vector space.

A (pre)ordered vector space $(X, \leqslant)$ is *Archimedean* if for all $x, y \in X$ $(y \geqslant 0)$ the condition $(\forall\, n \in \mathbb{N})(x \leqslant \frac{1}{n}y)$ implies $x \leqslant 0$. A cone (wedge) K in a vector space X is *Archimedean* if so is the (pre)ordered space $(X, \leqslant_K)$. Generalize the notion of Archimedean wedge as follows: a convex set $C \subseteq X$ is *Archimedean* whenever for all $x, y \in X$ the condition $(\forall\, n \in \mathbb{N})(x + \frac{1}{n}y \in C)$ implies $x \in C$. It is easy to see that, for wedges, the latter is equivalent to the above definition.

The main information on Archimedean cones can be found in [1].

The following proposition clarifies the notion of Archimedean set.

Proposition 1. *Given a convex set C in a vector space X, the following are equivalent:*

(a) *C is Archimedean;*

(b) *for all $x, y \in X$ the condition $(\exists\, \varepsilon > 0)(x +]0, \varepsilon] y \subseteq C)$ implies $x \in C$;*

(c) *the complement $X \setminus C$ coincides with its algebraic interior;*

(d) *the intersection of C with every line is closed;*

(e) *the intersection of C with every finite-dimensional subspace of X is closed.*

Note that every convex set which is sequentially closed in any vector topology is obviously Archimedean. In order to give another description of Archimedean sets, we introduce an auxiliary notion: A topological vector space is *sequentially total* if all linear functionals on it are sequentially continuous.

Lemma 1. *Let X be a sequentially total space and let $C \subseteq X$ be a convex set. Then C is Archimedean if and only if C is sequentially closed.*

The problem is of interest of describing the topological vector spaces which include nonclosed Archimedean cones. In what follows, we present the main results obtained on the way to solving this problem as well as its variations (in the statement, cones can be replaced by wedges, while closedness by sequential closedness). It is immediate that, in finite-dimensional spaces, all Archimedean wedges (moreover, all Archimedean convex sets) are closed.

A criterion is rather easily obtained for existence of Archimedean cones or wedges which are not sequentially closed.

Lemma 2. *Given a topological vector space X, the following are equivalent:*

(a) *X is sequentially total;*

(b) *every Archimedean wedge in X is sequentially closed;*

(c) *every Archimedean cone in X is sequentially closed.*

The main problem is solved for a wide class of topological vector spaces of uncountable dimension.

Theorem 1. *Every locally convex space of uncountable dimension includes a nonclosed Archimedean cone.*

For spaces of countable dimension, we currently succeeded only in obtaining a criterion for existence of a nonclosed Archimedean wedge.

Theorem 2. *A topological vector space X of countable dimension includes a nonclosed Archimedean wedge if and only if there is a discontinuous linear functional on X.*

The following theorems specify the boundaries of the class of spaces of countable dimension which include nonclosed Archimedean cones.

Theorem 3. *If a topological vector space contains a nonclosed linearly independent set, then it contains a nonclosed Archimedean cone.*

Theorem 4. *If X is a topological vector space of countable dimension and all linear functionals on X are continuous, then all Archimedean convex sets in X are closed.*

Hypothesis 1. A topological vector space X of countable dimension includes a nonclosed Archimedean cone if and only if there is a discontinuous linear functional on X.

Hypothesis 2. A topological vector space X of countable dimension includes a nonclosed Archimedean cone if and only if X includes a nonclosed linearly independent set.

The above hypotheses are not equivalent: there are examples of topological vector spaces of countable dimension in which all linearly independent sets are closed, while not all linear functionals are continuous.

The main results of the research are published in [2].

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Quasiconformal mappings on 3-dimensional manifolds with sub-Riemannian metrics

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We study quasiconformal analysis problems on some three dimensional Lie groups with sub-Riemannian metrics. We describe the group of sub-Riemannian isometries and 1-quasiconformal mappings. We show also that the set of quasiconformal mappings is infinite-dimensional. At the moment, only the case of Heisenberg group is well studied. The complete classification of left-invariant sub-Riemannian structures on three dimensional Lie groups in terms of the basic differential invariants is given in [1].

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Optimal control on parallelizable Riemannian manifolds

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The tangent bundle of the Lie group SE(3) as a parallelizable Riemannian manifold is used for working on an optimal control problem, the trajectory planning for air-vehicles. However, what makes hypersonic trajectory planning different from other air-vehicles is that the trajectory planning for hypersonic aircrafts must be done for both the attitude and position variables at the same time since the usual separation of time-scales assumption is practical for air-vehicles other than hypersonic aircrafts. An air-vehicle can be thought of as evolving on the tangent bundle of the Lie Group SE(3) with a Riemannian metric that is obtained from the total kinetic energy. For an optimal control problem on the tangent bundle of such a manifold, the frame coordinate is used to obtain first-order necessary conditions by employing calculus of variations. Quantities like Levi-Civita connection and Riemannian curvature tensor appear in the equations due to using of frame coordinates.

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The resulting equations using frame coordinates are singularity-free and, in comparison with what obtain through local coordinates representation, are by far simpler from a numerical perspective.

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Stability of Jordan zippers with an alternating signature

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Let $\{z_0, \ldots, z_m\}$ be the set of points in $\mathbb{R}^n$. A system $\mathcal{S} = \{S_1, \ldots, S_m\}$ of contracting similarities in $\mathbb{R}^n$ is called a self-similar zipper with vertices $\{z_0, \ldots, z_m\}$ and signature $\bar{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_m)$; $\varepsilon_k = 0, 1$, if for every $k = 1, \ldots, m$, $S_k((z_0, z_m)) = (z_{k-1+\varepsilon_k}, z_{k-\varepsilon_k})$. A zipper $\mathcal{S}$ is said to be Jordan, if its attractor γ is a Jordan arc with endpoints z_0, z_m, and for every $k = 2, \ldots, m$, $S_{k-1}(\gamma) \cap S_k(\gamma) = \{z_k\}$ [1].

In general, the property of zipper to be Jordan does not hold with small change of its parameters, so it's important to clarify the conditions under which this property is stable. One approach to solve this problem is based on Theorem of general position for fractal continuum [2]. We prove the following result for zippers with alternating signature:

Theorem 1. *Let Σ be the family of self-similar zippers in $\mathbb{R}^3$ with vertices $z_0 = \bar{0}, \ldots, z_m = \bar{e}_1$ and alternating signature $\bar{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_m)$, satisfying the following conditions:*

1. For every k, $q_k = \operatorname{Lip} S_k < 1/3$;

2. There exists an open set $W \subset V_{1/3}([z_0, z_m])$ such that

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a) for every k, $S_k(W) \subset W$ and

b) for every $k = 2, ..., m-1$, $S_k(W) \cap (V_{q_1/2}(z_0) \cup V_{q_m/2}(z_m)) = \varnothing$;

3. Similarity dimension of the system $\mathcal{S}$ is less than 3/2.

Then the subfamily $\Sigma' \subset \Sigma$ of Jordan zippers is open and dense in Σ.

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Metric properties of some Hölder mappings on Carnot groups

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In the research, we prove polynomial sub-Riemannian differentiability of classes of Hölder mappings on Carnot groups, describe intrinsic bases and deduce area formulas. It is well-known that mappings smooth in the classical sense are generally Hölder with respect to sub-Riemannian quasimetrics. Besides of this, if we consider a graph-mapping constructed with use of an intrinsically Lipschitz mapping, it will also belong to Hölder class only.

The main classes of mappings under consideration are smooth (non-contact) mappings, graphs of these mappings and graphs of Lipschitz mappings defined on nilpotent homogeneous groups or their particular case, Carnot groups. Let us describe our results on the example of graph-mapping of a Lipschitz mapping of Carnot groups.

Theorem 1 ([1, 2]). *Suppose that $\mathbb{G}$ is a Carnot group, $\widetilde{\mathbb{G}}$ and $\mathbb{U}$ are nilpotent homogeneous groups, $\mathbb{G}, \widetilde{\mathbb{G}} \subset \mathbb{U}$, and $\varphi : \Omega \to \widetilde{\mathbb{G}}$, where $\Omega \subset \mathbb{G}$. A graph-mapping $\varphi_\Gamma : \Omega \to \mathbb{U}$ is polynomially sub-Riemannian differentiable at the points of hc-differentiability of φ, and polynomial hc-differential $\widehat{D}_P\varphi_\Gamma(x)$ assigns to each y the collection $(\Delta_1(y,x), \ldots, \Delta_N(y,x), \widetilde{\Delta}_1(y,x), \ldots, \widetilde{\Delta}_N(y,x))$, where*

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$$\begin{cases} \Delta_i(y,x) = & y_i + \sum\limits_{\substack{|\kappa+\mu+\lambda|_h=\deg X_i,\\ \kappa,\lambda>0}} K^{X_i}_{\kappa,\mu,\lambda} y^\kappa(-\varphi(x))^\mu(\varphi(x))^\lambda \\ & + \sum\limits_{\substack{|\alpha+\beta+\gamma+\tau|_h=\deg X_i,\\ \alpha,\tau>0}} L^{X_i}_{\alpha,\beta,\gamma,\tau}(-\varphi(x))^\beta(\varphi(x))^\gamma y^\alpha(\widehat{D}\varphi(x)\langle y\rangle)^\tau, \\ \widetilde{\Delta}_j(y,x) = & (\widehat{D}\varphi(x)\langle y\rangle)_j + \sum\limits_{\substack{|\kappa+\mu+\lambda|_h=\deg \widetilde{X}_j,\\ \kappa,\lambda>0}} K^{\widetilde{X}_j}_{\kappa,\mu,\lambda} y^\kappa(-\varphi(x))^\mu(\varphi(x))^\lambda \\ & + \sum\limits_{\substack{|\alpha+\beta+\gamma+\tau|_h=\deg \widetilde{X}_j,\\ \alpha,\tau>0}} L^{\widetilde{X}_j}_{\alpha,\beta,\gamma,\tau}(-\varphi(x))^\beta(\varphi(x))^\gamma y^\alpha(\widehat{D}\varphi(x)\langle y\rangle)^\tau, \end{cases}$$

and $K^{X_i}_{\kappa,\mu,\lambda}$, $L^{X_i}_{\alpha,\beta,\gamma,\tau}$, $K^{\widetilde{X}_j}_{\kappa,\mu,\lambda}$ *and* $L^{\widetilde{X}_j}_{\alpha,\beta,\gamma,\tau}$ *are constants for all multi-indices* κ, μ, λ, α, β, γ, τ, $i=1,\dots,N$, $j=1,\dots,\widetilde{N}$.

Theorem 2 ([1, 3]). *Under the conditions of Theorem 1, let* $\varphi:\Omega\to\widetilde{\mathbb{G}}$ *be a contact mapping which is* $C^{\left[\frac{\max\{M,\widetilde{M}\}+p-1}{p}\right]}$*-smooth in variables of degree* p, $p=1,\dots,M$. *Then for each* $x\in\Omega$ *there exists a basis* $\{{}^xX_1,\dots,{}^xX_N, {}^x\widetilde{X}_1,\dots,{}^x\widetilde{X}_{\widetilde{N}}\}$ *such that the set*

$$(\widetilde{\theta}_{\varphi_\Gamma(x)}\circ\theta^{-1}_{\varphi_\Gamma(x)})\big(D_{\mathrm{diag}}\big(\widehat{D}_P\varphi_\Gamma(x)\big)(x)\big\langle \mathrm{Box}(x,r)\big\rangle\big),$$

coincides with $\widehat{D}_P\varphi_\Gamma(x)\langle\mathrm{Box}(x,r)\rangle$. *Here* $\theta_{\varphi_\Gamma(x)}$ *is a normal coordinate mapping in the initial basis, and* $\widetilde{\theta}_{\varphi_\Gamma(x)}$ *is a normal coordinate mapping in the new basis.*

Theorem 3 ([4]). *Under the conditions of Theorem 2, the following area formula for the graph-surface* $\varphi_\Gamma(\Omega)$ *holds:*

$$\int\limits_\Omega \mathcal{J}(\widehat{D}\varphi,E_N,x)\,d\mathcal{H}^\nu(x) = \int\limits_{\varphi_\Gamma(\Omega)} d\mathcal{H}^\nu_\Gamma(y),$$

where $\mathcal{J}(\widehat{D}\varphi,E_N,x)$ *is the Jacobian depending on* $\widehat{D}\varphi(x)$ *and identity* N*-matrix* E_N, *and* $\mathcal{H}^\nu_\Gamma$ *is the intrinsic surface measure on* $\varphi_\Gamma(\Omega)$.

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Ricci solitons on conformally flat metric Lie groups

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The works of many mathematicians are devoted to investigation of Einstein manifolds (see reviews [1, 2]). In recent years, different generalizations of such manifolds are studied, and one of them are Ricci solitons, which were first considered by R. S. Hamilton in [3] and are defined by the equation

$$r = \Lambda g + L_X g,$$

where r is the Ricci tensor, Λ is a real constant and $L_X g$ is the Lie derivative of the metric g in the direction of the complete differentiable vector field X.

The problem of finding Ricci solitons is quite difficult, therefore one can assume some restrictions either on a structure of the manifold or on the dimension of the manifold or on the class of metrics, or on the type of vector field X.

One of the natural restrictions is the assumption that the manifold is a homogeneous space and, in particular, a Lie group. Several results are known in this direction. For example, on three- and four-dimensional Lie groups with a left-invariant Riemannian metric there are no nontrivial homogeneous invariant Ricci solitons; a similar fact has also been proved for unimodular Lie groups with a left-invariant Riemannian metric of any finite dimension (see [4, 5]). However, for nonunimodular Lie groups of dimension > 4, the question of the existence of nontrivial solutions of the Ricci soliton equation remains open.

Algebraic Ricci solitons on Lie groups are an important example of Ricci solitons, they were first considered by J. Lauret and defined by the equation

$$\rho = \Lambda \cdot \mathrm{Id} + D,$$

where ρ is the Ricci operator, $\Lambda \in \mathbb{R}$ is a constant, I is the identity operator, D is some derivation of the Lie algebra. Any algebraic Ricci soliton on a connected and simply connected Lie group G with a left-invariant (pseudo)Riemannian metric corresponds to the homogeneous Ricci soliton (see [6, 7]).

Earlier conformally flat Ricci solitons on metric Lie groups were studied in [8], where a classification of conformally flat homogeneous invariant Ricci solitons on four-dimensional Lie groups was obtained; and also in [9] for arbitrary dimension, but under the additional restriction of diagonalizability of the Ricci operator.

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The main results of this paper include the following theorems.

Theorem 1. *Let G be an $n(\geqslant 4)$-dimensional Lie group with left-invariant (pseudo)Riemannian metric g, diagonalizable Ricci operator ρ and harmonic Weyl tensor. If (G,g) is an algebraic Ricci soliton, then (G,g) is a trivial Ricci soliton.*

Theorem 2. *Let G be an $n(\geqslant 4)$-dimensional Lie group with left-invariant conformally flat (pseudo)Riemannian metric g of nontrivial algebraic Ricci soliton and metric Lie algebra $\mathfrak{g}$. Then $\Lambda = 0$ and there is basis $\{e_1, \dots e_n\}$ in $\mathfrak{g}$, in which the derivation operator D and nontrivial scalar products are given by the following equations*

$$De_i = 0, \text{ where } i \neq n, \qquad De_n = e_{n-1};$$
$$\langle e_i, e_i \rangle = \varepsilon_i, \text{ if } i \leqslant n-2, \qquad \langle e_{n-1}, e_n \rangle = \varepsilon_0,$$

where $\varepsilon_i = \pm 1$, $i = \overline{0, n-2}$. Nontrivial Lie brackets are given by the following equations

$$[e_i, e_j] \subseteq \text{span}\,\{e_1, \dots, e_{n-1}\},\ i,j \leqslant n-2, \qquad [e_i, e_{n-1}] = 0,\ i \leqslant n-2,$$
$$[e_i, e_n] \subseteq \text{span}\,\{e_1, \dots, e_{n-1}\},\ i \leqslant n-2, \qquad [e_{n-1}, e_n] \subseteq \text{span}\,\{e_{n-1}\}.$$

In addition, this metric Lie algebra is locally symmetric ($\nabla R = 0$) if and only if $[e_{n-1}, e_n] = 0$.

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The spectrum of the curvature operators of 4-dimensional metric Lie groups

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The problem of the establishing of connections between topology and curvature of a Riemannian manifold is one of the important problems of Riemannian geometry. Some results of J. Milnor [1], V. N. Berestovskii [2], E. D. Rodionov and V. V. Slavskii [3, 4] about the connection between the Ricci, one dimensional, and sectional curvatures and topology of the homogeneous Riemannian spaces are well known.

The study of the properties of the curvature operators is interesting for understanding of the geometrical and topological structure of the homogeneous Riemannian manifolds, in particular, of metric Lie groups. Therefore, it is natural to study general properties of each operator. One of the variants is to investigate the spectrum of the curvature operator on the Lie group with left-invariant Riemannian metric, and in particular, signature of this operator.

The curvatures of left-invariant Riemannian metrics on Lie groups were studied by J. Milnor [1]. He found the possible signatures of the Ricci operator in the case of three-dimensional Lie groups with a left-invariant Riemannian metric. Similar results were obtained by D. N. Oskorbin, E. D. Rodionov, O. P. Khromova for the one-dimensional curvature operator and the sectional curvature operator [4–6].

The classification of four-dimensional real Lie algebras was received by G. M. Mubarakzyanov in [7]. Possible signatures of the Ricci operator on 4-dimensional Lie groups with left-invariant Riemannian metric were identified by Y. G. Nikonorov and A. G. Kremlev with the help of Mubarakzyanov's classification (see [8, 9]).

This paper continues the research, which was started by Y. G. Nikonorov and A. G. Kremlev. In this paper we study the realization of signatures of the one-dimensional curvature operator and the sectional curvature operator of four-dimensional Lie groups with left-invariant Riemannian metric.

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On unique determination of plane domains by relative conformal moduli of pairs of boundary components

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The boundary values of conformal mappings of plane finitely connected domains are studied. A complete description of the boundary values of such mappings in terms of the moduli (extremal lengths) of pairs of boundary components of domains is obtained in the case where the number of boundary components is not larger than three.

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On capacity for Sobolev–Lorentz classes of mappings

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The talk is based on the joint paper [1].

Let $p > 1$ and $k \in \mathbb{N}$ satisfy $kp < n$. Denote by $W^k_{p,1}$ the Sobolev–Lorentz space:

$$W^k_{p,1}(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R} \ : \ \nabla^k f \in L_{p,1}(\mathbb{R}^n),\ f \in L_{p_*}(\mathbb{R}^n)\},$$

where p_* is the usual Sobolev exponent $p_* = \frac{np}{n-kp}$, and $L_{p,1}$ means the Lorentz space:

$$\|f\|_{L_{p,q}} = \left(q \int\limits_0^{+\infty} t^{q-1} \big[\mathcal{L}^n\{x \in \mathbb{R}^n : |f(x)| > t\}\big]^{\frac{q}{p}}\, \mathrm{d}t \right)^{\frac{1}{q}}.$$

One has an evident identity $\|f\|_{L_p} = \|f\|_{L_{p,p}}$ so in particular $L_{p,p}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$. Further, for a fixed exponent p and $q_1 < q_2$ we have an estimate $\|f\|_{L_{p,q_2}} \le \|f\|_{L_{p,q_1}}$, and, consequently, an embedding $L_{p,q_1}(\mathbb{R}^n) \subsetneq L_{p,q_2}(\mathbb{R}^n)$. Recall, that for sets $U \subset \mathbb{R}^n$ of finite measure and $P > p > 1$, $q \in (1,p)$ we have

$$L_P(U) \subsetneq L_{p,1}(U) \subsetneq L_{p,q}(U) \subsetneq L_{p,p}(U) = L_p(U),$$

so the Lorentz space is like microscopic scale in Lebesgue spaces.

Here we shall mainly be concerned with the Lorentz space $L_{p,1}$, and in this case one may rewrite the norm as

$$\|f\|_{L_{p,1}} = \int\limits_0^{+\infty} \big[\mathcal{L}^n(\{x \in \mathbb{R}^n : |f(x)| > t\})\big]^{\frac{1}{p}}\, \mathrm{d}t.$$

For a set $A \subset \mathbb{R}^n$ consider the corresponding capacity:

$$\mathrm{Cap}_{k,p}(A) := \inf\big\{\|\nabla^k f\|_{L_{p,1}} : f \in W^k_{p,1}(\mathbb{R}^n), f \ge 1 \text{ on } U \supset A, U \text{ is an open set}\big\}.$$

One of the main results is as follows:

Theorem 1. *Under above assumptions, the equivalence*

$$c_1 \mathcal{H}^{n-kp}_\infty(A) \le \mathrm{Cap}_{k,p}(A) \le c_2 \mathcal{H}^{n-kp}_\infty(A)$$

holds, where the constants c_1, c_2 depend on n, k, p only and H^m_∞ means the usual m–dimensional Hausdorff content:

$$\mathcal{H}^m_\infty(E) = \inf\Big\{ \sum_{i=1}^\infty (\operatorname{diam} E_i)^m \ : \ E \subset \bigcup_{i=1}^\infty E_i \Big\}.$$

Recall that for usual capacity for Sobolev spaces

$$\mathrm{Cap}^S_{k,p}(A) := \inf\{\|\nabla^k f\|_{L_p} : f \in W^k_p(\mathbb{R}^n),\ f \geq 1 \text{ on } U \supset A,\ U \text{ is an open set}\}$$

we have only one sided inequality:

$$\mathrm{Cap}^S_{k,p}(A) \leq c_1\, \mathcal{H}^{n-kp}_\infty(A),$$

another inequality fails since $\mathrm{Cap}^S_{k,p}(A) = 0$ if $\mathcal{H}^{n-kp}(A) < \infty$.

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On convergence of multidimentional Mellin-Barnes integrals

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In 1921 Mellin [1] obtained Mellin-Barnes integral for the solution $y(x)$ of a general reduced algebraic equation

$$y^n + x_1 y^{n-1} + \ldots + x_{n-1} y - 1 = 0.$$

This integral has a nonempty domain of convergence defined by conditions on $\theta_j = \arg x_j$. The complete description of the domain of convergence of such an integral has been given recently in [2].

This paper aims to study the same questions in a general case. Consider a system of algebraic equations of the form

$$y_j^{m_j} + \sum_{\lambda \in \Lambda^{(j)}} x^{(j)}_\lambda y^\lambda - 1 = 0,\ j = 1, \ldots, n, \tag{1}$$

where $\Lambda^{(j)} \subset \mathbb{Z}^n$. Denote by $\Lambda := \bigsqcup_{j=1}^{n} \Lambda^{(j)}$ the disjunctive union of $\Lambda^{(j)}$, $|\Lambda| = N$. The coefficients of the system (1) run over the vector space $\mathbb{C}^\lambda \cong \mathbb{C}^N_x$, where the coordinates of $x = (x_\lambda)$ are indexed by elements $\lambda \in \Lambda$. The coordinates with indices $\lambda \in \Lambda^{(i)}$ we denote by $x^{(i)}_\lambda$, identifying $\mathbb{C}^\Lambda$ with the space $\mathbb{C}^{\Lambda^{(1)}} \times \ldots \times \mathbb{C}^{\Lambda^{(n)}}$.

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We consider the set Λ as matrix $\Lambda = \left(\Lambda^{(1)}, \ldots, \Lambda^{(n)}\right) = \left(\lambda^1, \ldots, \lambda^N\right)$, whose columns are the vectors $\lambda^k = \left(\lambda_1^k, \ldots, \lambda_n^k\right)$ of the exponents of the monomials of the system (1). Here the block $\Lambda^{(i)}$ of Λ corresponds to the i-th equation of the system (1), the enumeration of columns λ^k of $\Lambda^{(i)}$ is arbitary but fixed. The rows of Λ we denote by φ_j, $j = 1, \ldots, n$. The object of our interest is a branch of the solution $y(x) = (y_1(x), \ldots, y_n(x))$ of the system (1) that satisfies $y(0) = (1, \ldots, 1)$, which we call the principal solution. Following [3, 4], to a monomial $y^\mu = y_1^{\mu_1} \ldots y_n^{\mu_n}$ of the principal solution $y = y(x)$ we put into correspondence a Mellin-Barnes integral

$$\frac{1}{(2\pi i)^N} \int\limits_{\gamma + i\mathbb{R}^N} \frac{\prod\limits_{j=1}^{n} \prod\limits_{\lambda \in \Lambda^{(j)}} \Gamma\left(u_\lambda^{(j)}\right) \prod\limits_{j=1}^{n} \Gamma\left(\frac{\mu_j}{m_j} - \frac{1}{m_j}\langle \varphi_j, u\rangle\right)}{\prod\limits_{j=1}^{n} \Gamma\left(\frac{\mu_j}{m_j} - \frac{1}{m_j}\langle \varphi_j, u\rangle + \sum\limits_{\lambda \in \Lambda^{(j)}} u_\lambda^{(j)} + 1\right)} Q(u) x^{-u}\, du, \qquad (2)$$

where γ is from polyhedron

$$\{u \in \mathbb{R}^N_{>0} : \langle \varphi_j, u\rangle < \mu_j, j = 1, ..., n\},$$

and $Q(u)$ is a polynomial given by determinant

$$Q(u) = \frac{1}{m_1 \ldots m_n} \det \left\| \delta_i^j \left(\mu_j - \langle \varphi_j, u\rangle\right) + \langle \varphi_j^{(i)}, u^{(i)}\rangle \right\|_{i,j=1}^{n}.$$

The integral (2) is obtained as a result of the formal Mellin transform of $y^\mu(x)$ after linearization.

Consider the matrices of the form

$$\begin{pmatrix} \lambda_1^{(1)} & \cdots & \lambda_1^{(n)} \\ \vdots & \ddots & \vdots \\ \lambda_n^{(1)} & \cdots & \lambda_n^{(n)} \end{pmatrix}, \qquad (3)$$

composed by exponents of the monomials in the system (1), where each column is a vector $\lambda^{(j)} = \left(\lambda_1^{(j)} \ldots \lambda_n^{(j)}\right)^T$ from $\Lambda^{(j)}$.

We call a minor of the matrix (3) *principal* if the indices of chosen columns coincide with indices of rows.

Theorem. *Integral* (2) *corresponding to a principal solution of a system pf algebraic equations* (1) *has non-emty domain of convergence if and only if in each matrix of the form* (3) *all principal minors are positive.*

In the proof we use the description of the domain of convergence of a general Mellin-Barnes integral [5, section 4.4.1]). Besides an important role in the proof belongs to the partition theorem of Euclidean n-space [6, p. 134].

The proved theorem implies the following interesting fact.

Corollary. *Let* $A = \|a_{ij}\|_{i,j=1}^{n}$ *be a nonnegative* $n \times n$*-matrix, and* $a^{(i)}$ *be its rows. Concider the function*

$$g_A(v) = \sum_{i=1}^{n} \big(|v_i| + |\langle a^{(i)}, v\rangle| - |\langle a^{(i)}, v\rangle - v_i|\big).$$

All principal minors of A *are positive if and only if the function* $g_A(v)$ *is positive for all* $v \neq 0$.

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Absolutely trianalytic tori in Hyperkähler manifolds

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A Riemannian manifold (M, g) is called hyperkähler if it admits a triple of a complex structures I, J, K satisfying quaternionic relations and Kähler with a respect to g. A subvariety Z in hyperkähler manifold M is called absolutely trianalytic if it is complex-analytic with respect to all hyperkähler structures compatible with I. In late 90-s Kaledin and Verbitsky proved a non-existence of absolutely trianalytic subvarieties in the Hilbert scheme of point in $K3$ [1, 2]. Also

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they construct example of absolutely trianalytic subvariety deformation equivalent to the Hilbert scheme in the generalized Kummer variety [2].

The case of absolutely trianalytic tori in the hyperkähler manifolds has been firstly studied by Soldatenkov and Verbitsky [3]. They have proved non-existence of such tori in O'Grady examples. We study the case of the generalized Kummer manifold and prove the following

Theorem ([4]). *Let $K_n(T)$ be the generalized Kummer variety and $Z \subset K_n(T)$ an absolutely trianalytic submanifold of $K_n(T)$. Then Z is not a torus.*

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Kantorovich-Wright integration and representation of quasi-Banach lattices

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The aim of this talk is to develop an integration theory of scalar function with respect to positive measures with values in quasi-Banach lattices and to prove some general representation theorems for quasi-Banach lattice as spaces of integrable functions.

Plane domains which admit extension of Sobolev spaces

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The domain G is a subset of $\mathbb{R}^n$, $p \geq 1$, $W_k^p(G)$ – is a Sobolev space of functions $f : G \to \mathbb{R}$, where $k \in \mathbb{N}$, p is the integration degree.

G belongs to $\mathrm{Ext}_k^p(n)$, if for $f \in W_k^p(G)$ there exists an extension $F \in W_k^p(\mathbb{R}^n)$.

We discuss the problem: $G \in \mathrm{Ext}_1^p(2) \quad \Rightarrow \quad G \in \mathrm{Ext}_2^p(2)$?

This problem is equivalent to the extension problem for potential or divergence-free vector fields of class $W_1^p(G)$.

On linear independence of the Laurent series for rational functions in several variables

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Let V be an algebraic hypersurface in the complex algebraic torus $(\mathbb{C}^*)^n = (\mathbb{C} \setminus 0)^n$ and defined as zero locus of a polynomial $P(z)$

$$V = \{z \in (\mathbb{C}^*)^n : P(z_1, \ldots, z_n) = 0\}.$$

We're interested in different Laurent expansions of the rational function $1/P(z)$ centered at the origin. Relevance of the problem of such expansions is evident, for example, in study of the fundamental solutions of differential equations with the characteristic polynomial $P(z)$. It is convenient for description of the Laurent expansions to use the concept of amoebas.

Definition. The *amoeba* $\mathscr{A}_V$ of a hypersurface V is the image of V under the logarithmic map $\mathrm{Log} : (\mathbb{C}^*)^n \to \mathbb{R}^n$,

$$\mathrm{Log} : (z_1, \ldots, z_n) \mapsto (\log |z_1|, \ldots, \log |z_n|).$$

The complement $\mathbb{R}^n \setminus \mathscr{A}_V$ consists of a finite number of connected components $\{E\}$, which are open and convex. Each preimage $\mathrm{Log}^{-1}(E)$ is the domain of convergence for the corresponding Laurent expansion of the rational function $1/P(z)$ [1].

A number of these connected components and their structures are in close connection with the *Newton polytope* Δ_P of the polynomial P. The polytope is by definition convex hull in $\mathbb{R}^n$ of all the exponents of the monomials occurring in the polynomial. On the set $\{E\}$ there exists an injective *order function* $\nu : \{E\} \to \mathbb{Z}^n \cap \Delta_P$. So the connected components from the set $\{E\}$ can be labelled as E_ν.

In [2] M. Passare and A. Tsikh hypothesized about linear independence of Laurent expansions of a rational function. The report will be dealt with the resolution of this problem in the case $n = 2$ and V is a normalized Harnack polynomial.

Mikhalkin in [3] introduced the notion of a *Harnack curve* in the complex area. The polynomial $P(z, w)$ (the curve $V = \{P(z, w) = 0\}$) is determined as a Harnack polynomial (a Harnack curve) when its amoeba is of maximal area among all polynomials with the fixing Newton polygon Δ_P [4].

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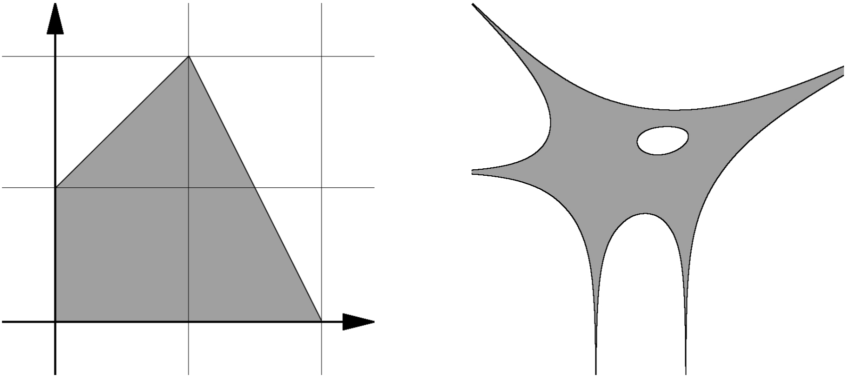

Figure 1: The Newton polygon (left) for the Harnack polynomial $P(z,w) = 1 - 4z - w + z^2 - 6zw - zw^2$ and the amoeba (right) of the normalized Harnack curve $P(z,w) = 0$.

We say that a Harnack curve V is *normalized* if its defining polynomial P has real coefficients and

- $(0,0)$ is a vertex of the Newton polygon Δ_P
- $\partial E_{(0,0)} \subset \mathrm{Log}(V \cap \mathbb{R}^2_+)$.

One proves following theorem.

Theorem. *If $V = \{P(z,w) = 0\}$ is a normalized Harnack curve, then different Laurent expansions of the function $1/P(z,w)$ are linear independent.*

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On a holonomy of non-holonomic 3D Lie Groups

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We study the 3D subRiemannian groups with algebra of vector fields satisfying the following conditions on Lie brackets:

$$[X,Y]=Z, \quad [Y,Z]=(\chi+\kappa)X, \quad [X,Z]=(\chi-\kappa)Y,$$

where vector fields X, Y form the orthonormal frame of the distribution of admissible directions. We define the holonomy flag as a set of embedded vector subspaces in tangent space generated by formal curvature tensors:

$$R(X,Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z,$$

where the connection ∇ is defined using the standard formula

$$\nabla_X Y=\frac{1}{2}[X,Y]+U(X,Y).$$

Here symmetric form $U: TM\times TM\to TM$ plays the role of Christoffel symbols. We describe all such connections for which the metric on the distribution is parallel along the admissible directions and has no torsion.

Theorem 1. *Let ∇ be the torsionless connection on a subRiemannian Heisenberg group compatible with metric*

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&g_{33}\end{pmatrix}$$

along the admissible directions X and Y. Then the zero step of the holonomy flag never vanish for any U-term of the connection.

If one will define a connection δ in an opposite to a Riemannian way then he will get a correct flag.

Theorem 2. *If for a connection δ on a subRiemannian Heisenberg group $\delta_Z Z=0$ then its holonomy flag vanishes.*

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Normalized manifolds over local Weil algebras and their cohomologies

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A local algebra $\mathbb{A}$ in the sense of A. Weil (Weil algebra) [1] is a finite-dimensional associative commutative unital algebra with maximal ideal $\mathfrak{m}(\mathbb{A})$ consisting of nilpotent elements such that $\mathbb{A}/\mathfrak{m}(\mathbb{A}) \cong \mathbb{R}$. Let $L_n^{\mathbb{A}}$ be an n-dimensional $\mathbb{A}$-module, i. e. a module isomorphic to the module of n-tuples of elements of the algebra $\mathbb{A}$. One can associate with a module $L_n^{\mathbb{A}}$ a number of $\mathbb{A}$-modules: the tensor product $L_n^{\mathbb{A}} \otimes_{\mathbb{R}} \mathbb{A}$, where $\mathbb{A}$ acts on $L_n^{\mathbb{A}} \otimes_{\mathbb{R}} \mathbb{A}$ by $\alpha(V \otimes \beta) = V \otimes \beta\alpha$; the module $\mathrm{Hom}_{\mathbb{A}}(L_n^{\mathbb{A}}, \mathbb{A})$ of $\mathbb{A}$-linear $\mathbb{A}$-valued 1-forms on $L_n^{\mathbb{A}}$; the module $\mathrm{Hom}_{\mathbb{R}}(L_n^{\mathbb{A}}, \mathbb{A})$ of $\mathbb{R}$-linear $\mathbb{A}$-valued 1-forms on $L_n^{\mathbb{A}}$, which is isomorphic to the module $\mathrm{Hom}_{\mathbb{A}}(L_n^{\mathbb{A}} \otimes_{\mathbb{R}} \mathbb{A}, \mathbb{A})$ of $\mathbb{A}$-linear $\mathbb{A}$-valued 1-forms on $L_n^{\mathbb{A}} \otimes_{\mathbb{R}} \mathbb{A}$. The submodule ${}^0L \subset L_n^{\mathbb{A}} \otimes_{\mathbb{R}} \mathbb{A}$ which annihilates the submodule $\mathrm{Hom}_{\mathbb{A}}(L_n^{\mathbb{A}}, \mathbb{A}) \subset \mathrm{Hom}_{\mathbb{R}}(L_n^{\mathbb{A}}, \mathbb{A})$ has dimension $(m-1)n$, where m is the dimension of $\mathbb{A}$.

A *normalization* of an n-dimensional $\mathbb{A}$-module $L_n^{\mathbb{A}}$ is the choice of an n-dimensional submodule ${}^1L$ in $L_n^{\mathbb{A}} \otimes_{\mathbb{R}} \mathbb{A}$ such that

$$L_n^{\mathbb{A}} \otimes_{\mathbb{R}} \mathbb{A} = {}^1L \oplus_{\mathbb{A}} {}^0L. \tag{1}$$

By a *normalized* n-dimensional $\mathbb{A}$-module we mean a pair $(L_n^{\mathbb{A}}, {}^1L)$. Decomposition (1) allows to represent the module $\mathrm{Hom}_{\mathbb{R}}(L_n^{\mathbb{A}}, \mathbb{A})$ of $\mathbb{R}$-linear forms on $L_n^{\mathbb{A}}$ as the direct sum

$$\mathrm{Hom}_{\mathbb{R}}(L_n^{\mathbb{A}}, \mathbb{A}) \cong \mathrm{Hom}_{\mathbb{A}}(L_n^{\mathbb{A}} \otimes_{\mathbb{R}} \mathbb{A}, \mathbb{A}) = {}^1L^* \oplus_{\mathbb{A}} {}^0L^*$$

of the submodule ${}^1L^* = \mathrm{Hom}_{\mathbb{A}}(L_n^{\mathbb{A}}, \mathbb{A})$ and its complement ${}^0L^*$ in $\mathrm{Hom}_{\mathbb{R}}(L_n^{\mathbb{A}}, \mathbb{A})$ which annihilates the submodule ${}^1L$.

A smooth manifold $M_n^{\mathbb{A}}$ of dimension n over m-dimensional Weil algebra $\mathbb{A}$ ($\mathbb{A}$-smooth manifold) is an mn-dimensional smooth manifold endowed with an atlas whose charts take values in the module $\mathbb{A}^n$ and transition functions are $\mathbb{A}$-smooth (which means that their tangent mappings are $\mathbb{A}$-linear) [2]. Tangent spaces $T_X M_n^{\mathbb{A}}$ to a $\mathbb{A}$-smooth manifold $M_n^{\mathbb{A}}$ carry natural structures of n-dimensional $\mathbb{A}$-modules isomorphic to $\mathbb{A}^n$.

Let $i_0 : M_n^{\mathbb{A}} \to T^{\mathbb{A}} M_n^{\mathbb{A}}$ be the embedding of $M_n^{\mathbb{A}}$ into the Weil bundle [1] of $M_n^{\mathbb{A}}$ as the zero section. Then the pullback $i_0^{-1}(TT^{\mathbb{A}} M_n^{\mathbb{A}}$ is a bundle over $M_n^{\mathbb{A}}$ naturally isomorphic to the bundle $TM_n^{\mathbb{A}} \otimes_{\mathbb{R}} \mathbb{A}$.

A *normalization* of an n-dimensional $\mathbb{A}$-smooth manifold $M_n^{\mathbb{A}}$ is a smooth distribution of n-dimensional submodules ${}^1T_X$ in $T_X M_n^{\mathbb{A}} \otimes_{\mathbb{R}} \mathbb{A}$, $i = 1, \ldots, n$, such that

$$T_X M_n^{\mathbb{A}} \otimes_{\mathbb{R}} \mathbb{A} = {}^1T_X \oplus_{\mathbb{A}} {}^0T_X. \tag{2}$$

An $\mathbb{A}$-smooth manifold $M_n^{\mathbb{A}}$ is called *normalized* if a normalization of $M_n^{\mathbb{A}}$ is chosen.

Denote by $\Lambda^1_X M_n^{\mathbb{A}}$ the $\mathbb{A}$-module of $\mathbb{A}$-valued $\mathbb{R}$-linear forms on $T_X M_n^{\mathbb{A}}$, which is naturally isomorphic to the $\mathbb{A}$-module of $\mathbb{A}$-linear forms on $T_X M_n^{\mathbb{A}} \otimes_{\mathbb{R}} \mathbb{A}$, and by $\Lambda^1_{\mathbb{A}-\mathrm{lin},X} M_n^{\mathbb{A}}$ the module of $\mathbb{A}$-linear forms on $T_X M_n^{\mathbb{A}}$. Decomposition (2) of $T_X M_n^{\mathbb{A}} \otimes_{\mathbb{R}} \mathbb{A}$ induces the decomposition

$$\Lambda^1_X M_n^{\mathbb{A}} = \Lambda^1_{\mathbb{A}-\mathrm{lin},X} M_n^{\mathbb{A}} \oplus \bar{\Lambda}^1_X M_n^{\mathbb{A}},$$

where $\bar{\Lambda}^1_X M_n^{\mathbb{A}}$ is the submodule in $\Lambda^1_X M_n^{\mathbb{A}}$ which annihilates ${}^1T_z$.

Let $X^i = x^{0i} + x^{\hat{a}i} e_{\hat{a}}$ (where e_k is a basis in $\mathbb{A}$, $e_{\hat{a}} \in \mathfrak{m}(\mathbb{A})$, $e_0 = 1$) be local coordinates on $M_n^{\mathbb{A}}$. In a neighbourhood of $X \in M_n^{\mathbb{A}}$ the submodule $\Lambda^1_X M_n^{\mathbb{A}}$ is spanned by 1-forms $dX^1, \dots, dX^n$ submodule $\bar{\Lambda}^1_z M_n^{\mathbb{A}}$ is spanned by

$$\theta^{i\hat{a}} = dx^{i\hat{a}} + \lambda^{i\hat{a}}_j dX^j, \quad \lambda^{i\hat{a}}_j \in \mathbb{A}.$$

Let $\Lambda^p_X M_n^{\mathbb{A}}$ denote the $\mathbb{A}$-module of $\mathbb{A}$-valued exterior forms on $T_X M_n^{\mathbb{A}}$ and $\Lambda^{r,s}_X M_n^{\mathbb{A}}$ the submodule in $\Lambda^{r+s}_X M_n^{\mathbb{A}}$ spanned by exterior products $\xi_1 \wedge \cdots \wedge \xi_r \wedge \eta_1 \wedge \cdots \wedge \eta_s$, where 1-forms $\eta_1, \dots, \eta_s$ are $\mathbb{A}$-linear and $\xi_1, \dots, \xi_n \in \bar{\Lambda}^1_X M_n^{\mathbb{A}}$. The union of submodules $\Lambda^{r,s}_X M_n^{\mathbb{A}}$ over all $z \in M_n^{\mathbb{A}}$ is a subbundle $\Lambda^{r,s} M_n^{\mathbb{A}}$ of $\Lambda^{r+s} M_n^{\mathbb{A}}$. Denote by $\Omega^{r,s} M_n^{\mathbb{A}}$ the $\mathbb{A}$-module of smooth sections of $\Lambda^{r,s} M_n^{\mathbb{A}}$. In terms of local coordinates on $M_n^{\mathbb{A}}$, a differential form $\alpha_{r,s}$ from $\Omega^{r,s} M_n^{\mathbb{A}}$ has the following expression:

$$\alpha_{r,s} = \sum \alpha_{i_1\hat{a}_1 \dots i_r\hat{a}_r,\, j_1 \dots j_s} \theta^{i_1\hat{a}_1} \wedge \dots \wedge \theta^{i_r\hat{a}_r} \wedge dX^{j_1} \wedge \dots \wedge dX^{j_s}.$$

The exterior differential $d\alpha_{r,s}$ is a form belonging to $\Omega^{r-1,s+2} M_n^{\mathbb{A}} \oplus \Omega^{r,s+1} M_n^{\mathbb{A}} \oplus \Omega^{r+1,s} M_n^{\mathbb{A}}$. Denote its component belonging to $\Omega^{r+1,s} M_n^{\mathbb{A}}$ by $\hat{d}\alpha_{r,s}$. The operator $\hat{d}$ satisfies the relation $\hat{d} \circ \hat{d} = 0$ and defines a series of complexes:

$$0 \longrightarrow \Omega^{0,s} M_n^{\mathbb{A}} \xrightarrow{\hat{d}} \Omega^{1,s} M_n^{\mathbb{A}} \xrightarrow{\hat{d}} \Omega^{2,s} M_n^{\mathbb{A}} \xrightarrow{\hat{d}} \cdots \xrightarrow{\hat{d}} \Omega^{n,s} M_n^{\mathbb{A}} \xrightarrow{\hat{d}} 0. \qquad (3)$$

Then the notion of a normalization is generalized to the case of a Weil bundle $T^{\mathbb{A}} M_n^{\mathbb{A}}$, which allows us to introduce complexes of $\mathbb{A}$-smooth forms on $T^{\mathbb{A}} M_n^{\mathbb{A}}$ similar to (3). The properties of the introduced complexes and their relations with complexes (3) are studied.

Complexes (3) can also be generalized to the case of $\mathfrak{g}^{\mathbb{A}}$-valued forms on an $\mathbb{A}$-smooth principal bundle $P(M^{\mathbb{A}}, G^{\mathbb{A}})$, where $G^{\mathbb{A}}$ is an $\mathbb{A}$-smooth Lie group and $\mathfrak{g}^{\mathbb{A}}$ its Lie algebra, and the obstruction to existence of an $\mathbb{A}$-smooth connection in $P(M^{\mathbb{A}}, G^{\mathbb{A}})$ can be represented in terms of cohomology classes of these complexes (the Atiyah class).

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Sub-Riemannian geodesics in SO(3) with application to vessel tracking in spherical images of retina

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In order to detect vessel locations in spherical images of retina we consider a minimization problem $\mathbf{P_{curve}}$ for a curve on a sphere with fixed boundary points and directions. We lift this problem to the sub-Riemannian (SR) problem $\mathbf{P_{mec}}$ in Lie group SO(3) and show that spherical projection of certain SR-minimizers provides a solution to problem $\mathbf{P_{curve}}$. We propose a method for computing SR-minimizers and validate it by comparison to exact solution in the case of uniform external cost. An experiment of vessel tracking in a spherical image of the retina shows a promising result.

Let $S^2 = \{\mathbf{n} \in \mathbb{R}^3 \mid \|\mathbf{n}\| = 1\}$ be a sphere of unit radius. We consider the problem $\mathbf{P_{curve}}$, which is for given boundary points $\mathbf{n}_0, \mathbf{n}_1 \in S^2$ and directions $\mathbf{n}_0' \in T_{\mathbf{n}_0}(S^2)$, $\mathbf{n}_1' \in T_{\mathbf{n}_1}(S^2)$, $\|\mathbf{n}_0'\| = \|\mathbf{n}_1'\| = 1$ to find a smooth curve $\mathbf{n}(\cdot) : [0,l] \to S^2$ that satisfies the boundary conditions

$$\mathbf{n}(0) = \mathbf{n}_0,\ \mathbf{n}(l) = \mathbf{n}_1, \quad \mathbf{n}'(0) = \mathbf{n}_0',\ \mathbf{n}'(l) = \mathbf{n}_1',$$

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and for given $\xi > 0$ minimizes the functional

$$\mathcal{L}(\mathbf{n}(\cdot)) := \int_0^l \mathfrak{C}(\mathbf{n}(s))\sqrt{\xi^2 + k_g^2(s)}\,\mathrm{d}s,$$

where s denotes the spherical arclength, $k_g(s)$ is the geodesic curvature on S^2 of $\mathbf{n}(\cdot)$ evaluated in time s, the total length l is free, and $\mathfrak{C} : S^2 \to [\delta, +\infty)$, $\delta > 0$, is a smooth function "external cost".

We call $\mathbf{P_{mec}}$ the following SR problem in SO(3):

$$\dot{R} = u_1 X_1 + u_2 X_2, \qquad R(0) = \mathrm{Id}, \quad R(T) = R_{fin},$$
$$\mathcal{L}(R(\cdot)) := \int\limits_0^T C(R(t))\sqrt{\xi^2 u_1^2(t) + u_2^2(t)}\,\mathrm{d}t \to \min,$$

where $R \in \mathrm{SO}(3)$, X_i are basis left-invariant vector fields in SO(3), $\xi > 0$ is a constant and terminal time $T > 0$ is free.

The external cost $C : \mathrm{SO}(3) \to [\delta, +\infty)$, $\delta > 0$, is a smooth function that is typically obtained by lifting the external cost $\mathfrak{C}$ from the sphere S^2 to the group $SO(3)$, i.e. $C(R) = \mathfrak{C}(R\,\mathbf{e}_1)$.

We call a spherical projection the projection map $\mathrm{SO}(3) \ni R \mapsto R\,\mathbf{e}_1 \in S^2$. We show that the spherical projection of certain minimizers of $\mathbf{P_{mec}}$ provides solution of problem $\mathbf{P_{curve}}$. More precisely this only holds for the minimizers whose spherical projection does not have a cusp. The spherical projection of a minimizer of $\mathbf{P_{mec}}$ is said to have a cusp if the corresponding control $u_1(t)$ changes a sign locally. Before the first cusp a minimizer of $\mathbf{P_{mec}}$ can be parameterized by spherical arclength $s \in [0, s_{\max})$.

Theorem 1. *Let $R(t)$, $t \in [0,T]$ be a minimizer of $\mathbf{P_{mec}}$ parameterized by SR-arclength $C(R(t))\sqrt{\xi^2 u_1^2(t) + u_2^2(t)} = 1$, and let the corresponding optimal control $(u_1(t), u_2(t))$ satisfy the inequality $u_1(t) > 0$ for all $t \in [0,T]$.*

Set $\quad \mathbf{n}_0 = \mathbf{e}_1, \quad \mathbf{n}_1 = R(T)\,\mathbf{e}_1, \quad \mathbf{n}_0' = \mathbf{e}_3, \quad \mathbf{n}_1' = R(T)\,\mathbf{e}_3.$

For such boundary conditions $\mathbf{P_{curve}}$ has a minimizer $\mathbf{n}(s) = R(t(s))\,\mathbf{e}_1$, along which the following equalities hold:

$$\begin{cases} u_1(t) = \frac{ds}{dt}(t), \\ u_2(t) = k_g(s(t))\,\frac{ds}{dt}(t), \end{cases} \qquad t(s) = \int_0^s \mathfrak{C}(\mathbf{n}(\sigma))\sqrt{\xi^2 + k_g^2(\sigma)}\,\mathrm{d}\,\sigma,$$

for $0 \le s \le l < s_{\max}$, and $T = t(l)$.

We obtain exact formulas for SR geodesics in problem $\mathbf{P_{mec}}$ in the uniform external cost case $C = 1$ and use them to validate our method to compute SR-minimizers $\mathbf{P_{mec}}$ for general external cost case $C \neq 1$. The method is summarized by the following theorem.

Theorem 2. *Let $\mathcal{W}(R)$ be a viscosity solution* [1] *of the eikonal system*

$$\begin{cases}\sqrt{\frac{(X_1|_R(\mathcal{W}))^2}{\xi^2} + (X_2|_R(\mathcal{W}))^2} = C(R), \quad \textit{for } \mathrm{Id} \neq R \in \mathrm{SO}(3),\\ \mathcal{W}(\mathrm{Id}) = 0.\end{cases}$$

A SR-minimizer $R(t) = R_b(\mathcal{W}(R_{fin}) - t)$ ending at R_{fin} is found by backward integration for $t \in [0, \mathcal{W}(R_{fin})]$ of the equation

$$\dot{R}_b(t) = -u_1(t)\; X_1|_{R_b(t)} - u_2(t)\; X_2|_{R_b(t)}\,, \quad R_b(0) = R_{fin},$$

where $u_1(t) = \frac{X_1|_{R_b(t)}(\mathcal{W})}{(\xi\, C(R_b(t)))^2}$ and $u_2(t) = \frac{X_2|_{R_b(t)}(\mathcal{W})}{(C(R_b(t)))^2}$.

Solving numerically this system for external cost C induced by a spherical image of the retina we obtain an effective method for vessel tracking. Comparison with tracking via SE(2)-geodesics [1] shows, that in general the SO(3)-geodesics more accurately follow the vessel locations. Furthermore, there is a visible difference between the geodesic curvature of solution to problem $\mathbf{P_{curve}}$ on a sphere and the curvature of its projection on a plane. As in retinal imaging applications curvature is considered as a relevant biomarker for detection of diabetic retinopathy and other systemic diseases, the data-driven SR geodesic model in SO(3) is a relevant extension of data-driven geodesic model in $SE(2)$ [1].

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Inverse optimal control: the sub-Riemannian case

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This is a joint work with Frédéric Jean (ENSTA ParisTech) and Igor Zelenko (Texas A&M University). Recent applications of optimal control theory to the study of human motion motivates the so-called *inverse optimal control problem*: given a set Γ of experimental data, and a class of optimal control problems – that is, a pair (control system $q' = f(q,u)$, class $\mathcal{C}$ of costs) – suitable to model the system, identify a cost function C in $\mathcal{C}$ such that the elements of Γ are minimizing trajectories of the optimal control problem associated with C.

Studying such an inverse problem requires to analyze its well-posedness, in particular, the question of the injectivity: given a control system, do the set of all minimizing trajectories of an associated optimal control problem determine in a unique way the corresponding infinitesimal cost in the class $\mathcal{C}$? If the answer is negative then the further analysis is to determine the classes of cost on which the set of minimizing trajectories is the same. Note that a cost can be determined only up to a multiplicative constant.

In this talk, we consider the case where the system $q' = f(q,u)$ is linear w.r.t. the control u and spans a Lie-bracket generating distribution, and where the class of costs is either the length functionals or the energy functionals associated with functions g which are quadratic w.r.t. the control u. The corresponding optimal control problem can be stated in the geometrical framework as the problem of finding the length minimizers (in the first case) and the energy minimizers (in the second case) in Riemannian or sub-Riemannian geometry. For the first kind of problem the injectivity of the inverse optimal control problem can be reduced to the problem of finding the classes of *projectively equivalent* metrics and for the second kind of *affine equivalent* metrics. Two sub-Riemannian or Riemannian metrics are said to be projectively equivalent at a point of the manifold if their normal geodesics coincide as unparameterized curves in some neighborhood of the point. They are called affine equivalent if their normal geodesics coincide as parameterized curves. The abnormal geodesics always coincide as they are completely determined by the distribution. In the talk we will discuss the results on projective equivalence in case of Riemannian metrics obtained by Levi-Civita in the 19th century, and more recent results of Zelenko in case of contact and quasi-contact sub-Riemannian metrics. We will then present the new results we have obtained for sub-Riemannian metrics on a generic distribution of step higher than 2. In this case the projectively equivalent metrics are conformal and the affine equivalent metrics are constantly proportional.

Regularity of the inverse of Sobolev–Orlicz mappings

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The article is devoted to the problem of describing the properties of regularity of the inverse mapping to some homeomorphism of the Sobolev–Orlicz class W^1_M, if we know properties of regularity of direct mapping. In addition, we found necessary and sufficient conditions for a homeomorphism $\varphi : D \to D'$, with D and D' domains in $\mathbb{R}^n$, to generate the bounded composition operator $\varphi^* : L^1_{M_1}(D') \to L^1_M(D)$ by the rule $\varphi^* f = f \circ \varphi$. As a consequence of these results, we prove a theorem about the conditions under which the inverse homeomorphism also generates the bounded composition operator of the other couple of Sobolev–Orlicz spaces defined by the first couple. Note that if the N-function M determining the Sobolev–Orlicz spaces is the power function then the problem is reduced to the case of Sobolev spaces W^1_p, which is well studied in [1]. Methods of article [1] are used to prove propositions of this work.

The main result is formulated in the assumption that N-functions M and M_1 are such that the function M_2 from the equations $M_1(u) = M(2M_3(u))$, $M_2(u) = M(2M_3^*(u))$ satisfies the Δ'-condition and the assumption that N-function M satisfies [2]

$$\int\limits_1^\infty \Big(\frac{t}{M(t)}\Big)^{\frac{1}{n-2}} dt < \infty. \tag{1}$$

Following theorems hold.

Theorem 1. *Suppose $\varphi : D \to D'$ has the following properties:*

1) $\varphi \in W^1_{M,\mathrm{loc}}(D)$;
2) φ *has finite codistortion;*
3) $\mathcal{K}_{\varphi,F_2} = \left\| \frac{|\operatorname{adj} D\varphi|}{(M_1^{-1}(|J(x,\varphi)|))^{n-1}} \mid L_{F_2} \right\| < \infty.$

Then the inverse homeomorphism has the following properties:

4) $\varphi^{-1} \in W^1_{F_1,\mathrm{loc}}(D')$;
5) φ^{-1} *has finite distortion;*
6) $K_{\varphi^{-1},F} = \left\| \frac{|D\varphi^{-1}|}{F^{-1}(|J(x,\varphi^{-1})|)} \mid L_{F_2} \right\| < \infty.$

N-functions F, F_2 are defined by equations:

$$F^{-1}(u) = u(M^{-1}(1/u))^{n-1}, \quad F_2(u) = M_2(u^{\frac{1}{n-1}}),$$

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and function F_1 is defined by equations:

$$F(u) = F_1(2F_3(u)), \quad F_2(u) = F_1(2F_3^*(u)).$$

Theorem 2. *Let M and M_1 satisfy Δ'-condition. If homeomorphism $\varphi : D \to D'$ generates a bounded composition operator $\varphi^* : L^1_{M_1}(D') \to L^1_M(D)$ and has finite codistortion then the inverse mapping $\varphi^{-1} : D' \to D$ generates a bounded composition operator ${\varphi^{-1}}^* : L^1_F(D) \to L^1_{F_1}(D')$ and has finite distortion.*

Also we prove a theorem about a bounded composition operator and by using this theorem we achieve another result about the properties of the composition operator generated by the inverse mapping to some homeomorphism of the Sobolev–Orlicz class.

Theorem 3. *A homeomorphism $\varphi : D \to D'$ generates a bounded composition operator $\varphi^* : L^1_{M_1}(D') \to L^1_M(D)$, if functions $M, M_1 \in \Delta_2$ and following statements hold:*

1) $\varphi \in \mathrm{ACL}(D)$*;*

2) φ *has finite distortion;*

3) $K = \left\| \frac{|D\varphi|}{M_1^{-1}(|J(x,\varphi)|)} \mid L_{M_2} \right\| < \infty$ ($K = \left\| \frac{|D\varphi|}{M_1^{-1}(|J(x,\varphi)|)} \mid L_\infty \right\|$ *if* $M = M_1$).

If $\varphi : D \to D'$ generates a bounded composition operator $\varphi^* : L^1_{M_1}(D') \to L^1_M(D)$ and a function $M_1 \in \Delta'$ then φ satisfies conditions 1), 2) *and the following condition:*

4) $K_\gamma = \left\| \frac{(M(|D\varphi|))^{1/\alpha}}{(|J(x,\varphi)|)^{1/\beta}} \mid L_\gamma \right\| < \infty$ ($K = \left\| \frac{M(|D\varphi|)}{|J(x,\varphi)|} \mid L_\infty \right\|$ *if* $M = M_1$).

Theorem 4. *Let $Q(u) = Cu^q(\ln u)^{a_1}(\ln\ln u)^{a_2}...(\ln...\ln u)^{a_n}$, $q > 1$, $a_i \in \mathbb{R}$ be a main part of N-function M and M_1 satisfy Δ_2-condition. Suppose M also satisfies* (1). *If homeomorphism $\varphi : D \to D'$ generates a bounded composition operator $\varphi^* : L^1_{M_1}(D') \to L^1_M(D)$ and has finite codistortion then the inverse mapping $\varphi^{-1} : D' \to D$ generates a bounded composition operator ${\varphi^{-1}}^* : L^1_{\alpha'}(D) \to L^1_{\beta'}(D')$ and has finite distortion.*

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Continuity results for TV-minimizers

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In this talk, we want to illustrate how geometry can help studying regularity of the minimizer of a generalized total variation. This idea have been already used in studying the least gradient problem [1–3], consisting, given a function g on $\partial\Omega$, in minimizing

$$\int_\Omega F(\nabla u)$$

with $u = g$ on $\partial\Omega$.

Similarly, geometry has been used as well in studying jumps introduced by minimizing the generalized Rudin Osher Fatemi functional

$$\int_\Omega F(\nabla u) + \frac{(u-g)^2}{2}, \tag{1}$$

see [4, 5].

In [6], we extend the results mentioned above to obtain continuity results for both problems. In this talk, we will present one of the results in [6] which deals with anisotropic least gradient problem in the spirit of [3].

Theorem 1. *Let φ be a norm in $\mathbb{R}^n$ which is $\mathcal{C}^2$ in $\mathbb{R}^n \setminus \{0\}$ and such that φ^2 is strongly convex. Let also Ω be a bounded Lipschitz open subset that satisfies some "φ-mean convexity property". Moreover, let g be continuous on $\partial\Omega$. Then, there is a unique minimizer u of*

$$u = \underset{\substack{v \in BV \\ v = g \text{ on } \partial\Omega}}{\arg\min} \int_\Omega \varphi(\nabla u) \tag{2}$$

where the equality $v = g$ on $\partial\Omega$ means, as in [3], *that*

$$\forall x \in \partial\Omega,\ \lim_{r\to 0} \operatorname*{ess\,sup}_{\substack{y\in\Omega \\ |x-y|\leqslant r}} |v(y) - g(x)| = 0.$$

In addition, this minimizer is continuous.

After introducing briefly the space BV of functions with bounded variation as well as finite perimeter sets, we introduce the anisotropic perimeter Per_φ and show that the level sets $\{u > t\}$ of the minimizers u of (2) are anisotropic minimal surfaces, that is minimizers of Per_φ.

Then, we show that two anisotropic minimal surfaces that are ordered with no contact in a neighborhood of $\partial\Omega$ cannot touch in the whole Ω. This property applied to $\{u > t\}$ and $\{u > s\}$ for $t \neq s$ proves that two level-sets cannot touch, which is exactly saying that the minimizer u of (2) is continuous.

The proof takes advantage of the regularity of minimizers of Per_φ up to a set of dimension at most $n-3$ [7].

If we have time, we will briefly discuss the results concerning the Rudin Osher Fatemi denoising model (1).

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Mappings with finite distortion in elastodynamics

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In this talk, we describe a new class of admissible mappings which are closed to the mappings with finite distortion. We apply the variational approximation scheme to the elastodynamic problem for a three dimensional torus using this new admissible class of mappings.

The so-called *variational approximation method* is a technique by which the sequence of variational problems for appropriate functionals are solved. Specifically, it allows the limit of a minimizing iteration sequence to be evaluated. The

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variational approximation scheme developed in [1] establishes a link between elastostatics and elastodynamics. The method is based on the observation that the solution to the elastodynamic system of equations

$$\frac{\partial}{\partial t}F_{i\alpha} = \frac{\partial}{\partial x_\alpha}v_i, \quad \frac{\partial}{\partial t}v_i = \sum_{\alpha=1}^{3}\frac{\partial}{\partial x_\alpha}S_{i\alpha}(F), \qquad i,\alpha = 1,2,3,$$

for the deformation differential $F = D_x y$ and the velocity $v = \partial_t y$, where $y\colon \Omega \times \mathbb{R}_+ \to \mathbb{R}^3$, $\Omega \subset \mathbb{R}^3$, is the displacement and S is the first Piola–Kirchgoff stress tensor, satisfies the *additional conservation laws*

$$\frac{\partial}{\partial t}\det F = \sum_{i,\alpha=1}^{3}\frac{\partial}{\partial x_\alpha}((\operatorname{cof} F)_{i\alpha}v_i),$$

$$\frac{\partial}{\partial t}(\operatorname{cof} F)_{k\gamma} = \sum_{i,j,\alpha,\beta=1}^{3}\frac{\partial}{\partial x_\alpha}(\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}F_{j\beta}v_i)$$

where ϵ_{ijk} stands for the sign of a permutation and $\operatorname{cof} F$ and $\det F$ are the cofactor matrix and the determinant of the matrix F respectively.

Further time discretization gives the variational problem

$$I(v,F,Z,w) = \int_\Omega \frac{1}{2}|v - v^0|^2 + G(F,Z,w)\,dx \to \min$$

for new variables $Z = \operatorname{cof} F$, $w = \det F$ with initial data $v^0(x)$, $F^0(x)$, $Z^0(x)$, $w^0(x)$ and the admissible set $\mathcal{C}$ involves "additional" constraints.

In this talk we present the results of [2] where we consider the variational approximation scheme using a new class of admissible mappings, prove existence and uniqueness theorems as well as derive the Euler–Lagrange equations (where we shall see that some additional requirements are necessary) in the case of smaller regularity ($p \geq 3$), finite distortion ($|F|^3/w \in L_s(\Omega)$, $s > 2$) and nonnegative Jacobian ($w \geq 0$) requirements.

These results are based on techniques and methods of papers [1, 3].

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Mapping connecting critical values of rational functions to thier critical values and poles

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Every rational function R in the complex plane $\mathbb{C}$ can be represented in the form

$$R(z) = C_0 \int_{z_0}^{z} \frac{\prod_{k=1}^{M}(\zeta - a_k)^{m_k - 1}\, d\zeta}{\prod_{j=1}^{N}(\zeta - b_j)^{n_j+1}} + C_1. \tag{1}$$

Here a_k are its critical points of order m_k and b_j are poles of order n_j. Denote by $A_k = R(a_k)$ critical values of R. It is well known that R defines a Riemann surface S, i.e. branched covering of the Riemann sphere non-ramified over $\mathbb{C} \setminus \{A_1, \ldots, A_M\}$. Given S, an important problem is to determine a rational function R uniformizing S. In essence, it is equivalent to the problem of finding parameters a_k and b_j in (1). Without loss of generality we can assume that $C_0 = 1$, $z_0 = a_1$, $C_1 = A_1$, and

$$\sum_{k=1}^{M}(m_k - 1)a_k - \sum_{j=1}^{N}(n_j + 1)b_j = 0. \tag{2}$$

Consider the set $\mathfrak{P}$ in $\mathbb{C}^{M+N}$ consisting of all $(a_1, \ldots, a_M, b_1, \ldots, b_N)$ satisfying (2) and the conditions

$$\operatorname{res}_{z=b_l} \frac{\prod_{k=1}^{M}(\zeta - a_k)^{m_k-1}}{\prod_{j=1}^{N}(\zeta - b_j)^{n_j+1}} = 0, \quad 1 \le l \le N,$$

and the mapping $f : \mathfrak{P} \to \mathbb{C}^M$ defined by

$$(a_1, \ldots, a_M, b_1, \ldots, b_N) \mapsto (A_1, \ldots, A_M).$$

Theorem. *The set $\mathfrak{P}$ is an M-dimensional submanifold in $\mathbb{C}^{M+N}$. The mapping f is non-degenerate at every point of $\mathfrak{P}$ and the differential of the locally inverse mapping is given by the formulas:*

$$da_l = \frac{H_l^{(m_l-1)}(a_l)}{(m_l - 1)!}\, dA_l + \sum_{k=1, k\ne l}^{M} \frac{G_{kl}^{(m_k-2)}(a_k)}{(m_k - 2)!}\, dA_k,\ db_j = \sum_{k=1}^{M} \frac{I_{kj}^{(m_k-2)}(a_k)}{(m_k - 2)!}\, dA_k,$$

where

$$H_l(x) = \frac{\prod_{j=1}^{N}(x - b_j)^{n_j+1}}{\prod_{k=1, k\ne l}^{M}(x - a_k)^{m_k-1}}, \quad G_{kl}(x) = \frac{H_k(x)}{x - a_l}, \quad I_{kj}(x) = \frac{H_k(x)}{x - b_j}.$$

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The theorem allows us to find approximately rational functions uniformizing given Riemann surface over the Riemann sphere (see [1] for the case of simple critical points).

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Symmetric Riemannian problem on the groups of proper motions of a sphere and the hyperbolic plane

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Let M be a two-dimensional surface of constant curvature $k = \pm 1$. So, M is a sphere or a hyperbolic plane. Consider it as a homogeneous space of the corresponding group of proper motions $M = G/\mathrm{SO}_2$, where G is SO_3 or $\mathrm{PSL}_2(\mathbb{R})$. We study the riemannian metric on G that is the lift of the metric on M. This metric is determined by its eigenvalues $I_1 = I_2$, $I_3 > 0$ at $\mathfrak{g} = T_{\mathrm{id}}G$. We investigate the global optimality of geodesics and find the cut locus and cut time for such metrics. When $I_3 \to +\infty$, the cut time and the cut locus converge to the cut time and the cut locus of the sub-Riemannian problems described by V. N. Berestovskii, I. A. Zubareva [1, 2] and U. Boscain, F. Rossi [3].

Let $\eta = k\frac{I_1}{I_3} - 1$ be the parameter reflecting oblateness of small spheres of our metric. Note that for $\mathrm{PSL}_2(\mathbb{R})$ we have $\eta \in (-\infty, -1)$ and for SO_3 we have $\eta \in (-1, +\infty)$. Let $e_1, e_2, e_3 \in \mathfrak{g}$ be eigenvectors of the metric. We fix the isomorphism of $\mathfrak{g}$ and $\mathfrak{g}^*$ via the Killing form. Now $p = p_1e_1 + p_2e_2 + p_3e_3 \in \mathfrak{g}^*$ is the initial momentum of a geodesic.

Proposition 1. *The geodesic starting at* id *with an initial momentum p is the product of two one-parametric subgroups:*

$$Q(t) = \exp\left(\frac{tp}{I_1}\right)\exp\left(\frac{t\eta p_3 e_3}{I_1}\right).$$

Theorem 1. *For the left-invariant metric on G with eigenvalues $I_1 = I_2$, $I_3 > 0$ the cut locus is:*

(1) *the plane P_{-1} of central symmetries of a hyperbolic plane if $\eta \in (-\infty, -\frac{3}{2}]$;*

(2) *the union of P_{-1} and the interval H_η of rotations of a hyperbolic plane around fixed point by angles $\pm\varphi$, where $\varphi \in [-2\pi(1+\eta), \pi]$, if $\eta \in (-\frac{3}{2}, -1)$;*

(3) *the union of the projective plane* P_1 *of central symmetries of a sphere and the interval* S_η *of rotations of a sphere around fixed axis by angles* $\pm\varphi$*, where* $\varphi \in [2\pi(1+\eta), \pi]$*, if* $\eta \in (-1, -\frac{1}{2})$*;*
(4) *the projective plane* P_1 *of central symmetries of a sphere if* $\eta \in [-\frac{1}{2}, +\infty)$*.*

Note that $G = \mathrm{PSL}_2(\mathbb{R})$ in cases (1–2) and $G = \mathrm{SO}_3$ in cases (3–4).

The proof of this theorem is based on considering the symmetry group of the Hamiltonian vector field of Pontryagin maximum principle, finding its fixed points (Maxwell points), verification that the Maxwell time is less or equal to the conjugate time, then considering an open set bounded by Maxwell time in the preimage of the exponential map, which is diffeomorphic to the complement of the closure of the Maxwell strata in G. This method was presented by Yu. L. Sachkov for the Euler elastic problem [4]. Cases (3–4) were obtained in our paper [5]. For computations we use quaternions and split-quaternions for SO_3 and $\mathrm{PSL}_2(\mathbb{R})$ respectively, so also we have results for cut loci on double coverings SU_2 and $\mathrm{SU}_{1,1} \simeq \mathrm{SL}_2(\mathbb{R})$ of SO_3 and $\mathrm{PSL}_2(\mathbb{R})$ respectively. We have computed the diameter of SO_3 and the injectivity radius for $\mathrm{PSL}_2(\mathbb{R})$.

Consider the left-invariant sub-Riemannian structure on G generated by the plane $\mathrm{span}\{e_1, e_2\}$ and the restriction of the Killing form to this plane.

Proposition 2. *Parametrization of geodesics, conjugate time and locus, cut time and locus in the sub-Riemannian problems on* SO_3 *and* $\mathrm{PSL}_2(\mathbb{R})$ *are obtained from the Riemannian ones by passing to the limits* $\eta \to -1+0$ *and* $\eta \to -1-0$ *respectively.*

When $\eta \to -1$, the intervals H_η and S_η converges to the punctured circle $S^1 \setminus \{\mathrm{id}\}$ that is the local component of the cut locus of the sub-Riemannian metric for both groups. The global component of the cut locus (P_{-1} or P_1) is the same for the Riemannian and sub-Riemannian metrics.

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On controllability in a mass transportation problem

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Consider a probability measure θ on $\mathbb{R}^n$ that drifts along a smooth vector field $x \mapsto f(x, u)$. The evolution of θ can be described by the continuity equation

$$\partial_t \mu + div\,\big(f(x, u)\mu\big) = 0 \tag{1}$$

with the initial condition $\mu(0) = \theta$. We address the following question: is it possible steer θ to a given measure, using u as a control?

In the talk we will discuss some connections of this issue with the problems of controllability on the group of diffeomorphisms and controllability of ensembles. For example, using the results of [1], one can proof the following sufficient condition for approximate controllability.

Theorem. *Let f have the form*

$$f(x, u) = u_1 f_1(x) + \cdots + u_m f_m(x),$$

where $f_1, \ldots, f_m$ are bracket generating vector fields on $\mathbb{R}^n$ with $n \neq 4$. Then, for any two nonatomic compactly supported probability measures θ_1, θ_2 and any $\varepsilon > 0$, there exist a feedback control $u = u(t, x)$ and a time moment T such that

$$\mathcal{W}_1\big(\mu(T; u, \theta_1), \theta_2\big) < \varepsilon,$$

where $\mu(\cdot, u, \theta_1)$ is the solution of (1) that corresponds to the initial condition $\mu(0) = \theta_1$ and the control law $u = u(t, x)$, $\mathcal{W}_1$ is the Wasserstein distance.

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Curves with affine-congruent arcs

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The curve in n-dimentional affine space A^n is called the curve with affine-congruent arcs (CwACA) if any two its oriented arcs are transformable into each other by an affinity.

Hypothesis. *The set of all CwACA in A^n coincides with the class of curves, defined in some affine frame by one of the parametrisations*

$$\vec{x} = (t, t^2, \dots, t^k, 0, \dots, 0), \quad k = 1, 2, \dots, n. \tag{1}$$

Certainly, every curve, affine-congruent of CwACA, is itself CwACA.

The author found:

1. CwACA in A^n, having a flattening of order k point, $k < n$, is contained in the k-dimensional plane.

2. Curves in A^n, defined by (1), are CwACA.

They lie on the boundary of their convex hulls.

3. The curve, defined in some affine frame by

$$\vec{x} = \left(t, \frac{t^2}{2!}, \dots, \frac{t^n}{n!}\right),$$

being affine-congruent of (1), when $k = n$, is CwACA, is nondegenerate (i. e. does not lie in any hyperplane in A^n), and is parameterized by affine arclength (i. e. $\det[\vec{x}', \vec{x}'', \dots, \vec{x}^{(n)}] = 1$). All its affine curvatures are

$$k_1 = k_2 = \ldots = k_n = 0 = const.$$

4. The hypothesis is true for $n = 2$ and $n = 3$.

For $n = 3$ the assertion is proved under the assumption of C^3-smoothness of the curve. For $n = 2$ a priori regularity of the curve is not required.

5. The validity of the hypothesis yields that straight lines are the only curves in A^n, any two arcs of which are similar-congruent.

The moduli and capacity on Riemann surface

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Let $\mathcal{R}$ be a Riemann surface glued from finitely or countably many domains in the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ so that the following conditions are satisfied: each point in $\mathcal{R}$ projects onto a point $w = \operatorname{pr} W$ in one of the glued domains, each point in $\mathcal{R}$ has a neighbourhood which is a univalent disk or a multivalent disk with the unique ramification point at the centre of the disk.

The operation of projection $W \to \operatorname{pr} W = w$ induces the Lebesgue 2-dimensional measure σ, the Hausdorff 1-dimensional measure $\mathcal{H}^1$, the Euclidean metric ds on $\mathcal{R}$.

Set $\mathcal{R}_\infty = \{W \in \mathcal{R} : \operatorname{pr} W = \infty\}$. Below, let G be an open set in $\mathcal{R}$ such that $\overline{G}$ is a compact set in $\mathcal{R}$.

A curve γ in Γ is defined as the image of the number interval (a, b) under a continuous mapping $W = W(t)$ into $\mathcal{R}$. Let $\gamma \subset G$ and $T = \{t \in (a, b) : W(t) \notin \mathcal{R}_\infty\}$. Then T consists of at most countable number of open intervals T_j, $j \geq 1$, and $W(t)$, $t \in T_j$, represents a curve $\gamma_j \subset G$. For all $j \geq 1$ γ_j is a locally rectifiable curve in G. A condenser in G is an ordered triple of sets (F_0, F_1, G) where F_0, F_1 are disjoint compact sets in $\overline{G}$. For a condenser (F_0, F_1, G) we denote as $\Gamma(F_0, F_1, G)$ a family of all locally rectifiable curves connecting F_0 and F_1 in G.

For $p \in (1, +\infty)$ we define the modulus of $\Gamma(F_0, F_1, G)$ by the equality

$$m_p(F_0, F_1, G) = \inf \int_G \rho^p \, d\sigma,$$

where the infimum is taken over all nonnegative Borel functions $\rho : G \to [0, +\infty]$ satisfying $\int_\gamma \rho \, ds \geq 1$ for all $\gamma \in \Gamma(F_0, F_1, G)$.

The capacity of the condenser (F_0, F_1, G) is defined by

$$C_p(F_0, F_1, G) = \inf \int_G |\nabla u|^p \, d\sigma = \inf \int_{G_0} |\nabla u|^p \, d\sigma,$$

where the infimum is taken over all functions $u : G \to [0, 1]$ such that u are equal to j on a neighbourhood of F_j, $j = 0, 1$; and are locally Lipschitz in G.

For a condenser (F_0, F_1, G) let λ be any compact set in $\mathcal{R}$ such that $\mathcal{R} \setminus \lambda = A_0 \cup A_1$, where A_0, A_1 are disjoint open sets, and $F_0 \subset A_0$, $F_1 \subset A_1$. The λ is called a set separating F_0 and F_1 in $\mathcal{R}$. Respectively, the set $\tau = \lambda \cap G$ is called the set separating F_0 and F_1 in G.

By using the technique of Ziemer [1], and the results from [2], we obtain the following assertions.

Theorem 1. *$C_p(F_0, F_1, G) < \infty$ and $m_p(F_0, F_1, G) = C_p(F_0, F_1, G)$.*

Theorem 2. *$C_p(F_0, F_1, G)^{\frac{1}{p}} \cdot M_q(F_0, F_1, G)^{\frac{1}{q}} = 1$.*

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Conformally flat metrics of positive curvature and quasiconformal mappings of the sphere

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Theorem 1. *Let* $ds^2 = \frac{dx^2}{f^2(x)}$ *be a conformally flat metric of positive* [1] *one-dimensional curvature on the unit sphere* $S^n \subset \mathbb{R}^{n+1}$*, and the function* f *is continued to the function* $f : \mathbb{R}^{n+1} \to \mathbb{R}$ *according homogeneity. Then the transformation* $H_f : S^n \to S^n$*, which is determined by the formula:*

$$H_f : \vec{x} \in S^n \to \vec{x} - 2f(x)\frac{\vec{\nabla} f}{|\vec{\nabla} f|^2} \in S^n,$$

where $\vec{\nabla} f$ *– gradient* f *in* R^{n+1}*, is a diffeomorphism of the sphere. The coefficient of quasiconformality of* H_f *is calculated by the formula:*

$$q(H_f) = \max_x \frac{\max_i\{k_i(x)\}}{\min_j\{k_j(x)\}},$$

where $\{k_i(x)\}$*,* $i = 1, \ldots, n$*, are the principal values of one-dimensional curvature at the point* $x \in S^n$*.*

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On the traces of Sobolev functions on the hypersurfaces

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Consider the unit cube $Q \subset R^n$ and the Sobolev space $W^1_p(Q)$. If $p \le n$ Sobolev space $W^1_p(Q)$ contains discontinuous functions and the space of traces on an arbitrary cross section S cube Q by hyperplane coincides with the Besov space $B^{1-1/p}_p(S)$, which also contains discontinuous functions. However, if $n-1 < p \le n$ for an arbitrary function $u \in W^1_p(Q)$ there is an equivalent quasi-continuous function $\tilde{u}$, which is Hölder with exponent $\alpha_p = 1-(n-1)/p$ on almost all sections

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orthogonal to an axis. This fact is a simple consequence of Fubini's theorem and embedding theorems for Sobolev spaces to space of continuous functions. It is obvious that there must be the relationship between the Hölder exponent $0 < \alpha \leq \alpha_p$ and the dimension of the set of the "bad" sections on which the function does not belong C^α.

Let $n-1 < p \leq n$, $0 < \alpha \leq \alpha_p$, $d > n-(1-\alpha)p$, the set $E \subset (-1,1)$ is d-set, i.e. there is such a measure ν on E, that $C_1 r^d \leq \nu(E \cap B(x,r)) \leq C_2 r^d$, $x \in E$. For every $t \in E$, consider the cross section $G_t = \{x \in Q \mid x_n = t\}$.

Theorem 1. *If $n-1 < p \leq n$, $0 < \alpha \leq \alpha_p$ and the function $u \in W_p^1(Q)$, then $u|_{G_t} \in C^\alpha(G_t)$ for ν-almost all $t \in E$.*

Note that if we change the Hölder exponent α from zero to α_p then the Hausdorff dimension d varies from $n-p$ to 1. When $p = n-1$ a function $u \in W_{n-1}^1(Q)$ may have irremovable discontinuity points on all sections G_t, $t \in (-1,1)$. In this case we should consider the Sobolev class of functions, which gradients belong to the Lorentz space $L_{n-1,1}(Q)$.

Theorem 2. *If the function u belongs to the Sobolev-Lorentz space $W_{n-1,1}^1(Q)$ then u is $(n-1)$-absolutely continuous on almost all sections Q_t.*

Following [1] the family of all Lipschitz hypersurfaces in the unit cube $Q \subset R^n$ denoted by Σ and the p-module family of hypersurfaces $\Gamma \subset \Sigma$ by symbol $M_p(\Gamma)$.

Family $\Gamma \subset \Sigma$ is called p-exceptional, if its p-module is zero, $M_p(\Gamma) = 0$. It is said that some property holds for p-almost all hypersurfaces family Γ, if the set hypersurfaces $S \in \Gamma$, for which the property is not satisfied, a p-exceptional.

Theorem 3. *Let $n-1 < q \leq p \leq n$. If the function $u \in W_p^1(Q)$, then it locally satisfies a Hölder condition with exponent $\alpha = 1 - \frac{n-1}{q}$ on p/q-almost all Lipschitz hypersurfaces $S \in \Sigma$.*

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Sobolev type embedding theorems and rigidity estimates in metric case

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In [1, 2] a new definition of Sobolev spaces in measure-metric case was formulated. The definition appeared to be convenient to prove embedding theorems and other important estimates in the most general situation including the case when the measure does not satisfy the doubling condition.

This approach is productive even in Euclidean case. To prove embedding theorems one can prove equivalence of the classical definition and the definition in [1, 2] and to use some estimates directly following from the definition in [1, 2]. For the case of domains with singularities and for some weighted cases this approach appeared to be simpler then the classical one.

This approach could be used also to prove some important estimates, in particular Gagliardo-Nirenberg estimates and some estimates that could be used to prove rigidity theorems in geometry. In the talk we consider some inequalities generalizing Sobolev imbedding theorems that could be used to prove rigidity of quasiisometries and other classes of mappings.

Let (X, ρ_X, μ) is a measure-metric space, μ is a borel measure, (Y, ρ_Y) is a separable metric space, $V \subset X$ is a completely bounded set. We consider a general case of mappings from V to Y, see also [3].

Definition 1. We say that a sequence $\Xi = \{\sigma_0, \sigma_1, \sigma_2, \dots, \sigma_j, \dots\}$ of partitions of a set V into disjoint μ-measurable sets E_i^j, $i = 1, \dots, n(j)$, satisfies d-condition, if for each i partition σ_{i+1} is subpartition of σ_i and the following inequalities hold $\operatorname{diam}(E_i^j) \leq C_1 10^{-j}$, $\mu(E_i^j) \geq C_2 10^{-jd}$. We suppose that $\sigma_0 = \{V\}$.

Definition 2. Let us fix a family of mappings $\mathfrak{A}$ of a set V to Y, such that for some constants C_1 and C_2 for any mapping $A \in \mathfrak{A}$ and a set $E_i^k \in \sigma_k$ the following inequality holds $\sup_{x \in E_i^k} |A(x)| \leq C_1 \inf_{x \in E_i^k} |A(x)| + C_2$. Denote by G_{σ_k} the set that consists of all functions on every set E_i^k of partition σ_k equal to a function from $A_i^k \in \mathfrak{A}$.

Definition 3. Let $p \in [1, \infty)$, $r \in (0, 1]$, a sequence $\Xi = (\sigma_0, \sigma_1, \dots, \sigma_j, \dots)$ satisfies d-condition, $u \in L_p(V, Y)$. Denote sets in partition σ_j by E_i^j, $i = 1, \dots, m(j)$. Let $A_i^j \in \mathfrak{A}$ be the best approximation in $\mathfrak{A}$ of a mapping u to the distance of $L_p(E_i^j, Y)$. Suppose that there exists a function $h^r \in L_p(V)$ such that for any set $E_i^j \in \sigma_j$ the following inequality holds

$$\int\limits_{E_i^j} \frac{d(u(x), A_i^j(x))^p}{10^{jrp}} \, d\mu(x) \le \int\limits_{E_i^j} (h^r(x))^p \, d\mu(x).$$

Then we will write $u \in W^{r,p}_{\Xi,\mathfrak{A}}(V)$. Any function $h^r(x)$ satisfying (1) we will call an upper gradient of mapping u of order r. By seminorm in space $W^{r,p}_{\Xi,\mathfrak{A}}(V)$ we will call the minimum of norms $\|h^r\|_{L_p(U)}$ among all functions $h^r(x)$ satisfying the inequality (1), i. e. among all upper gradient of mapping u of order r, and denote by $[u]_{W^{r,p}_{\Xi,\mathfrak{A}}(V)}$ or $[u]^{r,p}_{\Xi,\mathfrak{A}}$. Denote $\|u\|_{W^{r,p}_{\Xi,\mathfrak{A}}(V)} := \|u\|_{L_p(V,Y)} + [u]_{W^{r,p}_{\Xi,\mathfrak{A}}(V)}$.

Lemma 1. *Let $p \in [1, \infty)$, $r \in (0, 1]$, $\Xi = (\sigma_0, \sigma_1, \sigma_2, \dots, \sigma_j, \dots)$ satisfies d-condition. Suppose that $C_1 = 1$, $C_2 = 10^d$. Let $rp < d$. Let $u \in W^{r,p}_{\Xi,\mathfrak{A}}(V, \rho, \mu)$ and $[u]^{r,p}_{\Xi,\mathfrak{A}} \le 2^{-l}$, where $l = 2 + \frac{d}{d-rp}$. By g_k denote the function best approximating the function u among all functions from G_{σ_k} to the distance $L_p(V, Y)$, in particular the function $g_0(x)$ belongs to $\mathfrak{A}$. Then for any integer k, satisfying the inequality*

$$k - 1 \le \frac{p}{d - rp} \lg(A)$$

the set $\{x \in U \mid d(g_k(x), A_0(x)) \ge A/2\}$ is empty.

Lemma 2. *Suppose that the conditions of Lemma 1 are fulfilled. Fix an integer $i \ge 0$. Denote by V_i^k the set of all points $x \in V$ such that $d(g_k(x), A_0(x)) \in [10^i, 10^{i+1})$. Denote by $h^r(x)$ an upper gradient of function u of order r. Let $q \ge p$. Then the following inequality holds*

$$\int\limits_{V_i^k} d(g_k(x), A_0(x))^q \, d\mu(x) \le 2^p 10^{(q-p)i} 10^{-\frac{rp^2 i}{d-rp}} \int\limits_{V_i^k} (h^r(x))^p \, d\mu(x)$$

$$= 2^p 10^{\left(q - \frac{dp}{d-rp}\right)i} \int\limits_{V_i^k} (h^r(x))^p \; d\mu(x).$$

Theorem 1. *Let $p \in [1, \infty)$, $r \in (0, 1]$, $\Xi = (\sigma_0, \sigma_1, \sigma_2, \dots, \sigma_j, \dots)$ satisfy d-condition. Suppose that $rp < d$. Then $W^{r,p}_{\Xi,\mathfrak{A}}(V, \rho, \mu)$ is continuously embedded into $L_{\frac{dp}{d-rp}}(V, \mu)$ and the following inequality holds*

$$\left(\int\limits_V d(u(x), g_0(x))^q \, d\mu(x)\right)^{\frac{1}{q}} \le C[u]^{r,p}_{\Xi,\mathfrak{A}}, \text{ where } C = (C_1)^r \left(2^{lq}\left(\frac{10^d \mu(V)}{C_2} + 2^{(1-l)p}\right)\right)^{\frac{1}{q}}.$$

Remark. If $Y = \mathbb{R}$, $\mathfrak{A}$ is a set of constants Theorem 1 is an analogue of the classical Sobolev embedding theorem. If in addition (X, ρ, μ) is Euclidean space with Lebegue measure then it coincides with the classical Sobolev embedding theorem.

If $X = Y = \mathbb{R}^n$, $\mathfrak{A}$ is a set of isometries then from the theorem 1 directly follows the rigidity estimates for isometries, see also [4, 5, 6].

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Integrability and optimal synthesis in sub-Riemannian geometry

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The talk is devoted to the two recent results, one negative and one positive, in sub-Riemannian geometry.

For left-invariant sub-Riemannian structures on free nilpotent Lie groups we consider a problem of Liouville integrability of vertical subsystem of the Hamiltonian system from Pontryagin's maximum principle on orbits of coadjoint representation of maximum dimension. All cases of integrability and nonintegrability are fully described.

For the left-invariant sub-Riemannian problem on Engel group we constructed an optimal synthesis and cut locus, started investigation of a sphere. This is the first result of its kind on three-step Lie groups.

Isoperimetric inequalities in torsion problem

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In 1951 G. Pólya and G. Szegö [1] expressed a lower bound for the torsional rigidity of a convex plane domain in terms of the area and inradius of the domain. Only in 1962 Makai [2] proved an upper bound for the torsional rigidity in the same terms. On the other hand, in 1995 F. G. Avkhadiev [3] showed that the torsional rigidity and the Euclidean moment of inertia are comparable quantities for simply connected domains. In our report we discuss various isoperimetric inequalities classical and new in the torsion problem. As an application of our results we discuss approximate formulas for the torsional rigidity.

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Geometry of nonregular weighted Carnot-Carathéodory spaces and applications

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We give an overview of recent results on local geometry of weighted Carnot-Carathéodory spaces in a neighbourhood of a nonregular point and discuss possible applications to control systems $\dot{x} = f(x, u)$ nonlinear on control functions $u(t)$. The considered geometry is a generalization of classical sub-Riemannian geometry, which naturally arises in affine control problems $\dot{x} = \sum_{i=1}^{n} u_i(t)X_i(x)$. Here vector fields X_i span the horizontal subbundle of the considered sub-Riemannian space and are the "allowed" motion directions.

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Conformal maps and variational method of studying equilibrium forms

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Variational method is well known [1] to be highly useful in studying of separating surfaces of two-phase environments in the state of equilibrium. In [2] the classical theory is generalized from the axisymmetric case to the case when intermediate layer is taken into consideration. In [3] not only existence of intermediate layer is reckoned but its elasticity as well. In the proposed research we study equilibrium forms which do not possess axial symmetry. We prove the theorem on existence of almost global semigeodesical parameterisation of surfaces. This result allows to introduce a functional that in the case of axisymmetric surfaces considers the impact of the intermediate layer. We showed the the Euler’s condition for defining a form of the equilibrium surface is a generalization of the classical Laplace condition and is a non-linear equation principal part of which is defined by Laplace-Beltrami operator. The method we use can be used for studying generalized minimal surfaces [4].

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An open mapping theorem for the Navier-Stokes equations

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We consider the Navier-Stokes equations in the layer $\mathbb{R}^n \times [0, T]$ over $\mathbb{R}^n$ with finite $T > 0$. Using the standard fundamental solutions of the Laplace operator and the heat operator, we reduce the Navier-Stokes equations to a nonlinear Fredholm equation of the form $(I + K)u = f$, where K is a compact continuous operator in anisotropic normed Hölder spaces weighted at the point at infinity with respect to the space variables. Actually, the weight function is included to provide a finite energy estimate for solutions to the Navier-Stokes equations for all $t \in [0, T]$. On using the particular properties of the de Rham complex we conclude that the Fréchet derivative $(I + K)'$ is continuously invertible at each point of the Banach space under consideration and the map $I + K$ is open and injective in the space. In this way the Navier-Stokes equations prove to induce an open one-to-one mapping in the scale of Hölder spaces.

A duality principle in the weighted Sobolev spaces on the real line

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We analyse characterization of an embedding inequality of Sobolev type with help of a duality principle and boundedness criteria for the Hardy-Steklov integral operator in weighted Lebesgue spaces.

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Metrics ρ, quasimetrics ρ^s and pseudometrics $\inf \rho^s$

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Let $\rho(x, y) : M \times M \to [0, \infty)$ be a metric in a space M, $s > 0$. For $s > 1$ the function ρ^s is possibly not a metric, because it does not necessarily satisfy the triangle inequality. However, it is a q-quasimetric due to fulfilment of the "q-triangle inequality" $\forall x, y, z \in M \;\; \rho^s(x, z) \leq q \cdot (\rho^s(x, y) + \rho^s(y, z))$ for $q = 2^{s-1}$. Such quasimetrics are a particular case of f- quasimetrics (that is, such functions ρ that satisfy the "f-triangle inequality" $\rho(x, z) \leq f(\rho(x, y) + \rho(y, z))$, where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a function such that $f(t) \underset{t \to 0}{\to} f(0) = 0$). The topology on a f-quasimetric space is defined the same way as in a metric space: a set U is considered to be open, if its each point is included in U together with some ε-ball $B(x, \varepsilon) = \{y \mid \rho(x, y) < \varepsilon\}$.

Denote by $\inf \rho(A, B)$ for such ρ the infimum of the lengths of polygonal chains that connect two points $A, B \in (M, \rho)$. These polygonal chains are understood as the sets of points $\{A = z_0, \ldots, z_n = B\} \subset (M, \rho)$; the length of a polygonal chain is the sum of the lengths of its segments: $|L| = \sum_{i=1}^n \rho(z_{i-1}, z_i)$. Thus, $\inf \rho(A, B) = \inf\{\sum_{i=1}^n \rho(z_{i-1}, z_i) \mid n < \infty, z_0, \ldots, z_n \in M, z_0 = A, z_n = B\}$.

We construct a set $M \subset \mathbb{R}^2$, on which there is a continuum of pairwise different topologies induced by the metrics $\inf d^s$, $s \geq 1$.

Example (comb). Let us construct a set $M \subset \mathbb{R}^2$, on which there is a continuum of pairwise different topologies induced by the metrics $\inf d^s$, $s \geq 1$. For each $n = 0, 1, 2, \ldots$ consider a dashed ray $L_n = \{A_{n0}, A_{n1}, \ldots\}$ in $\mathbb{R}^2$, which goes upward from the point $A_n = A_{n0} = (n, 0) \in \mathbb{R}^2$, $A_{nk} = (n, k \cdot 2^{-n}) \in \mathbb{R}^2, k = 0, 1, 2, \ldots$ The base A_n of each ray L_n is joined with the point $z = (0, -1)$ by a dashed segment R_n consisting of 2^n equal parts. Define $M = \{z\} \cup \bigcup_{n \geq 1}(L_n \cup R_n)$.

Theorem 1. *Let $\alpha > 1$. Put $z_n = (n, [2^{(\alpha-1)n}])$ (here the square brackets denote the integer part of a number). The sequence z_n converges to z in the metric $\inf d^s$ if and only if $s > \alpha$. Therefore, for different s the topologies defined by the metric $\inf d^s$ on M are different: if $1 \leq s < \tilde{s}$, then the identity map $Id : (M, \inf d^s) \to (M, d_{\tilde{s}})$ is continuous, but the inverse map is discontinuous.*

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Direct approach to the field theory in the Calculus of Variations

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Consider an integral functional

$$J(u) = \int_a^b L(x, u(x), \dot{u}(x))\, dx,$$

where $L(x, u, v) \in C^3([a, b] \times \mathbb{R} \times \mathbb{R})$, $L_{vv} > 0$. Recall that C^1-regular minimizer of such functional satisfies the Euler-Lagrange equation

$$\frac{d}{dx} L_v(x, u(x), \dot{u}(x)) = L_u(x, u(x), \dot{u}(x)). \tag{1}$$

We prove the following two theorems in the spirit of the field theory.

Theorem 1. *Let $G = \{(x, u) : a \leq x \leq b, g(x) \leq u \leq f(x)\}$, where $f > g$ in $[a, b]$, $f, g \in C^3[a, b]$.*

If there is a family of C^1-regular extremals (solutions of (1)) which covers G without intersections, then each representative $u : [a, b] \to \mathbb{R}$ of the family is a minimizer of the functional in the class of functions $w \in W^{1,1}$, graphs of which belong to G and w satisfies the same boundary conditions as u.

Theorem 2. *Let L, G be as in the previous theorem. Assume that a family of Sobolev functions covers G without intersections, where all functions except one $u_0 : [a, b] \to \mathbb{R}$ are C^1-regular extremals of the functional and u_0 is not Lipschitz (the derivative is essentially unbounded). Then u_0 has Tonelli's regularity, i.e. it has everywhere classical derivative (possibly infinite) and the derivative is continuous as a function with values in $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$; in the open set of full measure, where $|\dot{u}| < \infty$, the function u_0 satisfies (1). For any Lipschitz $u : [a, b] \to \mathbb{R}$ which have the same boundary conditions and with graph in G we have*

$$J(u) > J(u_0). \tag{2}$$

If $L(x, u, v)$ has superlinear growth in v for (x, u) sufficienly close to the graph of u_0 then (2) holds also for any $u \in W^{1,1}[a, b]$ with the same boundary conditions as u_0 and with graph in the set G.

The spaces ordered by metrics

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By $|x - y|$ we mean the distance between x and y – elements of an arbitrary metric space.

The sequence $(x_1, \ldots, x_k)$ is λ-relative, if $x_i \neq x_{i+1}$ and satisfies the inequalities

$$\frac{|x_{i-1} - x_{i+1}|}{\min\{|x_i - x_{i-1}|, |x_i - x_{i+1}|\}} \leq \lambda.$$

Definition. Let A be a metric space and $P(A)$ be a set of ordered pairs of A, i.e. $P(A) = \{(x, y) \in A \times A, x \neq y\}$.

The λ-component or the component of λ-relative connectedness is a subset γ of pairs from $P(A)$ such that any two pairs in γ are connected by λ-relative sequence with an even number of members. The λ-component is called oriented if it contains no pair (x, y) and (y, x) simultaneously. Fixing a pair $(x, y) \in \gamma$ and assuming $x < y$, we define a partial order on A.

Definition. The sequence $(x_0, \ldots, x_k)$ in $\mathbf{R}$ is called alternating if for every $i = 1, \ldots, k - 1$ the inequalities $x_i < x_{i\pm1}$ or $x_i > x_{i\pm1}$ hold, that is, the signs of inequalities alternate.

Lemma 1. *Let $\lambda < 2$. Then any $\lambda-$relative sequence of $\mathbf{R}$ is alternating.*

The Cantor set C_α is built throwing the median segment of length αd out of segment of length d. Thus, C_α is the union of two similar sets with similarity coefficients $p = (1 - \alpha)/2$.

Theorem 1. *The set C_α is λ-relatively connected if and only if $\lambda \geq 2\alpha/(1 - \alpha)$.*

Theorem 2. *The set $A \subset R$ allows the extension of all M-bilipschitz functions $f : A \to R$ to $M - BL$ functions $F : R \to R$ for all $M < N$ if and only if $\overline{A}$ is λ-connected with $\lambda = 2/(N^2 - 1)$.*

Logarithmic Gauss mapping for complex surfaces and solutions to algebraic equations

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To every nonsingular point z of a complex surface V in the algebraic torus $(\mathbb{C} \setminus 0)^n$ the logarithmic Gauss mapping associates complex normal subspace to the logarithmic image $\log(V)$ in $\log(z)$. Using parameterization of classical discriminant set as inverse to logarithmic Gauss mapping we obtain rational expression for multiplicity roots of a general algebraic equation in the form

$$\frac{\Delta_2}{\Delta_1} = \cdots = \frac{\Delta_{n-1}}{\Delta_{n-2}},$$

where Δ_i is a derivative discriminant by i-th equation coefficient. The same result is obtained for general system of n algebraic equations of n variables.

These results are obtained in collaboration with I. A. Antipova and E. N. Mikhalkin.

On Whitney-type problem for Sobolev spaces

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Let $S \subset \mathbb{R}^n$ be a closed set such that for some $0 \leq d \leq n$ and $\varepsilon > 0$ the d-Hausdorff content $\mathcal{H}^d_\infty(S \bigcap B) \geq \varepsilon r^d$ for any ball B centered in S of volume $|B| \leq 1$. Let $W^l_p(\mathbb{R}^n)$ ($p \in (1, \infty)$, $l \in \mathbb{N}$) be the Sobolev space on $\mathbb{R}^n$. For each $p > n - d$ and each $l \in \mathbb{N}$ we give an intrinsic characterization of the restrictions $\{D^\alpha F|_S : |\alpha| \leq l - 1\}$ to S of $(l - 1)$-jets generated by functions $F \in W^l_p(\mathbb{R}^n)$. In particular, for $p > n - 1$ we characterize the trace space of the Sobolev space $W^1_p(\mathbb{R}^n)$ to the closure $\overline{\Omega}$ of arbitrary open path-connected set Ω.

Errors, condition numbers, and guaranteed accuracy of higher-dimensional spherical cubatures

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On the unit sphere S of $\mathbb{R}^n$ for $n \geq 2$ we consider the cubature formulas of the form

$$\frac{1}{\sigma_{n-1}} \int_S \varphi(\theta)\, dS \approx \sum_{j=1}^{N} c_j \varphi(\theta^{(j)}), \qquad \sum_{j=1}^{N} c_j = 1. \tag{1}$$

Here $\sigma_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ is the area of S, $\theta \in S$; $\theta^{(j)} \in S$ are the nodes of the formula, while c_j are its nonzero weights. The rules specifying the nodes $\theta^{(j)}$ and weights c_j of (1) are independent of the choice of the concrete integrand $\varphi(\theta)$. Assume that the set of integrands amounts to a Banach space $X = X(S)$ that is boundedly embedded into the space $C(S)$ of continuous functions on S. In other words, there exists a finite embedding constant, that is, the minimal positive number A_n satisfying $\sup\limits_{\theta \in S} |\varphi(\theta)| \leq A_n \|\varphi \mid X(S)\|$ for all $\varphi \in X(S)$. If the identically-one function belongs to $X(S)$, and has norm 1 then $A_n \geq 1$. It is natural to regard (1) as some standard approximation to the singular generalized function $M_S(x)$ whose action on a test function $\varphi(x)$ amounts to the spherical average over S of the function itself: $(M_S, \varphi) = \frac{1}{\sigma_{n-1}} \int\limits_S \varphi(\theta)\, dS$. As typical tools for this standard approximation we use linear combinations of shifts of Dirac's delta-function $\delta(x)$, that is, the cubature sums of the form

$$(\Sigma_N, \varphi) \equiv \left(\sum_{j=1}^{N} c_j \delta(x - \theta^{(j)}), \varphi(x) \right) = \sum_{j=1}^{N} c_j \varphi(\theta^{(j)})$$

for all $\varphi \in C(S)$. The error of approximation to the generalized function $M_S(x)$ by the cubature sum $\Sigma_N(x)$ is theoretically studied [1] by using estimates for the values at concrete elements φ in $X(S)$ of the error functional l_N of the formula which is defined as $(l_N, \varphi) = (M_S - \Sigma_N, \varphi)$ for all $\varphi \in X(S)$. The natural domain of the error functional l_N is $C(S)$ where l_N is linear and bounded. The error formula on an arbitrary function φ in $X(S)$ satisfies

$$|(l_N, \varphi)| \leq \|l_N \mid X(S)^*\| \cdot \|\varphi \mid X(S)\| \qquad \forall\, \varphi \in X(S). \tag{2}$$

But for specified values of the integrand φ at the nodes we can in reality only approximate the average (M_S, φ) by the cubature sum (Σ_N, φ) by calculating this cubature sum. In the general case this would require that we perform N

multiplications and N additions of real numbers, which is only approximately possible. Denote by $(\widetilde{\Sigma}_N, \varphi)$ the result of the calculation of the cubature sum and observe that in general this quantity need not coincide with the original sum (Σ_N, φ). In other words, together with the theoretical error a new practical error arises.

The practical error of a cubature formula is characterized not by the functional l_N, but by a certain analog of it, the functional $\widetilde{l}_N$: $(\widetilde{l}_N, \varphi) = (M_S - \widetilde{\Sigma}_N, \varphi)$ for all $\varphi \in X(S)$. The practical error functional $\widetilde{l}_N$ is nonlinear and fails to coincide with the original functional l_N. We cannot apply (2) to it. Nevertheless, if we wish to obtain the action (M_S, φ) numerically and with guaranteed accuracy then we should have some analog of (2) to estimate the practical error. In order to propose this analog, we must understand in more detail the reasons for the distinction between the practical and theoretical errors.

The first and principal reason for the difference consists in the necessity of approximating real numbers by the elements of some finite and discrete subset $\mathbf{F}$ of the real axis $\mathbb{R}$. The elements of $\mathbf{F}$ are called machine numbers (and only those are handled by computers). Specific standards of approximation to real numbers by machine numbers are described in the framework of the well-known model of finite precision arithmetic (or a representation of real numbers with floating point). In $\mathbf{F}$ we select the three fundamental positive constants: the machine zero threshold ε_0, the overflow threshold ε_∞, and the relative error ε_1. Below we assume that the ε-constants corresponding to the usual computational standard of floating point representations satisfy the constraints

$$\varepsilon_1 \le 1/2, \qquad 2\varepsilon_\infty^{1/3} \le \sqrt{\varepsilon_\infty \varepsilon_1}, \qquad \varepsilon_0 \le \sqrt{2}\varepsilon_1^{11/2}. \tag{3}$$

The universally accepted IEEE standard of binary arithmetic reserves 32 bits for machine number expressions of the usual or single precision (one bit for the binary expression of the sign s, 8 bits for the binary expression of the exponent e, and 23 bits for the binary digits of the mantissa f). Associated to given s, e, and f is the machine number $(-1)^s 2^{e-127}(1+f)$. In this case the ε-constants are defined as $\varepsilon_0 \approx 10^{-38}$, $\varepsilon_\infty \approx 4 \times 10^{38}$, $\varepsilon_1 \approx 12 \times 10^{-8}$.

In the general case the cubature sum (Σ_N, φ) amounts to the inner product of the vector $\vec{c} = (c_1, \dots, c_N)$ of its weights by the vector $\vec{\varphi} = (\varphi_1, \dots, \varphi_N)$ of values $\varphi_j = \varphi(\theta^{(j)})$ of the integrand at the nodes: $(\Sigma_N, \varphi) = (\vec{c}, \vec{\varphi})$ for all $\varphi \in X(S)$. Consequently, it is natural to regard $(\widetilde{\Sigma}_N, \varphi)$ as the output of some method for calculating the inner product (Σ_N, φ) in the arithmetic with the specified collection of the ε-constants. Henceforth we assume that both the values of the integrand at the nodes as the weights of the cubature formula are defined with the same machine precision characterized by the collection of constants $(\varepsilon_0, \varepsilon_1, \varepsilon_\infty)$. There is a well-known estimate for the deviation $(\widetilde{\Sigma}_N, \varphi)$ from the original cubature sum such as $|(\vec{c},\vec{\varphi}) - \widetilde{\Sigma}_N(\varphi)| \le (N+1)\varepsilon_1(1+h)\,\|\vec{c}\|_2\,\|\vec{\varphi}\|_2$, where $N\varepsilon_1 \le 2h(1+h)$, $h>0$, and $\|\cdot\|_2$ stands for the Euclidean norm of a vector.

Returning to the practical error, we propose for it an analog of (2), requiring its validity now not for all functions φ in $X(S)$, but only for those in the spherical layer $\mathbf{BL}_\varepsilon = \{\varphi \in X(S) \mid \sqrt{2}\varepsilon_1^{9/2} \le A_n\|\varphi \mid X(S)\| \le \varepsilon_\infty^{1/3}\}$. Require at the outset that the interior of this layer include the unit sphere of $X(S)$; so $\sqrt{2}\varepsilon_1^{9/2} \le A_n \le \varepsilon_\infty^{1/3}$. The left inequality certainly holds, for the first conditions in (3) yields $\sqrt{2}\varepsilon_1^{9/2} \le 1$, while the embedding constant A_n is always at least 1. The second inequality imposes an explicit upper bound on the norm of operators of the admissible embeddings of $X(S)$ into $C(S)$. Put

$$\mathbf{R_F} = \sup_{\sqrt{2}\varepsilon_1^{9/2} \le A_n\|\varphi|X(S)\| \le \varepsilon_\infty^{1/3}} \frac{|M_S(\varphi) - (\widetilde{\Sigma}_N,\varphi)|}{\|\varphi \mid X(S)\|}. \tag{4}$$

For every number N of nodes, basing on the triangle inequality $|M_S(\varphi) - (\widetilde{\Sigma}_N,\varphi)| \le |(l_N,\varphi)| + |(\Sigma_N,\varphi) - (\widetilde{\Sigma}_N,\varphi)|$, we conclude that $\mathbf{R_F}$ is nonnegative and finite. The definition (4) implies directly that for every function φ in the spherical layer $\mathbf{BL}_\varepsilon$ the practical error satisfies $|M_S(\varphi) - (\widetilde{\Sigma}_N,\varphi)| \le \mathbf{R_F}\|\varphi \mid X(S)\|$ for all $\varphi \in \mathbf{BL}_\varepsilon$. Here $\mathbf{R_F}$ plays the same role as the norm $\|l_N \mid X(S)^*\|$ in (2). Let us find out under which conditions the parameter $\mathbf{R_F}$ of (4) differs very little from the norm $\|l_N \mid X(S)^*\|$ (these cubature formulas are of a practical interest).

Theorem 1. *Suppose that the constants ε_0, ε_1, and ε_∞ corresponding to the standard of representation by machine numbers satisfy* (3)*, the embedding constant A_n satisfies $1 \le A_n \le \varepsilon_\infty^{1/3}$, while the weights $(c_1,\dots,c_N)$ of the cubature formula are such that $(\sum_{j=1}^N c_j^2)^{1/2} \le \sqrt{\varepsilon_\infty}/2$. Then for $N \le 1/\varepsilon_1$ the deviation of $\mathbf{R_F}$ defined by* (4) *from the norm $\|l_N \mid X(S)^*\|$ of the error functional satisfies*

$$\Big|\mathbf{R_F} - \|l_N \mid X(S)^*\|\Big| \le 2NA_n\varepsilon_1\Big(\sum_{j=1}^N |c_j| + 1\Big).$$

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Properties of a new class of spatial mappings and its applications

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1. It is known that in the theory of mappings with bounded distortion both outer and inner distortion coefficients are used. In some cases, it does not matter which one to apply (see, for instance, [1, Corollary 3]). However, one should bear in mind that the outer distortion factor characterises the geometric properties of the direct mapping, while the behaviour of the inner rate distortion reflects the geometric properties of the inverse mapping.

Relationship between these values is set in the following theorem. To formulate it we introduce the following objects:

1) non-negative locally integrable functions $\theta, \omega : \mathbb{R}^n \to (0, \infty)$ (referred to be *weighted*);

2) for $n \times n$-matrix A, the symbol $\operatorname{adj} A$ denotes an adjoint matrix, defined by the condition $A \operatorname{adj} A = I \det A$ if the determinant of A is different of zero, and by the continuity in the topology of $\mathbb{R}^{n\times n}$ otherwise;

3) for numbers $n-1 \leq q \leq p < \infty$ and a mapping $\varphi : \Omega \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$, of Sobolev class $W^1_{1,\mathrm{loc}}(\Omega)$, we define the (inner) operator distortion function:

$$\Omega \ni x \mapsto \mathcal{K}^{\theta,1}_{q,p}(x,\varphi) = \begin{cases} \dfrac{\theta^{\frac{n-1}{q}}(x)|\operatorname{adj} D\varphi(x)|}{|J(x,\varphi)|^{\frac{n-1}{p}}} & \text{at } x \in \Omega \setminus Z, \\ 0 \quad \text{otherwise.} \end{cases} \tag{1}$$

Hereinafter, $Z = \{x \in \Omega : J(x,\varphi) = 0\}$;

4) for a mapping $\psi : \Omega' \to \mathbb{R}^n$, $\Omega' \subset \mathbb{R}^n$, of Sobolev class $W^1_{1,\mathrm{loc}}(\Omega')$ we define the (outer) operator distortion function:

$$\Omega' \ni y \mapsto K^{1,\omega}_{s,r}(y,\psi) = \begin{cases} \dfrac{|D\psi(y)|}{\omega(y)^{\frac{1}{r}}|J(y,\psi)|^{\frac{1}{r}}} & \text{at } y \in \Omega' \setminus Z', \\ 0 \quad \text{otherwise.} \end{cases} \tag{2}$$

(here $n-1 < s \leq r \leq \infty$.

The value (1) is a generalisation of the inner distortion coefficient well-known in the quasiconformal analysis, and (2) generalises the outer distortion coefficient.

From [1, Theorem 3] and [2, Theorem 1.2] we deduce the following result:

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Theorem 1. *Let the homeomorphism $\varphi:\Omega\to\Omega'$ has the following properties:*

1) $\varphi\in W^1_{q,\mathrm{loc}}(\Omega)$, $n-1\le q<\infty$,

2) *φ has the finite codistortion:* $\operatorname{adj}D\varphi(x)=0$ *a. e. on the set* Z,

3) $\mathcal{K}^{\theta,1}_{q,p}(\cdot,\varphi)\in L_\varrho(\Omega)$ *where* $\frac{1}{\varrho}=\frac{n-1}{q}-\frac{n-1}{p}$, $n-1<q<p<\infty$.

Then

4) *the mapping $\varphi:\Omega\to\Omega'$ induces a bounded pull-back operator* $\varphi^*:\mathcal{L}_{\frac{p}{n-1}}(\Omega',\Lambda^{n-1})\to\mathcal{L}_{\frac{q}{n-1},\theta}(\Omega,\Lambda^{n-1})$ *of* $\mathcal{L}_{\frac{p}{n-1}}$*-forms of degree $n-1$ into the weighted space* $\mathcal{L}_{\frac{q}{n-1},\theta}(\Omega,\Lambda^{n-1})$ *of forms of degree $n-1$;*

5) *the inverse homeomorphism* $\varphi^{-1}\in W^1_{s,\mathrm{loc}}(\Omega')$ *where* $s=\frac{p}{p-n+1}$,

6) φ^{-1} *has the finite distortion:* $D\varphi^{-1}(y)=0$ *a. e. on* Z',

7) $K^{1,\omega}_{s,r}(\cdot,\varphi^{-1})\in L_\varrho(\Omega')$ *where* $r=\frac{q}{q-n+1}$, $s=\frac{p}{p-n+1}$, $\omega=\theta^{-\frac{n-1}{q-(n-1)}}\circ\varphi^{-1}$.

8) *the homeomorphism φ^{-1} induces a bounded operator* $\varphi^{-1*}:L^1_r(\Omega';\omega)\cap W^1_{\infty,\mathrm{loc}}\to L^1_s(\Omega)$ *by the composition rule.*

Furthermore, $\|\varphi^{-1*}\|\le\|K^{1,\omega}_{s,r}(\cdot,\varphi^{-1})\mid L_\varrho(\Omega')\|=\|\mathcal{K}^{\theta,1}_{q,p}(\cdot,\varphi)\mid L_\varrho(\Omega)\|$.

2. The last assertion motivates the definition of the next class of mappings.

Definition 1. A mapping $f:\Omega\to\mathbb{R}^n$ is called the *mapping with bounded θ-weighted (q,p)-distortion* (belongs to the class $\mathcal{ID}(\Omega;q,p;\theta,1)$) $n-1\le q\le p<\infty$, if:

1) f is continuous, open and discrete;

2) f belongs to Sobolev class $W^1_{q,\mathrm{loc}}(\Omega)$;

3) the Jacobian $J(x,f)\ge 0$ in Ω almost everywhere;

4) the mapping f has the finite codistortion;

5) the function of the local θ-weighted (q,p)-distortion

$$\Omega\ni x\mapsto\mathcal{K}^{\theta,1}_{q,p}(x,f)=\begin{cases}\dfrac{\theta^{\frac{1}{q}}(x)|\operatorname{adj}Df(x)|}{J(x,f)^{\frac{n-1}{p}}}, & \text{if } J(x,f)\ne 0,\\ 0 & \text{otherwise}\end{cases}\tag{3}$$

belongs to $L_\varrho(\Omega)$ where ϱ is found from the condition $\frac{1}{\varrho}=\frac{n-1}{q}-\frac{n-1}{p}$ ($\varrho=\infty$ at $q=p$).

We introduce the following notation $\mathcal{K}^{\theta,1}_{q,p}(f;\Omega)=\big\|\mathcal{K}^{\theta,1}_{q,p}(\cdot,f)\mid L_\varrho(\Omega)\big\|$.

We note that condition 2 of this definition can be relaxed. For example, condition 2 can be replaced by $f\in\mathcal{ID}(\Omega;n-1,n;1,1)$ (see [1] and [3]).

Following the paper [4, p.265] by E. Poleckiĭ, for continuous open and discrete mapping $f:\Omega\to\mathbb{R}^n$ preserving orientation, and normal domain $D\Subset\Omega$, on the image $V=f(D)$ we define Poleckiĭ function $g_D:V\to\mathbb{R}^n$ as

$$V\ni y\mapsto g_D(y)=\Lambda\sum_{x\in f^{-1}(y)\cap D}i(x,f)x\quad\text{where }\Lambda\in(0,\infty).\tag{4}$$

The function of the form (4) was introduced by E. Poletskiĭ in [4] for the mappings with bounded distortion.

Properties Poletskiĭ function are formulated in the following statement.

Theorem 2 ([5]). *Suppose that a continuous open and discrete mapping f belongs $W^1_{q,\mathrm{loc}}(\Omega;\mathbb{R}^n)$, $q > n-1$, $\det Df(x) \geq 0$, and f has the finite codistortion: $\operatorname{adj} Df(x) = 0$ on $Z = \{x \in \Omega : J(x,f) = 0\}$ almost everywhere (or $f \in \mathcal{ID}(\Omega; n-1, n; 1, 1)$). Then*

1) f is sense-preserving,

2) function g_D, defined by the equality (4), belongs to Sobolev class $W^1_1(V)$ and has the finite distortion, i. e. $Dg_D(y) = 0$ on the set of zeros of the Jacobian $\det Dg_D(y)$ almost everywhere.

Furthermore, $f(B_f \cap D) \subset \{y \in V : Dg_D(y) = 0\}$.

3. These results serve as a basis for finding estimates for the norms of the push-forward functions, generalising the results of [6, Section 1.2] on the mappings of the class $\mathcal{ID}(\Omega; q, p; \theta, 1)$. Using estimates for the norms of the push-forward functions, one can obtain estimates for a capacity similar to that of the work [6, Section 1.3]. Here we give only one result.

Theorem 3. *Let $f \in \mathcal{ID}(\Omega; q, p; \theta, 1)$, $n-1 < q \leq p < \infty$, and a weighted function $\omega(x) = \theta^{-\frac{n-1}{q-(n-1)}}(x)$ be locally integrable. If $E = (A, C)$ is a condenser in Ω, and $A \Subset \Omega$, and $C \subset A$ is a compact, then*

$$\big(\operatorname{cap}_s f(E)\big)^{1/s} \leq \mathcal{K}^{\theta,1}_{q,p}(f;\Omega)\big(\operatorname{cap}^{\omega}_r E\big)^{1/r}, \tag{5}$$

where $s = \frac{p}{p-(n-1)}$ and $r = \frac{q}{q-(n-1)}$.

Recall

Definition 2. An ordered triple $E = (F_0, F_1; D)$ of non-empty sets, where D is an open set in $\mathbb{R}^n$, and F_1, F_0 are closed subsets of $\overline{D}$, is called a *condenser* in $D \subset \mathbb{R}^n$. The value

$$\operatorname{cap}^{\omega}_p(E) = \operatorname{cap}^{\omega}_p(F_0, F_1; D) = \inf \int\limits_D |\nabla v|^p(x)\omega(x)\,dx$$

where the infimum is taken over all functions $v \in C(D) \cap W^1_\infty(D) \cap L^1_p(D,\omega)$ such that $v \geq 1$ $(v \leq 0)$ in a neighbourhood of F_1 (F_0), is called *ω-weighted p-capacity of the condenser $E = (F_0, F_1; D)$*.

If U is an open set, C is a compact in U, then the condenser $E = (\partial U, C; U)$ will be denoted by $E = (U, C)$.

If $\omega \equiv 1$ we consider a wider class of admissible functions: $v \in C(D) \cap L^1_p(D)$, for the definition of the capacity of condenser.

4. Theorems 1 and 3 provide a validity of many assertions formulated in papers [6, sections 2–4] and [7] for mappings belonging to $\mathcal{ID}(\Omega; q, p; \theta, 1)$.

Notice that the mappings of the class $\mathcal{ID}(\Omega; n, n; 1, 1)$ are just the mappings with bounded distortion [3, Corollary 5.4] (see [1, Corollary 3] for quasi-conformal homeomorphisms).

Notice that the homeomorphisms of the class $\mathcal{ID}(\Omega; n, n; 1, 1)$ are just the quasi-conformal mappings [1, Corollary 3]).

Note that in a number of studies the geometric condition of the form (5) with some constant instead of $\mathcal{K}^{\theta,1}_{q,p}(f; \Omega)$ is taken as the definition of the class of mappings under consideration. So many facts obtained under the assumption that only the geometric conditions (5) holds, are valid also for the class of mappings of $\mathcal{ID}(\Omega; q, p; \theta, 1)$. For example, as a corollary of Theorem 3 and the results of [8] we obtain sufficient conditions on the weighted function ω, under which a local homeomorphism of the class $\mathcal{ID}(\Omega; n, n; \theta, 1)$ is globally homeomorphic. Below we present one such generalisation of Lavrent'eva-Zorich Theorem.

Theorem 4. *Let a local homeomorphism $f \in \mathcal{ID}(\mathbb{R}^n; n, n; \theta, 1)$, and the weighted function $\omega(x) = \theta^{1-n}(x)$ is locally integrable and satisfies $\omega(x) = O([\ln |x|]^{n-1})$ as $|x| \to \infty$. Then f is homeomorphic and $f(\mathbb{R}^n) = \mathbb{R}^n$.*

A family $\mathcal{F}$ of continuous mappings $f : X \to X'$ of metric spaces is called *normal*, if in any sequence of mappings $f_m \in \mathcal{F}$ there is a subsequence f_{m_k} converging locally uniformly in X to a continuous function $F : X \to X'$.

A function $\varphi : D \to \mathbb{R}$ is said to have the finite average oscillation in a point $x_0 \in D$ if

$$\overline{\lim_{\varepsilon \to 0}} \frac{1}{|B(x_0, \varepsilon)|} \int\limits_{B(x_0,\varepsilon)} |\varphi(x) - \overline{\varphi}_\varepsilon| \, dx < \infty,$$

where $\overline{\varphi}_\varepsilon$ is the average value of φ on the ball $B(x_0, \varepsilon)$. We will say that $\varphi : D \to \mathbb{R}$ is a function of finite average oscillation in D ($\varphi \in \mathrm{FMO}(D)$) if φ has the finite average oscillation in every point $x \in D$.

Denote by $\mathcal{F}_\delta(D)$ the class of all homeomorphisms $f : D \to \overline{\mathbb{R}^n}$ of $\mathcal{ID}(\mathbb{D}; n, n; \theta, 1)$, $n \geq 2$, such that $h(\overline{\mathbb{R}^n} \setminus f(D)) \geq \delta > 0$ (here $h(A)$ is a spherical diameter of A).

From Theorem 3 and results of [9] we obtain sufficient conditions on the weighted function ω, under which the family $\mathcal{F}_\delta(D)$ is normal.

Theorem 5. *If $\omega = \theta^{1-n} \in FMO(D)$ then the class $\mathcal{F}_\delta(D)$ constitutes a normal family of mappings.*

5. The collection of homeomorphisms $f : \Omega \to \Omega'$ of the $\mathcal{ID}(\Omega; q, n; 1, 1) \cap W^1_n(\Omega)$ where the domains Ω and Ω' have Lipschitz boundaries, can be considered as a class of admissible deformations in the nonlinear elasticity theory. Note that the class of admissible deformations of the paper [10] is contained in the intersection $\mathcal{ID}(\Omega; q, n; 1, 1) \cap W^1_n(\Omega)$ for some $q > n-1$, and class deformations of [11] coincides with $\mathcal{ID}(\Omega; n-1, n; 1, 1) \cap W^1_n(\Omega)$. The new class of admissible deformations is a scale of families, depending on a continuous parameter $q \in [n-1, \infty]$. These families are naturally ordered by inclusion: a class with bigger q is contained in a class with less q. For a given material, this hierarchy allows us to choose an appropriate class class of deformations. We apply the method of paper [12] in the proof of the existence of an extreme strain of $\mathcal{ID}(\Omega; q, n; 1, 1) \cap W^1_n(\Omega)$ in a variational problem with some natural conditions on the growth of integrant.

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A uniqueness theorem for the hyperbolic Darboux equation

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Let $\mathcal{L}$ be the Laplace-Beltrami operator on a Riemannian manifold X. The partial differential equation

$$\mathcal{L}_x\big(f(x,y)\big) = \mathcal{L}_y\big(f(x,y)\big) \tag{1}$$

with $f = f(x,y) \in C^2(X \times X)$ is called the generalizing Darboux equation. Such equations are of considerable interest in their own right, but they are also important for many applications in geometric analysis and integral geometry. In particular, equations of type (1) are closely connected with the mean value operators on symmetric spaces.

We investigate zero sets of solutions of the generalized Darboux equation for the case where X is the real hyperbolic space.

We take X as the ball $B = \{x \in \mathbb{R}^n : |x| < 1\}$ with the Riemannian structure

$$ds^2 = \frac{dx_1^2 + \ldots + dx_n^2}{(1-|x|^2)^2}. \tag{2}$$

The Laplace-Beltrami operator for (2) is given by

$$L = \big(1-|x|^2\big)^n \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\Big(\big(1-|x|^2\big)^{2-n}\frac{\partial}{\partial x_j}\Big).$$

Thus equation (1) has the form

$$\big(1-|x|^2\big)^n \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\Big(\big(1-|x|^2\big)^{2-n}\frac{\partial}{\partial x_j}\Big) f =$$

$$= \big(1-|y|^2\big)^n \sum_{j=1}^{n} \frac{\partial}{\partial y_j}\Big(\big(1-|y|^2\big)^{2-n}\frac{\partial}{\partial y_j}\Big) f, \tag{3}$$

where $f = f(x,y) \in C^2(B \times B)$.

For $R \in (0,1)$ and $r \in [0,R)$ we set

$$M_{r,R,1} = \{(x,y) \in B \times B : r \le |x| \le R,\ |y| \le \operatorname{th}(\operatorname{arth}|x| - \operatorname{arth} r)\},$$

$$M_{r,R,2} = \{(x,y) \in B \times B : R \le |x| \le \operatorname{th}(2\operatorname{arth} R - \operatorname{arth} r),$$

$$|y| \le \operatorname{th}(2\operatorname{arth} R - \operatorname{arth} r - \operatorname{arth}|x|)\}.$$

Let $SO(n)$ be the rotation group of $\mathbb{R}^n$.

Theorem 1. *Let $f \in C^2(B \times B)$ satisfy* (3). *Suppose that $R \in (0,1)$ and $r \in [0,R)$ are given and the following conditions hold.*

(i) $f(x,y) = f(x,ky)$ *for all* $x, y \in B$, $k \in SO(n)$.

(ii) $f(x,0) = 0$ *if* $r \le |x| \le R$.

(iii) $f(x,y) = 0$ *for all* $x, y \in B$, $|x| = R$.

Then $f = 0$ in $M_{r,R,1} \cup M_{r,R,2}$. Moreover, if $r = 0$ then $f = 0$ in B.

We need to say a word about condition (i). It is a familiar fact that if $f(x,y)$ is a radial function of y and

$$f(x,y) = h(x,t), \qquad t = \operatorname{arth}|y|,$$

then equation (3) can be rewritten as

$$\left(1-|x|^2\right)^n \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\Big(\left(1-|x|^2\right)^{2-n} \frac{\partial}{\partial x_j}\Big) h = \frac{\partial^2 h}{\partial t^2} + 2(n-1)\operatorname{cth} 2t\, \frac{\partial h}{\partial t}.$$

This relation is a hyperbolic analog of the Darboux equation.

The following result shows that Theorem 1 cannot be essentially reinforced.

Theorem 2. *Suppose that $R \in (0,1)$ and $\varepsilon \in (0,R)$ are given. Then there exists a nonzero solution $f \in C^2(B \times B)$ of equation* (3) *such that*

$$f(x,0) = 0 \qquad \text{in} \qquad \{x \in B : |x| \le R - \varepsilon\}$$

and conditions (i) *and* (iii) *in Theorem 1 are valid.*

On conformal qc geometry, spherical qc manifolds and convex cocompact subgroups

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Qc manifolds are a kind of subRiemannian manifolds with quaternionic structures. We construct the qc Yamabe operators on qc manifolds, which are covariant under the conformal transformations. A qc manifold is scalar positive, negative or vanishing if and only if its qc Yamabe invariant is positive, negative or zero, respectively. On a scalar positive spherical qc manifold, we construct the Green function of the qc Yamabe operator, by which we can construct a conformally invariant tensor. It is a metric if the qc positive mass conjecture is true. When a spherical qc manifold is constructed from a convex cocompact subgroups of $Sp(n+1,1)$, we construct another invariant spherical qc metric. As a corollary, we prove that a spherical qc manifold is scalar positive, negative or vanishing if and only if the Poincare critical exponent of this discrete subgroup is less than, greater than or equal to $2n+2$, respectively.

On controllability and observability of descriptor fractional continuous-time linear systems with regular pencils

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Necessary and sufficient conditions of controllability and observability for solutions of the state equation of descriptor fractional continuous time linear systems with regular pencils are proposed. The derivation of the conditions is based on the construction of Gramian matrices. Several examples are presented to illustrate the concepts and results.

Visualization of the energy and variation of energy functionals for a planar quadratic Bézier curve with a monotonic curvature function

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Methods of optimal control theory, such as minimization of energy

$$E_{\mathrm{MEC}} = \int_0^l \kappa^2(s)ds \to \min \tag{1}$$

and variation of energy

$$E_{\mathrm{MVC}} = \int_0^l \kappa'^2(s)ds \to \min \tag{2}$$

functionals, play an important role in the field of geometric modelling [1]. These methods are used in order to increase the degree of smoothness of a free-form curves (B-splines, Bézier curves, NURBS, etc.), and for generating so-called *fair curves* which have a range of applications in computer-aided design [2, 3, 4].

In this work we first propose a novel visualization of two functionals (1-2) for the case of a quadratic Bézier curve with monotonic curvature function (Figs. 1, 2). The values of (1-2) are shown by the range of colors starting from blue (small values) to red (high values) color.

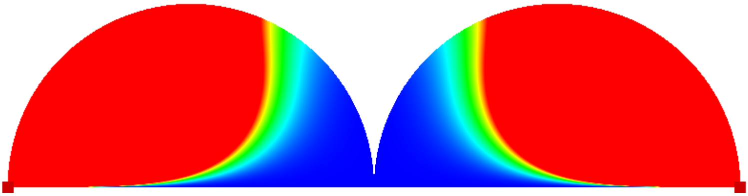

Figure 1: Visualization of energy functional values for the regions (two hemispheres) where a quadratic Bézier curve has a monotonic curvature function.

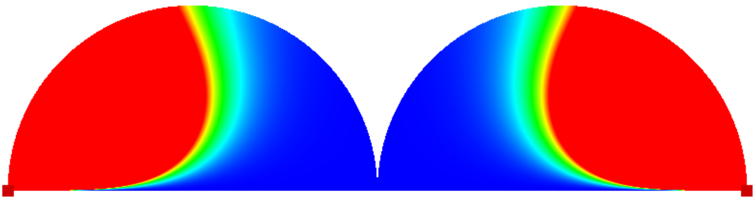

Figure 2: Visualization of variation of energy functional values for the regions (two hemispheres) where a quadratic Bézier curve has a monotonic curvature function.

The results of this work can be used by computer scientists or industrial designers who are interested in geometric modelling of shapes with a high order of smoothness.

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Finitely-additive measures on the invariant foliations of Anosov diffeomorphisms

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The talk is based on the joint work in progress with Alexander Bufetov *(CNRS, Steklov Institute of RAS, NRU Higher School of Economics)* and Sébastien Gouëzel *(CNRS, Université de Nantes, IRMAR)*.

Let $T : M \to M$ be a topologically mixing Anosov diffeomorphism of a compact smooth Riemannian manifold M. We study the asymptotics of the normalized averages of a function $f \in C^2(M)$ along unit unstable balls, iterated under T^n. Since the 70's it is well known that these normalized averages converge uniformly to the mean value of f on the whole manifold M with respect to the Bowen-Margulis measure of the maximal entropy; that is, for any $x \in M$, we have

$$\frac{1}{\mathrm{Leb}^u(T^n B^u(x))} \int_{T^n B^u(x)} f \, d\mathrm{Leb}^u \underset{(n\to\infty)}{\to} \int_M f \, d\mu_{\mathrm{BM}}.$$

We are interested in the description of next terms in this convergence.

We give the solution of this problem introducing the families of finitely-additive measures supported on the leaves of the unstable foliation and invariant under the stable holonomy. These finitely-additive measures are classified via the spectrum of the Perron-Frobenius transfer operator $\mathcal{L}$, acting in the Banach space $\mathfrak{B}$ obtained as the closure of the space of smooth differential forms of degree d_u (the dimension of the unstable foliation) under a certain weak norm.

More precisely, let $\mathbb{V}^{\alpha,p}(\mathbb{R}^d)$ be the space of L_p functions on $\mathbb{R}^d$ with the norm $\|\cdot\|_{\mathbb{V}^{\alpha,p}} := \|\cdot\|_{L_p} + |\cdot|_{\alpha,p}$, where

$$|\varphi|_{\alpha,p} = \sup_{0<\varepsilon\le\varepsilon_0} \varepsilon^{-\alpha} \left[\int_{\mathbb{R}^d} \mathrm{osc}(\varphi, B_\varepsilon(x))^p \, dx \right]^{\frac{1}{p}}.$$

If $\alpha p < 1$, then the characteristic function of the unit ball belongs to $\mathbb{V}^{\alpha,p}$. In a similar way one can introduce the space $\mathbb{V}^{k+\alpha,p}$, $k \in \mathbb{N}$, setting

$$\|\varphi\|_{\mathbb{V}^{k+\alpha,p}} := \|\varphi\|_{W^{k,p}} + \sup_{|\beta|=k} |\partial^\beta \varphi|_{\alpha,p}.$$

The norm on $\mathfrak{B}$ is introduced as follows:

$$\begin{aligned}\|\omega\| &= \|\omega\|_u + \|\omega\|_s;\\ \|\omega\|_u &= \sup_{W\in\Sigma^u} \sup_{\substack{\varphi\in\mathbb{V}_0^{\alpha,p}(W)\\ \|\varphi\|_{\mathbb{V}^{\alpha,p}}=1}} \left|\int_W \varphi\cdot\omega\right|;\\ \|\omega\|_s &= \sup_{W\in\Sigma^u} \sup_{\substack{v\in\mathfrak{F}\\ \|v\|_{C^\gamma}=1}} \sup_{\substack{\varphi\in\mathbb{V}_0^{1+\alpha,p}(W)\\ \|\varphi\|_{\mathbb{V}^{1+\alpha,p}}=1}} \left|\int_W \varphi\cdot L_v\omega\right|.\end{aligned}$$

Here Σ^u stands for the set of almost unstable leaves (that is, of those whose tangent space at any point belongs to the unstable cone), and L_v denotes the Lie derivative along the C^γ-Hölder vector field, where $\gamma \le 1$ is the Hölder degree of the stable holonomy.

It turns out that, under some additional assumptions, the Perron-Frobenius operator $\mathcal{L} : \omega \mapsto T^*\omega$, where $T^*\omega$ is the pullback of ω, has a nice spectrum:

Proposition 1. *Let $T : M \to M$ be a topologically mixing Anosov diffeomorphism with the dilatation constant $\nu > 1$. Then, for any α such that $\alpha < 1/p$ and $\alpha \le \gamma$, if*

$$\limsup_{n\to\infty} \frac{\log\|\mathrm{Jac}^u\, T^n\|}{n} < \alpha\cdot p\cdot\log\nu,$$

then the space $\mathfrak{B}$ admits the $\mathcal{L}$-invariant direct sum decomposition

$$\mathfrak{B} = \mathcal{F} \oplus \mathcal{S},$$

where F has a finite dimension, and the spectrum of $\mathcal{L}|_{\mathcal{F}}$ is given by a finite number of eigenvalues with finite multiplicities, and the norm of $\mathcal{L}|_S$ is bounded by $æ_0 \cdot e^{h_{top}}$, where $æ_0 < 1$ and h_{top} is the topological entropy of T. Moreover, the top eigenvalue of $\mathcal{L}$ is simple and equal $e^{h_{top}}$.

The families of finitely-additive measures correspond to a certain subspace in $\mathfrak{M}^u \subset \mathcal{F}$. To obtain this subspace precisely, one should put an additional restriction, taking into account the holonomy invariance property.

Proposition 2. *In assumptions of Proposition 1, let us assume that*

$$\|\mathcal{L}\omega\| \ge e^{h_{top}-\beta}\|\omega\|,$$

where

$$\beta = \alpha\log\nu - \frac{1}{p}\limsup_{n\to\infty}\frac{\log\|\mathrm{Jac}^u\, T^n\|}{n}$$

Then the set $\mathfrak{M}^u$ of holonomy invariant leafwise finitely-additive measures is isomorphic to the subspace of forms enjoying (1).

In the same way one can consider the finitely-additive measures on the stable foliation. There exists a natural duality between the stable and unstable finitely-additive measures, which is induced by the duality of currents. This observation leads us to the precise description asymptotics of (1).

Theorem. *In assumptions of Propositions 1 and 2, let* $\{\Phi_i^u\}$ *be the basis in* $\mathfrak{M}^u$*, and let* $\{\Phi_i^s\}$ *be the corresponding dual basis in* $(\mathfrak{M}^u)^* = \mathfrak{M}^s$*. Then, for any function* $f \in C^2(M)$*, there exists* $\kappa < 1$ *such that*

$$\frac{1}{\mathrm{Leb}^u(T^n B^u(x))} \int_{T^n B^u(x)} f\, d\mathrm{Leb}^u = \\ = \sum_{i=1}^{\dim \mathfrak{M}^u} \Phi_i^u(T^n B^u(x)) \cdot \int_M f\, d(\mathrm{Leb}^u \otimes \Phi_i^s) + O(\kappa^n).$$

Hypothesis. It is interesting to consider, as it was done in [3] for horocycle flows on Riemannian surfaces of constant negative curvature, the integral in the LHS of (1) as a random variable. The conjecture is that, after a centralization and a proper normalization, the limit probability distribution will have a compact support, unlike the Central Limit Theorem.

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