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BOOLEAN-VALUED UNIVERSE AS AN ALGEBRAIC SYSTEM. I: BASIC PRINCIPLES

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Abstract: The paper is devoted to the study of Boolean-valued algebraic systems of set-theoretic signature. The technique of partial elements of these systems is developed. Some formal apparatus is presented for using partial elements and Boolean-valued classes in the truth values of formulas. The predicative Boolean-valued classes are studied that admit quantification. Logical interrelations are described between the basic properties of Boolean-valued systems: the transfer, mixing, and maximum principles.

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This paper is the first part of the work that continues, develops, and supersedes the study of Boolean-valued algebraic systems of set-theoretic signature which was initiated in [1].

In § 1 we clarify the logical machinery which justifies the hypotheses and conclusions that are constituted by infinitely many formulas. In § 2, some basic information is presented related to the notion of Boolean-valued algebraic system. In § 3 we introduce and develop the technique of partial elements and their ascents to a Boolean-valued system and present formalization for using partial elements and Boolean-valued classes in the truth values of formulas. In § 4 we study the predicative Boolean-valued classes that admit quantification as well as the elements that represent the classes. In § 5, the mixing principle is interpreted in terms of the joins of antichains of partial elements. In § 6, the logical interrelations are described between the basic principles of Boolean-valued systems: transfer, mixing, and maximum.

The second part of this paper will be devoted to the notion of universe over a Boolean-valued extensional system. We will study the logical properties of transitive Boolean-valued subsystems, describe cumulative hierarchies in a Boolean-valued universe, and present a general tool of intensional completion which makes it possible to construct the examples of Boolean-valued systems with unusual properties. By means of the tool, we will show, for an arbitrary complete Boolean algebra, that none of the conditions listed in the definition of a Boolean-valued universe follows from the other conditions.

§ 1. Formalization of Infinite Assertions

Mathematical texts often use "declarations of hypotheses." A declaration means that, within a fragment of reasoning (in definitions and proofs) certain conditions are assumed to be valid or some variables are fixed and play the role of objects with certain properties. As an example serves the phrase "in what follows, B is a complete Boolean algebra" that the next section of this article starts with. Actually, the phrase "fixes" the letter B and adds a "temporary" axiom $\gamma(B)$ that formalizes the assertion "B is a complete Boolean algebra."

In most cases, the effect of declaring a hypothesis is quite clear at the informal level, but the use of infinite sets of formulas as hypotheses or conclusions requires accuracy.

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1.1. Infinite assertions are specific for the subject under consideration. Those are infinite sets of formulas. The examples of these assertions are "a system X is a model of a theory T" and "a system X meets the maximum principle."

Logical connectives with infinite assertions make sense due to the apparatus of formal deduction. For instance, if at least one of the assertions Γ or Δ is infinite, then the implication $\Gamma \Rightarrow \Delta$ itself has no sense; while the phrase "Γ implies Δ within the theory T" can be formalized as the deducibility $T, \Gamma \vdash \Delta$.

1.2. Let T be a theory (a set of sentences) of signature Σ and let Γ and Δ be arbitrary sets of formulas of signature Σ. (The sets Γ and Δ can be infinite and the formulas they contain may have free occurrences of variables.) The deducibility of the conclusion Δ from the hypothesis Γ within T is written as $T, \Gamma \vdash \Delta$ and defined by means of the notion of formal deduction in predicate calculus: For every formula $\delta \in \Delta$ there is a finite sequence of formulas $\varphi_1, \dots, \varphi_n$ such that $\varphi_n = \delta$ and each formula φ_i either belongs to $T \cup \Gamma$ or is obtained from the previous formulas $\varphi_1, \dots, \varphi_{i-1}$ by the classical deduction rules save the rules with quantifiers over the free variables occurred in the formulas of $\{\varphi_1, \dots, \varphi_n\} \cap \Gamma$.

1.3. By using the Completeness Theorem, it is easy to prove that the following are equivalent:

(a) $T, \Gamma \vdash \Delta$;

(b) $T, \Gamma \vdash \delta$ for all $\delta \in \Delta$;

(c) for every formula $\delta \in \Delta$ there is a finite set $\gamma_1, \dots, \gamma_n \in \Gamma$ such that $T \vdash (\forall\, \bar{v})(\gamma_1 \wedge \dots \wedge \gamma_n \Rightarrow \delta)$, where $\bar{v}$ is the list of free variables occurred in $\gamma_1, \dots, \gamma_n, \delta$;

(d) for every model X of T and every valuation $\nu : V \to X$ of the free variables V occurred in $\Gamma \cup \Delta$, the validity $X \vDash \gamma[\nu]$ of all $\gamma \in \Gamma$ imply the validity $X \vDash \delta[\nu]$ of all $\delta \in \Delta$.

1.4. Declaration of a finite hypothesis $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ has simpler formalization. In this case, we may replace the free variables of $\gamma := \gamma_1 \wedge \dots \wedge \gamma_n$ with new constants and stay within the formal deduction involving only sentences. The formalism of "fixing" objects $v_1, \dots, v_m$ subject to a condition $\gamma(v_1, \dots, v_m)$ is based on the following fact that is easily verified with the help of the Completeness Theorem.

Let $\bar{v} := v_1, \dots, v_m$ be the list of free variables occurred in a formula $\gamma(\bar{v})$. Consider the signature Σ^* obtained from Σ by adding the constants $\bar{c} := c_1, \dots, c_m$ and the theory T^* of signature Σ^* obtained from T by adding the axiom $\gamma(\bar{c})$.

(a) *For every formula $\delta(\bar{v})$ of signature Σ,*

$$T^* \vdash \delta(\bar{c}) \;\Leftrightarrow\; T, \gamma(\bar{v}) \vdash \delta(\bar{v}) \;\Leftrightarrow\; T \vdash (\forall\, \bar{v})\big(\gamma(\bar{v}) \Rightarrow \delta(\bar{v})\big).$$

(b) *If $T \vdash (\exists\, \bar{v})\, \gamma(\bar{v})$ then T^* is a conservative extension of T; i.e., for every sentence φ of signature Σ,*

$$T^* \vdash \varphi \;\Leftrightarrow\; T \vdash \varphi.$$

1.5. For instance, under the hypothesis "B is a complete Boolean algebra" the expression

$$\mathrm{ZFC} \vdash (\mathbb{V}^{(B)} \vDash \mathrm{ZFC}),$$

which symbolizes the provability in ZFC of the infinite assertion "$\mathbb{V}^{(B)}$ is a model of ZFC," is formally equivalent to the deducibility

$$\mathrm{ZFC},\ B \text{ is a complete Boolean algebra} \vdash (\mathbb{V}^{(B)} \vDash \mathrm{ZFC});$$

the latter in turn means that, for every sentence φ which is a theorem of ZFC, the following equivalent conditions hold:

(a) ZFC, B is a complete Boolean algebra $\vdash (\mathbb{V}^{(B)} \vDash \varphi)$;

(b) $\mathrm{ZFC} \vdash (\forall\, B)(B$ is a complete Boolean algebra $\Rightarrow \mathbb{V}^{(B)} \vDash \varphi)$;

(c) $\mathrm{ZFC}^* \vdash (\mathbb{V}^{(B)} \vDash \varphi)$,

where ZFC^* is the extension of ZFC obtained by adding the constant B and the axiom "B is a complete Boolean algebra."

1.6. In the sequel, by *formulas* we mean the formulas of set-theoretic signature $\{=,\in\}$ (or of a richer signature obtained as a result of extending the theory by formal definitions).

All formulas are understood in the context of set theory rather than a theory of classes. In particular, even if the symbol X is chosen to mean the class $\{x : \varphi(x)\}$, the expression $(\exists\, Y \subset X)\, \psi(Y)$, which is a shorthand for the formula $(\exists\, Y)\big((\forall\, x)(x \in Y \Rightarrow \varphi(x)) \wedge \psi(Y)\big)$, is interpreted as existence of a *set* (not a class) $Y \subset X$ possessing the property $\psi(Y)$.

A phrase of the form "φ is a formula with free variables $\bar{x} = x_1, \dots, x_n$" means that the list $\bar{x}$ contains all free variables occurred in φ. The cases in which $\bar{x}$ coincides with the totality of free variables of φ will be stipulated explicitly.

§ 2. Boolean-Valued Algebraic Systems

In what follows, B is a complete Boolean algebra.

2.1. Let X, $=_X$, $\in_X$ be some terms or classes defined in ZFC. Say that $(X, =_X, \in_X)$ is a *Boolean-valued* (more exactly, *B-valued*) *algebraic system* or, in short, *Boolean-valued system* or *B-system*, if the conjunction of the following formulas holds:

$X \neq \varnothing,\ \ =_X : X^2 \to B,\ \ \in_X : X^2 \to B,$
for all $x, y, z \in X$

$$=_X(x,x) = 1_B,\ \ =_X(x,y) = \,=_X(y,x),\ \ =_X(x,y) \wedge_B =_X(y,z) \leqslant_B \,=_X(x,z),$$
$$\in_X(x,y) \wedge_B =_X(y,z) \leqslant_B \,\in_X(x,z),\ \ \in_X(x,y) \wedge_B =_X(x,z) \leqslant_B \,\in_X(z,y).$$

We will write X instead of $(X, =_X, \in_X)$.

2.2. Assume that X is a B-system. For every formula $\varphi(\bar{x})$, where $\bar{x} = x_1, \dots, x_n$ is the list of all free variables in φ, introduce the n-ary function symbol φ_X, agree to write the term $\varphi_X(\bar{x})$ as $[\varphi(\bar{x})]_X$, and extend the theory by the definitions

$$[x = y]_X := \,=_X(x,y), \quad [x \in y]_X := \,\in_X(x,y),$$
$$[\varphi \vee \psi]_X := [\varphi]_X \vee_B [\psi]_X, \quad [\varphi \wedge \psi]_X := [\varphi]_X \wedge_B [\psi]_X,$$
$$[\neg\varphi]_X := \neg_B\, [\varphi]_X, \quad [\varphi \Rightarrow \psi]_X := \neg_B [\varphi]_X \vee_B [\psi]_X,$$
$$[(\exists\, x)\, \varphi]_X := \vee_B\{[\varphi]_X : x \in X\}, \quad [(\forall\, x)\, \varphi]_X := \wedge_B\{[\varphi]_X : x \in X\},$$

where φ and ψ are arbitrary formulas, and $\neg_B$ is the complement operation in B.

Formally, the above equalities correctly define the function symbols φ_X in the theory obtained from ZFC by adding the hypotheses "B is a complete Boolean algebra" and "X is a B-system." The definitions of φ_X extend the signature and axiomatics and lead to a conservative extension ZFC* of the initial theory. (See [2] for the details of the corresponding formalism.) For every formula $\varphi(\bar{x})$, the formula $(\forall\, \bar{x} \in X)\ [\varphi(\bar{x})]_X \in B$ is deducible in ZFC*. In the sequel, we will write ZFC instead of ZFC* and implicitly add all declared hypotheses and definitions to the axiomatics.

2.3. In the context $\bar{x} \in X$, say that a formula $\varphi(\bar{x})$ *is valid in a Boolean-valued system* X and write $X \vDash \varphi(\bar{x})$ provided that $[\varphi(\bar{x})]_X = 1_B$.

Call X a *Boolean-valued model of a set of sentences* Φ and write $X \vDash \Phi$ whenever $X \vDash \varphi$ for all $\varphi \in \Phi$. By a *Boolean-valued model of a theory* we mean a model of the set of theorems of the theory. Observe that, if Φ is infinite, then the condition $X \vDash \Phi$ is an infinite assertion (see § 1) constituted by the formulas $X \vDash \varphi$, $\varphi \in \Phi$.

As is known, in any Boolean-valued system, all tautologies (i.e., the theorems of the theory of predicates) are valid and the deduction rules preserve validity. More exactly, under the assumption that X is a B-system, the following hold:

(a) ZFC $\vdash (X \vDash \Phi)$, *where* Φ *is the totality of all tautologies;*

(b) *if some sets of sentences* Γ *and* Δ *are such that* ZFC $\vdash (X \vDash \Gamma)$ *and* $\Gamma \vdash \Delta$, *then* ZFC $\vdash (X \vDash \Delta)$.

Therefore, in the expression $X \vDash \mathrm{ZFC}$, the fragment ZFC may mean the totality of special axioms of ZFC or, equivalently, the totality of all theorems of ZFC.

2.4. The following simple relation, which ensues from 2.3 (a), will be often used without explicit reference.

Let X be a B-system and let $\varphi(x, \bar{y})$ be an arbitrary formula, where $\bar{y} = y_1, \dots, y_n$. Then it is provable in ZFC *that*

$$[\varphi(x_1, \bar{y})]_X \wedge [x_1 = x_2]_X = [\varphi(x_2, \bar{y})]_X \wedge [x_1 = x_2]_X;$$
$$[x_1 = x_2]_X \geqslant b \Rightarrow [\varphi(x_1, \bar{y})]_X \wedge b = [\varphi(x_2, \bar{y})]_X \wedge b$$

for all $x_1, x_2, \bar{y} \in X$ and $b \in B$.

2.5. In the case $\mathrm{ZFC} \vdash (X \vDash \mathrm{ZFC})$, the expressions $[\varphi(\bar{x})]_X$ and $X \vDash \varphi(\bar{x})$ make sense not only for the formulas φ of the initial signature $\{=, \in\}$ but also for the formulas containing any predicate or function symbols definable in ZFC. If φ is such a formula, then by $[\varphi(\bar{x})]_X$ we mean $[\psi(\bar{x})]_X$ where ψ is the result of *elimination* of the definable symbols; i.e., ψ is a formula of signature $\{=, \in\}$ such that $\mathrm{ZFC} \vdash (\varphi \Leftrightarrow \psi)$. In particular, when considering a Boolean-valued model X of ZFC, we may use such terms and formulas as $[x \cap y = \varnothing]_X$ or $X \vDash (f : x \to y)$.

Moreover, in the constructions of the form $[\cdots]_X$ and $X \vDash (\cdots)$, the informal use is allowed of the "external" terms defined in ZFC. For instance, in the context $f : A \to X$, the expression $[f(a_1) = f(a_2)]_X = b$ is a shorthand for the formula $(\exists\, x_1, x_2)\big(x_1 = f(a_1) \wedge x_2 = f(a_2) \wedge [x_1 = x_2]_X = b\big)$.

In the sequel, given an arbitrary B-system X, we will use the expressions $[x = \varnothing]_X$ and $[x \neq \varnothing]_X$ as abbreviations of the corresponding terms $[\neg(\exists\, y)(y \in x)]_X$ and $[(\exists\, y)(y \in x)]_X$.

2.6. Consider the following equivalence $\sim$ of elements of a B-system X:

$$x \sim y \Leftrightarrow [x = y]_X = 1_B \Leftrightarrow X \vDash (x = y).$$

The system X is called *separated* if $x \sim y \Leftrightarrow x = y$ for all $x, y \in X$. In the case of a 2-system, where $2 := \{0, 1\}$ is the simplest Boolean algebra; X is separated whenever the interpretation of equality in X is standard: $=_X(x, y) = 1 \Leftrightarrow x = y$.

The quotient $\widetilde{X} := X/{\sim}$ of X with respect to $\sim$ (see [3, 4.5]) is a separated B-system which is elementary equivalent to the initial system: ZFC proves the equality $[\varphi]_{\widetilde{X}} = [\varphi]_X$ for every sentence φ. Moreover,

$$\mathrm{ZFC} \vdash (\forall\, x_1, \dots, x_n \in X)\ [\varphi(\widetilde{x}_1, \dots, \widetilde{x}_n)]_{\widetilde{X}} = [\varphi(x_1, \dots, x_n)]_X$$

for every $\varphi(x_1, \dots, x_n)$, where $\widetilde{x} \in \widetilde{X}$ is the coset of $x \in X$.

2.7. Let X be a B-system. Given an arbitrary $x \in X$, define the function $[\cdot \in x]_X : X \to B$ by putting $[\cdot \in x]_X(z) = [z \in x]_X$ for all $z \in X$. Consider the following equivalence $\simeq$ between elements of X:

$$x \simeq y \Leftrightarrow [\cdot \in x]_X = [\cdot \in y]_X \Leftrightarrow X \vDash (\forall\, z)(z \in x \Leftrightarrow z \in y).$$

As is easily seen, $x \sim y \Rightarrow x \simeq y$. Say that X is *extensional* whenever $x \sim y \Leftrightarrow x \simeq y$ for all $x, y \in X$ or, which is the same, if the axiom of extensionality is valid in X:

$$X \vDash (\forall\, x, y)\big((\forall\, z)(z \in x \Leftrightarrow z \in y) \Rightarrow x = y\big).$$

A separated extensional system is characterized by the fact that its elements are uniquely determined by the truth values of the containment:

$$x = y \Leftrightarrow (\forall\, z \in X)\ [z \in x]_X = [z \in y]_X.$$

2.8. A B-system Y is a *subsystem* of a B-system X if

$$Y \subset X, \quad =_Y \subset =_X, \quad \in_Y \subset \in_X.$$

Say that a term or class f definable in ZFC is an *isomorphism between B-systems* X and Y and write $f : X \leftrightarrow_B Y$, if f is a bijection between X and Y subject to the condition

$$[f(x_1) = f(x_2)]_Y = [x_1 = x_2]_X, \quad [f(x_1) \in f(x_2)]_Y = [x_1 \in x_2]_X$$

for all $x_1, x_2 \in X$. Say that B-systems X and Y are *isomorphic* and write $X \leftrightarrow_B Y$, if there exists an isomorphism between X and Y.

§ 3. Intensionality

In what follows, B is a complete Boolean algebra and X is an arbitrary B-system. To make expressions less bulky, we will usually omit the indices B and X in the symbols $\wedge_B$, $[\dots]_X$, etc.

3.1. Introduce the equivalence $\sim$ on the class $X \times B$ as follows:

$$(x_1, b_1) \sim (x_2, b_2) \ \Leftrightarrow\ [x_1 = x_2] \geqslant b_1 = b_2.$$

Define the quotient ${}^{\%}X := (X \times B)/{\sim}$ by using the so-called Frege–Russell–Scott trick (see [3, 1.6.8]):

$$\begin{aligned} {}^{\%}X &:= \{\sim(x, b) : (x, b) \in X \times B\}; \\ \sim(x, b) &:= \big\{(y, b) : y \in X,\ [x = y] \geqslant b, \\ &\qquad (\forall\, z \in X)\big([x = z] \geqslant b \Rightarrow \operatorname{rank}(y) \leqslant \operatorname{rank}(z)\big)\big\}, \end{aligned}$$

where $\operatorname{rank}(y)$ is the rank of a set y in the von Neumann cumulative hierarchy. By this approach, the cosets $\sim(x, b)$ corresponding to pairs $(x, b) \in X \times B$ occur to be sets even in the case of a proper class X. We will denote the coset $\sim(x, b)$ by $x|_b$ and call it a *partial element* of X or, more exactly, the *part of x with domain b* or the *restriction of x onto b*. The domain b of a partial element $p = x|_b$ is denoted by $\operatorname{dom} p$. Given $p = x|_b$ and $c \in B$, put $p|_c := x|_{b \wedge c}$. Moreover, granted $P \subset X \cup {}^{\%}X$ and $c \in B$, put $P|_c := \{p|_c : p \in P\}$ and ${}^{\%}P := \{p|_b : p \in P,\ b \in B\} = \bigcup_{b \in B} P|_b$. If $p \in {}^{\%}X$ and $\bar{p} \in X \cup {}^{\%}X$; then write $p \sqsubset \bar{p}$, say that p is a *part* or *restriction* of $\bar{p}$, and call $\bar{p}$ an *extension* of p, whenever $(\exists\, b \in B)\ p = \bar{p}|_b$ or, which is the same, $p = \bar{p}|_{\operatorname{dom} p}$.

3.2. We will propose an agreement on using partial elements in the truth values of formulas. Let φ be a formula with free variables $\bar{x} = x_1, \dots, x_n$ and $\bar{y} = y_1, \dots, y_m$. Along with the terms of the form $[\varphi(\bar{x}, \bar{y})]$ which have sense in the context $\bar{x}, \bar{y} \in X$, we will also use the terms $[\varphi(\bar{x}, p_1, \dots, p_m)]$ for $p_1, \dots, p_m \in {}^{\%}X$ by defining them as follows: if $p_j = y_j|_{b_j}$ $(j = 1, \dots, m)$ then

$$[\varphi(\bar{x}, p_1, \dots, p_m)] := [\varphi(\bar{x}, y_1, \dots, y_m)] \wedge b_1 \wedge \dots \wedge b_m. \tag{1}$$

Agreement (1) is correct, since the right-hand side of (1) does not depend on the choice of representatives (y_j, b_j) of $y_j|_{b_j}$. Indeed, if $y_j|_{b_j} = y'_j|_{b_j}$ then $[y_j = y'_j] \geqslant b_j$ and so

$$[\varphi(\bar{x}, y_1, \dots, y_m)] \wedge b_1 \wedge \dots \wedge b_m = [\varphi(\bar{x}, y'_1, \dots, y'_m)] \wedge b_1 \wedge \dots \wedge b_m.$$

With (1) taken into account, for arbitrary partial elements $p_1, p_2 \in {}^{\%}X$, we have

$$p_1 = p_2 \ \Leftrightarrow\ \operatorname{dom} p_1 = \operatorname{dom} p_2 = [p_1 = p_2].$$

Moreover, due to (1), the agreement of 2.4 extends to the case of partial elements:

$$[p_1 = p_2] \geqslant b \ \Rightarrow\ [\varphi(p_1, \bar{q})] \wedge b \ =\ [\varphi(p_2, \bar{q})] \wedge b$$

for all $p_1, p_2, \bar{q} \in {}^{\%}X$ and $b \in B$.

3.3. A partial element $p \in {}^{\%}X$ is called *everywhere defined* or *global* if $\operatorname{dom} p = 1_B$ or, which is the same, $p = x|_{1_B}$ for some $x \in X$. As is easily seen,

$$x|_{1_B} = y|_{1_B} \Leftrightarrow [x = y] = 1_B \Leftrightarrow x \sim y$$

for all $x, y \in X$. In the sequel, we denote $x|_{1_B}$ by $\widetilde{x}$ or $x^{\sim}$ and write the relation $\widetilde{x} = p$ as $x \sim p$ or $p \sim x$. Moreover, given $Y \subset X$, we put $\widetilde{Y} := Y^{\sim} := \{\widetilde{y} : y \in Y\} = Y|_{1_B}$.

If X is separated then the equalities $\widetilde{x} = \widetilde{y}$ and $x = y$ are equivalent. In this case, we identify the elements $x \in X$ with the corresponding global partial elements $\widetilde{x} \in \widetilde{X}$ and thus assume that $X \subset {}^{\%}X$. This identification agrees with the rule 3.2 of using partial elements in the truth values of formulas: If $p_j = \widetilde{y}_j$ $(j = 1, \dots, m)$ then $[\varphi(\bar{x}, p_1, \dots, p_m)] = [\varphi(\bar{x}, y_1, \dots, y_m)]$.

3.4. Call a global partial element $\bar{p} \in X|_{1_B} = \widetilde{X}$ the *least extension* of a partial element $p \in {}^{\%}X$ if $p \sqsubset \bar{p}$ and $[\bar{p} = \varnothing] \geqslant \neg \operatorname{dom} p$ or, which is equivalent,

$$[\bar{p} \neq \varnothing] \leqslant [\bar{p} = p] = \operatorname{dom} p. \tag{2}$$

The term is justified by the fact that $\bar{p}$ is the inclusion least inside X among all possible extensions of p; i.e., $(\forall\, q \in \widetilde{X})(p \sqsubset q \Rightarrow X \vDash \bar{p} \subset q)$.

Relation (2) implies that, for all $z \in X$,

$$\begin{aligned}[z \in \bar{p}] &= [(z \in \bar{p}) \wedge (\bar{p} \neq \varnothing)] = [z \in \bar{p}] \wedge [\bar{p} \neq \varnothing] \wedge [\bar{p} = p] \\ &= [z \in p] \wedge [p \neq \varnothing] \wedge \operatorname{dom} p = [z \in p];\end{aligned}$$

and so, if X is extensional, then the least extension $\bar{p}$ of a partial element p is unique. In this case we denote $\bar{p}$ by $\operatorname{ext} p$. If, moreover, X is separated; then $\operatorname{ext} p \in X$ according to the agreement of 3.3.

3.5. The *ascent* of a set or class $P \subset {}^{\%}X$ is the function (a class function if X is a proper class) $P{\uparrow} : X \to B$ defined as follows:

$$P{\uparrow}(x) = \bigvee_{p \in P} [x = p], \quad x \in X.$$

Moreover, given $Y \subset X$ and $\Psi : Y \to B$, put $Y{\uparrow} := \widetilde{Y}{\uparrow}$ and $\Psi{\uparrow} := \{y|_{\Psi(y)} : y \in Y\}{\uparrow}$, i.e., $Y{\uparrow}, \Psi{\uparrow} : X \to B$,

$$Y{\uparrow}(x) = \bigvee_{y \in Y} [x = y], \quad \Psi{\uparrow}(x) = \bigvee_{y \in Y} [x = y] \wedge \Psi(y), \quad x \in X.$$

3.6. Lemma. *The following properties of a function $\Phi : X \to B$ are equivalent:*

(a) *$\Phi(x) \wedge [x = y] \leqslant \Phi(y)$ for all $x, y \in X$;*
(b) *$\Phi(x) \wedge [x = y] = \Phi(y) \wedge [x = y]$ for all $x, y \in X$;*
(c) *$[x = y] \leqslant \big(\Phi(x) \Leftrightarrow \Phi(y)\big)$ for all $x, y \in X$, where $(a \Leftrightarrow b) = (\neg a \vee b) \wedge (\neg b \vee a)$;*
(d) *$\Phi = P{\uparrow}$ for some set or class $P \subset {}^{\%}X$;*
(e) *$\Phi = \{x|_{\Phi(x)} : x \in X\}{\uparrow}$;*
(f) *$\Phi = \Psi{\uparrow}$, where $\Psi : Y \to B$ and $Y \subset X$ is a set or class;*
(g) *$\Phi = \Phi{\uparrow}$.*

A function $\Phi : X \to B$ subject to each of the equivalent conditions (a)–(g) is called *extensional* (see [3, 4.5.6; 4, 3.5]).

3.7. Given functions $\Phi, \Psi : Y \to B$ on a subclass $Y \subset X$, write $\Phi \leqslant \Psi$ whenever $\Phi(y) \leqslant \Psi(y)$ for all $y \in Y$.

Lemma. *Consider $Y \subset X$ and $\Psi : Y \to B$. The function $\Psi\!\uparrow : X \to B$ is the least extensional dominant of Ψ:*

(a) $\Psi \leqslant (\Psi\!\uparrow)|_Y$;

(b) *if $\overline{\Psi} : X \to B$ is extensional and $\Psi \leqslant \overline{\Psi}|_Y$ then $\Psi\!\uparrow \leqslant \overline{\Psi}$.*

$\triangleleft$ If $\overline{\Psi} : X \to B$ is extensional and $(\forall\, y \in Y)\ \Psi(y) \leqslant \overline{\Psi}(y)$ then, for all $x \in X$,

$$\Psi\!\uparrow(x) = \bigvee_{y\in Y} [x = y] \wedge \Psi(y) \leqslant \bigvee_{y\in Y} [x = y] \wedge \overline{\Psi}(y) = \overline{\Psi}\!\uparrow(x) = \overline{\Psi}(x). \ \triangleright$$

3.8. Extensional functions $\Phi : X \to B$ are also called *Boolean-valued classes* or *classes inside* X (see [3, 4.6.1; 4, 3.5]) and are employed in the truth values of formulas in a manner similar to the use of classes in the language of set theory. More exactly, the syntax of Boolean-valued truth values 2.2 is extended by the following definitions involving the symbols of Boolean-valued classes Φ and Ψ:

$$[x \in \Phi]_X := [\Phi(x)]_X := \Phi(x), \quad [x = \Phi]_X := [\Phi = x]_X := [(\forall\, y)(y \in x \Leftrightarrow y \in \Phi)]_X,$$
$$[\Phi \in x]_X := [(\exists\, y)(\Phi = y \wedge y \in x)]_X, \quad [\Phi = \Psi]_X := [(\forall\, y)(y \in \Phi \Leftrightarrow y \in \Psi)]_X,$$
$$[\Phi \in \Psi]_X := [(\exists\, x)(\Phi = x \wedge x \in \Psi)]_X.$$

As a result, the expressions of the form $[\varphi(x_1, \dots, x_m, \Phi_1, \dots, \Phi_n)]_X$ make sense, where φ is a formula, x_i are variables, and Φ_j are symbols of Boolean-valued classes. For instance, if $\Phi : X \to B$ is a Boolean-valued class, then

$$[\Phi \neq \varnothing] = [(\exists\, x)(x \in \Phi)] = \bigvee_{x\in X} [x \in \Phi] = \bigvee_{x\in X} \Phi(x) = \vee\Phi,$$

and if $Y \subset X$ and $P \subset {}^{\%}Y$, then

$$[P\!\uparrow \neq \varnothing] = \bigvee_{x\in X} P\!\uparrow(x) = \bigvee_{y\in Y} P\!\uparrow(y) = \bigvee_{p\in P} \operatorname{dom} p. \tag{3}$$

In the sequel, the assertion that a function $\Phi : X \to B$ is a Boolean-valued class (i.e., Φ is extensional) will be written as $\Phi \Subset X$.

3.9. At the logic level, extending the language of the Boolean-valued truth values by means of the symbols $\Phi_1, \dots, \Phi_n \Subset X$ of Boolean-valued classes means that the signature is enriched by the unary predicate symbols $\Phi_1, \dots, \Phi_n$; the algebraic system X is endowed with the Boolean-valued interpretations $[\Phi_i(x)]_X := \Phi_i(x)$; and the "syntactic sugar" is employed in the form of the synonyms and abbreviations listed in 3.8: $(x \in \Phi_i) := \Phi_i(x)$, $(x = \Phi_i) := (\forall\, y)(y \in x \Leftrightarrow y \in \Phi_i)$, etc. Owing to extensionality of the functions $\Phi_i \colon X \to B$, the axioms of predicate calculus of the enriched signature become valid in X, and the following strengthened version of 2.3 holds:

Let $\Sigma := \{=, \in, \Phi_1, \dots, \Phi_n\}$ be the enrichment of the initial signature $\{=, \in\}$ by the unary predicate symbols $\Phi_1, \dots, \Phi_n$. Then, under the assumption that X is a B-system, $\Phi_1, \dots, \Phi_n \Subset X$, and $[\Phi_i(x)]_X = \Phi_i(x)$ $(i = 1, \dots, n)$, the following hold:

(a) $\mathrm{ZFC} \vdash (X \vDash \varphi)$ *for every tautology φ of the theory of predicates of signature Σ;*

(b) *if Γ and Δ are sets of sentences of signature Σ, $\mathrm{ZFC} \vdash (X \vDash \Gamma)$, and $\Gamma \vdash \Delta$, then $\mathrm{ZFC} \vdash (X \vDash \Delta)$.*

3.10. By 3.9, there is a possibility of proving the validity of a formula in each B-system X by "reasoning inside X." Namely, let $\varphi_1, \dots, \varphi_k, \varphi$ be formulas with free variables $\bar{x} = x_1, \dots, x_n$ and $\overline{\Phi} = \Phi_1, \dots, \Phi_m$. Assume that, reasoning within the theory of predicates and treating $\overline{\Phi}$ as the predicate symbols, we can prove φ basing on some hypotheses $\varphi_1, \dots, \varphi_k$. Then we may assert that, for arbitrary Boolean-valued classes $\overline{\Phi} = \Phi_1, \dots, \Phi_m \Subset X$,

$$X \vDash (\forall\, \bar{x})\big(\varphi_1(\bar{x}, \overline{\Phi}) \wedge \dots \wedge \varphi_k(\bar{x}, \overline{\Phi}) \ \Rightarrow\ \varphi(\bar{x}, \overline{\Phi})\big)$$

and, in particular, for all $\bar{x} \in X$, the validity $X \vDash \varphi_1(\bar{x}, \overline{\Phi})$, ..., $X \vDash \varphi_k(\bar{x}, \overline{\Phi})$ implies $X \vDash \varphi(\bar{x}, \overline{\Phi})$.

3.11. Extend the agreements of 3.2 and 3.8 to the formulas that contain variables $\bar{x} = x_1, \dots, x_k$, partial elements $p = y|_b$, $p_j = y_j|_{b_j}$ $(j = 1, \dots, m)$, and Boolean-valued classes $\Phi, \overline{\Phi} = \Phi_1, \dots, \Phi_n$ as follows:

$$\begin{aligned} &[\Phi(p)] := \Phi(y) \wedge b, \\ &[\varphi(\bar{x}, p_1, \dots, p_m, \overline{\Phi})] := [\varphi(\bar{x}, y_1, \dots, y_m, \overline{\Phi})] \wedge b_1 \wedge \dots \wedge b_m. \end{aligned} \tag{4}$$

For instance, if $q \in {}^{\%}X$ and $P \subset {}^{\%}X$ then

$$[q \in P{\uparrow}] = \bigvee_{p \in P} [q = p].$$

3.12. Lemma. *Let φ be an arbitrary formula. If $\bar{x} = x_1, \dots, x_k \in X$, $\bar{p} = p_1, \dots, p_m \in {}^{\%}X$, $\overline{\Phi} = \Phi_1, \dots, \Phi_n \Subset X$, and $i \in \{1, \dots, k\}$ then the following function is extensional:*

$$x_i \in X \;\mapsto\; [\varphi(\bar{x}, \bar{p}, \overline{\Phi})] \in B.$$

◁ It suffices to observe that the extensionality of $\Phi_\xi : X \to B$ $(\xi \in \Xi)$ implies the extensionality of the pointwise connectives $\bigvee_{\xi \in \Xi} \Phi_\xi$, $\bigwedge_{\xi \in \Xi} \Phi_\xi$, and $\neg \Phi_\xi$. ▷

3.13. Lemma 3.12 justifies the correctness of (4). Indeed, the right-hand side of (4) does not depend on the choice of representatives (y_j, b_j) of the cosets $p_j = y_j|_{b_j}$, since $y_j|_{b_j} = y'_j|_{b_j}$ implies $[y_j = y'_j] \geqslant b_j$, whence, by extensionality of $y_j \mapsto [\varphi(\bar{x}, \bar{y}, \overline{\Phi})]$, it follows that

$$[\varphi(\bar{x}, y_1, \dots, y_m, \overline{\Phi})] \wedge b_1 \wedge \dots \wedge b_m = [\varphi(\bar{x}, y'_1, \dots, y'_m, \overline{\Phi})] \wedge b_1 \wedge \dots \wedge b_m.$$

3.14. By 3.12, we may substantially simplify the statements of the general assertions on the truth values of formulas: Instead of considering such expressions as $[\varphi(\bar{x}, \bar{p}, \overline{\Phi})]$ or $[(\exists\, x_i)\, \varphi(\bar{x}, \bar{p}, \overline{\Phi})]$ for an arbitrary formula $\varphi(\bar{x}, \bar{y}, \bar{z})$, elements $\bar{x} = x_1, \dots, x_k \in X$, partial elements $\bar{p} = p_1, \dots, p_m \in {}^{\%}X$, and Boolean-valued classes $\overline{\Phi} = \Phi_1, \dots, \Phi_n \Subset X$, it suffices to speak of the values $\Phi(x)$ or $[(\exists\, x)\, \Phi(x)]$ for some class $\Phi \Subset X$ (see, e.g., 3.18 and 5.11).

3.15. Let X and Y be B-systems. Given a correspondence $f \subset X \times Y$, define $f^{\%} \subset {}^{\%}X \times {}^{\%}Y$ as follows:

$$f^{\%} := \{(x|_b, y|_b) : (x, y) \in f,\, b \in B\}.$$

If $f : X \to Y$ then $f^{\%} : {}^{\%}X \to {}^{\%}Y$ and $\operatorname{dom} p = \operatorname{dom} f^{\%}(p)$ for all $p \in {}^{\%}X$. The next assertion is easily proven by induction on the complexity of a formula.

Lemma. *Let φ be an arbitrary formula. If f is an isomorphism between B-systems X and Y; then $f^{\%}$ is a bijection between ${}^{\%}X$ and ${}^{\%}Y$, and*

$$\begin{aligned} &[\varphi(x_1, \dots, x_k, p_1, \dots, p_m, P_1{\uparrow}, \dots, P_n{\uparrow})]_X \\ &\quad = \big[\varphi\big(f(x_1), \dots, f(x_k), f^{\%}(p_1), \dots, f^{\%}(p_m), f^{\%}(P_1){\uparrow}, \dots, f^{\%}(P_n){\uparrow}\big)\big]_Y \end{aligned}$$

for all $x_1, \dots, x_k \in X$, $p_1, \dots, p_m \in {}^{\%}X$, and $P_1, \dots, P_n \subset {}^{\%}X$.

3.16. Lemma. *If $\Phi \Subset X$ and $P \subset {}^{\%}X$ then $[\Phi \in P{\uparrow}] = \bigvee_{p \in P} [\Phi = p]$.*

◁ Let $P = \{x_i|_{b_i} : i \in I\}$, where $x_i \in X$ and $b_i \in B$. According to 3.8 we have

$$\begin{aligned} [\Phi \in P{\uparrow}] &= [(\exists\, x)(\Phi = x \,\wedge\, x \in P{\uparrow})] = \bigvee_{x \in X} [\Phi = x] \wedge \bigvee_{i \in I} [x = x_i] \wedge b_i \\ &= \bigvee_{i \in I} \bigvee_{x \in X} [\Phi = x] \wedge [x = x_i] \wedge b_i = \bigvee_{i \in I} [\Phi = x_i] \wedge b_i = \bigvee_{p \in P} [\Phi = p]. \end{aligned}$$ ▷

3.17. Lemma. *Let X be an arbitrary B-system.*

(a) *If $\Phi, \Psi \Subset X$ then $X \vDash (\Phi \subset \Psi) \Leftrightarrow \Phi \leqslant \Psi$.*

(b) *If $Y \subset X$ and $P \subset {}^{\%}Y$ then $X \vDash (P{\uparrow} \subset Y{\uparrow})$.*

(c) *If $Y \subset X$, $\Phi \Subset X$, and $P = \{y|_{\Phi(y)} : y \in Y\}$ then $X \vDash (P{\uparrow} = Y{\uparrow} \cap \Phi)$ and $[\Phi \subset Y{\uparrow}] = [\Phi = P{\uparrow}]$. In particular, if $X \vDash (\Phi \subset Y{\uparrow})$ then $X \vDash (\Phi = P{\uparrow})$ for some subclass $P \subset {}^{\%}Y$.*

(d) *If $Z \subset Y \subset X$ then $X \vDash \big(Y{\uparrow} \setminus Z{\uparrow} \subset (Y \setminus Z){\uparrow}\big)$.*

$\lhd$ Assertion (a) is obvious and (b) ensues from (a).

(c): For all $x \in X$,

$$[x \in P{\uparrow}] = \bigvee_{y \in Y} [x = y] \wedge \Phi(y) = \bigvee_{y \in Y} [x = y] \wedge \Phi(x) = [x \in Y{\uparrow} \cap \Phi],$$

i.e., $X \vDash (P{\uparrow} = Y{\uparrow} \cap \Phi)$ and so $[\Phi \subset Y{\uparrow}] = [\Phi = Y{\uparrow} \cap \Phi] = [\Phi = P{\uparrow}]$.

(d): For all $x \in X$,

$$\begin{aligned}
[x \in Y{\uparrow} \setminus Z{\uparrow}] &= \bigvee_{y \in Y} [x = y] \wedge [x \notin Z{\uparrow}] \\
&= \Big(\bigvee_{y \in Y \setminus Z} [x = y] \wedge [x \notin Z{\uparrow}]\Big) \vee \Big(\bigvee_{y \in Z} [x = y] \wedge [x \notin Z{\uparrow}]\Big) \\
&\leqslant \Big(\bigvee_{y \in Y \setminus Z} [x = y]\Big) \vee ([x \in Z{\uparrow}] \wedge [x \notin Z{\uparrow}]) = [x \in (Y \setminus Z){\uparrow}]. \ \rhd
\end{aligned}$$

3.18. Lemma. *Let X be a B-system, $P \subset {}^{\%}X$, and $\Phi \Subset X$. Then*

(a) *$[(\exists\, x \in P{\uparrow})\, \Phi(x)] = \bigvee_{p \in P} [\Phi(p)]$;*

(b) *$[(\forall\, x \in P{\uparrow})\, \Phi(x)] = \bigwedge_{p \in P} [\Phi(p)] \vee \neg \operatorname{dom} p$;*

(c) *$X \vDash (\forall\, x \in P{\uparrow})\, \Phi(x) \Leftrightarrow (\forall\, p \in P)\ [\Phi(p)] = \operatorname{dom} p$;*

(d) *if $Y \subset X$ then $X \vDash (\forall\, x \in Y{\uparrow})\, \Phi(x) \Leftrightarrow (\forall\, x \in Y)\, X \vDash \Phi(x)$.*

$\lhd$ (a): With the equality $[\Phi(x)] \wedge [x = p] = [\Phi(p)] \wedge [x = p]$ taken into account, we have

$$\begin{aligned}
[(\exists\, x \in P{\uparrow})\, \Phi(x)] &= \bigvee_{x \in X} [\Phi(x)] \wedge [x \in P{\uparrow}] \\
&= \bigvee_{x \in X} \bigvee_{p \in P} [\Phi(x)] \wedge [x = p] = \bigvee_{x \in X} \bigvee_{p \in P} [\Phi(p)] \wedge [x = p] \\
&= \bigvee_{p \in P} [\Phi(p)] \wedge \bigvee_{x \in X} [x = p] = \bigvee_{p \in P} [\Phi(p)] \wedge \operatorname{dom} p = \bigvee_{p \in P} [\Phi(p)].
\end{aligned}$$

Relation (b) is easily deduced from (a), (c) is a partial case of (b), and (d) is a partial case of (c). $\rhd$

3.19. Say that $x \in X$ *represents a Boolean-valued class* $\Phi \Subset X$ and write $x \simeq \Phi$ or $\Phi \simeq x$, if $[\cdot \in x] = \Phi$ (see 2.7). Therefore,

$$x \simeq \Phi \Leftrightarrow (\forall\, z \in X)\ [z \in x] = \Phi(z) \Leftrightarrow X \vDash (x = \Phi).$$

As is easily seen, for all $x \in X$, the function $[\cdot \in x] : X \to B$ is extensional; and so every element $x \in X$ represents the Boolean-valued class $[\cdot \in x]$. An element $x \in X$ *represents the ascent* $P{\uparrow}$ of a set or class $P \subset {}^{\%}X$ if $x \simeq P{\uparrow}$; i.e.,

$$(\forall\, z \in X)\ [z \in x] = [z \in P{\uparrow}] = \bigvee_{p \in P} [z = p].$$

In particular, if $y_i \in X$ and $b_i \in B$ $(i \in I)$ then

$$x \simeq \{y_i|_{b_i} : i \in I\}\uparrow \ \Leftrightarrow\ (\forall\, z \in X)\ [\![z \in x]\!] = \bigvee_{i\in I} [\![z = y_i]\!] \wedge b_i.$$

Given an arbitrary subclass $Y \subset X$, denote by $\mathscr{P}_X(Y)$ the totality of all elements that represent the ascents of subsets of ${}^{\%}Y$:

$$\mathscr{P}_X(Y) := \{x \in X : (\exists\, P \subset {}^{\%}Y)(x \simeq P\uparrow)\}.$$

Call a Boolean-valued system X *intensional* if the ascents of arbitrary sets of partial elements are represented in X: $(\forall\, P \subset {}^{\%}X)(\exists\, x \in X)(x \simeq P\uparrow)$.

3.20. The element $x \in X$ representing a Boolean-valued class $\Phi \Subset X$ is uniquely determined up to the equivalence $\simeq$ and, in case X is extensional, up to the equivalence $\sim$. In the latter case, we will identify the Boolean-valued class Φ with the corresponding coset $\widetilde{x}$. Therefore, if X is extensional and $x \simeq \Phi$ then $x \sim \Phi \in \widetilde{X}$. If X is extensional and separated, then the agreement of 3.3 takes effect, the Boolean-valued class Φ becomes an element of X, and the relation $x \simeq \Phi$ turns into the equality $x = \Phi$.

§ 4. Predicativity

Let B be a complete Boolean algebra and let X be an arbitrary B-system.

4.1. Define the *saturated descent* $\Phi\Downarrow$ *of a Boolean-valued class* $\Phi \Subset X$ as follows:

$$\Phi\Downarrow := \{p \in {}^{\%}X : [\![p \in \Phi]\!] = \operatorname{dom} p\} = \{x|_b : x \in X,\ b \leqslant \Phi(x)\}.$$

The *saturated descent* $x\Downarrow$ of $x \in X$ is defined as the saturated descent $[\![\cdot \in x]\!]\Downarrow$ of the Boolean-valued class $[\![\cdot \in x]\!]$. Observe that $\Phi\Downarrow$ and $x\Downarrow$ can be proper classes.

4.2. Call a class $P \subset {}^{\%}X$ *saturated* if P satisfies the two conditions:

$$\begin{aligned} &p \in P \ \Rightarrow\ p|_b \in P, && p \in {}^{\%}X,\ b \in B;\\ &(\forall\, a \in A)(x|_a \in P) \ \Rightarrow\ x|_{\vee A} \in P, && x \in X,\ A \subset B. \end{aligned}$$

Lemma. *The following properties of a class $P \subset {}^{\%}X$ are equivalent:*
(a) *P is saturated;*
(b) $(\forall\, x \in X)(\exists\, c \in B)(\forall\, b \in B)(x|_b \in P \ \Leftrightarrow\ b \leqslant c)$;
(c) $(\forall\, x \in X)(\forall\, b \in B)\big(b \leqslant P\uparrow(x) \ \Rightarrow\ x|_b \in P\big)$;
(d) $(\forall\, q \in {}^{\%}X)([\![q \in P\uparrow]\!] = \operatorname{dom} q \ \Rightarrow\ q \in P)$;
(e) $(\forall\, Q \subset {}^{\%}X)(Q\uparrow = P\uparrow \ \Rightarrow\ Q \subset P)$;
(f) *$P = \Phi\Downarrow$ for some Boolean-valued class $\Phi \Subset X$;*
(g) $P = P\uparrow\Downarrow$.

4.3. According to 3.6, a function $\Phi : X \to B$ is extensional if and only if $\Phi = P\uparrow$ for some set or class $P \subset {}^{\%}X$. The following assertion states that the saturated descent $\Phi\Downarrow$ plays the role of such a P.

Lemma. *If $\Phi \Subset X$ then $\Phi = \Phi\Downarrow\uparrow$. In particular, $P\uparrow\Downarrow\uparrow = P\uparrow$ and $x \simeq x\Downarrow\uparrow$ for all $P \subset {}^{\%}X$ and $x \in X$.*

◁ Given $x \in X$ and using the containment $x|_{\Phi(x)} \in \Phi\Downarrow$, we have

$$\Phi(x) = [\![x = x]\!] \wedge \Phi(x) = [\![x = x|_{\Phi(x)}]\!] \leqslant \bigvee_{p\in\Phi\Downarrow} [\![x = p]\!].$$

On the other hand, due to extensionality of Φ,

$$[\![x = y|_b]\!] = [\![x = y]\!] \wedge b \leqslant [\![x = y]\!] \wedge \Phi(y) \leqslant \Phi(x)$$

for all $y \in X$ and $b \leqslant \Phi(y)$. ▷

4.4. The following assertion ensues from 4.2 (e) and 4.3.

Corollary. *Let $\Phi \Subset X$ and $P \subset {}^{\%}X$. The following are equivalent:*
(a) *P is saturated and $P\!\uparrow = \Phi$;*
(b) *$P = \max\{Q \subset {}^{\%}X : Q\!\uparrow = \Phi\}$;*
(c) *$P = \Phi\!\Downarrow$.*

Therefore, if $P \subset {}^{\%}X$ then $P\!\uparrow\!\Downarrow$ is the largest among the classes $Q \subset {}^{\%}X$ subject to the equality $Q\!\uparrow = P\!\uparrow$, and $P\!\uparrow\!\Downarrow$ is also the only saturated class among these Q. In this connection, it is natural to call $P\!\uparrow\!\Downarrow$ the *saturated hull* or the *saturation* of P.

4.5. Theorem. *The following properties of a Boolean-valued class $\Phi \Subset X$ are equivalent:*
(a) *$\Phi = P\!\uparrow$ for some set $P \subset {}^{\%}X$;*
(b) *$\Phi = P\!\uparrow$ for some saturated set $P \subset {}^{\%}X$;*
(c) *$\Phi = (\Phi|_Y)\!\uparrow$ for some set $Y \subset X$;*
(d) *$\Phi = \Psi\!\uparrow$ for some function $\Psi : Y \to B$, where $Y \subset X$ is a set;*
(e) *the class $\Phi\!\Downarrow$ is a set.*

$\triangleleft$ (a)$\Rightarrow$(c): If Φ meets (a) then there are families $(y_i)_{i\in I} \subset X$ and $(b_i)_{i\in I} \subset B$ such that, for all $x \in X$,

$$\Phi(x) = \bigvee_{i\in I} [x = y_i] \wedge b_i.$$

Then, for all $i \in I$,

$$\Phi(y_i) = \bigvee_{j\in I} [y_i = y_j] \wedge b_j \geqslant [y_i = y_i] \wedge b_i = b_i,$$

whence, for every $x \in X$ we have

$$\Phi(x) = \bigvee_{i\in I} [x = y_i] \wedge b_i \leqslant \bigvee_{i\in I} [x = y_i] \wedge \Phi(y_i) \leqslant \Phi(x)$$

and so $Y := \{y_i : i \in I\}$ meets (c).

The implication (c)$\Rightarrow$(d) is obvious.

(d)$\Rightarrow$(a): If $Y \subset X$ and $\Psi\colon Y \to B$ satisfy (d) then $\Phi = \{y|_{\Psi(y)} : y \in Y\}\!\uparrow$.

(a)$\Rightarrow$(e): Let $\Phi = P\!\uparrow$, where $P \subset {}^{\%}X$ is a set. Then

$$\bigvee_{p\in P} [q = p] = [q \in P\!\uparrow] = [q \in \Phi] = \operatorname{dom} q \tag{5}$$

for all $q \in \Phi\!\Downarrow$. Denote by F the set of functions $f : P \to B$ for which there exists $q \in \Phi\!\Downarrow$ such that

$$(\forall\, p \in P)\ f(p) = [q = p]. \tag{6}$$

The element $q \in \Phi\!\Downarrow$ is uniquely determined by (6). Indeed, if $q_1, q_2 \in \Phi\!\Downarrow$ and $[q_1 = p] = [q_2 = p]$ for all $p \in P$ then, with (5) taken into account,

$$\operatorname{dom} q_1 = \bigvee_{p\in P} [q_1 = p] = \bigvee_{p\in P} [q_2 = p] = \operatorname{dom} q_2,$$

$$[q_1 = q_2] \geqslant \bigvee_{p\in P} [q_1 = p] \wedge [q_2 = p] = \bigvee_{p\in P} [q_1 = p] = \operatorname{dom} q_1,$$

and so $q_1 = q_2$. Consequently, there is a function $g : F \to \Phi\!\Downarrow$ that sends each $f \in F$ to the unique element $q \in \Phi\!\Downarrow$ subject to (6). It remains to observe that g is surjective, since every $q \in \Phi\!\Downarrow$ satisfies (6) for the function $f : p \mapsto [q = p]$.

The implication (e)$\Rightarrow$(b) follows from 4.3, and the implication (b)$\Rightarrow$(a) is trivial. $\triangleright$

4.6. Say that a Boolean-valued class $\Phi \Subset X$ is *predicative* if Φ possesses each of the equivalent properties 4.5 (a)–(e). The term is based on the fact that the classes that are uniquely determined by sets admit quantification in the first-order predicate language: A phrase starting with the words "for every predicative Boolean-valued class" is not an infinite assertion (see § 1) and can be written as a single formula within predicate theory.

If Φ and Ψ are Boolean-valued classes and $\Phi \leqslant \Psi$ then $\Phi\Downarrow \subset \Psi\Downarrow$; and so the predicativity of Ψ implies the predicativity of Φ.

4.7. Corollary. *The following properties of X are equivalent:*
(a) *X is intensional;*
(b) *the ascents of all saturated subsets of ${}^{\%}X$ are represented in X;*
(c) *all predicative Boolean-valued classes are represented in X.*

4.8. Corollary. *The following properties of $x \in X$ are equivalent:*
(a) *$x \simeq P\uparrow$ for some set $P \subset {}^{\%}X$;*
(b) *$x \simeq P\uparrow$ for some saturated set $P \subset {}^{\%}X$;*
(c) $(\exists\, Y \subset X)(\forall\, z \in X)\ [z \in x] = \bigvee_{y \in Y} [z = y] \wedge [y \in x]$;
(d) *$x \simeq \Psi\uparrow$ for some function $\Psi : Y \to B$, where $Y \subset X$ is a set;*
(e) *the class $x\Downarrow$ is a set.*

The elements $x \in X$ subject to (a)–(e) will be called *predicative*. Therefore, $\mathscr{P}_X(X)$ is the totality of all predicative elements of X (see 3.19). Say that the system X is *predicative*, if all elements of X are predicative; i.e., $\mathscr{P}_X(X) = X$.

4.9. If a Boolean-valued system is intensional and predicative, then it is said to satisfy the *ascent principle*:

$$(\forall\, P \subset {}^{\%}X)(\exists\, x \in X)(x \simeq P\uparrow),$$
$$(\forall\, x \in X)(\exists\, P \subset {}^{\%}X)(x \simeq P\uparrow).$$

4.10. Lemma. *Let $x \in X$ be a predicative element and let $Y \subset X$. Then $[x \subset Y\uparrow] = [x = P\uparrow]$ for some subset $P \subset {}^{\%}Y$. In particular,*

$$X \vDash (x \subset Y\uparrow) \ \Leftrightarrow\ (\exists\, P \subset {}^{\%}Y)\ x \simeq P\uparrow.$$

$\triangleleft$ The statement follows from 3.17 (c), since the class $P = \{y|_{[y \in x]} : y \in Y\}$ is included in $x\Downarrow$ and so P is a set. $\triangleright$

§ 5. Cyclicity

Let B be a complete Boolean algebra and let X be an arbitrary B-system.

Recall that elements $a, b \in B$ are said to be *disjoint*, in writing $a \perp b$, whenever $a \wedge b = 0_B$. For $A \subset B$, the relation $(\forall\, a \in A)(a \perp b)$ is abbreviated as $A \perp b$. An *antichain* in a Boolean algebra is a set or family of pairwise disjoint elements, while a *partition of unity* is a maximal antichain (i.e., a chain whose supremum equals 1_B).

5.1. The following simple observations are repeatedly employed in the sequel (see, e.g., 5.4 and 5.8).

Lemma. (a) *If $(d_i)_{i \in I} \subset B$ is an antichain, $(b_i)_{i \in I} \subset B$, $(\forall\, i \in I)(b_i \leqslant d_i)$, and $\bigvee_{i \in I} b_i = \bigvee_{i \in I} d_i$; then $(\forall\, i \in I)(b_i = d_i)$.*
(b) *If $A \subset B$, $\bar{a} \in B$, and $A \leqslant \bar{a}$; then $\vee A = \bar{a} \Leftrightarrow (\forall\, b \in B)(A \perp b \Rightarrow \bar{a} \perp b)$.*

5.2. Define the *descent* $\Phi{\downarrow}$ *of a Boolean-valued class* $\Phi \Subset X$ as follows:

$$\begin{aligned}\Phi{\downarrow} &= \{p \in {}^{\%}X : [\![p \in \Phi]\!] = \operatorname{dom} p = \vee\Phi\}\\ &= \{p \in \Phi{\Downarrow} : [\![p \in \Phi]\!] = \vee\Phi\}\\ &= \{x|_{\vee\Phi} : x \in X,\ \Phi(x) = \vee\Phi\},\end{aligned}$$

where $\vee\Phi := \bigvee_{x\in X}\Phi(x) = [\![\Phi \neq \varnothing]\!]$. The *descent* $x{\downarrow}$ of $x \in X$ is defined as the descent $[\![\cdot \in x]\!]{\downarrow}$ of the Boolean-valued class $[\![\cdot \in x]\!]$ (see 2.7). Observe that $\Phi{\downarrow}$ and $x{\downarrow}$ can be proper classes.

If $X \models (\Phi \neq \varnothing)$ and X is separated; then, according to the agreement of 3.3,

$$\Phi{\downarrow} = \{x \in X : X \models (x \in \Phi)\}.$$

5.3. Say that a subset $P \subset {}^{\%}X$ is an *antichain* if the elements of P have pairwise disjoint domains: $\operatorname{dom} p_1 \perp \operatorname{dom} p_2$ for all $p_1, p_2 \in P$, $p_1 \neq p_2$. A partial element $q \in {}^{\%}X$ is the *join* of an antichain $P \subset {}^{\%}X$ provided that

$$(\forall\, p \in P)\ p \sqsubset q \quad \text{and} \quad \operatorname{dom} q = \bigvee_{p\in P} \operatorname{dom} p.$$

As is easily seen, q is uniquely determined by P, which fact justifies the notation $\sqcup P$ for the join of P. The join $\sqcup\{p_1, p_2\}$ of a two-element antichain $\{p_1, p_2\} \subset {}^{\%}X$ will be written as $p_1 \sqcup p_2$, while the join $\sqcup\{p_i : i \in I\}$ of the antichain defined by a family $(p_i)_{i\in I} \subset {}^{\%}X$, as $\bigsqcup_{i\in I} p_i$.

5.4. Inside X, the ascent of an antichain $P \subset {}^{\%}X$ contains at most one element: $X \models \big((\exists\, x)(x \in P{\uparrow}) \Rightarrow (\exists!\, x)(x \in P{\uparrow})\big)$.

Lemma. *If $P \subset {}^{\%}X$ is an antichain, $x \in X$, and $b := [\![P{\uparrow} \neq \varnothing]\!]$; then the following eight relations are equivalent:*

$$x|_b = \sqcup P, \quad [\![x = \cup(P{\uparrow})]\!] \geqslant b, \quad [\![x \in P{\uparrow}]\!] = b, \quad [\![P{\uparrow} = \{x\}]\!] = b,$$
$$x|_b \in P{\uparrow}{\downarrow}, \quad P{\uparrow}{\downarrow} = \{x|_b\}, \quad \bigvee_{p\in P}[\![x = p]\!] = b, \quad (\forall\, p \in P)\ [\![x = p]\!] = \operatorname{dom} p.$$

5.5. Corollary. *If $P \subset {}^{\%}X$ is an antichain and $P{\uparrow}{\downarrow} \neq \varnothing$, then the join $\sqcup P$ exists and $P{\uparrow}{\downarrow} = \{\sqcup P\}$.*

5.6. Lemma. *If X is an extensional B-system, $P \subset {}^{\%}X$ is an antichain, and $x \in X$ then the following are equivalent:*

(a) $x \sim \operatorname{ext} \sqcup P$;

(b) $X \models (x = \cup(P{\uparrow}))$;

(c) $[\![x \neq \varnothing]\!] \leqslant [\![P{\uparrow} \neq \varnothing]\!] \leqslant [\![x \in P{\uparrow}]\!]$;

(d) $(\forall\, z \in X)\ [\![z \in x]\!] = \bigvee_{p\in P}[\![z \in p]\!]$.

$\triangleleft$ The equivalence of (a)–(c) is straightforward from the definitions.

(a)$\Rightarrow$(d): For all $z \in X$, due to the relations $[\![z \in x]\!] \leqslant [\![x \neq \varnothing]\!] \leqslant \bigvee_{p\in P} \operatorname{dom} p$ and $x|_{\operatorname{dom} p} = p$, we have

$$[\![z \in x]\!] = \bigvee_{p\in P}[\![z \in x]\!] \wedge \operatorname{dom} p = \bigvee_{p\in P}[\![z \in p]\!].$$

(d)$\Rightarrow$(c): If $p \in P$ then $(\forall\, z \in X)\ [\![z \in x]\!] \wedge \operatorname{dom} p \geqslant [\![z \in p]\!]$ according to (d). Whence, with 5.1 (a) taken into account, it follows that $(\forall\, z \in X)\ [\![z \in x]\!] \wedge \operatorname{dom} p = [\![z \in p]\!]$ and so $[\![x = p]\!] \geqslant \operatorname{dom} p$ by extensionality of X. Consequently,

$$[\![x \neq \varnothing]\!] = \bigvee_{z\in X}[\![z \in x]\!] = \bigvee_{z\in X}\bigvee_{p\in P}[\![z \in p]\!] \leqslant \bigvee_{p\in P}\operatorname{dom} p \leqslant \bigvee_{p\in P}[\![x = p]\!] = [\![x \in P{\uparrow}]\!]. \ \triangleright$$

5.7. An antichain $P \subset {}^{\%}X$ is *maximal* whenever $\bigvee_{p\in P} \operatorname{dom} p = 1_B$; i.e., the domains of the elements of P form a partition of unity in B. The joins of maximal antichains are global partial elements of X; and, if X is separated, are identified with the corresponding elements of X (see 3.3).

If $x \in X$, $\{y_i|_{d_i} : i \in I\}$ is a maximal antichain, and $x \sim \bigsqcup_{i\in I} y_i|_{d_i}$; then x is called a *mixing* of $(y_i)_{i\in I} \subset X$ with respect to the partition of unity $(d_i)_{i\in I} \subset B$. As is easily seen,

$$x \sim \bigsqcup_{i\in I} y_i|_{d_i} \ \Leftrightarrow\ (\forall\, i \in I)\ [x = y_i] \geqslant d_i.$$

In a separated system, the mixing of $(y_i)_{i\in I}$ with respect to $(d_i)_{i\in I}$ is unique and denoted by $\operatorname{mix}_{i\in I} d_i y_i$.

5.8. Inside X, the ascent of a maximal antichain $P \subset {}^{\%}X$ is a singleton: $X \vDash (\exists!\, x)(x \in P{\uparrow})$.

Lemma. *Given a maximal antichain $P \subset {}^{\%}X$ and an element $x \in X$, the following eight relations are equivalent:*

$$x \sim \sqcup P, \quad X \vDash (x = \cup(P{\uparrow})), \quad X \vDash (x \in P{\uparrow}), \quad X \vDash (P{\uparrow} = \{x\}),$$
$$\widetilde{x} \in P{\uparrow}{\downarrow}, \quad P{\uparrow}{\downarrow} = \{\widetilde{x}\}, \quad \bigvee_{p\in P} [x = p] = 1_B, \quad (\forall\, p \in P)\ [x = p] = \operatorname{dom} p.$$

5.9. A subclass $Y \subset X$ is called *cyclic* whenever, for every family $(y_i)_{i\in I} \subset Y$ and every partition of unity $(d_i)_{i\in I} \subset B$, there exists a mixing $y \sim \bigsqcup_{i\in I} y_i|_{d_i}$, $y \in Y$. If the entire B-system X is cyclic, then X is said to satisfy the *mixing principle*.

Lemma. *The following properties of a nonempty class $Y \subset X$ are equivalent:*
(a) *Y is cyclic;*
(b) *every antichain $P \subset {}^{\%}Y$ has a join $\sqcup P \in {}^{\%}Y$;*
(c) *every maximal antichain $P \subset {}^{\%}Y$ has a join $\sqcup P \in {}^{\%}Y$.*

5.10. Lemma. *Let $Y \subset X$. The following properties of an element $x \in X$ are equivalent:*
(a) *there exist a family $(y_i)_{i\in I} \subset Y$ and a partition of unity $(d_i)_{i\in I} \subset B$ such that $x \sim \bigsqcup_{i\in I} y_i|_{d_i}$;*
(b) *$x \sim \sqcup P$ for some maximal antichain $P \subset {}^{\%}Y$;*
(c) *$X \vDash (x \in Y{\uparrow})$;*
(d) *$\widetilde{x} \in Y{\uparrow}{\downarrow}$;*
(e) *$\bigvee_{y\in Y} [x = y] = 1_B$.*

◁ The implications (a)⇒(b) and (c)⇒(d)⇒(e) are obvious. Furthermore, (b)⇒(c) follows from 5.8 and 3.17 (b), while (e)⇒(a) is justified by the exhaustion principle [3, 2.1.10 (1)]. ▷

The totality of elements $x \in X$ possessing each of the equivalent properties (a)–(e) is denoted by $\operatorname{mix} Y$. Note that, if the system is not separated, then $\operatorname{mix} Y$ can be a proper class even if Y is a set.

5.11. Lemma. *If $Y \subset X$ then $Y{\uparrow} = (\operatorname{mix} Y){\uparrow}$. In particular, for every Boolean-valued class $\Phi \Subset X$,*

$$\bigvee_{y\in Y} \Phi(y) = [(\exists\, y \in Y{\uparrow})\, \Phi(y)] = \big[\big(\exists\, x \in (\operatorname{mix} Y){\uparrow}\big)\, \Phi(x)\big] = \bigvee_{x\in \operatorname{mix} Y} \Phi(x).$$

◁ The inequality $Y{\uparrow} \leqslant (\operatorname{mix} Y){\uparrow}$ is obvious. On the other hand, for all $x \in X$ and $z \in \operatorname{mix} Y$,

$$[x = z] = [x = z] \wedge \bigvee_{y\in Y} [z = y] = \bigvee_{y\in Y} [x = z] \wedge [z = y] \leqslant \bigvee_{y\in Y} [x = y] = Y{\uparrow}(x),$$

whence $(\operatorname{mix} Y){\uparrow}(x) = \bigvee_{z\in \operatorname{mix} Y} [x = z] \leqslant Y{\uparrow}(x)$. ▷

5.12. Lemma. *If $Y \subset X$ then* $\operatorname{mix}\operatorname{mix} Y = \operatorname{mix} Y$.

$\lhd$ With 5.11 taken into account, for all $x \in X$ we have

$$x \in \operatorname{mix}\operatorname{mix} Y \ \Leftrightarrow\ X \vDash \big(x \in (\operatorname{mix} Y)\!\uparrow\big) \ \Leftrightarrow\ X \vDash (x \in Y\!\uparrow) \ \Leftrightarrow\ x \in \operatorname{mix} Y. \ \rhd$$

5.13. Corollary. *The following properties of a subclass $Y \subset X$ are equivalent:*

(a) *for every family $(y_i)_{i\in I} \subset Y$ and every partition of unity $(d_i)_{i\in I} \subset B$, there exists a mixing $x \sim \bigsqcup_{i\in I} y_i|_{d_i}$, $x \in X$.*

(b) *each antichain $P \subset {}^{\%}Y$ has a join $\sqcup P \in {}^{\%}X$;*

(c) *each maximal antichain $P \subset {}^{\%}Y$ has a join $\sqcup P \in {}^{\%}X$;*

(d) *$Y \subset \overline{Y}$ for some cyclic class $\overline{Y} \subset X$;*

(e) $\operatorname{mix} Y$ *is a cyclic class.*

A class Y subject to each of the equivalent conditions (a)–(e) will be called *precyclic.*

The cyclicity of a precyclic class Y is tantamount to the relation $\widetilde{Y} = (\operatorname{mix} Y)^{\sim}$, which in the case of a separated system turns into the equality $Y = \operatorname{mix} Y$. If Y is a precyclic subset of a separated system X, then $\operatorname{mix} Y$ is the inclusion least cyclic subset of X including Y. In this case, $\operatorname{mix} Y$ is called the *cyclic hull* of Y.

5.14. Lemma. *If X satisfies the mixing principle; then, for all families $(x_i)_{i\in I} \subset X$ and $(b_i)_{i\in I} \subset B$, there are $Y \subset \operatorname{mix}\{x_i : i \in I\}$ and $b \in B$ such that*

$$\{x_i|_{b_i} : i \in I\}\!\uparrow = (Y|_b)\!\uparrow.$$

$\lhd$ In the case of $I = \varnothing$, the assertion is obvious. Suppose that $I \neq \varnothing$.

Put $b = \bigvee_{i\in I} b_i$, $I^\bullet = I \cup \{I\}$, $b_I = \neg b$, and consider an arbitrary element $x_I \in \{x_i : i \in I\}$. By the exhaustion principle, there exists a partition of unity $(d_i)_{i\in I^\bullet} \subset B$ such that $d_i \leqslant b_i$ for all $i \in I^\bullet$. Observe that $\bigvee_{i\in I} d_i = b$. Owing to cyclicity of X, there is an element $x \sim \bigsqcup_{i\in I^\bullet} x_i|_{d_i}$ determined by the relations

$$[x = x_i] \geqslant d_i \quad \text{for all } i \in I^\bullet. \tag{7}$$

For the same reason, for each $i \in I$ there is an element $y_i \sim x_i|_{b_i} \sqcup x|_{\neg b_i}$ in X subject to the inequalities

$$[y_i = x_i] \geqslant b_i, \quad [y_i = x] \geqslant \neg b_i. \tag{8}$$

Put $Y := \{y_i : i \in I\}$. Obviously, $Y \subset \operatorname{mix}\{x_i : i \in I\}$. Show that $\{x_i|_{b_i} : i \in I\}\!\uparrow = (Y|_b)\!\uparrow$, i.e.,

$$\bigvee_{i\in I} [z = x_i] \wedge b_i \ = \ \bigvee_{i\in I} [z = y_i] \wedge b \quad \text{for all } z \in X.$$

For every $z \in X$, with (8) taken into account, we have

$$\bigvee_{i\in I} [z = x_i] \wedge b_i \ = \ \bigvee_{i\in I} [z = y_i] \wedge b_i \ \leqslant\ \bigvee_{i\in I} [z = y_i] \wedge b.$$

On the other hand, due to (7), the following holds:

$$[z = x] \wedge b \ = \ \bigvee_{i\in I} [z = x] \wedge d_i \ = \ \bigvee_{i\in I} [z = x_i] \wedge d_i \ \leqslant\ \bigvee_{i\in I} [z = x_i] \wedge b_i,$$

whence, according to (8), we conclude that, for all $i \in I$,

$$\begin{aligned}
[z = y_i] \wedge b \ &= \ ([z = y_i] \wedge b_i) \vee ([z = y_i] \wedge b \wedge \neg b_i)\\
&= \ ([z = x_i] \wedge b_i) \vee ([z = x] \wedge b \wedge \neg b_i)\\
&\leqslant \ ([z = x_i] \wedge b_i) \vee \bigvee_{j\in I} [z = x_j] \wedge b_j \ = \ \bigvee_{j\in I} [z = x_j] \wedge b_j,
\end{aligned}$$

and, therefore, $\bigvee_{i\in I} [z = y_i] \wedge b \ \leqslant\ \bigvee_{i\in I} [z = x_i] \wedge b_i$. $\rhd$

5.15. Corollary. *Suppose that X is a B-system satisfying the mixing principle.*

(a) *For all $Z \subset X$ and $P \subset {}^{\%}Z$, there are $Y \subset \operatorname{mix} Z$ and $b \in B$ such that $P\!\uparrow = (Y|_b)\!\uparrow$.*

(b) *For every predicative element $x \in X$, there exist $Y \subset X$ and $b \in B$ such that $x \simeq (Y|_b)\!\uparrow$. In this event, $b = [x \neq \varnothing] = [y \in x]$ for all $y \in Y$.*

(c) *If $(\forall\, Y \subset X)(\forall\, b \in B)(\exists\, x \in X)\ x \simeq (Y|_b)\!\uparrow$ then X is intensional.*

§ 6. The Maximum Principle

Let B be a complete Boolean algebra and let X be an arbitrary B-system.

6.1. Say that a Boolean-valued class $\Phi \Subset X$ *attains its maximum on* $Y \subset X$ whenever the set $\{\Phi(y) : y \in Y\}$ has a top or, which is the same,

$$(\exists\, z \in Y)\ \Phi(z) = [(\exists\, y \in Y{\uparrow})\, \Phi(y)].$$

Say that Φ *attains its maximum*, if Φ attains its maximum on X, i.e., $\Phi{\downarrow} \neq \varnothing$.

Let $\varphi(x, \bar z)$ be a formula whose free variables are contained in the list $x, \bar z$, where $\bar z = z_1, \dots, z_n$. The system X is said to satisfy the *maximum principle for* $(\exists\, x)\varphi$ provided that, for every $\bar z \in X$, the Boolean-valued class $x \mapsto [\varphi(x, \bar z)]$ attains its maximum, i.e.,

$$(\forall\, \bar z \in X)(\exists\, y \in X)\ [\varphi(y, \bar z)] = [(\exists\, x)\, \varphi(x, \bar z)].$$

Say that X satisfies the *maximum principle* if X satisfies the maximum principle for every formula $(\exists\, x)\varphi$. (The latter is an infinite assertion; see § 1.)

6.2. Lemma. *A Boolean-valued class $\Phi \Subset X$ satisfies the relation $\Phi = \Phi{\downarrow}{\uparrow}$ if and only if*

$$(\forall\, p \in \Phi{\downarrow})([x = p] \perp b) \;\Rightarrow\; \Phi(x) \perp b$$

for all $x \in X$ and $b \in B$.

◁ If $x \in X$ and $p \in \Phi{\downarrow}$ then $p = y|_{\Phi(y)}$ for some $y \in X$ and, by extensionality of Φ,

$$[x = p] = [x = y|_{\Phi(y)}] = [x = y] \wedge \Phi(y) = [x = y] \wedge \Phi(x) \leqslant \Phi(x).$$

Since $\Phi{\downarrow}{\uparrow}(x) = \bigvee_{p \in \Phi{\downarrow}} [x = p]$, it remains to refer to 5.1 (b). ▷

6.3. Lemma. *Let $\Phi \Subset X$. For $x \in X$, define the extensional function $\Phi_x : X \to B$ by putting*

$$\Phi_x(y) = (\neg\Phi(x) \vee [x = y]) \wedge \Phi(y), \quad y \in X.$$

Then the following are equivalent:

(a) $(\forall\, p \in \Phi{\Downarrow})(\exists\, \bar p \in \Phi{\downarrow})(p \sqsubset \bar p)$;

(b) $(\forall\, x \in X)(\exists\, y \in X)(\Phi(x) \leqslant [x = y],\ \Phi(y) = \vee\Phi)$;

(c) $(\forall\, x \in X)(\Phi_x{\downarrow} \neq \varnothing)$.

If Φ possesses any of the equivalent properties (a)–(c) *then*

$$\Phi = \Phi{\downarrow}{\uparrow}. \tag{9}$$

◁ Show first that $\vee\Phi_x = \vee\Phi$. Indeed, the inequality $\vee\Phi_x \leqslant \vee\Phi$ is obvious, while, on the other hand, for all $y \in X$,

$$\vee\Phi_x \geqslant \Phi_x(x) \vee \Phi_x(y) = \Phi(x) \vee ((\neg\Phi(x) \vee [x = y]) \wedge \Phi(y)) = \Phi(x) \vee \Phi(y) \geqslant \Phi(y).$$

(a)⇔(b): It suffices to observe that

$$x|_b \in \Phi{\Downarrow} \;\Leftrightarrow\; b \leqslant \Phi(x), \quad y|_{\vee\Phi} \in \Phi{\downarrow} \;\Leftrightarrow\; \Phi(y) = \vee\Phi,$$
$$x|_{\Phi(x)} \sqsubset y|_{\vee\Phi} \;\Leftrightarrow\; \Phi(x) \leqslant [x = y].$$

(b)⇒(c): If $\Phi(x) \leqslant [x = y]$ and $\Phi(y) = \vee\Phi$ then

$$\Phi_x(y) = (\neg\Phi(x) \vee [x = y]) \wedge \Phi(y) \geqslant (\neg\Phi(x) \vee \Phi(x)) \wedge \Phi(y) = \Phi(y) = \vee\Phi = \vee\Phi_x;$$

and so Φ_x attains its maximum at y.

(c)⇒(b): Let $x \in X$ and let $y \in X$ be a maximum point of Φ_x; i.e.,

$$(\neg\Phi(x) \vee [x=y]) \wedge \Phi(y) = \vee\Phi_x = \vee\Phi. \tag{10}$$

Since $\Phi(y) \leqslant \vee\Phi$, from (10) it follows that $\Phi(y) = \vee\Phi$. Consequently, $(\neg\Phi(x) \vee [x=y]) \wedge \vee\Phi = \vee\Phi$, whence, $\neg\Phi(x) \vee [x=y] \geqslant \vee\Phi \geqslant \Phi(x)$ and so $\Phi(x) \leqslant [x=y]$.

Suppose that Φ meets (b) and demonstrate that $\Phi = \Phi{\downarrow}{\uparrow}$. According to 6.2, it suffices to consider $x \in X$ and $b \in B$ subject to the condition

$$(\forall\, y \in Y)\big(\Phi(y) = \vee\Phi \;\Rightarrow\; [x=y] \wedge \vee\Phi \perp b\big) \tag{11}$$

and show that $\Phi(x) \perp b$. By (b), there is $y \in X$ such that $\Phi(x) \leqslant [x=y]$ and $\Phi(y) = \vee\Phi$. Then, with (11) taken into account, we have

$$\Phi(x) = \Phi(x) \wedge \vee\Phi \leqslant [x=y] \wedge \vee\Phi \perp b. \;\triangleright$$

Observe that (a)–(c) are not equivalent to the equality $\Phi = \Phi{\downarrow}{\uparrow}$. For instance, if $B = \mathscr{P}(\{0,1,2\})$, $X = \{(1,0,0),(0,1,0),(1,1,1)\}$, $[x=y] = \{i \in \{0,1,2\} : x(i) = y(i)\}$, $[x \in y] = \{i \in \{0,1,2\} : x(i) \in y(i)\}$ (where $0 \in \{0\} = 1$), $\Phi(1,0,0) = \Phi(0,1,0) = \{0,1,2\}$, and $\Phi(1,1,1) = \{0,1\}$; then $\Phi = \Phi{\downarrow}{\uparrow}$, but for $x = (1,1,1)$ there is no $y \in X$ subject to the conditions $\Phi(x) \leqslant [x=y]$ and $\Phi(y) = \vee\Phi$.

6.4. Corollary. *Suppose that the B-system X satisfies the maximum principle for the formula $(\exists\, y)\big(y \in x \wedge (z \in x \Rightarrow z=y)\big)$. Then*
(a) *$x \simeq x{\downarrow}{\uparrow}$ for all $x \in X$;*
(b) *$x \in X$ is predicative if and only if the class $x{\downarrow}$ is a set* (see 4.8).

6.5. Theorem. *The following properties of a nonempty subclass $Y \subset X$ are equivalent:*
(a) *Y is cyclic;*
(b) *each Boolean-valued class $\Phi \Subset X$ attains its maximum on Y;*
(c) *each predicative Boolean-valued class $\Phi \Subset X$ attains its maximum on Y.*

$\triangleleft$ (a)⇒(b): Put $b := \bigvee_{y\in Y} \Phi(y)$. By the exhaustion principle, there exist an antichain $(d_i)_{i\in I} \subset B$ and a family $(y_i)_{i\in I} \subset Y$ such that $\bigvee_{i\in I} d_i = b$ and $d_i \leqslant \Phi(y_i)$ for all $i \in I$. Owing to cyclicity of Y, there is an element $z \in Y$ subject to the condition $z|_b = \bigsqcup_{i\in I} y_i|_{d_i}$. Show that $\Phi(z) = b$. Indeed, $\Phi(z) = \bigvee_{y\in Y} [z=y] \wedge \Phi(z) \leqslant \bigvee_{y\in Y} \Phi(y) = b$. On the other hand, for all $i \in I$, with account taken of the inequalities $[z=y_i] \geqslant d_i$, $\Phi(y_i) \geqslant d_i$, we have $\Phi(z) \geqslant \Phi(y_i) \wedge [z=y_i] \geqslant d_i$ and, hence, $\Phi(z) \geqslant \bigvee_{i\in I} d_i = b$.

The implication (b)⇒(c) is trivial.

(c)⇒(a): Let $P \subset {}^{\%}Y$ be a maximal antichain. According to (c), there exists an element $z \in Y$ such that $P{\uparrow}(z) = \bigvee_{y\in Y} P{\uparrow}(y)$. By (3), we have $\bigvee_{y\in Y} P{\uparrow}(y) = 1_B$. Consequently, $[z \in P{\uparrow}] = 1_B$; whence, it follows by 5.8 that $z \sim \sqcup P$. $\triangleright$

6.6. Theorem. *The following properties of a B-system X are equivalent:*
(a) *X satisfies the mixing principle;*
(b) *each Boolean-valued class $\Phi \Subset X$ attains its maximum;*
(c) *each predicative Boolean-valued class $\Phi \Subset X$ attains its maximum;*
(d) *$\Phi = \Phi{\downarrow}{\uparrow}$ for every Boolean-valued class $\Phi \Subset X$;*
(e) *$\Phi = \Phi{\downarrow}{\uparrow}$ for every predicative Boolean-valued class $\Phi \Subset X$;*
(f) *$P{\uparrow}{\downarrow} \neq \varnothing$ for every class $P \subset {}^{\%}X$;*
(g) *$P{\uparrow}{\downarrow} \neq \varnothing$ for every set $P \subset {}^{\%}X$;*
(h) *$P{\uparrow}{\downarrow} \neq \varnothing$ for every maximal antichain $P \subset {}^{\%}X$.*

$\triangleleft$ The equivalence of (a)–(c) is established in 6.5. The implications (d)⇒(f) and (e)⇒(g) are rather obvious: if $P{\uparrow}{\downarrow} = \varnothing$ then $P{\uparrow} = P{\uparrow}{\downarrow}{\uparrow} = \varnothing{\uparrow}$ and, hence, $P{\uparrow}{\downarrow} = \varnothing{\uparrow}{\downarrow} = \{x|_{0_B}\} \neq \varnothing$. The implication (b)⇒(d) follows from 6.3 (c)⇒(9), the implication (h)⇒(a) follows from 5.5, while (d)⇒(e) and (f)⇒(g)⇒(h) are trivial. $\triangleright$

6.7. Lemma. *If $Y \subset X$ is a nonempty precyclic class then*

$$(\forall\, x \in X)(\exists\, y \in \operatorname{mix} Y)\ [\![x = y]\!] = [\![x \in Y\!\uparrow]\!].$$

$\lhd$ According to 6.5, the Boolean-valued class $y \mapsto [\![x = y]\!]$ attains its maximum on $\operatorname{mix} Y$ and, moreover, $(\operatorname{mix} Y)\!\uparrow = Y\!\uparrow$ (see 5.11). $\rhd$

6.8. Lemma. *Let X be an extensional B-system, let $(d_i)_{i\in I} \subset B$ be an antichain, let $(P_i)_{i\in I}$ be a family of subsets of ${}^{\%}X$, and let $(x_i)_{i\in I} \subset X$. If $(\forall\, i \in I)\ x_i \simeq P_i\!\uparrow$ and $x \simeq (\bigcup_{i\in I} P_i|_{d_i})\!\uparrow$ then $x \sim \operatorname{ext} \bigsqcup_{i\in I} x_i$.*

$\lhd$ The claim follows from 5.6, since, for all $z \in X$,

$$[\![z \in x]\!] = \Big[\!\!\Big[z \in \Big(\bigcup_{i\in I} P_i|_{d_i}\Big)\!\uparrow\Big]\!\!\Big] = \bigvee_{i\in I} \bigvee_{p\in P_i} [\![z = p|_{d_i}]\!] = \bigvee_{i\in I} \bigvee_{p\in P_i} [\![z = p]\!] \wedge d_i$$
$$= \bigvee_{i\in I} [\![z \in P_i\!\uparrow]\!] \wedge d_i = \bigvee_{i\in I} [\![z \in x_i]\!] \wedge d_i = \bigvee_{i\in I} [\![z \in x_i|_{d_i}]\!]. \ \rhd$$

6.9. Corollary. *Let X be an extensional B-system. If $Y \subset X$ and $(\forall\, P \subset {}^{\%}Y)(\exists\, x \in X)(x \simeq P\!\uparrow)$ then $\mathscr{P}_X(Y)$ is a cyclic class. In particular, if X is extensional and intensional, then the class $\mathscr{P}_X(X)$ of all predicative elements of X is cyclic.*

6.10. Corollary. (a) *If X satisfies the mixing principle, then X satisfies the maximum principle* [5, 1.10; 3, 6.1.7].

(b) *If X is intensional and satisfies the maximum principle for the formula $(\exists\, x)(x \in y)$, then X satisfies the mixing principle* [5, 1.12; 3, 6.1.9].

(c) *If X is extensional and satisfies the ascent principle, then X satisfies the mixing and maximum principles* [5, 1.11; 3, 6.1.8].

$\lhd$ Item (a) is a consequence of 3.12 and 6.6, (b) follows from 5.5, and (c) follows from 6.9 and (a). $\rhd$

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