ARCHIMEDEAN AND DIRECTIONALLY CLOSED CONES

ALEXANDER GUTMAN

In what follows, X is a vector space over $\mathbb{R}$. A nonempty subset $W \subseteq X$ is called a *wedge* whenever $\alpha W \subseteq W$ for all $\alpha \geqslant 0$. A wedge W is a *cone* if $W \cap (-W) = \{0\}$. As is known, in every (pre)ordered vector space $(X, \leqslant)$ the set $X^+ := \{x \in X : x \geqslant 0\}$ is a cone (wedge). Conversely, if W is a cone (wedge) in X then the relation $\leqslant_W$ defined by the rule $x \leqslant_W y \Leftrightarrow y - x \in W$ makes X into a (pre)ordered vector space with $X^+ = W$.

A (pre)ordered vector space $(X, \leqslant)$ is said to satisfy the *axiom of Archimedes* whenever for all $x \in X$ and $y \in X^+$ the condition $(\forall\, n \in \mathbb{N})(x \leqslant \frac{1}{n} y)$ implies $x \leqslant 0$. A cone (wedge) W in a vector space X is called *Archimedean* if the corresponding (pre)ordered vector space $(X, \leqslant_W)$ satisfies the axiom of Archimedes.

See [1] for the basic information on Archimedean cones. The relation between Archimedean and closed cones is studied in [2].

Say that a set $C \subseteq X$ is *closed along the direction of* $y \in X$ if for every family $(\alpha_i)_{i \in I} \subseteq \mathbb{R}$ and every $\alpha \in \mathbb{R}$, $x \in X$, the conditions $\inf_{i \in I} \alpha_i = \alpha$ and $(\forall\, i \in I)(x + \alpha_i y \in C)$ imply $x + \alpha y \in C$. A convex set $C \subseteq X$ is closed along the direction of $y \in X$ if and only if for all $x \in X$ the condition $\inf\{\alpha \in \mathbb{R}^+ : x + \alpha y \in C\} = 0$ implies $x \in C$.

Theorem 1. *The following properties of a convex subset $C \subseteq X$ are equivalent:*
(1) *C is Archimedean;*
(2) *C is closed along all directions;*
(3) *the intersection of C with every straight line is closed;*
(4) *the intersection of C with every finite-dimensional subspace of X is closed;*
(5) *$X \backslash C$ is algebraically closed (i.e., coincides with its algebraic interior).*

Theorem 2. *Let $W \subseteq X$ be a wedge and let f be a linear functional on X such that $f \geqslant 0$ on W and $f(y) > 0$ for some $y \in W$. The wedge W is Archimedean if and only if W is closed along the direction of y and the set $\{x \in W : f(x) = 1\}$ is Archimedean.*

We will now present criteria for existence of f and y satisfying the hypotheses of Theorem 2.

Theorem 3. *Let $\overline{W}$ be the closure of a wedge $W \subseteq X$ in the weak topology induced by the space $X^{\#}$ of all linear functionals on X. The following are equivalent:*
(1) *there exist $f \in X^{\#}$ and $y \in W$ such that $f \geqslant 0$ on W and $f(y) > 0$;*
(2) *there exists $y \in W$ such that $-y \notin \overline{W}$;*
(3) *$\operatorname{lin} W \nsubseteq \overline{W}$, where $\operatorname{lin} W$ is the linear span of W.*

Every locally convex space of uncountable dimension includes a nonclosed Archimedean cone (see [2]). It remain unknown which locally convex spaces of countable dimension include nonclosed Archimedean cones.

References

[1] C. D. Aliprantis, R. Tourky, *Cones and Duality* // Providence, RI: American Mathematical Society, 2007.
[2] A. E. Gutman, E. Yu. Emelyanov, A. V. Matyukhin, *Nonclosed Archimedean cones in locally convex spaces* // Vladikavk. Math. J., **17**:3 (2015), 36–43. [in Russian]

Sobolev Institute of Mathematics, 4 Koptyuga ave., Novosibirsk, 630090, Russia;
Novosibirsk State University, 1 Pirogova st., Novosibirsk, 630090, Russia

Email address: gutman@math.nsc.ru

The work was supported by the program of fundamental scientific researches of the SB RAS № I.1.2., project № 0314-2016-0005.