International Conference on Geometric Analysis

in honor of the 90th anniversary of academician Yu. G. Reshetnyak

22-28 of September 2019

Abstracts

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Тезисы докладов

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Сборник содержит тезисы некоторых докладов, представленных на Международной конференции по геометрическому анализу, проводимой в честь 90-летия академика Ю. Г. Решетняка (22–28 сентября 2019 года). Темы докладов относятся к современным направлениям в геометрии, теории управления и анализе, а также к приложениям методов метрической геометрии и анализа к смежным областям математики и прикладным задачам.

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The digest contains abstracts of some of the talks presented on the International Conference on Geometric Analysis in honor of the 90th anniversary of academician Yu. G. Reshetnyak (22–28 of September, 2019). Topics of talks concern modern trends in geometry, control theory and analysis, as well as applications of the methods of the metric geometry and analysis to related fields of mathematics and applied problems.

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The volume of a compact hyperbolic tetrahedron in terms of its edge lengths

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A *compact hyperbolic tetrahedron* T is a convex hull of four points in the hyperbolic space $\mathbb{H}^3$. Let us denote the vertices of T by numbers $1, 2, 3$ and 4. Then denote by ℓ_{ij} the length of the edge connecting i-th and j-th vertices. We put θ_{ij} for the dihedral angle along the corresponding edge. It is well known that T is uniquely defined up to isometry either by the set of its dihedral angles or the set of its edge lengths. A *Gram matrix* $G(T)$ of tetrahedron T is defined as $G(T) = \langle -\cos\theta_{ij}\rangle_{i,j=1,2,3,4}$, we assume here that $-\cos\theta_{ii} = 1$. An *edge matrix* $E(T)$ of hyperbolic tetrahedron T is defined as $E(T) = \langle \cosh \ell_{ij}\rangle_{i,j=1,2,3,4}$, where $\ell_{ii} = 0$.

More than 100 years ago Italian matematician G. Sforza found a formula for the volume of a compact hyperbolic tetrahedron T in terms of its Gram matrix (see [1]). The new proof of the Sforza's formula was recently given in [2].

In the present work we present an exact formula for the volume of a compact hyperbolic tetrahedron T in terms of its edge matrix.

Theorem. *Let T be a compact hyperbolic tetrahedron given by its edge matrix $E = E(T)$ and $c_{ij} = (-1)^{i+j}E_{ij}$ is ij-cofactor of E. We assume that all the edge lengths are fixed exept ℓ_{34} which is formal variable. Then the volume $V = V(T)$ is given by the formula*

$$V = -\frac{1}{2}\int_0^{\ell_{34}} \frac{c_{14}c_{33}(c_{24}c_{34} - c_{23}c_{44}) + c_{13}c_{44}(c_{23}c_{34} - c_{24}c_{33})}{c_{33}c_{44}\det E\sqrt{c_{33}c_{44} - c_{34}^2}}\, t\sinh t\, dt,$$

where cofactors c_{ij} and edge matrix determinant $\det E$ are functions in one variable ℓ_{34} denoted by t.

References

1. Sforza G. *Spazi metrico-proiettivi*, Ricerche di Estensionimetria Integrale, Ser. III, VIII (Appendice) (1907), 41–66.
2. Abrosimov N. V., Mednykh A. D. *Volumes of polytopes in constant curvature spaces*, Fields Institute Communications 70 (2014), 1–26.

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Polynomial first integrals of magnetic geodesic flows on the 2-torus

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Searching for Riemannian metrics on 2-surfaces with integrable geodesic flows is a classical problem. We will discuss the question of polynomial integrability of such flows on the 2-torus in presence of a magnetic field. This talk is based on joint results with M. Bialy, A. E. Mironov, A. A. Valyuzhenich.

References

1. Agapov S. V., Bialy M., Mironov A. E. *Integrable magnetic geodesic flows on 2-torus: New examples via quasi-linear system of PDEs*, Comm. Math. Phys. 351:3 (2017), 993–1007.
2. Agapov S., Valyuzhenich A. *Polynomial integrals of magnetic geodesic flows on the 2-torus on several energy levels*, Disc. Cont. Dyn. Syst. – A, 39:11 (2019), 6565–6583.

On integrable domains and surfaces

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The notion in the title goes back to motivated by celestial mechanics Newton's Lemma about ovals (Principia, Lemma XXVIII, 1687), claimed that no oval in the plane is algebraically integrable: the area cut off the figure by a straight line is a solution of no algebraic equation in the position of the line. In 1987, V. I. Arnold proposed to generalize Lemma for volume functions in higher dimensions and conjectured that the only algebraically integrable domains in $\mathbb{R}^n$, with smooth boundaries, are ellipsoids in odd-dimensional spaces. V. Vassiliev proved that there is no such domains in even-dimensional spaces. The problem in odd dimensions remains open, in full generality. In a broader context, it asks: whether (to what extent) integrability property characterizes quadratic surfaces? In my talk, I will touch upon the history of the problem and recent related results.

References

1. Agranovsky M. *On polynomially integrable domains in Euclidean spaces*, Complex Analysis and Dynamical Systems, Trends in Mathematics, Birkhauser (2018), 1–22; arXiv:1701.05551.

2. Agranovsky M. *On algebraically integrable bodies*, Contemporary Math., Functional Analysis and Geometry. Selim Krein Centennial, American Math. Society, Providence, RI (2019), 33–44; arXiv:1705.06063.
3. Agranovsky M. *Locally polynomially integrable surfaces and finite stationary phase expansions*, J. d'Analyse Math., in press; arXiv:1901.03976.
4. Arnold V. I., Vassiliev V. A. *Newton's Principia read 300 years later*, Notices AMS, 36:9 (1989), 1148–1154.
5. Koldobsky A., Merkurjev A., Yaskin V. *On polynomially integrable convex bodies*, Advances in Mathematics 320 (2017), 876–886; arXiv:1702.00429.
6. Newton's theorem about ovals, Wikipedia.

Fundamental extension function for bilipschitz maps

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We study the possibility of extension of $(1+\varepsilon)$-bilipschitz maps $f\colon A \to R^n$, $A \subset R^n$ and ε is small enough. The extended map should be $(1+\varphi(\varepsilon))$-bilipschitz. The case of extension to $F\colon R^n \to R^n, F|_A = f$ was considered in [1,2]. We gave in [2] a necessary and sufficient condition for A to have linear extension property, that is $\varphi(\varepsilon) = C\varepsilon$. If A is a straight line, then $\varphi(\varepsilon) = C\sqrt{\varepsilon}$.

Hypothesis 1. *We believe that every convex monotone function $\varphi(\varepsilon) \geq \varepsilon$ has a class of sets $A \subset R^n$ with $\varphi(\varepsilon)$-extension property.*

Here we study extensions $F\colon R^{n+1} \to R^{n+1}$. It looks surprising, but there is only one possibility: $\varphi(\varepsilon) = C/\log\varepsilon$.

Theorem 1. *There are constants $\varepsilon_0 > 0$ and $C > 1$ such that for every $A \subset R$ and $\varepsilon \leq \varepsilon_0$ every $1+\varepsilon$-bilipschitz map $f\colon A \to R$ has a $(1+\varphi(\varepsilon))$-bilipschitz extension $F\colon R^2 \to R^2$ with $\varphi(\varepsilon) = -C/\log\varepsilon$.*

A sequence $(x_1, \ldots, x_k)$ is called α-related if $x_i \neq x_{i+1}$ for each $i = 1, \ldots, k-1$ and

$$\alpha^{-1} \leq \frac{|x_{i-1} - x_i|}{|x_{i+1} - x_i|} \leq \alpha.$$

A set A is α-relatively connected if every two pairs of points in A can be connected by α-related sequence.

Theorem 2. *The function $\varphi(\varepsilon)$ in Th. 1 is sharp. There is $C_1 < C$ satisfying the following property: let $A \subset R$ be not α-relatively connected for any $\alpha > 1$. Then one can find a sequence (ε_i) with $\varepsilon_i \to 0$ and $(1+\varepsilon_i)$-bilipschitz maps $f_i\colon A \to R$ such that for every $(1+\delta)$-bilipschitz extension $G_i\colon R^2 \to R^2$ of f_i we have*

$$\delta \geq -C_1/\log \varepsilon_i.$$

It was proved in [3] that in the case A is α-relatively connected for some $\alpha > 1$ this is not true.

Hypothesis 2. *The same result is true for $A \subset R^n, f\colon A \to R^n$ and extension $F\colon R^{n+1} \to R^{n+1}, F|_A = f$.*

References

1. Väisälä J. *Bilipschitz and quasisymmetric extension properties*, Ann. Acad. Sci. Fenn. Ser. A I Math. 11 (1986), 239–274.
2. Alestalo P., Trotsenko D. A., Väisälä J. *Linear bilipschitz extension property*, Sib. Math. J. 44 (2003), 959–968.
3. Trotsenko D.A. *An extendability condition for bilipschitz functions*, Sib. Math. J. 57:6 (2016), 1082–1087.

The spectrum of the Laplacian in a domain bounded by a flexible polyhedron in $\mathbb{R}^d$ does not always remain unaltered during the flex

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Being motivated by the theory of flexible polyhedra, we study the Dirichlet and Neumann eigenvalues for the Laplace operator in special bounded domains of Euclidean d-space, $d \geqslant 3$. The boundary of such a domain is an embedded simplicial complex which allows a continuous deformation (a flex), under which each simplex of the complex moves as a solid body and the change in the spatial shape of the domain is achieved through a change of the dihedral angles only.

The main result is that both the Dirichlet and Neumann spectra of the Laplace operator in such a domain do not necessarily remain unaltered during the flex of its boundary. The talk is based on the author's preprint [1].

References

1. Alexandrov V. *The spectrum of the Laplacian in a domain bounded by a flexible polyhedron in $\mathbb{R}^d$ does not always remain unaltered during the flex*, preprint, arXiv:1809.00322 [math.MG] (2018).

Motion control of mobile wheeled robot along Euler's elastica

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The work is devoted to the optimal control problem for mobile wheel robot moving on a plane surface. The design, the functional and the algorithms for controlling mobile robots are often determined by conditions of applied problems. However, despite different modifications of mobile wheeled robots, each of which has its own advantages, disadvantages and limitations, one can carry out research for generalized models and specify the control algorithms later to the specific model.

The model of a wheeled robot we consider here corresponds to the model of a Chaplygin sleigh in which the configuration space coincides with the group of motions of Euclidean plane:

$$\mathrm{SE}(2) = \{q = (x, y, \theta) \in \mathbb{R}^2_{x,y} \times S^1_\theta\}.$$

The position of the robot $q = (x, y, \theta)$, is defined by the pair $(x, y) \in \mathbb{R}^2_{x,y}$, which specifies the coordinates of the center of mass of the driving wheel pair, and by the angle $\theta \in S^1$, which specifies the direction of motion of the wheels.

In these coordinates, the nonholonomic constraints of the control system are described by the following kinematic relations:

$$\begin{cases} \dot{x} = v\cos\theta, \\ \dot{y} = v\sin\theta, \\ \dot{\theta} = w, \end{cases}$$

where v, w are linear and angular velocities of the robot.

One of the control methods suitable for solving such a problem is motion along optimal trajectories in the form of Euler elasticae.

Movement along the optimal elastica corresponds to minimizing the following integral:

$$\int_0^T \frac{\alpha w^2(t) + \beta v^2(t)}{v(t)} dt \to \min$$

with some $\alpha > 0$ and $\beta > 0$.

When linear velocity $v(t)$ is constant the control of the mobile robot that implements its motion along Euler elasticae is optimal in sense of minimization of maneuvers or steering action.

For experimental investigations, a prototype of the mobile wheeled robot with a spherical supporting wheel and two driving wheels has been created. Several practical issues of the implementation of the control algorithm for motion along Euler elasticae were studied. In particular, analysis of acceleration and deceleration along the elastica, and of the influence of the measurement errors. We estimate the control algorithms from the experimental trajectories of motion which are restored by means of Vicon motion capture systems using reflective markers located on the body of the mobile robot. Results of experimental investigations are presented in [1].

Developed control algorithm can be used as basis for algorithms for motion along other different classes of optimal curves, e. g. sub-Riemannian and sub-Finsler geodesics which appear in time-minimization problems for a mobile wheel robot which also might tow passive trailers.

References

1. Ardentov A. A., Karavaev Yu. L., Yefremov K. S. *Euler Elasticas for Optimal Control of the Motion of Mobile Wheeled Robots: the Problem of Experimental Realization*, Regul. Chaotic Dyn. 24:3 (2019), 312–328.

On the energy characteristics of the anomalous transfer processes

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We consider an anomalous diffusion models in which space-time nonlocalities are generated by singular zones forming sub- and superdiffusion transfer regimes. The presented models are realized as a random walk in the Euclidean space with respect to metric topologies that are nonequivalent to the Euclidean metric topology. The estimates obtained in [1] for the mean squared displacement $r(t)$ of a walking particle have the form

$$\frac{\langle r^2(t)\rangle}{t^\alpha} = O(1) \quad \text{and} \quad \frac{t^\alpha}{\langle r^2(t)\rangle} = O(1), \quad t \to +\infty,$$

where α is a constant in the range $(0, 2)$. We relate "long flights" of particles and "time delays" generated by time-space nonlocalities to singular zones as follows.

In the case of superdiffusion where $\alpha > 1$, the singular zone is a continuum of zero Lebesgue measure in the Euclidean space on which the position of a walking particle is fixed at each instant of discrete time. In this case, "long flights" are precisely related to the walk in a rarified set (of the Cantor dust type) and to hops of this walk of nonzero length at each instant of discrete time. In the case of subdiffusion where $\alpha < 1$, the singular zone is a countable everywhere dense set of "traps" in the Euclidean space, i. e., a set of points at each of which the particle is located for a nonzero random time period in the general case.

We note that the superdiffusion process is realized using a topology with respect to which the uninterrupted continuum becomes singular (of the Cantor set type). In contrast, the subdiffusion process in turn corresponds to a topology with respect to which the singular continuum becomes uninterrupted. These topologies are crucial here: first, they coincide with the metric topology of the Euclidean space in the absence of singular zones; second, not only random walk process but also other random processes can be considered with respect to these topologies. We regard stationary processes as such processes. First, stationary processes are in a certain sense oscillatory processes, for which the concepts of energy and energy density can be defined. Second, we can define a stationary process for the (uninterrupted or singular) continuum such that almost all its trajectories are everywhere dense in this continuum; in what follows, we use the term the stationary process concentrated in the continuum. Such a "loading" of the continuum by the stationary process allows modeling its energy state and, in addition, establishing the law of the variation of this state when the continuum is transformed from uninterrupted into singular. We note that the energy of the stationary process is defined as the second moment of this process; in this case, the spectral density shows the frequency distribution of this energy.

Therefore, to find the energy characteristics of the abovementioned deformation of a diffusion process into an anomalous diffusion process, we consider stationary processes concentrated in uninterrupted and singular continua. As a dynamical model of the deformation of a classical walk into an anomalous diffusion process, we propose the dynamical model of the deformation of stationary processes (see [2]).

References

1. Arkashov N. S., Seleznev V. A. *On a random walk model on sets with self-similar structure*, Siberian Math. J. 54:6 (2013), 968–983.

2. Arkashov N. S., Seleznev V. A. *Energy characteristics of the anomalous diffusion process*, Theoretical and Mathematical Physics 199:3 (2019), 894–908.

Hadamard's global homeomorphism theorem under relaxed smoothness conditions

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Global homeomorphism theorems provide sufficient conditions for local homeomorphisms to be global. Let us recall a classical assertion of this type.

Hadamard's Theorem (see, for example, [1, Theorem 5.3.10]). *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable mapping, the linear operator $\frac{\partial f}{\partial x}(x) : \mathbb{R}^n \to \mathbb{R}^n$ be invertible for every $x \in \mathbb{R}^n$, and there exist $c \geq 0$ such that*

$$\left\| \frac{\partial f}{\partial x}(x)^{-1} \right\| \leq c \quad \forall\, x \in \mathbb{R}^n.$$

Then f is a homeomorphism.

In our research, we obtained an analogous result for continuous mappings. In order to formulate it, recall some definitions.

Let a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ and a number $\alpha > 0$ be given. Denote by $B(x, r)$ a closed ball in $\mathbb{R}^n$ centered at a point $x \in \mathbb{R}^n$ with a radius $r > 0$. The mapping f is said to be α-covering at a point $x \in \mathbb{R}^n$ if

$$\forall\, \varepsilon > 0 \quad \exists\, r \in (0, \varepsilon) : \quad B(f(x), \alpha r) \subset f(B(x, r)).$$

Theorem. *If for every $x \in \mathbb{R}^n$ the mapping f is α-covering at x and injective in a neighbourhood of x, then f is a homeomorphism and f^{-1} is α^{-1}-Lipschitzian.*

This result implies Hadamard's global homeomorphism theorem and some of its analogs.

References

1. Ortega J. M., Rheinboldt W. C. *Iterative solutions of nonlinear equations in several variables*, Academic Press, NY, 1970.

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On regular covering/packing of the Euclidean plane with circles

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Minimum covering (maximum packing) problems of the plane with circles of the same radius are sufficiently studied [1, 2]. When the circles have a different radius, one should consider more complex Pareto optimal problems related to more sophisticated geometrical structures. In this regard, classes of regular coverings (packings) and the level of complexity are strictly determined in the present work based on the properties of a minimal model fragment. The concept of the complexity of regular covering COV-n (packing PAC-n) is connected to the number of circle n (different in radius and location) participating in covering (packing) one minimal typical fragment. We begin with a general definition of a regular $C(n)$–system which includes packings, coverings and "incomplete coverings".

Definition 1. *A set of circles on a plane forms the $C(n)$–system if the following conditions are fulfilled.*

1) *The plane is split into T tiles (typical fragments).*

2) *Each tile is related to a a number of similarly located nodes, each being an inner or a boundary point of the tile.*

3) *The circles have their centters in the corresponding nodes of the tile.*

By definition, the $C(n)$–system has a repeating pattern of circles on the plane. The way of splitting the plane into typical fragments is the main structural feature of the $C(n)$–system. Next we look at three regular splitting structures: triangular (A), quadrangular (B). Due to their symmetry, such structures allow finding the best solutions for packings and coverings.

The structure of the $C(n)$–system is determined by mutual location of the circles and the ratio of their radii. If a node is at the fragment boundary, then regularity implies the accordance of its location for adjacent tiles, where accordance means the presence of the central symmetry (if the node is at the vertex of the tile-polygon) or the axial symmetry (if the node is located on the common side of the adjacent tiles).

Definition 2. *Let n be the number of various circle types of a regular $C(n)$-system and let T be a typical fragment. In the following denotion $(T : k_1/p_1, k_2/p_2, \ldots, k_n/p_n)$, k_i/p_i is the proportion of the circle of the i–type which is involved in the covering of fragment T.*

The special cases of a $C(n)$–system which are coverings or packings will be referred to as

$$COV(T : k_1/p_1, k_2/p_2, \dots, k_n/p_n) \ \ \textit{and} \ \ PAC(T : k_1/p_1, k_2/p_2, \dots, k_n/p_n),$$

respectively.

Definition 3. *Covering density d_n and packing density D_n are calculated as a ratio of total circle areas to area of T:*

$$d_n = \frac{\pi \sum_{i=1}^{n} R_i^2 k_i/p_i}{S(T)} = D_n.$$

Several important models of coverings with circles of varied radius were studied in [3]. The efficient classes COV-2 of triangular structural A are $COV(T : 1/12, 1/6)$ and $COV(T : 1/12, 1/2)$. In both cases, minimal covering fragment is the rectangular triangle with angles 30° and 60° (Fig. 1). The smallest density for each class is determined by optimal ratios between the radii of two circles.

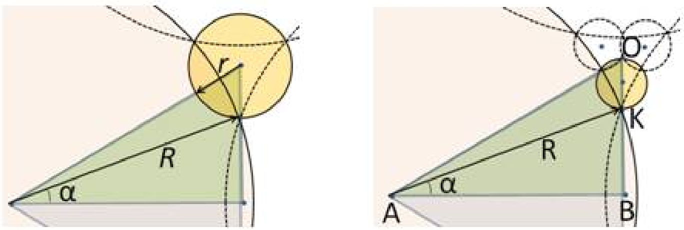

Figure 1: Minimal fragments $COV(T : 1/12, 1/6)$ and $COV(T : 1/12, 1/2)$.

In the first case,

$$min(d_2) = \frac{11\pi}{18\sqrt{3}} \approx 1,1084, \quad \frac{R_1}{R_2} = \frac{1}{\sqrt{31}} \approx 0,1796.$$

In the second case,

$$min(d_2) = \frac{3\pi}{5\sqrt{3}} \approx 1,0883, \quad \frac{R_1}{R_2} = \frac{1}{2\sqrt{21}} \approx 0,1091.$$

One can compare the two models in Figure 1 and see due to what there was the corresponding advantage.

Classes $COV(T : 1/8, 1/8)$ and $COV(T : 1/8, 1/2)$ are based on the quadrangular structure B and have the same density:

$$min(d_2) = \frac{3\pi}{8} \approx 1,1781, \quad \frac{R_1}{R_2} = \frac{1}{\sqrt{5}} \approx 0,4472 \text{ or } \frac{R_1}{R_2} = \frac{1}{2\sqrt{5}} \approx 0,2236.$$

The best packing density in the given class PAC-2 is reached in case $PAC(T : 1/12, 1/2)$ and equals

$$max(D_2) = \frac{5\pi(59 - 24\sqrt{6})}{\sqrt{12}} \approx 0,9624, \quad \frac{R_1}{R_2} = 5 - 2\sqrt{6} \approx 0,1010.$$

Circular coverings of flat regions are the basic models in the wireless sensor networks design. When optimizing a covering, its density is subjected to decrease, as it leads to smaller energy consumption for sensor network. It is important that a number of various circles are not very big, and a covering structure is simple. In other words, it is necessary to set a task of building covering of minimal density at a limited complexity of a covering class.

The investigations showed that the increase of structural complexity to three and more does not give considerable advantages in optimizing the covering (packing) density. The number of nodes (sites of devices or objects location) per fixed area is considerably increased and this is a negative factor. The analogous remarks are fair for tasks in packaging regions by three or more circle species.

References

1. Fejes Tóth L. *Lagerungen in der Ebene, auf der Kugel und in Raum*, Springer–Verlag, Berlin, 1953, 2 edn. 1972.
2. Rogers C. *Packing and Covering*, Cambridge University Press, 1964.
3. Zalubovsky V., Astrakov S., Erzin A., Choo H. *Energy-efficient Area Coverage by Sensors with Adjustable Ranges*, Sensors, 9 (2009), 2446–2460.

Kernel of Hörmander-type operator with vector fields of low smoothness

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We define *Carnot manifold with $C^{k,\alpha}$-smooth vector fields*, $k \in \mathbb{N}$, $\alpha \in [0,1]$, as a smooth connected manifold $\mathbb{M}$ with a fixed family of $C^{K,\alpha}$-smooth distributions

$$H_1 \subsetneq H_2 \subsetneq \ldots \subsetneq H_m = T\mathbb{M}$$

such that $H_{k+1} = H_k + \bigcup\limits_{i+j=k} [H_i, H_j]$, where $[\cdot,\cdot]$ is the Lie bracket. As shown in [1], any two points on such a manifold can be connected by an absolutely continuous curve γ such that $\dot{\gamma} \in H_1$ a. e. Consequently, this kind of structure can be seen as a certain low-smooth generalization of a Carnot–Carathéodory metric space. Carnot–Carathéodory spaces are tightly connected to the theory of hypoelliptic operators. As shown by L. Hörmander [2], if smooth vector fields $X_1, \ldots, X_m \in T\mathbb{M}$ generate the whole tangent space by means of Lie bracket then the operator

$$\mathcal{L} = X_1^2 + \ldots + X_n^2 \tag{1}$$

is hypoelliptic, i. e. the solution to the equation $\mathcal{L}u = f$ is C^∞-smooth as long as the right hand side is C^∞-smooth.

We study differential operator (1) where $C^{2,\alpha}$-smooth vector fields $X_1, \ldots, X_n$ form the basis of the distribution H_1. It is shown that the kernel of the operator (1), i. e. the distributional solution γ_y of the equation $\mathcal{L}\gamma_y = -\delta_y$ is a locally integrable function $\gamma(x,y)$, continuous when $x \neq y$.

References

1. Basalaev S. G., Vodopyanov S. K. *Approximate differentiability of mappings of Carnot–Carathéodory spaces*, Eurasian Math. J. 4:2 (2013), 10–48.
2. Hörmander L. *Hypoelliptic second order differential equations*, Acta Math. 119 (1967), 147–171.

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On Kolmogorov's ε-entropy for a compact set of infinitely differentiable aperiodic functions (Babenko's problem)

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The asymptotics of Kolmogorov's ε-entropy for a compact set of infinitely differentiable aperiodic functions that are boundedly embedded in the space of continuous functions on a finite interval is calculated.

References

1. Babenko K. I. *Basics of Numerical Analysis*, RKhD, Izhevsk (2002) [in Russian].
2. Belykh V. N. *On Kolmogorov's ε-entropy for a compact set of infinitely differentiable aperiodic functions (Babenko's problem)*, Doklady Math. 98:2 (2018), 416–420.
3. Belykh V. N. *The absolute ε-entropy of a compact set of infinitely differentiable aperiodic functions*, Siberian Math. J. 59:6 (2018), 947–959.

Finite homogeneous metric spaces

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We discuss the class of finite homogeneous metric spaces, its subclasses of normal, generalized normal and strongly generalized normal homogeneous, Clifford–Wolf homogeneous spaces, finite groups with left-invariant or bi-invariant metrics and relations between them.

Similar classes were studied for Riemannian manifolds in [1–3].

There are the following tools which use for constructing the considered spaces:

(1) homogeneous spaces G/H of finite (including indecomposable or simple) groups G by their subgroup H, endowed with invariant metrics;

(2) compact convex sets (including regular and semi-regular polyhedra [4, 5]) in Euclidean spaces with a group of isometries that are transitive on the set of vertices;

(3) vertex-symmetric (vertex-transitive, in other terminology) connected finite graphs [6] with the natural metric.

Definition 1. A finite metric space (M, d) is called homogeneous if for every points $x, y \in M$, there is an isometry f of the space (M, d) on itself such that $f(x) = y$.

Definition 2. A finite homogeneous metric space (M, d) is called *normal homogeneous* if for the group G in Proposition 1, there exists its subgroup Γ transitive on M and a bi-invariant metric σ on Γ such that the canonical projection $\pi : (\Gamma, \sigma) \to (\Gamma/(\Gamma \cap H), d) = (M, d)$ is a submetry [7].

Definition 3. A finite homogeneous metric space (M, d) is called *generalized normal homogeneous* if for every points $x, y \in M$, there is an isometry f (that is called a δ-shift at the point x) of the space (M, d) onto itself such that $f(x) = y$ and $d(x, f(x)) \geq d(z, f(z))$ for all $z \in M$.

Definition 4. A finite homogeneous metric space (M, d) is called *strongly generalized normal homogeneous* if for every points $x, y \in M$, $x \neq y$, there is an isometry f of the space (M, d) onto itself such that f has no fixed point, $f(x) = y$, and $d(x, f(x)) \geq d(z, f(z))$ for all $z \in M$.

Definition 5. A finite homogeneous metric space (M, d) is called *Clifford–Wolf homogeneous* (shortly, CW-homogeneous) if for every points $x, y \in M$ there is an isometry f (that is called a CW-translation) of the space (M, d) such that $f(x) = y$ and $d(x, f(x)) = d(z, f(z))$ for any point $z \in M$.

Notation 1. FGBM, FGLM, FCWHS, FSGNHS, FGNHS, FNHS, FHS denote respectively the classes of finite groups with bi-invariant metrics, finite groups with left-invariant metrics, finite CW-homogeneous spaces, finite strongly generalized normal homogeneous spaces, finite generalized normal homogeneous spaces, finite normal homogeneous spaces, and finite homogeneous spaces.

Theorem 1. *The following inclusions and equality are fulfilled:*

$$FGBM \subset FCWHS \subset FSGNHS \subset FGNHS = FNHS \subset FHS,$$

$$FGBM \subset FGLM \subset FHS.$$

Remark. All the above inclusions in Theorem 1 are strict.

Definition 6. A metric space (M, d) is called a *direct metric product of metric spaces* (M_1, d_1) and (M_2, d_2), if $M = M_1 \times M_2$ and

$$d((x_1, x_2), (y_1, y_2)) = [d_1(x_1, y_1)^\alpha + d_2(x_2, y_2)^\alpha]^{1/\alpha}, \quad 1 \leq \alpha < +\infty,$$

or

$$d((x_1, x_2), (y_1, y_2)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\},$$

$x_1, y_1 \in M_1$, $x_2, y_2 \in M_2$.

Theorem 2. If spaces (M_1, d_1) and (M_2, d_2) are contained in the same class of metric spaces introduced in Notation 1, then their direct metric product belongs to the same class.

Definition 7. A space (M, d), belonging to one of the classes indicated in Notation 1, is called *indecomposable* in this class, if it cannot be represented as a direct product of metric spaces (M_1, d_1) and (M_2, d_2) from the same class, provided that $|M_1| \geq 2$, $|M_2| \geq 2$. Otherwise, (M, d) is said to be decomposable in this class.

Theorem 3. *If $(M, d) \in FHS$, $|M| \geq 2$, and $|Isom(M, d)| = |M|!$, then (M, d) belongs to any class from Notation 1.*

Moreover, for any choice of one of these classes,

1) (M, d) is indecomposable in this class for all groups $M = G$ in the case of FGBM *or* FGLM *classes if and only if $|M|$ is a prime number.*

2) If $|M|$ is not simple and M is represented as any particular group G, if (M, d) is considered in the classes FGBM or FGLM, then (M, d) is uniquely represented as a direct metric product of spaces indecomposable in this class, up to a permutation of the factors.

Let us consider some $k, n \in \mathbb{N}$, $1 \leq k \leq n/2$. *The Kneser graph* $KG_{n,k} = (V, E)$ is a graph whose vertices are all k-element subsets of the set $\mathbb{Z}_n$ (or any other n-element set), and the edges are pairs of such subsets with empty intersection, i. e. $V = \{A \subset \mathbb{Z}_n \,|\, |A| = k\}$ and $E = \{\{A, B\} \,|\, A, B \in V,\ A \cap B = \emptyset\}$. It is clear that $|V| = C_n^k = \frac{n!}{k!(n-k)!}$.

For a given Kneser graph $KG_{n,k} = (V, E)$ with $k \geq 2$, we define a finite metric space $(M = V, d)$. Consider some positive numbers $\alpha_1 \neq \alpha_2$ such that $\alpha_1 < 2\alpha_2 < 4\alpha_1$ and define the metric $d = d_{n,k,\alpha_1,\alpha_2}$ as follows: $d(A, B) = \alpha_1$ for $A \cap B = \emptyset$ and $d(A, B) = \alpha_2$ for $A \cap B \neq \emptyset$ $(A, B \in V,\ A \neq B)$.

Theorem 4. *The metric space $(M, d = d_{n,k,\alpha_1,\alpha_2})$, $\alpha_1 > \alpha_2$, is a strongly generalized normal homogeneous for $(n, k) = (7, 3)$ and for $(n, k) = (3m, m + 1)$, where $m \geq 3$ and $m \not\equiv -1 \pmod 3$ but is not Clifford-Wolf homogeneous.*

References

1. Berestovskii V. N., Nikonorov Yu. G. *Clifford–Wolf homogeneous Riemannian manifolds*, J. Differ. Geom. 82:3 (2009), 467–500.
2. Berestovskii V. N., Nikonorov Yu. G. *Generalized normal homogeneous Riemannian metrics on spheres and projective spaces*, Ann. Global Anal. Geom. 45:3 (2014), 167–196.

3. Берестовский В. Н., Никоноров Ю. Г. *Римановы многообразия и однородные геодезические*, Владикавказ: ЮМИ ВНЦ РАН и РСО-А, 2012.

4. Berger M. *Geometry, Universitext.* Berlin: Springer-Verlag, 2009.

5. Wenninger M J., *Polyhedron models.* London-New York: Cambridge University Press, 1971.

6. Harary F. *Graph theory.* Reading, Mass.-Menlo Park Calif.-London: Addison-Wesley Publishing Co., 1969.

7. Berestovskii V. N., Guijarro L. *A metric characterization of Riemannian submersions*, Ann. Global Anal. Geom. 18:6 (2000), 577–588.

(Co)adjoint representation and normal geodesics of left-invariant (sub-)Finsler metrics on Lie groups

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Every left-invariant (sub-)Finsler metric $d = d_F$ on a connected Lie group G with Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ is defined by a subspace $\mathfrak{p} \subset \mathfrak{g}$, generating $\mathfrak{g}$, and some norm F on $\mathfrak{p}$. A distance $d(g,h)$ for $g, h \in G$ is defined as the infimum of lengths $\int_0^T |\dot{g}(t)|\,dt$ of piecewise smooth paths $g = g(t)$, $0 \leq t \leq T$, such that $dl_{g(t)^{-1}}\dot{g}(t) \in \mathfrak{p}$ and $g(0) = g$, $g(T) = h$; T is not fixed, $|\dot{g}(t)| = F(dl_{g(t)^{-1}}\dot{g}(t))$. If $\mathfrak{p} = \mathfrak{g}$ then d is a left-invariant Finsler metric on G; if $F(v) = \sqrt{\langle v, v\rangle}$, $v \in \mathfrak{p}$, where $\langle \cdot,\cdot\rangle$ is some scalar product on $\mathfrak{p}$, then$d$ is a left-invariant sub-Riemannian metric on G, and d is a left-invariant Riemannian metric, if additionally $\mathfrak{p} = \mathfrak{g}$.

We prove the following results below.

Theorem 1. *1. Any normal extremal $g = g(t) : \mathbb{R} \to G$ (parameterized by arc length and with origin $e \in G$), of left-invariant (sub-)Finsler metric d on a Lie group G, defined by a norm F on the subspace $\mathfrak{p} \subset \mathfrak{g}$ with closed unit ball U, is a Lipschitz integral curve of the following vector field*

$$v(g) = dl_g(u(g)), \quad u(g) = \psi_0(Ad(g)(w(g)))w(g), \quad w(g) \in U,$$

$$\psi_0(Ad(g)(w(g))) = \max_{w\in U}\psi_0(Ad(g)(w)) = \max_{w\in U} Ad(g)^*(\psi_0)(w),$$

where $\psi_0 \in \mathfrak{g}^*$ is some fixed covector with $\max_{v\in U}\psi_0(v) = 1$.

2. (Conservation law) In addition, $\psi(t)(g(t)^{-1}g'(t)) \equiv 1$ for all $t \in \mathbb{R}$, where $\psi(t) := (Ad(g(t)))^*(\psi_0)$.

Remark 1. *Similar results are proved in book [1] by Velimir Jurdjevich.*

Remark 2. *Every extremal with origin g_0 is obtained by the left shift l_{g_0} from some extremal with origin e.*

Remark 3. *In (sub-)Riemannian case, the vector $u(g)$ is characterized by condition $\langle u(g), v\rangle = \psi_0(Ad(g)(v))$ for all $v \in \mathfrak{p}$. In Riemannian case, every extremal is a normal geodesic, and we can assume that ψ_0 is an unit vector in $(\mathfrak{p} = \mathfrak{g}, (\cdot,\cdot))$, setting $\psi_0(v) = (\psi_0, v)$, $v \in \mathfrak{g}$. Moreover, $\dot{g}(0) = \psi_0$.*

Theorem 2. *If $v(g_0) \neq 0$, $g_0 \in G$, then an integral curve of the vector field $v(g)$, $g \in G$, with origin g_0 is a normal extremal parameterized proportionally to arc length with the proportionality factor $|dl_{g_0^{-1}}(v(g_0))|$.*

Remark 4. *Theorem 2 holds for left-invariant Riemannian metrics on (connected) Lie groups. In this case, $v(g_0) \neq 0$ for all $g_0 \in G$.*

Let us choose a basis $\{e_1, \dots, e_n\}$ in $\mathfrak{g}$, assuming that $\{e_1, \dots, e_r\}$ is an orthonormal basis for the scalar product $\langle\cdot,\cdot\rangle$ on $\mathfrak{p}$ in case of left-invariant (sub-)Riemannian metric. Define a scalar product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$, considering $\{e_1, \dots, e_n\}$ as its orthonormal basis. Then each covector $\psi \in \mathfrak{g}^*$ can be considered as a vector in $\mathfrak{g}$, setting $\psi(v) = \langle\psi, v\rangle$ for every $v \in \mathfrak{g}$. If $\psi = \sum_{i=1}^n \psi_i e_i$, $v = \sum_{k=1}^n v_k e_k$, then $\psi(v) = \psi \cdot v$, where ψ and v are corresponding vector-row and vector-column, $\cdot$ is the matrix multiplication.

If $g(t)$, $t \in \mathbb{R}$, is a normal geodesic of a left-invariant (sub-)Riemannian metric d on a Lie group G, then $u(g(t))$ is the orthogonal projection onto $\mathfrak{p}$ of the vector $(Ad(g(t)))^*(\psi_0)$ in the notation of Theorem 2 for the scalar product $\langle\cdot,\cdot\rangle$ introduced above on $\mathfrak{g}$. This fact implies

Theorem 3. *Every normal parameterized by arclength geodesic of left-invariant (sub-)Riemannian metric on a Lie group G issued from the unit is a solution of the following system of differential equations*

$$\dot{g}(t) = dl_{g(t)}u(t), \quad u(t) = \sum_{i=1}^r \psi_i(t)e_i, \quad |u(0)| = 1,$$
$$\dot{\psi}_j(t) = \sum_{k=1}^n \sum_{i=1}^r c_{ij}^k \psi_i(t)\psi_k(t), \quad \psi(0) = \psi_0,$$

where $j = 1, \dots, n$, c_{ij}^k are structure constants of Lie algebra $\mathfrak{g}$ in its basis $\{e_1, ..., e_n\}$.

Corollary 2. $|\dot{g}(t)| = |u(t)| \equiv 1$, $t \in \mathbb{R}$.

The only Lie groups which do not admit left-invariant sub-Finsler metrics are commutative Lie groups and Lie groups G_n, $n \geq 2$, which can be described as connected Lie groups whose every left-invariant Riemannian metric has constant negative sectional curvature [2]. The group operations for $G_n = \mathbb{R}^n_+ = \{(y,x) \in \mathbb{R}^n : x > 0\}$, $n \geq 2$, have a form

$$(y_1, x_1) \cdot (y_2, x_2) = x_1(y_2, x_2) + (y_1, 0), \quad (y, x)^{-1} = x^{-1}(-y, 1).$$

Then it is clear that $e = (0,1)$ is the unit in G_n and the group G_n is a simply transitive isometry group of the famous Poincare's model of the Lobachevsky space L^n in $\mathbb{R}^n_+$ with metric $ds^2 = (\sum_{k=1}^{n-1} dy_k^2 + dx^2)/x^2$ of constant sectional curvature -1 and corresponding coordinate orthonormal basis $\{e_1, \ldots, e_n\}$ in $\mathfrak{g}_n = (G_n)_e$. All nonzero structure constants in $\{e_1, \ldots, e_n\}$ are equal to $c^i_{ni} = -c^i_{in} = 1$, $i = 1, \ldots, n-1$. Using independently Theorems 2 and 4, we obtain the same results: Every geodesic $g(t) = (y(t), x(t))$, $t \in \mathbb{R}$, in (G_n, d), parameterized by arclength, with origin e are well-known semicircles passing through e and orthogonal to $\mathbb{R}^{n-1}$:

$$x(t) = 1/(\cosh t - \varphi_n \sinh t), \quad y_i(t) = \varphi_i \sinh t/(\cosh t - \varphi_n \sinh t),$$
$$\psi_i(0) = \varphi_i, \quad i = 1, \ldots, n, \quad \sum_{i=1}^{n} \varphi_i^2 = 1.$$

The Lie group $SE(2)$ is isomorphic to the group of matrices of a form

$$\begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix}; \quad A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}, \quad v = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$$

The Lie group $SH(2)$ is obtained from $SE(2)$ by changing $\cos\varphi$ and $\sin\varphi$ respectively by $\cosh\varphi$ and $\sinh\varphi$. Using Theorem 4 for $SO(3)$ and Theorem 2 for Lie groups $SE(2)$ and $SH(2)$, we proved that for left-invariant sub-Riemannian metrics, arbitrary on $SO(3)$ and some natural on $SE(2)$ and $SH(2)$, every parameterized by arclength geodesic $g(t)$, $t \in \mathbb{R}$, $g(0) = e$, has correspondingly the following form:

$$\dot{g}(t) = g(t)u(t), \; u(t) = \cos(\omega(t)/2)e_1 + \sin(\omega(t)/2)e_2,$$
$$\ddot{\omega}(t) = (a^2 - b^2)\sin\omega(t), \; a^2 \leq b^2;$$
$$\dot{\varphi}(t) = \cos(\omega(t)/2), \; \dot{x}(t) = \sin(\omega(t)/2)\cos\varphi(t), \; \dot{y}(t) = \sin(\omega(t)/2)\sin\varphi(t);$$
$$\dot{\varphi}(t) = \sin(\omega(t)/2), \; \dot{x}(t) = \cos(\omega(t)/2)\cosh\varphi(t), \; \dot{y}(t) = \cos(\omega(t)/2)\sinh\varphi(t);$$
$$\ddot{\omega}(t) = -\sin\omega(t).$$

References

1. Jurdjevich V., *Geometric control theory*, Cambridge Univ. Press, 1997.
2. Milnor J., *Curvatures of left invariant metrics on Lie groups*, Adv. Math. 21 (1976), 293–329.

Computing sums and intersections of local Morrey spaces

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It is shown that the calculation of the sums and the intersection of Morrey's spaces can be reduced to calculating the sum and intersections of spaces of functions included by a parameter in the definition Morrey's spaces. This reduction allows us to obtain new extrapolation theorems for Morrey's spaces.

Let $\{B_\beta\}$, $(\beta \in (\underline{\beta}, \overline{\beta}),\ -\infty \leq \underline{\beta} < \overline{\beta} \leq \infty)$ be a collection of Banach spaces, continuously embedded in a complete topological separable space V is given. Such a collection is called a V-collection.

For a V-collection and a measurable the function $\{\xi : (\underline{\beta}, \overline{\beta}) \to R_+\}$ we denote a new space $\sum\{\xi, B_\beta\}$ by norm

$$\|b| \sum\{\xi, B_\beta\}\| = \inf\{\Sigma\xi(\beta_i)\|b_{\beta_i}|B_{\beta_i}\| : b = \Sigma b_{\beta_i}, \text{ the series converges in } V, \quad \&$$

$$\underline{\beta} < \ldots < \beta_{-i} < \beta_{-i+1} \ldots < \beta_0 < \beta_1 < \ldots < \beta_i < \ldots < \overline{\beta}\},$$

(the symbol $\|b|B\|$ denotes the norm of an element b in the space B).

The space $\sum\{\xi, B_\beta\}$ is a Banach space.

Analogously, by $\bigcap\{\xi, B_\beta\}$ we denote a new Banach space, the norm of which is given by $\|b| \bigcap\{\xi, B_\beta\| = \sup_{\beta \in (\underline{\beta}, \overline{\beta})} \xi(\beta)\|b|B_\beta\|$.

Let μ be the Lebesgue measure on R^n, let $S(\mu)$ be the space of measurable functions $f : R^n \to R$, and let $\chi(D)$ stand for the characteristic function of a set $D \subseteq R^n$. A Banach space X of measurable functions on R^n is said to be ideal if it follows from the condition $f \in X$, the measurability of g, and the validity of the inequality $|g(t)| \leq |f(t)|$ for almost all $t \in R^n$ that $g \in X$ and $\|g|X\| \leq \|f|X\|$.

Let $S(Z)$ be the space of number sequences $a = \{a_i\}_{-\infty}^{\infty}$ with coordinate-wise convergence, $\{e^i\}_{-\infty}^{\infty}$ is a standard basis in $S(Z)$. A Banach space $l \subset S(Z)$ is called ideal, if $x = \sum_{-\infty}^{\infty} e^i x_i \in l$, $y = \sum_{-\infty}^{\infty} e^i y_i \in S(Z)$ and for all $i \in Z$ the inequality $|y_i| \leq |x_i|$ implies that $y \in l$ and $\|y|l\| \leq \|x|l\|$.

Classic examples of ideal Banach spaces are the spaces of Lebesque, Orlicz, Lorentz, Marcinkiewicz and others symmetric spaces.

We denote by Υ the set of non-negative number sequences $\tau = \{\tau_i\}$ each of which satisfies the conditions $\forall i : \tau_i < \tau_{i+1}$, $\lim_{i \to -\infty} \tau_i = 0$.

Let $\{U(0,\tau)\}$ be a collection homothetic, star-shaped with respect to the point 0 sets $U(0,\tau) \subset R^n$, for which $\mu(U(0,1)) > 0$. For every sequence $\tau \in \Upsilon$ we

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construct a family of sets $U(0,\tau_i)$ and a family of disjoint annuli $R(0,\tau_{i-1},\tau_i) = U(0,\tau_i) \setminus U(0,\tau_{i-1})$.

Definition 1. *Let an ideal space X on R^n, an ideal space l sequences with the standard basis $\{e^i\}$, a sequence $\tau \in \Upsilon$ and a collection $\{U(0,\tau)\}$ be given. The discrete local Morrey space $LM^{\tau}_{l,X}$ consists of those $f \in L^{1,loc}(R^n)$ for which the following norm is finite:*

$$\|f|LM^{\tau}_{l,X}\| = \|\Sigma_i e^i \|f\chi(U(0,\tau_i))|X\||l\|.$$

By the approximate local Morrey space $\overline{LM^{\tau}_{l,X}}$ we mean the set of functions $f \in L^{1,loc}(R^n)$ for each of which the following norm is finite:

$$\|f|\overline{LM^{\tau}_{l,X}}\| = \|\sum_i e^i \|f\chi(R(0,\tau_{i-1},\tau_i)|X\||l\|.$$

From the fact that X and l are ideal spaces, it follows that the spaces $LM^{r}_{l,X}$, $\overline{LM^{r}_{l,X}}$ are also ideal and, therefore, Banach spaces.

It follows immediately from the definition that the following embedding hold:

$$\overline{LM^{r}_{l,X}} \subseteq LM^{r}_{l,X}, \tag{1}$$

where the constants of the embedding is equal to unity.

The following crucial theorem gives natural conditions on the embedding inverse to (1).

Theorem B [1]**.** *Let spaces $\overline{LM^{r}_{l,X}}$, $LM^{r}_{l,X}$, be constructed from spaces X and l, a collection $\{U(0,\tau)\}$ and a sequence $\tau \in \Upsilon$. We introduce an operator $P : l \to l$ by the equation*

$$P(\Sigma_i e^i x_i) = \Sigma_i e^i y_i, \quad y_i = \Sigma_{j \le i} x_j.$$

When $\|P|l \to l\| = c_0 < \infty$, the spaces $\overline{LM^{r}_{l,X}}$ and $LM^{r}_{l,X}$ have the same set of elements, and the following inequalities hold:

$$\|f|\overline{LM^{r}_{l,X}}\| \le \|f|LM^{r}_{l,X}\| \le c_0 \|f|\overline{LM^{r}_{l,X}}\|.$$

An important point of the theorem B is that the conditions are imposed on the space l and do not depend on the sequence $\tau \in \Upsilon$.

If in Definition 1 we replace R^n, $L^{1,loc}(R^n)$, $U(0,\tau_i)$ by Ω, $L^{1,loc}(\Omega)$ and $U(0,\tau_i) \bigcap \Omega$, then we obtain definitions of Morrey spaces on the set Ω.

The following two theorems are basic.

Theorem 1. *We fix the ideal space X, the collection $\{U(0,\tau)\}$, the sequence $\tau \in \Upsilon$ and the measurable function $\xi : (\beta_0,\beta_1) \to R_+$. Let a set of ideal sequence* *spaces be given* $\{l_\beta\}$, *(*$\beta \in (\beta_0, \beta_1)$*). We construct for each* $\{l_\beta\}$ *Morrey local space* $M^\tau_{l_\beta,X}$ *and Morrey approximation local space* $M^\tau_{l_\beta,X}$*, (*$\beta \in (\beta_0, \beta_1)$*). Put* $\bar{l} = \sum\{\xi, l_\alpha\}$.

Then,

$$\sum\{\xi, \overline{M^\tau_{l_\beta,X}}\} = \overline{M^\tau_{\bar{l},X}},$$

and the norms in these spaces coincide.

When for the operator P *defined in theorem B the condition is satisfied*

$$\sup_{\beta\in(\beta_0,\beta_1)} \|P|l_\beta \to l_\beta\| = c_1 < \infty, \tag{2}$$

then,

$$\sum\{\xi, \overline{M^\tau_{l_\beta,X}}\} = \overline{M^\tau_{\bar{l},X}},$$

and for each $f \in M^\tau_{\bar{l},X}$ *inequalities are true*

$$\|f|\sum\{\xi, M^\tau_{l_\beta,X}\}\|Le\|f|M^\tau_{\bar{l},X}\| \le c_1\|f|\sum\{\xi, M^\tau_{l_\beta,X}\}\|.$$

Theorem 2. *Let the assumptions of Theorem 1 be fulfilled. Put* $\underline{l} = \bigcap\{\xi, l_\beta\}$.

Then,

$$\bigcap\{\xi, M^\tau_{l_\beta,X}\} = M^\tau_{\underline{l},X},$$

and the norms in these spaces coincide.

When the condition (2) is satisfied, then,

$$\bigcap\{\xi, \overline{M^\tau_{l_\beta,X}}\} = \overline{M^\tau_{\underline{l},X}}$$

and for each $f \in M^\tau_{\underline{l},X}$ *inequalities are true*

$$\|f|\bigcap\{\xi, M^\tau_{l_\beta,X}\}\| \le \|f|M^\tau_{\underline{l},X}\| \le c_1\|f|\bigcap\{\xi, M^\tau_{l_\beta,X}\}\|.$$

From theorems 1–2 we obtain the new extrapolation theorems for operators in local Morrey space.

References

1. Berezhnoĭ E. I. *A discrete version of local Morrey spaces*, Izvestiya: Mathematics 81:1 (2017), 1–28 (Izvestiya RAN: Ser. Mat. 81:1 (2017), 3–31 (Russian)).

Strong solutions to the Boltzmann equation in the case of a one-dimensional spatial variable

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The initial value problem for the Boltzmann equation is stated as follows

$$\partial_t f + v \cdot \nabla_x f = Q(f, f), \qquad f(x, v, 0) = f_0(x, v). \tag{1}$$

We are interested in strong (classical) solutions which express the conservation of mass in the phase space of a single particle in a gas, $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$. The solution $f = f(x, v, t)$ is the "molecular velocity distribution function" which by its physical meaning is the nonnegative local mass density in the phase space (x, v), at time $t \in \mathbb{R}$. The right-hand side $Q(f, f)$, known as the collision operator, can be described as the effect of a certain "force" acting on a given particle as a result of its interaction with other particles in a gas. The precise form of $Q(f, f)$ is due to Maxwell [4] and Boltzmann [2].

The collision operator $Q(f, f)$ is as follows:

$$Q(f, f)(x, v, t) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \left(f'_* \, f' - f_* \, f \right) K(|v - v_*|, \cos\vartheta) \, d\sigma \, dv_*,$$

where $f = f(x, v, t)$, $f_* = f(x, v_*, t)$, $f' = f(x, v', t)$, $f'_* = f(x, v'_*, t)$, $\cos\vartheta = \frac{(v-v_*)\cdot\sigma}{|v-v_*|}$, and the primed quantities denote the values corresponding to the instant before the collision while the unprimed to those after. In particular,

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \qquad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma \tag{2}$$

are the well-known formulas describing the parametrization of the velocities of the colliding particles in terms of v, v_* and the unit vector σ in the direction of $v' - v'_*$. Notice that the velocities in (2) satisfy identically the following conditions of conservation of momentum and kinetic energy:

$$v' + v'_* = v + v_*,$$
$$|v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2.$$

We consider solutions which satisfy periodic boundary conditions $f(x+\gamma, v, t) = f(x, v, t)$, $\gamma \in \mathbb{Z}^3$, which is to say that $x \in \mathbb{T}_x := \mathbb{R}^3_x/\mathbb{Z}^3$. We further restrict $f(x, v, t)$ to be independent of (x_2, x_3). We remark that the dependence on v variable is fully three dimensional.

The macroscopic *density* associated with a solution f is given by

$$\rho(x,t) = \int_{\mathbb{R}^3_v} f(x,v,t)\,dv.$$

The *total mass* and *total energy* of f are defined respectively by the integrals

$$A(f) = \int_{\mathbb{T}_x} \rho(x,t)\,dx, \quad E(f) = \int_{\mathbb{T}_x}\int_{\mathbb{R}^3_v} f(x,v,t)|v|^2\,dv\,dx.$$

The *relative entropy* of f with respect to the Maxwellian $M(v)$ is given by

$$H(f|M) = \int_{\mathbb{T}_x}\int_{\mathbb{R}^3_v} f\log(\tfrac{f}{M}) - f + M\,dv\,dx, \text{ where } M(v) = \tfrac{a}{\sqrt{2\pi b}^3} e^{-|v-c|^2/2b},$$

for $a, b > 0$ and $c \in \mathbb{R}^3$. The Maxwellian functions $f(x,v,t) = M(v)$ are the equilibrium solutions of (1). We remark that if $H(f|M)$ is finite for some Maxwellian, then it is for all others.

Our first hypothesis on the collision kernel $K(w,\xi)$ is about boundedness, linear bound rate near the origin, and a very mild condition of decrease at infinity.

Hypothesis 1. *There exist constants* $K_0, \varepsilon > 0$ *such that*

$$0 \le K(w,\xi) \le \frac{K_0 w}{1 + w\log^{1+\varepsilon}(1+w)}. \tag{\textbf{H1}}$$

Definition 1. A *strong solution* $f(x,v,t)$ of the Boltzmann equation (1) on a time interval $[0,T)$ is a distributional solution with the property that $f \in C\big([0,T); L^1(\mathbb{T}_x \times \mathbb{R}^3_v)\big) \cap L^1\big((0,T); L^\infty(\mathbb{T}_x; L^1(\mathbb{R}^3_v))\big)$.

Under the condition $\sup K < +\infty$, which follows from (**H1**), this coincides with the definition used in [3].

The free-streaming flow is defined as follows: $S_t : f_0(x,v) \mapsto f_0(x - tv, v)$, $t \ge 0$. This flow is unbounded on $L^\infty_x(L^1_v)$. Therefore we introduce a slightly more restrictive function class, adapted to the problem (1).

Definition 2. For functions $f(x,v)$ on phase space, define the norm

$$\|f\|_X := \sup_{x\in\mathbb{T}_x}\ \sup_{p\in\mathbb{R}^+} \int_{\mathbb{R}^3_v} |\,(S_p f)(x,v)\,|\,dv.$$

Denote by X the space of functions f which are independent of (x_2, x_3) and periodic in x_1 for which this norm is finite.

Theorem 1. *Consider the Boltzmann equation (1) whose collision kernel satisfies (**H1**). Given initial data $f_0(x, v) \in X$ such that $f_0 \geq 0$ and $H(f_0|M)$ is finite for a Maxwellian $M(v)$, then a strong solution $f(x, v, t) \in X$ exists locally in time. If for some Maxwellian $M_0(v)$ the quantity $H(f_0|M_0)$ is small enough, this strong solution exists globally in time.*

For our next Theorem we impose an additional small relative velocity cutoff and a lower bound on the collision kernel.

Hypothesis 2. *We suppose there exists a constant R such that*

$$K(w, \xi) = 0 \quad \textit{for} \quad w < R \qquad \textit{and} \qquad K(w, \xi) \geq \beta(\xi) \sup_{\xi \in (-1,1)} K(w, \xi), \tag{\textbf{H2}}$$

where $\beta(\xi)$ is positive on a set of positive measure.

Theorem 2. *Assume that the additional hypothesis (**H2**) is imposed on the Boltzmann collision kernel. Suppose that initial data $0 \leq f_0(x, v) \in X$ is given such that*

$$A(f_0) + E(f_0) < +\infty.$$

Then there exists a strong solution $f(x, v, t)$ for all $t \in \mathbb{R}^+$.

References

1. Biryuk A., Craig W., Panferov V. *Strong solutions of the Boltzmann equation in one spatial dimension.* Comptes Rendus Mathematique 342:11 (2006), 843–848.
2. Boltzmann L., *Weitere Studien über das Wärmegleichgewicht unter Gasmolekulen*, Sitzungberichte der kaiserlichen Akademie der Wissenschaften in Wien, Klasse IIa, (1872).
3. Lions P.-L., *Compactness in Boltzmann's equation via Fourier integral operators and applications. I, II.* J. Math. Kyoto Univ. 34 (1994), 391–427, 429–461.
4. Maxwell J. C., *On the dynamical theory of gases*, Phil. Trans. R. Soc. London, 157 (1867), 49–88.

Some properties of cyclic recurrent submanifolds

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Let F^n be n-dimensional ($n \geq 2$) smooth submanifold in $(n+p)$-dimensional ($p \geq 2$) space of constant curvature $M^{n+p}(\tilde{c})$. Denote by $\tilde{g}$ the Riemannian metric on $M^{n+p}(\tilde{c})$, by g the induced Riemannian metric on F^n, by $\tilde{\nabla}$ and ∇ the Riemannian connections on $M^{n+p}(\tilde{c})$ and F^n consistent with $\tilde{g}$ and g, respectively. Let b be the second fundamental form F^n in $M^{n+p}(\tilde{c})$. Denote by D and $R^{\perp}$ the normal connection and the tensor of normal curvature of F^n in $M^{n+p}(\tilde{c})$, respectively, by $\overline{\nabla} = \nabla \oplus D$ the connection of van der Waerden–Bortolotti.

Defintion 1. *The second fundamental form $b \neq 0$ is called cyclic recurrent if on F^n there exists 1-form μ such that*

$$\overline{\nabla}_X b(Y,Z) = \mu(X)b(Y,Z) + \mu(Y)b(Z,X) + \mu(Z)b(X,Y)$$

for all vector fields X, Y, Z tangent to F^n.

Submanifolds F^n in $M^{n+p}(\tilde{c})$ with cyclic recurrent second fundamental form $b \neq 0$ (cyclic recurrent submanifolds, for short) are natural generalizations of Darboux surfaces in three-dimensional Euclidean space E^3 (see [1]).

Certain properties of hypersurfaces F^n with cyclic recurrent non-parallel second fundamental form in Euclidean spaces E^{n+1} were obtained in [2].

The following statements hold.

Theorem 1. *Let the pointwise codimension of the surface F^2 at each point $x \in F^2$ in the space of constant curvature $M^4(\tilde{c})$ be equal to 1, and the surface F^2 have no asymptotic directions. Then if the surface $F^2 \subset M^4(\tilde{c})$ has cyclic recurrent second fundamental form $b \neq 0$ then F^2 belongs to totally geodesic hypersurface in $M^4(\tilde{c})$.*

Theorem 2. *Let the surface F^2 in the space of constant curvature $M^4(\tilde{c})$ have flat normal connection ($R^{\perp} \equiv 0$) and have no axial points. Then if the surface $F^2 \subset M^4(\tilde{c})$ has cyclic recurrent second fundamental form $b \neq 0$ then the following conditions hold:*

1) F^2 belongs to the hypersurface $F^3 \subset M^4(\tilde{c})$ of constant curvature $\widetilde{c} + c_0^2$, $c_0 = const > 0$,

2) F^2 has constant zero inner curvature $K_i \equiv 0$ and non-zero mean curvature vector of constant length $h_0 = const > 0$, $\widetilde{c} + h_0^2 > 0$, *3) indicatrix of normal curvature of the surface F^2 at each point $x \in F^2$ is straight segment of constant length $2\sqrt{\widetilde{c} + h_0^2}$, and this indicatrix and mean curvature vector form constant non-zero angle $\alpha_0 = const \neq 0$ where $c_0^2 = h_0^2 \sin^2 \alpha_0$.*

Theorem 3. *Let the normal connection of the surface F^2 in the space of constant curvature $M^4(\tilde{c})$ be not flat ($R^{\perp} \neq 0$), and the surface F^2 have no asymptotic directions. Then if the surface $F^2 \subset M^4(\tilde{c})$ has cyclic recurrent second fundamental form b then the following conditions hold:*

1) F^2 belongs to the space $M^4(\widetilde{c})$ of constant positive curvature $\tilde{c} = 3\lambda_0^2$, $\lambda_0 = const > 0$,

2)F^2 has constant positive inner curvature $K_i = \lambda_0^2 > 0$,

3) indicatrix of normal curvature of the surface F^2 at each point $x \in F^2$ is a circle of radius $\lambda_0 = const > 0$ with the center at the point x.

References

1. Bodrenko I. I. *Generalized Darboux surfaces in the spaces of constant curvature.* LAP Lambert Acad. Publ., Saarbrucken, Germany. 2013.
2. Bodrenko I. I. *A generalization of Bonnet's theorem on Darboux surfaces.* Math. Notes. 95:6 (2014), 760–767.

Group of motions of a special four-dimensional extension of pseudo-Euclidean three-dimensional geometry

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In the work [1] within the frame of solving the embedding problem of Euclidean geometry V. A. Kyrov found a special four-dimensional extension of the pseudo-Euclidean three-dimensional geometry. This geometry is defined by the following function of the pair of points:

$$f(i, j) = ((x_i - x_j)^2 + (y_i - y_j)^2 - (z_i - z_j)^2)e^{2w_i + 2w_j}, \tag{1}$$

where (x_i, y_i, z_i, w_i) and (x_j, y_j, z_j, w_j) are the coordinates of the points i and j in R^4.

The purpose of the work is to find explicit expressions for the groups of motions of a special four-dimensional extension of a pseudo-Euclidean three-dimensional geometry.

Let us consider the effective and differentiable action of the Lie group G in $U \subset R^4$ [2, 3], i. e. differentiable mapping

$$\lambda : U \times G \to U',$$

where $U' \subset R^4$. The action λ_a, defined by an arbitrary element $a \in G$, is called *movement* the space R^4 with the function of the pair of points f, if for any $i, j \in U$ such that $\langle i, j\rangle \in S_f$, $\langle \lambda_a(i), \lambda_a(j)\rangle \in S_f$, the following equality is satisfied

$$f\big(\lambda_a(i), \lambda_a(j)\big) = f(i, j),$$

moreover $S_f \subseteq R^4 \times R^4$ is the domain of the function f.

The set of all the movements defined this way forms a ten-dimensional Lie group of transformations. The basis operators of the Lie algebra of this transformation are as follows [1]:

$$\begin{gathered}\partial_x,\ \partial_y,\ \partial_z,\ y\partial_x - x\partial_y,\ y\partial_z + z\partial_y,\ x\partial_z + z\partial_x,\ 2x\partial_x - 2y\partial_y - 2z\partial_z + \partial_w,\\ (y^2 - z^2 - x^2)\partial_x - 2xy\partial_y - 2xz\partial_z + x\partial_w,\\ (x^2 - z^2 - y^2)\partial_y - 2yx\partial_x - 2yz\partial_z + y\partial_w,\\ (-x^2 - y^2 - z^2)\partial_z - 2zx\partial_x - 2zy\partial_y + z\partial_w, \end{gathered} \tag{2}$$

The method of finding consists in applying the exponential mapping [4, 5]:

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \mathrm{Exp}(tX) \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix},$$

where t is a real parameter, X is an arbitrary Lie algebra operator of the movement group,

$$\mathrm{Exp}(tX) = 1 + tX + \frac{t^2X^2}{2!} + \cdots$$

Theorem. *The group of motions of a special four-dimensional extension of pseudo-Euclidean three-dimensional geometry with the function of the pair of points* (1) *in the explicit form is given by:*

$$\begin{aligned} x' &= \frac{((x^2+y^2-z^2)(a_{11}a + a_{12}b - a_{13}c) + a_{11}x + a_{12}y + a_{13}z)e^{-2\delta}}{(a^2+b^2-c^2)(x^2+y^2-z^2) + 2ax + 2by + 2cz + 1} + \alpha,\\ y' &= \frac{((x^2+y^2-z^2)(a_{21}a + a_{22}b - a_{23}c) + a_{21}x + a_{22}y + a_{23}z)e^{-2\delta}}{(a^2+b^2-c^2)(x^2+y^2-z^2) + 2ax + 2by + 2cz + 1} + \beta,\\ z' &= \frac{((x^2+y^2-z^2)(a_{31}a + a_{32}b - a_{33}c) + a_{31}x + a_{32}y + a_{33}z)e^{-2\delta}}{(a^2+b^2-c^2)(x^2+y^2-z^2) + 2ax + 2by + 2cz + 1} + \gamma,\\ w' &= w + \tfrac{1}{2}\ln((a^2+b^2-c^2)(x^2+y^2-z^2) + 2ax + 2by + 2cz + 1) + \delta, \end{aligned} \tag{3}$$

where $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in SO(1,2)$.

The proof of the theorem consists in finding, using the exponential map, of one-parameter subgroups, corresponding to basic operators of Lie algebras of motion groups (2), composition of which allows finding explicit expressions for local actions of Lie groups (3) of a special four-dimensional extension of pseudo-Euclidean three-dimensional geometry with function of the pair of points (1).

References

1. Kyrov V. A. *The analytical method for embedding multidimensional pseudo-Euclidean geometries*, Sib. Elektron. Mat. Izv. 15 (2018), 741–758. [in Russian]
2. Bredon G. *Introduction to compact transformation groups..* Moscow: Nauka, 1980. [in Russian]
3. Pontryagin L. S. *Continuous groups.* Moscow: Nauka, 1973. [in Russian]
4. Postnikov M. M. *Lie groups and Lie algebras.* Moscow: Nauka, 1982. [in Russian]
5. Kyrov V. A., Bogdanova R. A. *The groups of moutions of some three-dimensional maximal mobility geometries*, Siberian Math. J. 59:2 (2018), 323–331.

General nondegenerate solution of one system of functional equations

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The work solves the following system of two functional equations:

$$\left.\begin{array}{l} \bar{x}\bar{\xi} + \bar{\mu} = \chi^1(x+\xi, y+\eta, \mu, \nu), \\ \bar{x}\bar{\eta} + \bar{y}\bar{\xi} + \bar{\nu} = \chi^2(x+\xi, y+\eta, \mu, \nu), \end{array}\right\} \tag{1}$$

where $\bar{x} = \bar{x}(x,y)$, $\bar{y} = \bar{y}(x,y)$, $\bar{\xi} = \bar{\xi}(\xi,\eta,\mu,\nu)$, $\bar{\eta} = \bar{\eta}(\xi,\eta,\mu,\nu)$, $\bar{\mu} = \bar{\mu}(\xi,\eta,\mu,\nu)$, $\bar{\nu} = \bar{\nu}(\xi,\eta,\mu,\nu)$.

Solutions of the system (1) are nondegenerate, if they satisfy these two conditions

$$\frac{\partial(\bar{x},\bar{y})}{\partial(x,y)} \neq 0, \quad \frac{\partial(\bar{\xi},\bar{\eta},\bar{\mu},\bar{\nu})}{\partial(\xi,\eta,\mu,\nu)} \neq 0. \tag{2}$$

The system of functional equations (1) and conditions (2) of nondegeneracy of its solutions naturally appear in the frame of the embedding hypothesis [1] of the dimetric phenomenologically symmetric geometry of two sets (DPS GTS) of mininal rank $(2,2)$ with metric function $g = (g^1, g^2) = (x+\xi, y+\eta)$ in DPS GTS of the rank $(3,2)$ with metric function $f = (f^1, f^2) = (x\xi+\mu, x\eta+y\xi+\nu)$, taken from the full classification mentioned in the works [2]. When embedding reversible replacement of local coordinates $(x,y) \to (\bar{x}, \bar{y})$ and $(\xi, \eta, \mu, \nu) \to (\bar{\xi}, \bar{\eta}, \bar{\mu}, \bar{\nu})$ is allowed in two-dimensional and four-dimensional manifolds, on which metric function f defines DPS GTS of the rank $(3,2)$.

The unknown in this system (1) are six functions $\bar{x}, \bar{y}, \bar{\xi}, \bar{\eta}, \bar{\mu}, \bar{\nu}$, which components χ^1 and χ^2 of the scaling function $\chi = (\chi^1, \chi^2)$ are defined univalently.

Theorem. *The system of functional equations* (1) *has two types of nondegenerate solutions depending on the structure of their first function* $\bar{x} = \bar{x}(x,y)$*:*

1. General exponential nondegenerate solution

$$\begin{aligned}
&\overline{x} = h\exp(ax+by) + g, \quad \overline{y} = (h(cx+\gamma y)+\beta)\exp(ax+by) + \alpha,\\
&\overline{\xi} = \overline{\xi}(\mu,\nu)\exp(a\xi+b\eta), \quad \overline{\eta} = ((c\xi+\gamma\eta)\overline{\xi}(\mu,\nu) + \overline{\eta}(\mu,\nu))\exp(a\xi+b\eta),\\
&\overline{\mu} = -g\overline{\xi}(\mu,\nu)\exp(a\xi+b\eta) + \overline{\mu}(\mu,\nu),\\
&\overline{\nu} = -((g(c\xi+\gamma\eta)+\alpha)\overline{\xi}(\mu,\nu) + g\overline{\eta}(\mu,\nu))\exp(a\xi+b\eta) + \overline{\nu}(\mu,\nu);
\end{aligned}$$

2. Common linear nondegenerate solution

$$\begin{aligned}
&\overline{x} = ax+by+g, \;\; \overline{y} = acx^2/2 + bcxy + b\gamma y^2/2 + (h+gc)x + (\beta+g\gamma)y + \alpha,\\
&\overline{\xi} = \overline{\xi}(\mu,\nu), \;\; \overline{\eta} = (c\xi+\gamma\eta)\overline{\xi}(\mu,\nu) + \overline{\eta}(\mu,\nu), \;\; \overline{\mu} = (a\xi+b\eta)\overline{\xi}(\mu,\nu) + \overline{\mu}(\mu,\nu),\\
&\overline{\nu} = (ac\xi^2/2 + bc\xi\eta + b\gamma\eta^2/2 + h\xi + \beta\eta)\overline{\xi}(\mu,\nu) + (a\xi+b\eta)\overline{\eta}(\mu,\nu) + \overline{\nu}(\mu,\nu),
\end{aligned}$$

in every solution limits for $a, b, c, g, h, \alpha, \beta, \gamma$ *and coefficients* $\overline{\xi}(\mu,\nu)$, $\overline{\eta}(\mu,\nu)$, $\overline{\mu}(\mu,\nu)$, $\overline{\nu}(\mu,\nu)$ *are defined by the conditions* (2)*, in particular,* $a^2+b^2 \neq 0$, $\overline{\xi}(\mu,\nu) \neq 0$.

References

1. Bogdanova R. A., Mikhailichenko G. G. *Successive embedding of the rank* $(n+1,2)$ *of dimetric phenomenologically symmetric geometries of two sets*, Geometry days in Novosibirsk – 2018: Abstracts of the International Conference. Novosibirk: Sobolev Institute of Mathematics, 2018. P. 40–43.

2. Mikhailichenko G. G. *Bimetric physical structures of rank* $(n+1,2)$, Siberian Math. J. 34:3 (1993), 513–522.

Extremal function in the Szego and Fekete problem

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Denote by S the class of normalized univalent functions $f : E \to \mathbb{C}$, $f(z) = z + c_2 z^2 + c_3 z^3 + \ldots$ defined on the unit disk $E = \{z \in \mathbb{C} : |z| < 1\}$. The paper solves the classical Szego and Fekete problem [1–3] about finding the bound of the functional

$$I : S \to \mathbb{R}, \quad I(f) = \mathrm{Re}\big(c_3 - \gamma c_2^2\big), \quad \gamma \in \mathbb{R}.$$

and its extremal functions.

Using standard variational technique [4] we obtain that extremal functions satisfy the differential equation

$$\frac{qf(z)+1}{f^4(z)} f'^2(z) = \frac{z^4 + \overline{q}z^3 + bz^2 + qz + 1}{z^4},$$

where $q = 2(1-\gamma)c_2$, $b = 2I(f)$.

Qualitative analysis of the differential equation gives that right-side of equation has the form

$$\frac{qf(z)+1}{f^4(z)} f'^2(z) = \frac{(z-\mu)^2(z-\eta)^2}{z^4} \tag{1}$$

or the form

$$\frac{qf(z)+1}{f^4(z)} f'^2(z) = \frac{(z-\mu)^2(z-\xi)(z-\frac{1}{\overline{\xi}})}{z^4}, \tag{2}$$

where $|\mu| = 1$, $|\eta| = 1$, $|\xi| < 1$.

Theorem 1. *For $\gamma \in (0,1)$ maximum $\max\limits_{f \in S} I(f) = 1 + 2e^{\frac{-2\gamma}{1-\gamma}}$ yields the solution of the equation (1). The expansion of this extremal function in a neighborhood of zero has real coefficients. Besides this function satisfies the equation*

$$2e^{-\frac{\gamma}{1-\gamma}} \ln \frac{1 + \sqrt{1 - 4e^{-\frac{\gamma}{1-\gamma}} f}}{1 - \sqrt{1 - 4e^{-\frac{\gamma}{1-\gamma}} f}} - \frac{\sqrt{1 - 4e^{-\frac{\gamma}{1-\gamma}} f}}{f} = -\frac{1}{z} + z - 2e^{-\frac{\gamma}{1-\gamma}} \ln z,$$

and maps unit disk onto the plane with three analytical curves excluded. The curves meet at one point $\frac{1}{4}e^{\frac{\gamma}{1-\gamma}}$ under equal angles, one of them goes to infinity.

For $\gamma < 0$ maximum $\max\limits_{f \in S} I(f) = 3 - 4\gamma$ yields the solution of the equation (2). The extremal function is a Koebe function and it has expansion in a neighborhood of zero with real coefficients.

For $\gamma > 1$ maximum $\max_{f \in S} I(f) = -3 + 4\gamma$ yields the solution of the equation (2). The extremal function is a Koebe function and it has expansion in a neighborhood of zero with coefficients $c_{2n-1} \in \mathbb{R}$, $ic_{2n} \in \mathbb{R}$.

The result was obtained jointly with I. A. Kolesnikov, S. A. Kopanev, G. D. Sadritdinova.

References

1. Szego G. G., Fekete M. *Eine Bemerkung uber ungerade schlichte Funktionen*, J. London Math. Soc. 8:2(30) (1933), 85–89.
2. Goluzin M. G. *Some questions of the theory of univalent functions*, The Steklov Mathematical Institute proc. 27 (1949).
3. Alexandrov I. A. *Extreme properties of a class $S(w_0)$*, Quest. of the Geom. Theory of Func., Proc. of Tomsk Univ. 169 (1963), 24–58.
4. Alexandrov I. A., Kolesnikov I. A., Kopanev S. A., Kopaneva L. S. *Method of internal variations in the theory of univalent functions* Ed. Tomsk State University, 2017.

Measures of non-compactness and Sobolev-Lorentz spaces

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The measure of non-compactness is defined for any continuous linear mapping $T\colon X \to Y$ between two Banach spaces X and Y as

$$\beta(T) := \inf\left\{ r > 0 \;\; \begin{matrix} T(B_X) \text{ can be covered by finitely} \\ \text{many open balls with radius } r \end{matrix} \right\}.$$

It can easily be shown that $0 \leq \beta(T) \leq \|T\|$ and that $\beta(T) = 0$, if and only if the mapping T is compact.

Prof. Stanislav Hencl has proved in his paper that the measure of non-compactness of the known embedding $W_0^{k,p}(\Omega) \to L^{p^*}(\Omega)$, where kp is smaller than the dimension, is equal to its norm.

In this thesis we prove that the measure of non-compactness of the embedding between function spaces is under certain general assumptions equal to the norm of that embedding. We apply this theorem to the case of Lorentz spaces to obtain that the measure of non-compactness of the embedding

$$W_0^k L^{p,q}(\Omega) \to L^{p^*,q}(\Omega)$$

is for suitable p and q equal to its norm.

On the vertical similarly homogeneous $\mathbb{R}$-trees

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We study the class of similarly homogeneous locally complete $\mathbb{R}$-trees with some additional requirements. In particular, vertical and strictly vertical $\mathbb{R}$-trees are defined. The metric classification of strictly vertical $\mathbb{R}$-trees is made: it is shown that every such $\mathbb{R}$-tree is isometric to some model $\mathbb{R}$-tree.

The similarly homogeneous locally complete spaces with intrinsic metric were introduced and studied in [1]. The examples of using of $\mathbb{R}$-trees were described in [2, 3] and some other papers. Some properties and criteria of $\mathbb{R}$-trees were proven in [4].

The space X is called *similarly homogeneous* if the group $Sim(X)$ of its similarities is transitive. That is, for any $x, y \in X$ there exists a similarity φ mapping x to $\varphi(x) = y$. If the group $Isom(X)$ is transitive (for any $x, y \in X$ the isometry φ, mapping x to y exists), the space X is called *homogeneous*.

The space (X, d) is called *locally complete* if for any point $x \in X$ there exists a number $r > 0$ such that the closed ball $B(x, r)$ is complete in the metric d.

Theorem 1 [1, Theorem 2.1]**.** *The locally complete similarly homogeneous space X is homogeneous if and only if it is complete.*

All spaces considered below are similarly homogeneous and non-homogeneous. Consequently, the function $c(x)$ is always finite. If $\varphi : X \to X$ is a similarity with the coefficient $k > 0$ then $c(\varphi(x)) = k \cdot c(x)$ for any point $x \in X$.

The criterion of $\mathbb{R}$-tree is the following.

Theorem 2 [5, Theorem 1]. *Let X be a geodesic space. If X is an $\mathbb{R}$-tree, then for every point $o \in X$ there exists unique partial order $\preceq_o$ on X such that the pair $(X, \preceq_o)$ is upper-semilinear $\vee$-semilattice with root o. Every nonempty subset $A \subset X$ has its supremum in this order. Conversely, if X admits a partial order, such that X becomes a semilinear metric semilattice such that the directions of the semilattice and the semilinearity coincide, then X is an $\mathbb{R}$-tree.*

Let X be an $\mathbb{R}$-tree. The *branching number* in the point $x \in X$ is by definition the cardinal number $\mathfrak{B}_x(X) = \mathfrak{B} - 1$ where $\mathfrak{B}$ is the cardinality of the set of connected components in $X \backslash \{x\}$. Each connected component in $X \backslash \{x\}$ is called a *branch* with the root x. If $\mathfrak{B} = 1$, the point x is called *terminal point*.

A continuous function $f : [a, b] \to \mathbb{R}$ is called *saw-like* if: 1) f is not constant in any interval $(c, d) \subset [a, b]$; 2) if f is monotone in the interval $(c, d) \subset [a, b]$ then $f|_{(c,d)}$ is linear with the angle coefficient ± 1.

Let (X, d) be a locally complete similarly homogeneous non-homogeneous $\mathbb{R}$-tree and $c : X \to \mathbb{R}_+$ corresponding function of the completeness radius. X is called *vertical* if the function $c(\gamma(t))$ is saw-like on each segment $[xy] \subset X$ naturally parameterized as $\gamma : [0, d(x, y)] \to X$ so that $\gamma(0) = x$, $\gamma(d(x, y)) = y$ and $d(x, \gamma(t)) = t$ for all $t \in [0, d(x, y)]$. X is called *strictly vertical* if the function $c(\gamma(t))$ mentioned above has at most one local extremum in each segment $[xy] \subset X$.

Let (X, d) be a strictly vertical $\mathbb{R}$-tree differ from $\mathbb{R}_+$. We say that X has *branching up* (correspondingly, *branching down*) if each local extremum of the completeness radius in the arbitrary segment in X is the local minimum (correspondingly, local maximum).

Let (X, d) be a strictly vertical $\mathbb{R}$-tree with branching up. Given points $x, y \in X$ we say that y lies in the upper branch with the root x if x is the minimum point of the completeness radius in the segment $[xy]$ and denote this relation $x \nearrow y$.

Let G be a group with the unity e. Given a number $a > 0$ the function $\varphi : [0, a) \to G$ is called *piecewise constant from the right* if for any $t \in [0, a)$ there exists $\varepsilon(t) > 0$ such that $\varphi(t + \tau) = \varphi(t)$ for all positive $\tau < \varepsilon(t)$.

We define a metric d on the set $X_+(G)$. Set $d(\varphi, \psi) = |a - a'|$ if $\varphi \nearrow \psi$ or $\psi \nearrow \varphi$, $D(\varphi) = [0, a)$ and $D(\psi) = [0, a')$.

Lemma. *The metric space $(X_+(G), d)$ is locally complete. The completeness radius at the point $(\varphi, a) \in X_+(G)$ is $c(\varphi) = a$.*

We have the following

Theorem 3. *The space $X_+(G)$ is the similarly homogeneous, non-homogeneous strictly vertical $\mathbb{R}$-tree with branching up and the branching number $|G|$.*

Let $(X, d), (X', d')$ be locally complete similarly homogeneous non-homogeneous spaces with interior metrics. We say that they are *inverse to each other* if there exists a homeomorphism $I : X \to X'$ called the inversion of X onto X' with coefficient $R > 0$ such that the completeness radii $c(x)$ of the arbitrary point $x \in X$ and $c(x')$ of its image $x' = I(x)$ satisfy the equality $c(x) \cdot c(x') = R^2$.

Next, we prove

Theorem 4. *Let (Y, d) be a strictly vertical $\mathbb{R}$-tree with branching down. Then it is inverse to a strictly vertical $\mathbb{R}$-tree (Y', d') with the same branching number and branching up.*

The precise result for the case of $\mathbb{R}$-trees with branching down is the following

Theorem 5. *Let T be a strictly vertical $\mathbb{R}$-tree with the branching number $\mathfrak{B}(T)$. If T has the branching type up then it is isometric to $X_+(G)$ where G is a group with cardinality $|G| = \mathfrak{B}(T)$. If T has the branching type down then it is isometric to $X_-(G)$.*

Consequently, two strictly vertical $\mathbb{R}$-trees are isometric if and only if their branching types and branching numbers coincide.

References

1. Berestovskii V. N. *Similarly homogeneous locally complete spaces with an intrinsic metric*, Russ. Math. (Iz.VUZ) 48:4 (2004), 1–19.
2. Bestvina M. $\mathbb{R}$-*trees in topology, geometry and group theory*, Handbook of Geometric Topology, Ed. by R. J. Daverman and R. B. Sher. Elsevier Science, North-Holland, Amsterdam, London, New York, 2002. P. 55–91.
3. Berestovskii V. N., Plaut C. *Covering* $\mathbb{R}$-*trees,* $\mathbb{R}$-*free groups and dendrites*, Adv. Math. 224 (2010), 1765–1783.
4. Andreev P. D., Berestovskii V. N. *Dimensions of* $\mathbb{R}$-*trees and self-similar fractal spaces of non-positive curvature*, Sib. Adv. Math. 17:2 (2006), 79–90.
5. Andreev P. D. *Semilinear metric semilattices on* $\mathbb{R}$-*trees*, Russ. Math. (Iz. VUZ) 51:1 (2007), 1–10.

Inverse problem for Möbius geometry on the circle: SRA-free condition by Zolotov for self-contracted curves and nondegeneracy of zz-distance for Möbius structures on the circle

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SRA-free condition for metric spaces (that is, spaces without Small Rough Angles) was introduced by Zolotov to study rectifiability of self-contracted curves in various metric spaces. We give a Möbius invariant version of this notion which allows to show that zz-distance associated with a respective Möbius structure on the circle is nondegenerate. This result is an important part of a solution to the inverse problem of Möbius geometry on the circle.

The *inverse problem* of Möbius geometry asks to describe Möbius structures which are induced by hyperbolic spaces. We are interested in the inverse problem for Möbius structures on the circle $X = S^1$.

We list a set of axioms for a Möbius structure M on the circle $X = S^1$.

(T) Topology: M-topology on X is that of S^1.

(M(α)) Monotonicity: Fix $0 < \alpha < 1$. Given a 4-tuple $q = (x, y, z, u) \in X^4$ such that the pairs (x, y), (z, u) separate each other, we have

$$|xy| \cdot |zu| \geq \max\{|xz| \cdot |yu| + \alpha|xu| \cdot |yz|, \alpha|xz| \cdot |yu| + |xu| \cdot |yz|\}$$

for some and hence any semi-metric from M.

(P) Ptolemy: for every 4-tuple $q = (x, y, z, u) \in X^4$ we have

$$|xy| \cdot |zu| \leq |xz| \cdot |yu| + |xu| \cdot |yz|$$

for some and hence any semi-metric from M.

A Möbius structure M on the circle X that satisfies axioms T, M(α), P is said to be *strictly monotone*.

Axiom M(α) is motivated by the work [1] of V. Zolotov.

We consider the set Hm of unordered harmonic pairs of unordered pairs of points in X as a required filling of a Möbius structure M in the inverse problem of Möbius geometry. A *zig-zag* path, or zz-path, $S \subset \mathrm{Hm}$ is defined as finite sequence of segments σ_i on lines in Hm, where consecutive segments σ_i, σ_{i+1} have a common end $q = \sigma_i \cap \sigma_{i+1} \in \mathrm{Hm}$ with axes determined by σ_i, σ_{i+1}.

We define a distance δ on Hm as

$$\delta(q, q') = \inf_S |S|,$$

where the infimum is taken over all zz-paths $S \subset \mathrm{Hm}$ from q to q', and $|S|$ is the length of S defined as the sum of the lengths of its sides.

Theorem 1. *Let M be a ptolemaic Möbius structure of the circle X which satisfies axiom (M(α)) for some $0 < \alpha < 1$, such that the M-topology is the topology of S^1. Then the distance δ on* Hm *is nondegenerate, $\delta(q, q') \neq 0$ if and only if $q \neq q'$, the δ-metric topology on* Hm *coincides with one induced from X^4, and the metric space* (Hm, δ) *is complete. In particual,* (Hm, δ) *is a proper geodesic metric space.*

We use notation $\operatorname{reg} \mathcal{P}_n$ for the set of ordered nondegenerate n-tuples of points in $X = S^1$, $n \in \mathbb{N}$. For $q \in \operatorname{reg} \mathcal{P}_n$ and a proper subset $I \subset \{1, \ldots, n\}$ we denote by $q_I \in \operatorname{reg} \mathcal{P}_k$, $k = n - |I|$, the k-tuple obtained from q (with the induced order) by crossing out all entries which correspond to elements of I. We denote the cross-ratio

$$q \mapsto \mathrm{cr}_1(q) = \frac{|x_1x_3||x_2x_4|}{|x_1x_4||x_2x_3|}$$

for $q = (x_1, x_2, x_3, x_4) \in \operatorname{reg} \mathcal{P}_4$.

(I) Increment Axiom: for any $q \in \operatorname{reg} \mathcal{P}_7$ with cyclic order $\operatorname{co}(q) = 1234567$ such that q_{247} and q_{157} are harmonic, we have

$$\operatorname{cr}_1(q_{345}) > \operatorname{cr}_1(q_{123}).$$

Theorem 2. *Assume that a Möbius structure M on $X = S^1$ is strictly monotone, i. e., it satisfies axioms (T), (M(α)), (P), and satisfies Increment axiom. Then* (Hm, δ) *is a complete, proper, hyperbolic geodesic metric space with δ-metric topology coinciding with that induced from X^4.*

References

1. Zolotov V. *Subsets with small angles in self-contraced curves*, arXiv:1804.00234, 2018.

Modular functional spaces

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The purpose of this note is to show how modulars in the sense of [1–5] generate functional spaces, which turn out to be interrelated in a more closer sense than can possibly be seen from their definitions.

1. A *modular* w on a set X is a one-parameter family $w = \{w_\lambda : \lambda \in (0, \infty)\}$ of functions $w_\lambda : X \times X \to [0, \infty]$ satisfying, for all $x, y, z \in X$, the following three conditions (axioms): (i) $x = y$ if and only if $w_\lambda(x, y) = 0$ for all $\lambda \in (0, \infty)$; (ii) $w_\lambda(x, y) = w_\lambda(y, x)$ for all $\lambda \in (0, \infty)$; (iii) $w_{\lambda+\mu}(x, y) \le w_\lambda(x, z) + w_\mu(z, y)$ for all $\lambda, \mu \in (0, \infty)$. If, in place of (i), we have only $w_\lambda(x, x) = 0$ for all $\lambda \in (0, \infty)$, then w is said to be a *pseudomodular* on X, and if (i) is supplemented by $w_\lambda(x, y) \ne 0$ for all $\lambda > 0$ and $x, y \in X$, $x \ne y$, then the modular w is called *strict*. The (pseudo)modular w is said to be *convex* if (iii) above is replaced by (iv) $w_{\lambda+\mu}(x, y) \le \dfrac{\lambda}{\lambda+\mu} w_\lambda(x, z) + \dfrac{\mu}{\lambda+\mu} w_\mu(z, y)$ for all $\lambda, \mu \in (0, \infty)$.

If w is a (pseudo)modular on X, then the function $\lambda \mapsto w_\lambda(x, y)$, mapping $(0, \infty)$ into $[0, \infty]$, is *nonincreasing* for all $x, y \in X$ (cf. [3, 5]).

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2. Examples. Suppose X is a metric space with metric d (below $x, y \in X$).

(a) Given $p \in (0, \infty)$, $w_\lambda(x, y) = d(x, y)/\lambda^p$ and $w_\lambda(x, y) = (d(x, y)/\lambda)^p$ are strict modulars on X, which are convex for $p \geq 1$.

(b) If $w_\lambda(x, y) = \infty$ for $0 < \lambda < d(x, y)$ and $w_\lambda(x, y) = 0$ for $\lambda \geq d(x, y)$, then $w = \{w_\lambda\}_{\lambda>0}$ is a nonstrict convex modular on X.

3. A (pseudo)modular w on X gives rise to three equivalence relations on X:
(a) $x \overset{*}{\sim} y$ if there is $\lambda = \lambda(x, y) \in (0, \infty)$ such that $w_\lambda(x, y) < \infty$;
(b) $x \overset{0}{\sim} y$ if $w_\lambda(x, y) \to 0$ as $\lambda \to \infty$;
(c) $x \overset{\mathrm{fin}}{\sim} y$ if $w_\lambda(x, y) < \infty$ for all $\lambda \in (0, \infty)$.

For a fixed element $x^\circ \in X$, we denote by X_w^*, X_w^0 and X_w^{fin} the equivalence classes of x° with respect to relations (a), (b) and (c), respectively, and call them *modular spaces*. If these spaces consist of functions, we call them *modular functional spaces*. Clearly, $X_w^0 \subset X_w^*$ and $X_w^{\mathrm{fin}} \subset X_w^*$, and if w is convex, then $X_w^0 = X_w^*$. Modular spaces are metrizable by infinitely many 'natural' metrics [5] (in contrast with the classical theory of modulars on linear spaces [6, 7] where, due to its limitations, only a few 'natural' norms and F-norms are known).

4. Modular sequence spaces. Let (M, d) be a metric space with metric d and $X = M^{\mathbb{N}}$ be the set of all sequences in M. If $x \in X$, we set as usual $x = \{x_n\}$ with $x_n \in M$ for all $n \in \mathbb{N}$. For $x^\circ = \{x_n^\circ\} \in X$, we write

$$x \in c(x^\circ) \text{ if } \lim_{n\to\infty} d(x_n, x_n^\circ) = 0, \quad \text{and} \quad x \in \ell_\infty(x^\circ) \text{ if } \sup_{n\in\mathbb{N}} d(x_n, x_n^\circ) < \infty.$$

Consider the *strict nonconvex* (continuous in λ) modular on X defined by

$$w_\lambda(x, y) = \sup_{n\in\mathbb{N}} \left(\frac{d(x_n, y_n)}{\lambda}\right)^{1/n}, \quad \lambda \in (0, \infty),\ x = \{x_n\},\ y = \{y_n\} \in X.$$

Theorem 1. (a) $X_w^0 \subset c(x^\circ) \subset \ell_\infty(x^\circ) \subset X_w^* = X_w^{\mathrm{fin}}$ (*with strict inclusions*);

(b) $x \in X_w^0$ *if and only if* $\lim_{n\to\infty}(d(x_n, x_n^\circ))^{1/n} = 0$ *if and only if the sequence* $\{\lambda^n d(x_n, x_n^\circ)\}_{n=1}^\infty$ *is bounded in* $\mathbb{R}$ *for all* $\lambda \in (0, \infty)$;

(c) $x \in X_w^*$ *if and only if* $\sup_{n\in\mathbb{N}}(d(x_n, x_n^\circ))^{1/n} < \infty$ *if and only if the sequence* $\{\lambda^n d(x_n, x_n^\circ)\}_{n=1}^\infty$ *is bounded in* $\mathbb{R}$ *for some* $\lambda \in (0, \infty)$.

In the next two sections, we assume that $I = [a, b]$ is a closed interval in $\mathbb{R}$, (M, d) is a metric space with metric d and $X = M^I$ is the set of all functions $x : I \to M$ mapping I into M. For $I_i = [a_i, b_i] \subset I$, we set $|I_i| = b_i - a_i$, and, given $n \in \mathbb{N}$, denote by $\{I_i\}_1^n \prec I$ a non-ordered collection of n nonoverlapping intervals $I_i \subset I$, $i = 1, \ldots, n$. The *joint increment* of two functions $x, y \in X$ on the interval $I_i = [a_i, b_i] \subset I$ is the quantity

$$|(x, y)(I_i)|_d = \sup_{u\in M} \bigl|d(x(b_i), u) + d(y(a_i), u) - d(x(a_i), u) - d(y(b_i), u)\bigr|.$$

Note that the function $(x,y) \mapsto |(x,y)(I_i)|_d$ is a *pseudometric* on X.

5. AC and BV functions. The classical subspaces of X of *absolutely continuous* (AC, for short) functions and functions of *bounded Jordan variation* (BV, for short) are introduced as follows:

(a) $x \in \mathrm{AC}(I;M)$ if $\forall\, \varepsilon > 0\ \exists\, \delta(\varepsilon) > 0$ such that $\sum_{i=1}^{n} d(x(a_i), x(b_i)) < \varepsilon$ for all $n \in \mathbb{N}$ and $\{I_i\}_1^n \prec I$ (with $I_i = [a_i, b_i]$) such that $\sum_{i=1}^{n} |I_i| < \delta(\varepsilon)$;

(b) $x \in \mathrm{BV}(I;M)$ if $\exists\, C \geq 0$ such that $\sum_{i=1}^{n} d(x(t_i), x(t_{i-1})) \leq C$ for all $n \in \mathbb{N}$ and all partitions $a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$ of $I = [a,b]$.

Now, we introduce a *nonconvex pseudomodular* on X by

$$w_\lambda(x,y) = \sup\left\{ \sum_{i=1}^{n} |(x,y)(I_i)|_d : n \in \mathbb{N} \text{ and } \{I_i\}_1^n \prec I \text{ such that } \lambda \sum_{i=1}^{n} |I_i| < 1 \right\}$$

for all $\lambda \in (0,\infty)$ and $x, y \in X$. If $x^\circ \in X$ is a constant function, we have:

Theorem 2. $X_w^0 = \mathrm{AC}(I;M) \subset X_w^* = X_w^{\mathrm{fin}} = \mathrm{BV}(I;M)$.

6. Bounded and regulated functions. Classically, a function $x \in X$ is *bounded* (in symbols, $x \in \mathrm{B}(I;M)$) if its oscillation $\sup_{s,t \in I} d(x(s), x(t))$ is finite, and $x \in X$ is said to be *regulated* (in symbols, $x \in \mathrm{Reg}(I;M)$) if

$$\lim_{s,t \to \tau - 0} d(x(s), x(t)) = 0 \text{ at each point } a < \tau \leq b \text{ and}$$
$$\lim_{s,t \to \tau' + 0} d(x(s), x(t)) = 0 \text{ at each point } a \leq \tau' < b$$

(if (M,d) is a complete metric space, this is equivalent to the existence of one-sided limits $x(\tau - 0) \in M$ and $x(\tau' + 0) \in M$, respectively).

Now, given $\lambda \in (0,\infty)$ and $x, y \in X$, we set

$$w_\lambda(x,y) = \sup\left\{ n \in \mathbb{N} : \exists\, \{I_i\}_1^n \prec I \text{ such that } \min_{1 \leq i \leq n} |(x,y)(I_i)|_d > \lambda \right\}$$

with $\sup \varnothing = 0$. The family $w = \{w_\lambda\}_{\lambda > 0}$ is a *nonstrict nonconvex pseudomodular* on X, for which the function $\lambda \mapsto w_\lambda(x,y)$ is continuous from the right on $(0,\infty)$ for all $x, y \in X$. If $x^\circ \in X$ is a constant function, we have:

Theorem 3. $X_w^{\mathrm{fin}} = \mathrm{Reg}(I;M) \subset X_w^* = X_w^0 = \mathrm{B}(I;M)$.

It is to be noted that the last pseudomodular w is useful for obtaining a powerful and the most general pointwise selection principle for sequences $\{x_n\} \subset M^I$.

References

1. Chistyakov V. V. *Metric modulars and their application*, Dokl. Math. 73 (1) (2006) 32–35.

2. Chistyakov V. V. *Modular metric spaces generated by F-modulars* // Folia Math. 15 (1) (2008) 3–24.
3. Chistyakov V. V. *Modular metric spaces, I: Basic concepts; II: Application to superposition operators* // Nonlinear Anal. 72 (1) (2010) 1–14; 15–30.
4. Chistyakov V. V. *Fixed points of modular contractive maps* // Dokl. Math. 86 (1) (2012) 515–518.
5. Chistyakov V. V. *Metric Modular Spaces: Theory and Applications.* Springer Briefs in Mathematics, Springer, Cham, 2015.
6. Musielak J. *Orlicz Spaces and Modular Spaces.* Lecture Notes in Math. Vol. 1034. Springer, Berlin, 1983.
7. Nakano H. *Modulared Semi-Ordered Linear Spaces.* – Maruzen, Tokyo, 1950.

Structural stability of dynamic inequalities on two-dimensional sphere

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The notion of rough dynamical system was introduced by A. A. Andronov and L. S. Pontryagin for the case of differentiable vector fields on a two-dimensional disk, which has no tangency with the boundary, in paper [1]. The necessary and sufficient conditions for such a field to be rough were also found in this paper. Later on such notion was defined for more general vector fields and for some objects of another nature and gets the name *structural stability* (see, for example, [2, 3, 5, 6, 8, 9, 12]).

For control system (and dynamic inequality) the notion of structural stability is the same as for vector fields, just under the trajectory of a point one needs to consider the union of the positive and negative orbits of this point. In this case, the analogue of singular point of vector field is a *local transitivity zone*, which is the union of all points having local transitivity property (a system has *local transitivity property* at a point if for any neighborhood of this point there exist

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time $T > 0$ and another neighborhood of the point such that any two points from the second neighborhood are attainable from one another for a time less than T and along the admissible trajectory lying in the first neighborhood). The role of closed trajectory plays *nonlocal transitivity zone*, which coincides with intersection of the positive and negative orbits of each of its points. A generic smooth control system on two-dimensional sphere (or a closed orientable surface) is structurally stable [5, 6]. Note that for control systems this stability includes also the stability of both local transitivity properties and the nonlocal ones [4]. Some property of objects we call *generic* if it takes place for open everywhere dense subset in the space of objects endowed by an appropriate topology. Here we deal with smooth of sufficiently smooth fine Whitney topology.

The problem to analyze the orbits of smooth dynamical inequalities with locally bounded derivatives was posed by A. D. Myshkis in [11]. Such an inequality is defined by a smooth real function on the tangent bundle to the manifold, in each fiber of which the set of admissible velocities (in which the value of this function is non-positive) is bounded. Such an inequality could have regions with no any admissible velocities at all and the ones, in which the motion with velocities of limiting directions does not satisfy the conditions of existence and uniqueness theorem of the integral curve. For example, there could appear sliding motion that can not be eliminated by a small perturbation of the inequality under consideration [7]. This makes the problem of the structural stability for generic dynamic inequalities on surfaces much more complicated than for control systems. Note, that for three dimensional control systems or dynamic inequalities the property of structural stability is not generic.

The stability of the local controllability properties for a generic dynamic inequalities on surfaces was proved in [7].

Admissible velocities of a smooth *simplest dynamic inequality* on any Riemannian manifold are determined by the sum of the smooth vector field (=drift) and the velocities modulo not exceeding $\sqrt{f}$ with some smooth function f. For example, on the plane with coordinates x, y such an inequality looks as

$$(\dot{x} - a(x, y))^2 + (\dot{y} - b(x, y))^2 \leq f(x, y),$$

where (a, b) is the drift and f characterizes own capacities to move for the object under control.

We show that the problem of structural stability of generic dynamic inequality with locally bounded derivatives on two-dimensional sphere is equivalent to such a problem on the plane when near the infinity the inequality either has local transitivity property or has no any admissible velocities at all. This result together with the ones from [9] implies the structural stability of generic smooth simplest dynamic inequalities on two-dimensional sphere.

References

1. Andronov A. A., Pontryagin L. S. *Rough Systems*, Doklady Academy Nauk SSSR 14:55 (1937), 247–250. [in Russian]
2. Arnold V. I. *Geometrical methods in the theory of ordinary differential equations*, New York: Springer-Verlag, 1983.
3. Arnold V. I., Varchenko A. N., Gusein-Sade S. M. *Singularities of differentiable mapping*, vol. 1. Monographs in Mathematics, 82. Boston: Birkhäuser, 1985.
4. Davydov A. A. *Singularities of fields of limiting directions of two-dimensional control systems*, Math. USSR-Sb. 64:2 (1989), 471–493.
5. Davydov A. A. *Structural stability of control systems on orientable surfaces*, Math. USSR-Sb. 72:1 (1992), 1–28.
6. Davydov A. A. *Qualitative theory of control systems.* Translations of Mathematical Monographs 141 (1994). Providence, RI.
7. Davydov A. A. *Local controllability of typical dynamical inequalities on surfaces*, Proc. Steklov Inst. Math. 209 (1995), 73–106.
8. Garcia R., Gutierrez C., Sotomayor J. *Structural Stability of Asymptotic Lines on Surfaces Immersed in* $\mathbb{R}^3$, Bull. Sci. Math. 123 (1999), 599–622.
9. Grishina Yu. A., Davydov A. A. *Structural Stability of Simplest Dynamical Inequalities*, Proc. Steklov Inst. Math. 256 (2007), 80–91.
10. Davydov A. A. *Normal forms of linear second order partial differential equations on the plane*, Sci. China Math. 61:11 (2018), 1947–1962.
11. Myshkis A. D. *On differential inequalities with locally bounded derivatives*, Notes of mekh-math department and Kharkiv mathematical society 30 (1964), 152–163.
12. Peixoto M. M. *Structural stability on two-dimensional manifolds*, Topology 1:2 (1962), 101–120.

Deformations of polygaskets

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Let $P \subset \mathbb{R}^2$ be a finite polygon homeomorphic to a disk, $\mathcal{V}_P = \{A_1, \ldots, A_{n_P}\}$ be the set of its vertices. We consider a system of similarities $\mathcal{S} = \{S_1, \ldots, S_m\}$ in $\mathbb{R}^2$ such that:
(D1) for any $i \in I$ set $P_i = S_i(P) \subset P$;
(D2) for any $i \neq j, i, j \in I, P_i \bigcap P_j = \mathcal{V}_{P_i} \bigcap \mathcal{V}_{P_j}$ and $\#(\mathcal{V}_{P_i} \bigcap \mathcal{V}_{P_j}) < 2$;
(D3) $\mathcal{V}_P \subset \bigcup\limits_{i \in I} S_i(\mathcal{V}_P)$.

Following R. Strichartz, the system $\mathcal{S}$ will be called a P-polygasket if it satisfies the conditions **(D1–D3)**, and a generalized P-polygasket, if it satisfies the conditions **(D2, D3)** only.

Let $\mathcal{S}$ be a generalized P-polygasket. The vertex $A \subset \mathcal{V}_P$ is called a cyclic vertex, if there is such multiindex $\mathbf{i} = i_1 i_2 \ldots i_k$, that $S_{\mathbf{i}}(A) = A$. Let A be a cyclic vertex and γ_{AB} be its invariant arc and $S_{\mathbf{i}}(A) = A$. Let $B' = S_{\mathbf{i}}(B)$. We denote by α the total change of argument of $z - A$ when z travels along γ_{AB} from B to B'. This gives unique representation $S_{\mathbf{i}}(z) = q_{\mathbf{i}} e^{i\alpha}(z - A) + A$. The number $\lambda_A = \frac{\alpha}{\ln r}$ is called the parameter of the cyclic vertex A. Generalized P-polygasket $\mathcal{S}$ satisfies the *parameter matching condition*, if for any $B \in \cup_{i=1}^m \mathcal{V}_{P_i}$ and for any cyclic vertices A, A' such that for some $\mathbf{i}, \mathbf{j} \in I^*$, $S_{\mathbf{i}}(A) = S_{\mathbf{j}}(A') = B$, the equality $\lambda_A = \lambda_{A'}$ holds.

A generalized P'-polygasket $\mathcal{S}' = \{S'_1, \ldots, S'_m\}$ is called a δ-deformation of a P-polygasket $\mathcal{S} = \{S_1, \ldots, S_m\}$, if there is a bijection $f : \bigcup_{k=1}^m \mathcal{V}_{P_k} \to \bigcup_{k=1}^m \mathcal{V}_{P'_k}$, such that
a) $f|_{\mathcal{V}_P}$ extends to a homeomorphism $\tilde{f} : P \to P'$;
b) $|f(x) - x| < \delta$ for any $x \in \bigcup_{k=1}^m \mathcal{V}_{P_k}$;
c) $f(S_k(x)) = S'_k(f(x))$ for any $k \in I$ and $x \in \mathcal{V}_P$.

Our main result is the following

Theorem. *Let $\mathcal{S}$ be a P-polygasket. There is such $\delta > 0$ that for any δ-deformation $\mathcal{S}'$ of the system $\mathcal{S}$, satisfying parameter matching condition, the attractor $K(\mathcal{S}')$ is homeomorphic to $K(\mathcal{S})$.*

References

1. Strichartz R. S. *Isoperimetric estimates on Sierpinski gasket type fractals*, Trans. Amer. Math. Soc. 351 (1999), 1705–1752.

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Condenser capacities and symmetrization in geometric function theory

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The present talk will discuss the evolution of the Hayman's symmetrization approach in several areas. First, we will give a brief survey of results involving the Schwarzian derivative and depending on the geometry of the image of a domain under a holomorphic map [1]. The author's results obtained previously by using the theory of condenser capacity and symmetrization constitute the core of the talk [2]. Inequalities for univalent and multivalent holomorphic functions are considered both at the interior and the boundary points of the domain of definition. Second, we will give a definition of a new version of the circular symmetrization of condensers on the Riemann surfaces [3]. One theorem about the behavior of a condenser capacity under symmetrization will be formulated and some its applications to multivalent functions of certain classes will be considered [4, 5].

References

1. Dubinin V. N. *Geometric estimates for the Schwarzian derivative*, Russian Mathematical Surveys 72:3 (2017), 479–511.

2. Dubinin V. N. *Condenser capacities and symmetrization in geometric function theory*, Basel: Birkhauser/ Springer, 2014.

3. Dubinin V. N. *Circular symmetrization of condensers on Riemann surfaces*, Sb. Math. 206:1 (2015), 61–86.

4. Dubinin V. N. *On one extremal problem for complex polynomials with constraints on critical values*, Siberian Math. J. 55:1 (2014), 63–71.

5. Dubinin V. N. *Lemniscate zone and distortion theorems for multivalent functions II*, Math. Notes 104:5 (2018), 683–688.

The equality of capacity and modulus of condenser in some Banach function spaces

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Let (R, μ) be a metric measure space with Radon measure μ. Define a Banach function space X on (R, μ) as in [1, 1.1]. The norm on X is denoted by $\|\cdot\|_X$.

We shall say that X has an absolutely continuous norm if for any $\rho \in X$ and for any sequence of sets E_n, $\mu\Big(\bigcap\limits_{m=1}^{\infty}\bigcup\limits_{n=m}^{\infty} E_n\Big) = 0$, the sequence $\|\rho\chi_{E_n}\|_X \to 0$, $n \to \infty$.

Next we define a modulus of family of paths (see [2]).

The path is the non-constant continuous Lipschitz map γ from interval (a, b) or segment $[a, b]$ in R. Let γ be parametrized by its arclength $s \in S$. Define for $\rho \in X$ $\int\limits_{\gamma} \rho\, ds = \int\limits_{S} \rho(\gamma(s))\, ds$.

Let Γ be a family of locally rectifiable paths in R. The modulus of this family is $M(\Gamma) = \inf \|\rho\|_X$ where the infimum is taken over all nonnegative $\rho \in X$ such that $\int\limits_{\gamma} \rho\, ds \geq 1$ for all $\gamma \in \Gamma$.

Let G be an open bounded set in R; E_0, E_1 are disjoint compacts in $\bar{G}$. The triple of sets (E_0, E_1, G) is called a condenser. The function $\rho \in X$ is called the upper gradient of real-valued function u on G if for any $x, y \in G$ and any path γ joining x and y in G we have $|u(x) - u(y)| \leq \int\limits_{\gamma} \rho\, ds$.

The capacity of condenser is defined as $Cap(E_0, E_1, G) = \inf \|\rho\|_X$ where the infimum is taken over all upper gradients of all real-valued functions u such that $u = i$ on intersection of some neighbourhood of E_i with G, $i = 0, 1$. Note that this definition is somewhat different from that given in [3, chapter 7].

The modulus of paths joining E_0 and E_1 in G we call the modulus of condenser (E_0, E_1, G) and denote by $M(E_0, E_1, G)$.

Also we suppose that the Hardy-Littlewood maximal operator $M : X \to X$ is bounded. The boundedness of this operator was researched in many papers. For example, the easily verified conditions on rearrangement-invariant spaces are in [4]. In particular, such spaces are Lorentz spaces $L^{p,q}$ and some Orlicz spaces [1]. It is proved the generalization of modulus and capacity equality in [5] to a Banach function spaces:

Theorem. *Let X be a Banach function space with absolutely continuous norm, maximal operator $M : X \to X$ is bounded and (E_0, E_1, G) is condenser on X. Then $Cap(E_0, E_1, G) < \infty$ and $M(E_0, E_1, G) = Cap(E_0, E_1, G)$.*

Corollary. *Under the conditions of the above theorem, the modulus of condenser is continuous, i. e. for any sequence of sets* $E_{in} \downarrow E_i$, $E_{in} \subset \bar{G}, i = 0, 1$, *we have* $M(E_{0n}, E_{1n}, G) \to M(E_0, E_1, G)$ *as* $n \to \infty$.

References

1. Bennett C., Sharpley R. *Interpolation of operators*, Academic Press, Pure and Applied Mathematics 129 (1988).
2. Honzlová Exnerová V., Malý J., Martio O. *Modulus in Banach function spaces*, Ark. Mat. 55:1 (2017), 105–130.
3. Heinonen J. *Lectures on Analysis on Metric Spaces*, Springer, 2011.
4. Mastyło M., Pérez C. *The Hardy-Littlewood maximal type operators between Banach function spaces*, Indiana Univ Math. J. 61:3 (2012), 883–900.
5. Ohtsuka M. *Extremal length and precise functions*, GAKUTO International Series, Gakkōtosho, 2003.

Volumes of right-angled polyhedra in Lobachevsky space

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We consider polyhedra that are realized with right, $\pi/2$, dihedral angles in a three-dimensional space of constant negative curvature H^3, also known as a hyperbolic space or Lobachevsky space. Moreover, we consider only two types of of such polyhedra: compact - those for which all vertices are finite, ideal - those for which all vertices lie on the absolute of the Lobachevsky space. The conditions on the combinatorics of such polyhedra can be obtained from Andreev's theorem [1].

With a certain accuracy, we calculated volumes of more than 40 thousand compact and 100 thousand ideal polyhedra, which significantly expands the results of work [2]. We formulated hypotheses about polyhedra with maximum and minimum volume for a fixed number of faces. We improved Atkinson's upper bounds [3] on the volume of rectangular hyperbolic polyhedra, additionally considering the size of the faces in the polyhedron.

References

1. Andreev E. M. *On convex polyhedra in Lobachevsky spaces*, Math. USSR Sbornik, 10(3) (1970), 413–440.
2. Inoue T. *Exploring the List of Smallest Right-Angled Hyperbolic Polyhedra*, Experimental Mathematics, DOI:10.1080/10586458.2019.1593897
3. Atkinson C. K. *Volume estimates for equiangular hyperbolic Coxeter polyhedra*, Algebraic & Geometric Topology, 9 (2009), 1225–1254.

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Maximal cones in vector spaces

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In what follows, X is a vector space over $\mathbb{R}$. A *cone* is a nonempty set $K \subset X$ that is closed under linear combinations with positive coefficients and contains at most one element of each pair x, $-x$. Every cone $K \subset X$ defines the order relation $\leqslant_K$ on X by $x \leqslant_K y \Leftrightarrow y - x \in K$, which makes X into an ordered vector space. Conversely, if $(X, \leqslant)$ is an ordered vector space then $X^+ := \{x \in X : x \geqslant 0\}$ is a cone.

A cone is *maximal* whenever it cannot be enlarged to another cone. A cone $K \subset X$ is maximal if and only if, for each $x \in X$, either $x \in K$ or $-x \in K$. The latter is also equivalent to the fact that the corresponding order $\leqslant_K$ is total, i. e., $x \leqslant_K y$ or $y \leqslant_K x$ for all $x, y \in X$.

A maximal cone K is called *basic* if there exists a totally ordered Hamel basis $(B, \leqslant)$ in X such that $K = \big\{x \in X \backslash \{0\} : x_B(\min[x_B]) > 0\big\} \cup \{0\}$, where $x_B \colon B \to \mathbb{R}$ is the function of coefficients in the expansion $x = \sum_{b \in B} x_B(b) b$ of x via the basis B and $[x_B] := \{b \in B : x_B(b) \neq 0\}$ is the support of x_B.

Let $(X, \leqslant)$ be a totally ordered vector space. Vectors $x, y \in X^+ \backslash \{0\}$ are called *Archimedean equivalent* if $x \leqslant \alpha y$ and $y \leqslant \beta x$ for some numbers $\alpha, \beta > 0$. Call a set $D \subset X$ *discrete* if D is constituted by pairwise nonequivalent vectors. The Archimedean equivalence classes are ordered as follows: $[x] < [y]$ whenever $x < \alpha y$ for all $\alpha > 0$.

The results presented below are proved in [1].

Theorem 1. *A maximal cone $K \subset X$ is dense in X with respect to the strongest locally convex topology if and only if the set of Archimedean equivalence classes in $(X, \leqslant_K)$ has no least element.*

Theorem 2. *Let $(X, \leqslant)$ be a totally ordered vector space. The maximal cone X^+ is basic if and only if there exists a discrete Hamel basis in X.*

Theorem 3. *All maximal cones in a vector space X are basic if and only if the dimension of X is at most countable.*

References

1. Gutman A. E., Emelyanenkov I. A. *Lexicographic structures on vector spaces*, Vladikavk. Math. J. 21:3 (2019). [in Russian]

On Killing vector fields on second-order symmetric Lorentzian manifolds

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We study the Killing equation on k-order symmetric Lorentzian manifolds. Solutions of this equation form a Lie algebra called the algebra of Killing fields. Our consideration is focused primarily on the dimension of the Lie algebra of Killing fields. The Lorentzian manifolds we consider in this article are the generalized Cahen-Wallach spaces, which are convinient to use because of the coordinate system they have.

Using this coordinates we describe the general solution of the Killing equation on locally indecomposable second-order symmetric Lorentzian manifolds, which are generalized Cahen-Wallach spaces, as was proved by A.S. Galaev and D.V. Alekseevsky [1, 2]. Finally, we give explicit description of all possible dimensions of the algebra of Killing fields on second-order symmetric Lorentzian manifolds of small dimensions.

References

1. Galaev A. S., Alexeevskii D. V. *Two-symmetric Lorentzian manifolds*, J. Geom. Physics 61:12 (2011), 2331–2340.
2. Blanco O. F., Sanchez M., Senovilla J. M. *Structure of second-order symmetric Lorentzian manifolds*, J. European Math. Soc. 15 (2013), 595–634.

On the change of variable in L^p-spaces with a changing internal structure

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In [1, 2], one of the possible approaches to the study of the problem of the transfer of matter in a porous medium was successively developed. The physical processes are described by a weakly coupled system of two parabolic equations. One of them describes the macroscopic level (the medium as a whole), and the other describes the microscopic level (processes occurring in a single pore). The

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porous medium itself is modeled as the set V (macro level) and the distribution of domains $\{Y_v\}_{v\in V}$ (micro level). As a solution space, $L^p(V, W^{1,r}(Y_v))$ is chosen, i.e. the space of functions defined on V and valued in the family of spaces $W^{1,r}(Y_v)$ (a special case of the L^p-direct integral of Banach spaces). We study the composition operator

$$C_\varphi : L^p(V, W^{1,r}(Y_v)) \to L^q(U, W^{1,s}(X_u))$$

in spaces of such type, inducing by the mapping $\varphi(u, x) = (\psi(u), \xi_u(x))$. If conditions, under which the change of variables preserves space, is established, it makes possible to study problems connected with the deformation of the initial medium.

The composition operator on Lebesgue and Sobolev spaces was studied in detail in the works of S. K. Vodopyanov and A. D. Ukhlov (for example, [3, 4]). A similar problem on mixed norm Lebesgue spaces is considered in [5].

The main result is the following

Theorem. *Suppose that the mapping $\psi : U \to V$ has approximative partial derivatives on U. Operator $C_\varphi : L^p(V, W^{1,r}(Y_v)) \to L^q(U, W^{1,s}(X_u))$, $p \geq q$, $r \geq s$ is bounded if and only if distortion function*

$$\mathcal{K}(v) = \left(\sum_{u\in\psi^{-1}(v)\setminus\Sigma_\psi} \frac{\|\mathcal{H}_u\|^q_{L^{\frac{rs}{r-s}}(Y_{\psi(u)})}}{|J_\psi(u)|} \right)^{\frac{1}{q}}$$

belongs to $L^{\frac{pq}{p-q}}(V)$. Here $\mathcal{H}_u$ is a family of another distortion functions connected with mappings $\xi_u : X_u \to Y_{\psi(u)}$. The norm of the operator C_φ is equivalent to $\|\mathcal{K}\|_{L^{\frac{pq}{p-q}}(V)}$.

References

1. Showalter R. E., Walkington N. J. *Micro-structure models of diffusion in fissured media*, J. Math. Anal. Appl. 155:1 (1991), 1–20.
2. Meier S., Böhm M. *A note on the construction of function spaces for distributed miscrostructure models with spatially varying cell geometry*, Int. J. Numer. Anal. Model. 5 (2008), 109–125.
3. Vodop'yanov S. K., Ukhlov A. D.-O. *Superposition operators in Sobolev spaces*, Russian Math. (Iz. VUZ) 46:10 (2002), 9–31.
4. Vodop'yanov S. K., Ukhlov A. D.-O. *Set Functions and Their Applications in the Theory of Lebesgue and Sobolev Spaces. I*, Siberian Adv. Math. 14:4 (2004), 78–125.
5. Evseev N. A., Menovshchikov A. V. *The Composition Operator on Mixed-Norm Lebesgue Spaces*, Math. Notes 105:6 (2019), 812–817.

A family of harmonic polynomials whose factors are linear or quadratic

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In 1890, Stieltjes in a letter to Hermite formulated two questions: first, are there two distinct polynomials P_n and P_m that have a nontrivial common root? and second, are $P_n(x)$ for even n and $\frac{P_n(x)}{x}$ for odd n irreducible? To the best of my knowledge, both questions remain open. The irreducibility over rational numbers was proved in many particular cases (see [3] and references in this paper). The factorization of real harmonic polynomials is obviously connected with their nodal sets (i. e., the set of zeroes). It is proved in the paper [5] by Logunov and Malinnikova that the ratio of harmonic functions whose nodal sets are equal extends to a real analytic function and that a kind of the Harnack principle holds for any pair of such functions. Thus the property that an infinite family of linearly independent harmonic functions vanish on a given hypersurface is very restrictive. It was not known for a long time if this can be true for a nontrivial round cone in $\mathbb{R}^n$. In [4, Example 4.3], an example of such a cone in $\mathbb{R}^4$ was given. However, for $\mathbb{R}^3$ the problem is not resolved. In [1], Agranovsky stated a question on the stronger property of the cone $P_2(x,y,z) = x^2 + y^2 - 2z^2 = 0$: does there exist a harmonic homogeneous polynomial of degree greater than 5 which is divisible by P_2? The answer is unknown. The results of the paper [6] by Mangoubi and Weller Weiser clarify the situation. The quadratic factors of harmonic polynomials are described in [2] in terms of polynomial solutions to a Fuchsian ordinary differential equation and in terms of solutions to a system of algebraic equations.

The talk concerns a family of harmonic polynomials which admit factorization by linear and quadratic polynomials and some related results. Let $\mathcal{H}_n$ be the space of homogeneous of degree n complex valued harmonic polynomials on $\mathbb{R}^3$. The restriction $\mathcal{H}_n$ onto the unit sphere S^2 is the eigenspace of the Laplace–Beltrami operator on S^2 for the eigenvalue $n(n+1)$, $\dim \mathcal{H}_n = 2n+1$. Let $o = (0,0,1)$ be the base point in S^2 and T_o be its stable subgroup in SO(3). The eigenfunctions of T_o in $\mathcal{H}_n$ have the form $\zeta^{n-k} p_{n,k}$ or $\bar{\zeta}^{n-k} p_{n,k}$, where $\zeta = x + yi$, $\bar{\zeta} = x - yi$, and $p_{n,k}$ is a real T_o-invariant polynomial of degree k. The polynomials $p_{n,k}$ depend only on $\zeta\bar{\zeta} = x^2 + y^2$ and z. They are either even or odd and are closely related to the associated Legendre functions P_n^m. We consider factorization of $p_{n,4}$ and $\frac{p_{n,5}}{z}$ over $\mathbb{Q}$. The problem may be stated as follows:

when the polynomials

$$\zeta^{n-2}p_{n+2,4} = (x+yi)^{n-2}(x^2+y^2-A_nz^2)(x^2+y^2-B_nz^2),$$
$$\zeta^{n-2}p_{n+3,5} = (x+yi)^{n-2}z(x^2+y^2-C_nz^2)(x^2+y^2-D_nz^2)$$

are harmonic for rational A_n, B_n, C_n, D_n? It reduces to the Diophantine equations $2n^2+n=3m^2$ and $2n^2+3n=5m^2$, respectively. In the first case we have the sequences n_k and a_k in $\mathbb{N}$ such that the setting $n=n_k$, $A_{n_k}=a_k$, and $B_{n_k}=a_{k+1}$ makes the polynomial harmonic and there is no other harmonic polynomial $\zeta^{n-2}p_{n+2,4}$. Thus the consecutive harmonic polynomials of this type have a common quadratic factor. The sequences n_k, a_k can be defined by the generating functions

$$G_n(t) = \tfrac{1+t}{(1-t)(1-10t+t^2)},$$
$$G_a(t) = \tfrac{4t}{(1-t)(1-10t+t^2)},$$

respectively. In the second case the answer is more complicated but it is similar to that of the first.

References

1. Agranovsky M. *On some injectivity problem for the Radon transform on a paraboloid*, Contemp. Math. 251 (2000), 1–14.
2. Agranovsky M., Krasnov Y. *Quadratic divisors of harmonic polynomials in* $\mathbb{R}^n$, J. Anal. Math. 82 (2000), 379–395.
3. Cullinan J., Farshid H., *On the Galois groups of Legendre polynomials*, Indag. Math. 25:3 (2014), 534–552.
4. Logunov A., Malinnikova E. *On ratios of harmonic functions*, Advances in Math. 274 (2015), 241–262.
5. Logunov A., Malinnikova E. *Ratios of harmonic functions with the same zero set*, Geom. Funct. Anal. 26 (2016), 909–925.
6. Mangoubi D., Weller Weiser A. *Harmonic functions vanishing on a cone*, Isr. J. Math. 230:2 (2019), 563–581.
7. Stieltjes T. J., Letter No. 275 of Oct. 2, 1890. In: *Correspondence de Hermite et Stieltjes*, vol. 2, Gauthier-Villars, Paris (1905).

On volumes of the convex hulls of leaves for foliations with semigroup property

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Let $\mathfrak{F}$ be a smooth foliation with compact leaves of a Euclidean space $\mathcal{E}$. It have the semigroup property (SP for short) if the family of the convex hulls $\widehat{F}$ of its leaves F is a semigroup with respect to the Minkowski addition. Our aim is to clarify connection between SP and the metric properties of the leaves. The volumes of $\widehat{F}$ and their mixed volumes are among the most important metric quantities.

In all examples of foliations with SP which we know, there is a linear subspace $\mathcal{A}$ of $\mathcal{E}$, a Coxeter group W acting in it, and a simplicial cone C such that every leaf has the unique common point with C. The correspondence between this point and its leaf defines the isomorphism between C and the semigroup of the convex hulls. Furthermore, the restriction of the function $V(p) = \mathrm{Vol}(\widehat{F}_p)$ to C is a polynomial.

It is proved in [1] that SP holds for the family $\mathfrak{F}_G$ of orbits of a compact connected linear Lie group $G \subseteq \mathrm{SO}(\mathcal{E})$ if and only if G is polar. Then it is isometric to the orbit foliation of the isotropy representation of a Riemannian symmetric space. We say that these foliations are homogeneous.

A foliation $\mathfrak{F}$ is called isoparametric if its normal bundle is flat and the second fundamental form has constant eigenvalues along any local parallel normal vector field. The space $\mathcal{A}$ can be the normal space N_p to a principal leaf F at every its point p. Then the stable subgroup of N_p in the holonomy group of the normal bundle of F is the Coxeter group W and SP holds for these foliations due to Terng's convexity theorem which can be stated as follows: the orthogonal projection of the leaf F onto N_p is a convex polytope that is the convex hull of the set $F \cap N_p$ which is finite.

The codimension of a principal leaf of $\mathfrak{F}$ is called the rank of $\mathfrak{F}$. By Thorbergsson's theorem (see [3]), $\mathfrak{F}$ is homogeneous if rank $(\mathfrak{F}) \geq 3$. However, the non-homogeneous isoparametric foliations of ranks 1 and 2 exist. The case of hypersurfaces is trivial. We compute all possible functions $V(p)$ in the case of rank 2 and prove some their properties in general setting. For example, in the case of the adjoint representation of SU(3) we have the following formula for the volume in the basis of fundamental weights:

$$\frac{s^8 + 16s^7t + 112s^6t^2 + 448s^5t^3 + 700s^4t^4 + 448s^3t^5 + 112s^2t^6 + 16st^7 + t^8}{10080}$$

The mixed volumes of convex hull of distinct leaves can be obtained by the polarization of the polynomial $V(p)$. According to SP, the coefficient at $s^k t^{8-k}$ is equal to the multiplied by $\binom{8}{k}$ mixed volume of k bodies $\widehat{F}_{\varpi_1}$ and $8-k$ bodies $\widehat{F}_{\varpi_2}$, where ϖ_1, ϖ_2 are the fundamental weights (see [4, 5.1.26]).

Due to BKK-theory we know the explicit formula for Bézout's number (i.e. the number of solutions) for systems of equations $f_k = 0$ with matrix elements of finite dimensional representations of a complex reductive Lie group. It involves volumes and mixed volumes of the convex hulls of integral orbits of the coadjoint representations (see [5]). We are grateful to Boris Kazarnovskiĭ for making us aware of this fact and for useful comments. These results were extended and generalized in several directions but to the best of our knowledge the case of isoparametric foliations is not covered yet.

References

1. Gichev V. M. *Polar representations of compact groups and convex hulls of their orbits*, Diff. Geom. Appl. 28 (2010), 608–614.
2. Palais R. S., Terng C.-I. *Critical Point Theory and Submanifold Geometry*, Springer-Verlag Berlin Heidelberg, 1988.
3. Thorbergsson G. *Isoparametric foliations and their buildings*, Annals of Math. 133:2 (1991), 429–446.
4. Schneider R. *Convex bodies, Brunn-Minkowski theory*, Cambridge, 1993.
5. Казарновский Б. Я. *Многогранники Ньютона и формула Безу для матричных функций конечномерных представлений*, Функц. анализ и его прил. 21:4 (1987), 73–74.

Composition operators on Sobolev spaces and the spectrum of elliptic operators

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This work is devoted to applications of the geometric theory of composition operators on Sobolev spaces to spectral problems of the A-divergent form elliptic operators with the Neumann boundary condition:

$$L_A = -\operatorname{div}[A(z)\nabla f(z)], \quad z = (x, y) \in \Omega, \quad \left.\frac{\partial f}{\partial n}\right|_{\partial\Omega} = 0, \tag{1}$$

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in quasiconformal regular domains $\Omega \subset \mathbb{C}$ with $A \in M^{2\times 2}(\Omega)$. We denote, by $M^{2\times 2}(\Omega)$, the class of all 2×2 symmetric matrix functions $A(z) = \{a_{kl}(z)\}$, $\det A = 1$, with measurable entries satisfying the uniform ellipticity condition

$$\frac{1}{K}|\xi|^2 \leq \langle A(z)\xi, \xi\rangle \leq K|\xi|^2 \text{ a.e. in } \Omega, \tag{2}$$

for every $\xi \in \mathbb{C}$, where $1 \leq K < \infty$.

Recall that a simply connected domain $\Omega \subset \mathbb{C}$ is called an A-quasiconformal β-regular domain, $\beta > 1$, if

$$\iint\limits_{\mathbb{D}} |J(w, \varphi^{-1})|^{\beta} \, dudv < \infty,$$

where $\varphi : \Omega \to \mathbb{D}$ is an A-quasiconformal mapping [3]. Since (see, for example, [1]) A-quasiconformal mappings $\varphi : \Omega \to \mathbb{D}$ are defined up to conformal automorphisms of $\mathbb{D}$, this definition doesn't depend on a choice of φ and depends on the quasihyperbolic geometry of Ω only. A class of quasiconformal regular domains includes Lipschitz domains, Gehring domains and also some fractal domains like snowflakes.

The suggested method is based on the connection between composition operators on Sobolev spaces and quasiconformal mappings, that refines (in the case $n = 2$) results of [2]. As applications we obtain the solvability of the spectral problem (1) and lower estimates of the first non-trivial Neumann eigenvalues in A-quasiconformal β-regular domains [3]:

Theorem. *Let A be a matrix satisfies the uniform ellipticity condition (2) and Ω be an A-quasiconformal β-regular domain. Then the spectrum of the Neumann divergence form elliptic operator L_A in Ω is discrete, and can be written in the form of a non-decreasing sequence:*

$$0 = \mu_0(A, \Omega) < \mu_1(A, \Omega) \leq \mu_2(A, \Omega) \leq \ldots \leq \mu_n(A, \Omega) \leq \ldots,$$

and $$\frac{1}{\mu_1(A, \Omega)} \leq \frac{4}{\sqrt[\beta]{\pi}} \left(\frac{2\beta - 1}{\beta - 1}\right)^{\frac{2\beta-1}{\beta}} \left(\iint\limits_{\mathbb{D}} |J(w, \varphi^{-1})|^{\beta} \, dudv\right)^{\frac{1}{\beta}},$$

where $J(w, \varphi^{-1})$ is a Jacobian of the quasiconformal mapping $\varphi^{-1} : \mathbb{D} \to \Omega$.

References

1. Astala K., Iwaniec T., Martin G. *Elliptic partial differential equations and quasiconformal mappings in the plane*, Princeton University Press (2008).
2. Vodop'yanov S. K., Gol'dstein V. M. *Lattice isomorphisms of the spaces W_n^1 and quasiconformal mappings*, Siberian Math. J. 16:2 (1975), 174–189.
3. Gol'dshtein V., Pchelintsev V., Ukhlov A. *Composition operators on Sobolev spaces and eigenvalues of divergent elliptic operators*, arXiv:1903.11301.

Estimates of operators on the cones of functions with monotonicity properties and optimal embeddings

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General properties are consider of operators acting from rearrangement invariant spaces into generalized Morrey-type spaces. We extract wide class of operators that preserve non-negativity and monotonicity of functions and prove two-sided estimates for their norms. As corollaries we obtain corresponding results for operators of embedding and for Hardy–Littlewood maximal operators. For operators commuting with a shift operator the results are extended to the case of global Morrey-type spaces. As an application of these approaches we establish a criterion of the embedding for a weighted Lorentz space into a Morrey-type space.

References

1. Goldman M. L., Bakhtigareeva E. *Some General Properties of Operators in Morrey-Type Spaces*, Springer Proceedings in Mathematics and Statistics, vol. 291. Springer, Cham.

Geometry of phase portrait of one piecewise linear gene network model

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We study the following piece-wise dynamical system constructed in [1]:

$$\frac{dX_1}{dt} = L_1(X_4) - X_1; \quad \frac{dX_2}{dt} = \Gamma_2(X_1) - X_2;$$

$$\frac{dX_3}{dt} = \Gamma_3(X_2) - X_3; \quad \frac{dX_4}{dt} = \Gamma_4(X_3) - X_4. \tag{1}$$

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Here, Γ_i are step-functions: $\Gamma_i(X_{i-1}) = \alpha_i > 0$ for $0 \le X_{i-1}$; $\Gamma_i(X_{i-1}) = -1$ for $0 \ge X_{i-1} \ge -1$, $i = 2, 3, 4$, and the step-function L_1 is decreasing: $L_1(X_4) = \alpha_1 > 0$ for $-1 < X_4 < 0$, $L_1(X_4) = -1$ for $0 \le X_4$. Decreasing function L_1 and increasing functions Γ_i correspond to negative, respectively positive, feedbacks in gene network described by the system (1).

Lemma. *Trajectories of all points of* $Q := [-1, \alpha_1] \times [-1, \alpha_2] \times [-1, \alpha_3] \times [-1, \alpha_4]$ *do not leave* Q *as* $t \to \infty$.

The coordinate planes $X_i = 0$ subdivide the domain Q to 16 blocks. We enumerate them by binary multi-indices $\{\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4\}$, where $\varepsilon_i = 0$, if $X_i < 0$, and $\varepsilon_i = 1$, if $X_i > 0$, see [2]. Here and below $i = 1, 2, 3, 4$.

It was shown in [1] that the system (1) has a stable cycle C which passes through the blocks following the arrows of the State Transition Diagram:

$$\begin{array}{ccccccc} \{1111\} & \longrightarrow & \{0111\} & \longrightarrow & \{0011\} & \longrightarrow & \{0001\} \\ \uparrow & & & & & & \downarrow \\ \{1110\} & \longleftarrow & \{1100\} & \longleftarrow & \{1000\} & \longleftarrow & \{0000\} \end{array} \tag{2}$$

Let W_1 be the interior of the union of the blocks of (2). Trajectories of the points of interiors of these blocks can pass to one adjacent block only, according to the arrow. Hence, W_1 is a positively invariant domain of the system (1). Denote by W_3 the interior of the union of remaining 8 blocks, connected by the diagram

$$\begin{array}{ccccccc} \{1101\} & \longrightarrow & \{0101\} & \longrightarrow & \{0100\} & \longrightarrow & \{0110\} \\ \uparrow & & & & & & \downarrow \\ \{1001\} & \longleftarrow & \{1011\} & \longleftarrow & \{1010\} & \longleftarrow & \{0010\} \end{array} \tag{3}$$

As above, the arrows show possible transitions of trajectories from block to block.

Remark. *Trajectories of the points of interiors of blocks in* (3) *can pass to three adjacent blocks. Diagrams* (2) *and* (3), *and passages from blocks of* (3) *to blocks of* (2) *define an oriented graph on the* 1*-dimensional sceleton of* Q.

$$\begin{aligned} \text{Let } F_0 &= \{1101\} \cap \{0101\},\ X_1 = 0; & F_1 &= \{0101\} \cap \{0100\},\ X_4 = 0; \\ F_2 &= \{0100\} \cap \{0110\},\ X_3 = 0; & F_3 &= \{0110\} \cap \{0010\},\ X_2 = 0; \\ F_4 &= \{0010\} \cap \{1010\},\ X_1 = 0; & F_5 &= \{1010\} \cap \{1011\},\ X_4 = 0; \\ F_6 &= \{1011\} \cap \{1001\},\ X_3 = 0; & F_7 &= \{1001\} \cap \{1101\},\ X_2 = 0; \end{aligned}$$

be the common faces of adjacent bocks of the domain W_3. Denote by

$$\varphi_0 : F_0 \to F_1,\ \ \varphi_1 : F_1 \to F_2,\ \ \dots\ \ \varphi_j : F_j \to F_{j+1},\ \ \dots\ \ \varphi_7 : F_7 \to F_8 = F_0$$

the shifts of the points of interiors of these faces along their trajectories of the system (1) in each of the blocks of the diagram (3), and let $\Phi : F_0 \to F_0$ be the composition of these shifts.

Trajectories of some points of the faces F_j intersect the faces of the blocks listed in (2) and then remain in the interior of the invariant domain W_1. So, we assume that the mappings $\varphi_j : F_j \to F_{j+1}$ are defined on preimages of F_{j+1}. Here and below $j = 0, 1, \dots, 7$. In interior of each of the blocks $\{\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4\}$, the system (1) is solved explicitly.

Denote by $Oy_1^{(0)}y_2^{(0)}y_3^{(0)}$ the coordinate system in the plane $X_1 = 0$ containing the face F_0 such that the positive octant of this coordinate system contains the face F_0, and the axes are enumerated as follows: $y_1^{(0)} = X_2 > 0$, $y_2^{(0)} = -X_3 > 0$, $y_3^{(0)} = X_4 > 0$ in the interior of F_0. Let $P^{(0)} \in \operatorname{int} F_0$ be a point with coordinates $y_1^{(0)}$, $y_2^{(0)}$, $y_3^{(0)}$. Simple calculations, see for example [3], show that the trajectory T of the point $P^{(0)}$ intersects the face $F_8 = F_0$ at the point

$$\Phi(P^{(0)}) = \frac{MP^{(0)}}{H(y_1^0, y_2^0, y_3^0)}, \tag{4}$$

where H is a non-homogeneous linear function of coordinates of the point $P^{(0)}$, $H(F_0) > 0$, and the inverse matrix M^{-1} of M has strictly positive elements. Hence, the matrix M has a unit eigenvector e_1 with positive coordinates, and its corresponding eigenvalue λ_1 is positive.

The formula (4) describe projective transformations $F_0 \to F_0$ defined on some subset of the face F_0, as it was remarked above. The rays in $\Phi^{-1}(F_0)$ containing the origin O are mapped by Φ to the rays containing O.

Let ℓ be the ray codirectional with e_1. After one round along the diagram (3) trajectories of all points of ℓ return to ℓ and compose an invariant octahedral surface S with one vertex O. Let ζ be a positive coordinate on the ray ℓ, then the map $\Phi : \ell \to \ell$ is described by

$$\Phi(\zeta) = \lambda_1\zeta \cdot (\beta + D\zeta)^{-1}, \quad \text{where} \quad D > 0, \quad \text{and} \quad \beta := \alpha_1\alpha_2\alpha_3\alpha_4. \tag{5}$$

Now, $\det M = \beta^3$, $\operatorname{tr} M = 3\beta$, and $P(\lambda) = \lambda^3 - 3\beta\lambda^2 + I_2\lambda - \beta^3$ is the characteristic polynomial of the matrix M. If all its eigenvalues are real then they coincide.

Direct calculations show that $I_2 - 3\beta^2 > 0$ for all positive values of α_i. Hence, $P(\lambda)$ grows monotonically with λ, so the equation $P(\lambda) = 0$ has a unique real root λ_1 in the interval $(0, \beta)$. Thus, trajectories of all points of the surface S follow the arrows of the diagram (3), but they are not periodic, they tend to O as $t \to \infty$.

Theorem 1. *Trajectories of all points of $Q \setminus S$ are attracted by the cycle C of the system* (1).

This theorem follows from properties of the matrices M^k and M^{-k} related to iterations of the map Φ.

Considerations of the diagrams (2) and (3) on the boundary of Q imply:

Theorem 2. *The cycle $C \subset W_1$ has a nontrivial link in Q with the surface S, this configuration is reduced to the Hopf link.*

References

1. Glass L., Pasternack J. S. *Stable oscillations in mathematical models of biological control systems*, J. of Mathematical Biology 6 (1978), 207 – 223.
2. Ayupova N. B., Golubyatnikov V. P. *On two classes of nonlinear dynamical systems: The four-dimensional case*, Siberian Math. J. 56:2 (2015), 231 – 236.
3. Golubyatnikov V. P., Ivanov V. V. *Cycles in the odd-dimensional models of circular gene networks*, J. Applied and Industrial Math. 12:4 (2018), 648–657.

Isolation in the moduli space of Kleinian groups

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Let $\mathrm{Isom}^+(\mathbb{H}^3)$ be the set of orientation preserving hyperbolic isometries of hyperbolic 3-space $\mathbb{H}^3$, a *Kleinian group* is a discrete and non-elementary subgroup of $\mathrm{Isom}^+(\mathbb{H}^3)$. It's known that $\mathrm{Isom}^+(\mathbb{H}^3)$ is topologically isomorphic to both the group Möb $(\overline{\mathbb{C}})$ of orientation preserving Möbius transformations on the Riemann sphere $\overline{\mathbb{C}}$ and the projective special linear group $\mathrm{PSL}(2, \mathbb{C})$.

We define the triple complex parameters for each two-generator group $\langle f, g\rangle$:

$$(\gamma(f,g), \beta(f), \beta(g)) = \big(tr([f,g]) - 2, \quad tr^2(f) - 4, \quad tr^2(g) - 4\big)$$

where $[f, g]$ is the commutator. Every two-generator group $\langle f, g\rangle$ can be determined uniquely (up to conjugation) by its complex numbers with $\gamma(f, g) \neq 0$.

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In this talk, we present our methods to produce universal constraints in the moduli space of Kleinian groups and generate a number of precise inequalities for Kleinian groups including Jørgensen's famous inequality: Let $\langle f, g\rangle$ be a Kleinian group, then

$$|\gamma(f,g)| + |\beta(f)| \geq 1.$$

References

1. Buser P., Karcher H. *Gromov's almost flat manifolds*, Asterique 81, (1981).
2. Cao C. *Some trace inequalities for discrete groups of Möbius transformations*, Proceedings of the AMS 123:12 (1995).
3. Gehring F. W., Martin G. J. *Iteration theory and inequalities for Kleinian groups*, Bull. Amer. Math. Soc. 21 (1989), 57–63.
4. Gehring F. W., Martin G. J. *Stability and extremality in Jørgensen's inequality*, Complex Variables 12 (1989), 277–282.
5. Gehring F. W., Martin G. J. *Inequalities for Mbius transformations and discrete groups*, J. Reine Angew. Math. 418 (1991), 31–76.
6. Gehring F. W., Martin, G. J. *Commutators, collars and the geometry of Möbius groups*, Jounal D'Analyse Mathematique 63 (1994), 175–219.
7. Hagelberg M., MacLachlan C., Rosenberger G. *On discrete generalised triangle groups*, Proceedings of the Edinburgh Mathematical Society 38:3 (1995), 397–412.
8. Jorgensen T. *On discrete groups of Möbius transformations*, Amer. J. Math 98:3 (1976), 739–749.
9. Martin G. J. *On discrete Möbius groups in all dimensions: a generalization of Jørgensen's inequality*, Acta Math. 163 (1989), 253–289.
10. Martin G. J. *The Geometry and Arithmetic of Kleinian Groups*, Handbook of Group Actions, Volume I, International Press of Boston, Inc. (2015), 411–494.
11. Maclachlan C., Martin G. *The (p, q)-arithmetic hyperbolic lattices; $p, q \geq 6$*, arXiv:1502.05453.

Distance functions between sets and theorems of completeness in (q_1, q_2)-quasimetric spaces

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Let q_1, q_2 be positive numbers and X any set. A function $\rho : X \times X \to \mathbb{R}^+$ satisfying the identity axiom is called a (q_1, q_2)-quasimetric [1] if the following (q_1, q_2)-generalized triangle inequality holds:

$$\rho(x, y) \leq q_1 \rho(x, z) + q_2 \rho(z, y) \quad \forall x, y, z \in X.$$

The pair (X, ρ) is called a (q_1, q_2)-quasimetric space.

The distance between a set $A \subset X$ to a set $B \subset X$ is defined by the formula

$$dist(A, B) = \inf\{\rho(a, b) \mid a \in A, \ b \in B\}.$$

The Hausdorff deviation $h^+(A, B)$ of a set B from a set A is defined by the formula

$$h^+(A, B) = \sup_{a \in A} dist(a, B).$$

If A, B are closed then $h^+(A, B) = 0 \Leftrightarrow A \subseteq B$. Moreover

$$h^+(A, B) \leq q_1 h^+(A, C) + q_2 h^+(C, B) \quad \forall A, B, C \subset X.$$

For arbitrary closed sets $A, B \subset X$ we put

$$h(A, B) = \max\{h^+(A, B), \ h^+(B, A)\}.$$

Let $\mathcal{M}(X)$ denote the set of closed subsets of X. We proved the following

Theorem 1. *Let (X, ρ) be a complete (q_1, q_2)-quasimetric space, where $\rho(\cdot\,, \cdot)$ is lower semicontinuous in the second argument. Then $(\mathcal{M}(X), h)$ is complete.*

Some generalizations of Theorem 1 were established.

References

1. Arutyunov A. V., Greshnov A. V. (q_1, q_2)*-quasimetric spaces. Covering mappings and coincidence points*, Izvestiya: Mathematics, 82:2 (2018), 245–272.

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Cumulative structure of a Boolean-valued model of set theory

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A Boolean-valued algebraic system of set-theoretic signature $\{=, \in\}$ is a nonempty class X endowed with Boolean-valued interpretations of the signature symbols which are functions $=_X, \in_X \colon X^2 \to B$ taking values in a complete Boolean algebra B and satisfying the analogs of the classical axioms of equality, such as

$$\in_X(x, y) \wedge =_X(y, z) \leqslant \in_X(x, z)$$

(see [1, 3.1]). By means of the operations in B of supremum $a \vee b$, infimum $a \wedge b$, and complement $\neg b$, as well as suprema $\sup A$ and infima $\inf A$ of subsets $A \subset B$, the truth value $[\varphi(\bar{x})]_X \in B$ in X at $\bar{x} = x_1, \ldots, x_n \in X$ is recursively defined for an arbitrary formula φ of the first-order language of signature $\{=, \in\}$. In the case $[\varphi(\bar{x})]_X = 1_B$, the assertion $\varphi(\bar{x})$ is said to be *true in* X, which fact is denoted as $X \vDash \varphi(\bar{x})$.

A simple and natural example of a Boolean-valued algebraic system is the class $\mathbb{V}^S$ of all functions defined on a nonempty set S with the interpretations

$$=_{\mathbb{V}^S}(x, y) = \{s \in S : x(s) = y(s)\},$$
$$\in_{\mathbb{V}^S}(x, y) = \{s \in S : x(s) \in y(s)\},$$

which take values in the Boolean algebra $\mathcal{P}(S)$ of all subsets of S. In this case, the truth value of every formula can be calculated pointwise:

$$[\varphi(x_1, \ldots, x_n)]_{\mathbb{V}^S} = \big\{s \in S : \varphi\big(x_1(s), \ldots, x_n(s)\big)\big\}.$$

A more general function example of a Boolean-valued system is the class $C(Q, V^Q)$ of continuous sections of a bundle V^Q of models of set theory over an extremally disconnected compact space Q (see [2; 3, Ch. 6]).

Let X be a Boolean-valued system with truth algebra B. A *Boolean-valued class* in X is a function $\Phi \colon X \to B$ subject to the relation

$$\Phi(x) \wedge [x = y]_X \leqslant \Phi(y)$$

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for all $x, y \in X$ (see [1, 3.5; 3, 4.6.1]). Such natural agreements as

$$[x \in \Phi]_X = \Phi(x), \quad [x = \Phi]_X = [(\forall\, y)(y \in x \Leftrightarrow y \in \Phi)]_X$$

make it possible to use Boolean-valued classes inside the truth value expressions, analogously to the use of classes in the language of set theory. Say that an *element* $x \in X$ *represents a Boolean-valued class* Φ and write $x \simeq \Phi$ if $X \vDash (x = \Phi)$.

Let B^X be the class of all functions $F\colon \operatorname{dom} F \to B$ on subsets $\operatorname{dom} F \subset X$. The *ascent* of $F \in B^X$ is the Boolean-valued class $F{\uparrow}\colon X \to B$ defined as follows:

$$F{\uparrow}(x) = \bigvee_{y \in \operatorname{dom} F} [x = y]_X \wedge F(y).$$

An element $x \in X$ is the *mixing* of a family $(x_i)_{i \in I} \subset X$ with respect to a partition of unity $(d_i)_{i \in I} \subset B$ whenever $[x = x_i]_X \geqslant d_i$ for all $i \in I$. The symbol $\operatorname{mix} Y$ denotes the totality of various mixings of elements of a subset $Y \subset X$.

A Boolean-valued algebraic system X with truth algebra B is called a *Boolean-valued universe* (see [1, 3.4]) or a *B-valued universe*, if it meets the following five conditions:

(1) $(\forall\, x, y \in X)\big(X \vDash (x = y) \Rightarrow x = y\big)$;
(2) $(\forall\, F \in B^X)(\exists\, x \in X)(x \simeq F{\uparrow})$;
(3) $(\forall\, x \in X)(\exists\, F \in B^X)(x \simeq F{\uparrow})$;
(4) $X \vDash (\forall\, x, y)\big((\forall\, z)(z \in x \Leftrightarrow z \in y) \Rightarrow x = y\big)$;
(5) $X \vDash (\forall\, x)\big((\exists\, y)(y \in x) \Rightarrow (\exists\, y \in x)(\forall\, z \in x)(z \notin y)\big)$.

As is known (see [1, 3]), for every complete Boolean algebra B, there exists a B-valued universe $\mathbb{V}^{(B)}$ which occurs a model of ZFC: if φ is a theorem of ZFC then the assertion $\mathbb{V}^{(B)} \vDash \varphi$ is also a theorem of ZFC.

In the research paper [4] under announcement, we show that every B-valued universe X has the following multilevel structure analogous to the von Neumann cumulative hierarchy:

$X_0 = \varnothing$;
$X_{\alpha+1} = \big\{x \in X : x \simeq F{\uparrow},\ F \in B^{X_\alpha}\big\}$ for every ordinal α;
$X_\alpha = \bigcup_{\beta<\alpha} X_\beta$ for every limit ordinal α;
$X = \bigcup_{\alpha \in \mathrm{Ord}} X_\alpha$, where Ord is the class of ordinals.

Another cumulative structure is obtained if we consider the ascents of constant functions only and add mixings at the limit steps:

$Y_0 = \varnothing$;
$Y_{\alpha+1} = \big\{x \in X : x \simeq (D \times \{b\}){\uparrow},\ D \subset Y_\alpha,\ b \in B\big\}$ for every ordinal α;
$Y_\alpha = \operatorname{mix} \bigcup_{\beta<\alpha} Y_\beta$ for every limit ordinal α;
$X = \bigcup_{\alpha \in \mathrm{Ord}} Y_\alpha$.

Such cumulative hierarchies clarify the structure of Boolean-valued systems and, in particular, make it possible to easily prove the uniqueness of a Boolean-valued universe up to isomorphism.

In addition, [4] contains a general tool for adding ascents to Boolean-valued systems which builds the hierarchy $(X_\alpha)_{\alpha\in\mathrm{Ord}}$ for an arbitrary system X_0 satisfying (4) and enlarges X_0 to a system $X = \bigcup_{\alpha\in\mathrm{Ord}} X_\alpha$ subject to (2)–(4), as well as to (1) and (5) as soon as X_0 meets the corresponding requirements. This makes it possible to construct examples of Boolean-valued systems with unusual properties. By means of the tool, we show in [4] that each of the five conditions (1)–(5) listed in the definition of a Boolean-valued universe, is essential and does not follow from the other conditions.

References

1. Solovay R. M., Tennenbaum S. *Iterated Cohen extensions and Souslin's problem*, Annals of Math. 94:2 (1971). 201–245.
2. Gutman A. E., Losenkov G. A. *Function representation of the Boolean-valued universe*, Siberian Adv. Math. 8:1 (1998). 99–120.
3. Kusraev A. G., Kutateladze S. S. *Introduction to Boolean Valued Analysis.* Moscow: Nauka, 2005. [in Russian]
4. Gutman A. E. *The Boolean-valued universe as an algebraic system*, Siberian Math. J. 60:5 (2019). (to appear)

Binary correspondences and an algorithm for solving an inverse problem of chemical kinetics

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Binary correspondences are used for formalization of problems, their basic components, properties, and constructions [1–3]. As an illustration, a singularly perturbed system of ordinary differential equations is considered which describes a process in chemical kinetics [4, 5]:

$$\begin{aligned} \dot{x}(t) &= f\big(x(t), y(t), t, \varepsilon\big), \\ \varepsilon\, \dot{y}(t) &= g\big(x(t), y(t), t, \varepsilon\big), \end{aligned}$$

where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, $t \in \mathbb{R}$, ε is a small parameter, f, g are sufficiently smooth functions. Formulas for the solution of the inverse problem are presented for the case $\varepsilon = 0$, the conditions of unique solvability are indicated, and realizability of the conditions is clarified.

An iteration algorithm is proposed for finding an approximate solution to the inverse problem for the case $\varepsilon \neq 0$. At each step of the algorithm, the solution of the inverse problem for the above-considered case $\varepsilon = 0$ is combined with the solution of the direct problem which is reduced to the proof of the existence and uniqueness of a solution in the case $\varepsilon \neq 0$. The conjecture is stated on convergence of the algorithm and an approach to its justification is developed which is based on the Banach fixed-point theorem.

References

1. Gutman A. E., Kononenko L. I. *Formalization of inverse problems and its applications*, Sib. J. Pure. Appl. Math. 17:4 (2017). 49–56.
2. Gutman A. E., Kononenko L. I. *The inverse problem of chemical kinetics as a composition of binary correspondences*, Sib. Electron. Math. Rep. 15 (2018). 48–53.
3. Gutman A. E., Kononenko L. I. *Binary correspondences and the inverse problem of chemical kinetics*, Vladikavkaz Math. J. 20:3 (2018). 37–47.
4. Vasil'eva A. V., Butuzov V. F. *Singularly perturbed equations in critical cases.* Moscow: Moscow Gos. Univ., 1978. [in Russian]
5. Goldstein V. M., Sobolev V. A. *Qualitative analysis of singularly perturbed systems.* Novosibirsk: Sobolev Institute of Mathematics, 1988. [in Russian]

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F-polynomials and connected sums of virtual knots

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Virtual knots were introduced by Louis Kaufman in [1] as an essential generalization of classical knot theory. Diagrams of virtual knots have both classical and virtual crossings. There is a way to make a new diagram of a knot from two other diagrams called connected sum: select a point on each of the diagrams and splice them together. It is known that connected sum of virtual knots, unlike connected sum of classical knots, is not well defined. In [2] a family of polynomial invariants called F-polynomials was introduced. It's obtained by assigning two weights for each classical crossing c in the diagram of the knot: index value $\mathrm{Ind}(c)$ defined in [3] and ∇J_n which is a difference of n-writhe and $-n$-writhe defined in [4]. In this work these polynomials are used to prove following result:

Theorem. *Let $\mathcal{K}$, $\mathcal{M}$ be two oriented virtual knots, such that $\nabla J_n(\mathcal{M}) \neq 0$. Then there exist an infinite family of different connected sums of these knots $\{M\#K_n\}_{n=1}^{\infty}$ where M and K_n are diagrams of $\mathcal{M}$ and $\mathcal{K}$ correspondingly.*

References

1. Kauffman L. *Virtual Knot Theory*, European J. Combinatorics, 20:7 (1999), 663–691.
2. Kaur K., Prabhakar L., Vesnin A. *Two-variable polynomial invariants of virtual knots arising from flat virtual knot invariants*, J. Knot Theory and Its Ramifications 27:13 (2018), paper no. 1842015.
3. Cheng Z., Gao H. *A polynomial invariant of virtual links*, J. Knot Theory and Its Ramifications 22(12) (2013), 1341002.
4. Satoh S., Taniguchi K. *The writhes of a virtual knot*, Fundamenta Mathematicae 225 (2014), 327–341.

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Interpolation of Riemannian manifolds

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In Manifold Learning one often deals with the following situation: given a "point cloud" in a high-dimensional Euclidean space that supposedly approximates a low-dimensional smooth submanifold, one has to determine if there is indeed an approximating submanifold and study its topological and geometric properties.

We [1] extend this circle of problems to the context of Riemannian manifolds and metric space approximations. This leads to the following problem: given a discrete metric space X, determine if there is a Riemannian manifold M^n with bounded geometry such that X and M are close in Gromov-Hausdorff or quasi-isometric sense. "Bounded geometry" here means bounded curvature and injectivity radius. Our main result is the following.

Theorem. *For every $n \in \mathbb{N}$, $n \geq 2$, there is a constant $C = C(n) > 0$ such that the following holds. Let X be a metric space and let $K, r > 0$ be such that Kr^2 is sufficiently small. Suppose that every ball of radius r in X lies within Gromov-Hausdorff distance $\delta := Kr^3$ from the Euclidean r-ball in $\mathbb{R}^n$.*

Then there exists a Riemannian manifold M^n with sectional curvature bound $|\operatorname{Sec}_M| \leq CK$ and injectivity radius bound $\operatorname{inj}_M \geq r/2$ approximating X in the following sense: there is a bijection between X and a $C\delta$-net in M distorting the distances not greater than r by at most $C\delta$.

If the metric of X is δ-intrinsic, this implies that M and X are quasi-isometric with parameters $(1 + CKr^2, CKr^3)$ and the Gromov-Hausdorff distance between them is bounded by $CKr^2 \operatorname{diam}(X)$.

The parameters in the theorem are optimal up to a multiplicative constant depending only on n. The manifold M can be constructed by an algorithmic procedure.

The theorem implies the following characterization of Alexandrov spaces with two-sided curvature bounds.

Corollary. *For a complete geodesic metric space X and $n \in \mathbb{N}$, the following two conditions are equivalent:*

1. *There exists $K > 0$ such that for all $x \in X$ and $r > 0$,*

$$d_{GH}(B_r^X(x), B_r^n) \leq Kr^3$$

where d_{GH} is the Gromov-Hausdorff distance and B_r^n is the r-ball in $\mathbb{R}^n$.

2. *X is an n-dimensional manifold, its metric has two-sided bounded curvature in the sense of Alexandrov, and its injectivity radius is bounded away from 0.*

Furthermore, if the first condition holds then X has Alexandrov curvature bounds between $-CK$ and CK and injectivity radius at least $1/(C\sqrt{K})$, where C is a positive constant depending only on n.

References

1. Fefferman C., Ivanov S., Kurylev Y., Lassas M., Narayanan H. *Reconstruction and interpolation of manifolds I: The geometric Whitney problem*, to appear in Foundations of Computational Mathematics.

Fejer sums and the von Neumann ergodic theorem

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The Fejer sums for measures on the circle and the norms of the deviations from the limit in the von Neumann ergodic theorem both are calculating, in fact, with the same formulas (by integrating of the Fejer kernels) — and so, this ergodic theorem, in fact, is a statement about the asymptotic of the growth of the Fejer sums at zero point of the spectral measure of corresponding dynamical system.

It gives a possibility to rework well-known estimates of the rates of convergence in the von Neumann ergodic theorem into the estimates of the Fejer sums in the point for measures on the circle — for example, we obtain natural criteria of polynomial growth and polynomial degree of these sums.

And vice versa, numerous in the literature estimates of the deviations of Fejer sums in the point allow to obtain estimates of the rate of convergence in this ergodic theorem. For example, we obtain from the results of S. N. Bernstein in harmonic analysis more than hundred years old the estimates of the rates of convergence in the von Neumann ergodic theorem for many popular in the applications dynamical systems — with sharp senior coefficient of the asymptotic.

Self-intersections of self-similar sets under translations and extensions of copies

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Let $\mathcal{S} = \{S_1, \ldots, S_m\}$ be a system of contracting similarities in $\mathbb{R}^n$. The unique nonempty compact set $K = K(\mathcal{S})$ such that $K = \bigcup_{i=1}^{m} S_i(K)$, is called an *attractor* of the system $\mathcal{S}$, or a *self-similar set*, generated by the system $\mathcal{S}$; sets $S_i(K)$ are called *copies* of the self-similar set K. The system $\mathcal{S}$ is said to satisfy the *strong separation condition* (SSC), iff $S_i(K) \cap S_j(K) = \varnothing$ for all distinct $i, j \in I = \{1, \ldots, m\}$. It is well-known that if SSC is satisfied, Hausdorff dimension $d = \dim_H K$ of self-similar set K is a solution of the following equation:

$$\sum_{i=1}^{m} (\mathrm{Lip}\ S_i)^d = 1. \tag{1}$$

Suppose the system $\mathcal{S}_t = \{S_1^t, \ldots, S_m^t\}$ of contracting similarities in $\mathbb{R}^n$ depends on parameter $t \in D$ and let K_t be its attractor. Fix some $j \in I$ and suppose $S_i^t = S_i$ do not depend on t for all $i \neq j$.

We study [1] the problem of intersection between copies $S_i(K_t)$ and $S_j^t(K_t)$, $i \neq j$ of the set K_t for the cases when either $S_j^t(x) = G(x) + t$, where $t \in D = \mathbb{R}^n$, or $S_j^t(x) = tG(x) + h$, where $t \in D = [a, b] \subset (0, 1)$ and G is isometry.

Under some constraints on Lip S_i^t, $i \in I$ we prove, using our General Position Theorem [2], that if the solution d of the equation (1) for the system $\mathcal{S}_t$ does not exceed some s for all $t \in D$, then in both the above cases we have:

$$\dim_H\{t \in D : \ S_i(K_t) \cap S_j^t(K_t) \neq \varnothing\} \leq 2s, \ i \in I, \ i \neq j.$$

We apply this result to find the conditions under which system $\mathcal{S}_{(t_1,\ldots,t_m)} = \{S_1^{t_1}, \ldots, S_m^{t_m}\}$ of contracting similarities in $\mathbb{R}^n$, where $t_1, \ldots, t_m \in D$ are either translation vectors or similarity coefficients of the corresponding mappings, satisfy SSC and therefore its attractor's K Hausdorff dimension $d = \dim_H K$ satisfy the equation (1).

References

1. Kamalutdinov K. G. *Self-intersections in parameterized self-similar sets under translations and extensions of copies*, Trudy Instituta Matematiki i Mekhaniki URO RAN 25:2 (2019), 116–124.
2. Kamalutdinov K., Tetenov A. *Twofold Cantor sets in* $\mathbb{R}$, Sib. Elektron. Mat. Izv. 15 (2018), 801–814.

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On a generalization of Zernike polynomials in a ball

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Let $\mathbb{B}^3$ and $\mathbb{S}^2$ be the unit ball and the unit sphere in $\mathbb{R}^3$, respectively, i. e. $\mathbb{B}^3 = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| < 1\}$ and $\mathbb{S}^2 = \partial\mathbb{B}^3 = \{\boldsymbol{\xi} \in \mathbb{R}^3 : |\boldsymbol{\xi}| = 1\}$, where $|\cdot|$ denotes the Euclidean norm. Throughout the paper we adopt the convention to denote in bold type the vectors in $\mathbb{R}^3$, and in simple type the scalars in $\mathbb{R}$. By the greek letters $\boldsymbol{\theta}$, $\boldsymbol{\eta}$, $\boldsymbol{\xi}$ and so on we denote the units vectors $\mathbb{S}^2$.

In the paper, we describe an approach for constructing a generalize Zernike polynomials $Z_{N\ell}^{[m](N+2k)}$ or simply m-Zernike polynomials. For $m = 0$ we get the well-known Zernike polynomials $Z_{N\ell}^{(N+2k)} \equiv Z_{N\ell}^{[0](N+2k)}$, see [1]. This approach can be considered as one of the options for generalized Zernike polynomials. The properties of an auxiliary polynomials $\mathbb{P}_{m+n}^{[m]}$ are studied and then polynomial bases for the Sobolev spaces $H_0^m(\mathbb{B}^3)$ and $H_0^m(\mathbb{B}^3) \cap H_0^{2m}(\mathbb{B}^3)$ are constructed.

A generalized m-Zernike polynomials in a ball are defined for integer m, N, $k \in \mathbb{N} \cup 0$, $N + 2k \geq m - 1$ as

$$Z_{N\ell}^{[m](N+2k)}(\mathbf{x}) := \int_{\mathbb{S}^2} Y_{N\ell}(\boldsymbol{\eta}) \mathbb{P}_{N+2k}^{[m-1]}(\mathbf{x} \cdot \boldsymbol{\eta})\, \mathrm{d}\boldsymbol{\eta}, \ \ \mathbf{x} \in \mathbb{B}^3. \tag{1}$$

The superscript $N + 2k$ in $Z_{N\ell}^{[m](N+2k)}$ indicates the degree of the polynomial. The $Y_{N\ell}$ are complex-valued spherical harmonics. In definition (1) the polynomials of one variable $P_{m+N}^{[m]}(t)$, $t \in [-1, 1]$ are determined by using m-multiple integration [2]

$$\begin{aligned} \mathbb{P}_{m+N}^{[m]}(t) &:= \mathbf{J}^m P_N(t) = \int_1^t \mathrm{d}s_1 \int_1^{s_1} \mathrm{d}s_2 ... \int_1^{s_{m-1}} P_N(s_m)\, \mathrm{d}s_m \\ &= \frac{1}{(m-1)!} \int_1^t (t-s)^{m-1} P_N(s)\, \mathrm{d}s = \frac{(t-1)^m}{m!}\, {}_2F_1\left(\begin{matrix} -N, N+1 \\ m+1 \end{matrix} \,\middle|\, \frac{1-t}{2} \right), \end{aligned} \tag{2}$$

where the ${}_2F_1$ is the Gauss hypergeometric function. For $m = 0$ we have $\mathbb{P}_N^{[0]}(t) = P_N(t)$ and for $m = -1$ it is assumed that $\mathbb{P}_N^{[-1]}(t) = C_N^{(3/2)}(t)$ in view of the fact that $\mathbf{J}^{-1} P_N(t) = \frac{d}{dt} P_N(t) = C_N^{(3/2)}(t)$, P_N, $C_N^{(3/2)}$ are Legendre and Gegenbauer polynomials of degree N, respectively.

The formula (2) is the classic Cauchy formula to solve the initial problem

$$\frac{d^m}{\mathrm{d}t^m} u(t) = P_N(t), \quad u(1) = \frac{d}{\mathrm{d}t} u(1) = \ldots = \frac{d^{m-1}}{\mathrm{d}t^{m-1}} u(1) = 0.$$

The following recurrence relations are obvious,

$$\mathbb{P}^{[m]}_{m+N}(t) = \int_1^t \mathbb{P}^{[m-1]}_{m-1+N}(s)\mathrm{d}s = \mathbf{J}\mathbb{P}^{[m-1]}_{m-1+N}(t), \quad \mathbb{P}^{[m]}_{m+N}(t) = \mathbf{J}^{-1}\mathbb{P}^{[m+1]}_{m+1+N}.$$

It is clear that many properties of the polynomials $\mathbb{P}^{[m]}_{m+N}$ follow from the properties of Legendre polynomials. This is the difference recurrence formula

$$\mathbb{P}^{[m]}_{m+N}(t) = \frac{\mathbb{P}^{[m-1]}_{m+N}(t) - \mathbb{P}^{[m-1]}_{m+N-2}(t)}{2N+1}, \ N \geq 1.$$

The following theorem gives an expansion of the polynomials $\mathbb{P}^{[m]}_{m+N}$ on the Legendre polynomials (see [2]).

Theorem 1. *1) If $N \leq m-1$, then*

$$\mathbb{P}^{[m]}_{m+N}(t) = \sum_{k=0}^{\left[\frac{m-N+1}{2}\right]-1} (-1)^{m-k} c^{[m]}_{N,k} P_{m-N-2k-1}(t) + \sum_{k=\left[\frac{m-N+1}{2}\right]}^{m} (-1)^{m-k} c^{[m]}_{N,k} P_{N-m+2k}(t).$$

2) If $N \geq m$, then $\mathbb{P}^{[m]}_{m+N}(t) = \sum_{k=0}^{m} (-1)^{m-k} c^{[m]}_{N,k} P_{N-m+2k}(t)$.

Here the coefficients $c^{[m]}_{N,k}$ are equal to

$$c^{[m]}_{N,k} = \frac{C^k_m (N-m+2k+1/2)\Gamma(N-m+k+1/2)}{2^m \Gamma(N+k+3/2)}, \ 0 \leq k \leq m. \tag{3}$$

The constructed generalized Zernike polynomials (1) can be used to solve the classical Dirichlet and Riquier boundary value problems for the m–polyharmonic equation in the unit ball, $\Delta^m u(\mathbf{x}) = f(\mathbf{x})$. The following theorems serve as the basis for solving these problems.

Theorem 2. *The m–Zernike polynomials $\{Z^{[m](N+2k)}_{N\ell}, \ k \geq m\}$ form a basis in $H^m_0(\mathbb{B}^3)$, on the boundary of $\partial\mathbb{B}^3$ normal derivatives up to the order of $m-1$ vanish*

$$Z^{[m](N+2k)}_{N\ell}(\boldsymbol{\xi}) = \frac{d}{dr} Z^{[m](N+2k)}_{N\ell}(\boldsymbol{\xi}) = \ldots = \frac{d^{m-1}}{dr^{m-1}} Z^{[m](N+2k)}_{N\ell}(\boldsymbol{\xi}) = 0.$$

Theorem 3. *The $2m$–Zernike polynomials $\{Z^{[2m](N+2k+2m)}_{N\ell}, \ 0 \leq k \leq m-1\}$ admit decomposition of the form*

$$Z^{[2m](N+2k+2m)}_{N\ell}(\mathbf{x}) = \underbrace{Q_{N+2m-2}(\mathbf{x})}_{\Delta^m = 0} + \underbrace{\sum_{s=m-k}^{m} (-1)^{m-s} c^{[m]}_{N+2k+1,s} Z^{[m](N+2k+2s)}_{N\ell}(\mathbf{x})}_{\Delta^m = Z^{(N+2k)}_{N\ell}},$$

where the coefficients are determined by the formula (3).

This expansion allows us to define corrected $2m$–Zernike polynomials

$$\widetilde{Z}_{N\ell}^{[2m](N+2k+2m)}(\mathbf{x}) := \sum_{s=m-k}^{m} (-1)^{m-k} c_{N,k}^{[m]} Z_{N\ell}^{[m](N+2k+2s)}(\mathbf{x}),\ 0 \le k \le m-1.$$

Theorem 4. *The corrected* $2m$*–Zernike polynomials* $\{\widetilde{Z}_{N\ell}^{[2m](N+2k+2m)},\ 0 \le k \le m-1\}$ *satisfy the homogeneous Dirichlet boundary value problem for an inhomogeneous* m*–polyharmonic equation*

$$\Delta^m u(\mathbf{x}) = Z_{N\ell}^{(N+2k)}(\mathbf{x}) \ \textit{in}\ \mathbb{B}^3, \tag{4}$$

$$u(\boldsymbol{\xi}) = \frac{d}{dr}u(\boldsymbol{\xi}) = \ldots = \frac{d^{m-1}}{dr^{m-1}}u(\boldsymbol{\xi}) = 0 \ \textit{on}\ \mathbb{S}^2 = \partial\mathbb{B}^3. \tag{5}$$

Theorem 5. *The* $2m$*–Zernike polynomials* $\{Z_{N\ell}^{[2m](N+2k+2m)},\ k \ge m\}$ *form an orthogonal basis of* $H_0^{2m}(\mathbb{B}^3)$ *and satisfy the Dirichlet boundary value problem* (4)–(5) *and also satisfy the Riquier boundary value problem*

$$\Delta^m u(\mathbf{x}) = Z_{N\ell}^{(N+2k)}(\mathbf{x}) \text{ in } \mathbb{B}^3,$$
$$u(\boldsymbol{\xi}) = \Delta u(\boldsymbol{\xi}) = \ldots = \Delta^{m-1} u(\boldsymbol{\xi}) = 0 \text{ on } \mathbb{S}^2 = \partial\mathbb{B}^3.$$

References

1. Derevtsov E. Yu., Kazantsev S. G., Schuster Th. *Polynomial bases for subspaces of vector fields in the unit ball. Method of ridge functions.* J. Inverse and Ill–Posed Problem 15:1 (2007). 19–55.

2. Kazantsev S. G. *Factorization of the Green's operator in the Dirichlet problem for* $(-1)^m(d/dt)^{2m}$ (in print).

Envelopes in function spaces with respect to convex cones or sets

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We discuss the existence of an envelope of a function from a certain subclass of function space. Here we restrict ourselves to considering the model space $L^1_{\text{loc}}(D)$ of functions locally integrable with respect to *Lebesgue measure* λ in an open connected subset, i. e., *domain* D from the *d-dimensional Euclidean space* $\mathbb{R}^d$, where $d \in \mathbb{N} = \{1, 2, \dots\}$, and $\mathbb{R}$ is the *real line*. Let H be a subset in $L^1_{\text{loc}}(D)$, and $F\colon D \to \mathbb{R}_{\pm\infty} := \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$ be a extended numerical function belonging to $L^1_{\text{loc}}(D)$. We say that there exists a *lower envelope for F with respect to H* if there is a function $h \in H$ such that $h \leq F$ on D. Denote by $\operatorname{sbh}(D)$, $C(D)$, and $C^k(D)$ with $k \in \mathbb{N} \cup \{\infty\}$ the classes of *subharmonic, continuous,* and *k times continuously differentiable functions* on D, respectively [1]. The class $\operatorname{sbh}(D)$ contains the minus-infinity function $\boldsymbol{-\infty}\colon x \mapsto -\infty$ identically equal to $-\infty$; $\operatorname{sbh}_*(D) := \operatorname{sbh}(D) \setminus \{\boldsymbol{-\infty}\}$.

The *Alexandroff compactification* of $\mathbb{R}^d$ is denoted by $\mathbb{R}^d_\infty := \mathbb{R}^d \cup \{\infty\}$. Given a subset S of $\mathbb{R}^d_\infty$, the *closure* $\operatorname{clos} S$ and the *interior* $\operatorname{int} S$ will always be taken relative $\mathbb{R}^d_\infty$. For $S' \subset S \subset \mathbb{R}^d_\infty$ we write $S' \Subset S$ if $\operatorname{clos} S' \subset \operatorname{int} S$.

Let $\operatorname{Borel}(S)$ be the *class of all Borel subsets in* $S \in \operatorname{Borel}(\mathbb{R}^d_\infty)$. We denote by $\operatorname{Meas}(S)$ the *class of all Borel signed measures on* $\operatorname{Borel}(S)$; $\operatorname{Meas}_{\text{cmp}}(S)$ is the class of measures $\mu \in \operatorname{Meas}(S)$ with a compact *support* $\operatorname{supp}\mu \Subset S$; $\operatorname{Meas}^+(S) := \{\mu \in \operatorname{Meas}(S)\colon \mu \geq 0\}$, $\operatorname{Meas}^+_{\text{cmp}}(S) := \operatorname{Meas}_{\text{cmp}}(S) \cap \operatorname{Meas}^+(S)$.

Definition (of linear and affine balayage of measures [2, Definition 7.1; 3, 4]). Let $H \subset \operatorname{sbh}(D)$, $\nu \in \operatorname{Meas}^+(D)$, and $\mathcal{M} \subset \operatorname{Meas}^+(D)$. We say that a measure $\mu \in \mathcal{M}$ is *a linear balayage of the measure ν with respect to H in $\mathcal{M}$* and write $\nu \preceq_{H,\mathcal{M}} \mu$ if

$$\int_D h\,\mathrm{d}\nu \leqslant \int_D h\,\mathrm{d}\mu \quad \textit{for all functions } h \in H.$$

Let $c \in \mathbb{R}$, and $\mathbf{1}\colon x \mapsto 1$ be the function identically equal to 1 on $D \ni x$. We say that an affine function $\mu + c := \mu + c \cdot \mathbf{1} \in \mathcal{M} + \mathbb{R}\mathbf{1}$ is *an affine balayage of the measure ν with respect to H in $\mathcal{M} + \mathbb{R}\mathbf{1}$*, and write $\nu \curlyeqprec_{H,\mathcal{M}} \mu + c$, if

$$\int h\,\mathrm{d}\nu \leqslant \int h\,\mathrm{d}\mu + c \quad \textit{for all functions } h \in H.$$

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Proposition [2, Proposition 8.1; 4, Proposition 7.1]. *Let* $\varnothing \neq H \subset \operatorname{sbh}_*(D)$, $\mathcal{M} \subset \operatorname{Meas}^+_{\mathrm{cmp}}(D)$, *and* $0 \neq \nu \in \operatorname{Meas}^+_{\mathrm{cmp}}(D)$ *be such that* $-\infty < \int_D h \,\mathrm{d}\nu$ *for all* $h \in H$. *Let the restriction of* F *to each compact subset* $K \Subset D$ *is* $\mu\bigm|_K$*-measurable for every measure* $\mu \in \mathcal{M}$, *where* $\mu\bigm|_K$ *is the restriction of* μ *to* K. *If there exists a lower envelope* $h \leq F$ *on* D *for* F *with respect to* $H \ni h$, *then*

$$-\infty < \inf \left\{ \int_D F \,\mathrm{d}\mu \colon \nu \preceq_{H,\mathcal{M}} \mu \right\}, \tag{2lin}$$

$$-\infty < \inf \left\{ \int_D F \,\mathrm{d}\mu + c \colon \nu \preccurlyeq_{H,\mathcal{M}} \mu + c \right\}. \tag{2aff}$$

This Proposition is somewhat reversible if H is convex. For simplicity and brevity, we formulate such an almost inverse statement only for $F \in C(D)$.

Theorem (general case in [2, Theorem 6], special case in [4, Theorem 7.1]). *Let* $F \in C(D)$, $\mathbb{R}\mathbf{1} \subset H \subset \operatorname{sbh}_*(D)$, $0 \neq \nu \in \operatorname{Meas}^+_{\mathrm{cmp}}(D)$, $\operatorname{supp}\nu \subset U_0 \Subset D$, *where* U_0 *is a domain*, $\mathcal{M} := \bigl\{\mu \in \operatorname{Meas}^+_{\mathrm{cmp}}(D \setminus U_0) \colon \mathrm{d}\mu = m \,\mathrm{d}\lambda, \text{ where } m \in C^\infty(D)\bigr\}$. *Suppose that one of the following two conditions is fulfilled:*

[H1] *for any locally bounded from above sequence of functions* $(h_k)_{k\in\mathbb{N}} \subset H$, *the upper semi-continuous regularization of the upper limit* $\limsup_{k\to\infty} h_k$ *belong to* H *provided that* $\limsup_{k\to\infty} h_k \not\equiv -\infty$ *on* D;

[H2] H *is sequentially closed in* $L^1_{\mathrm{loc}}(D)$.

[L] *If* H *is convex cone, and the condition* (2lin) *is fulfilled, then there exists a lower envelope* $h \leq F$ *on* D *for* F *with respect to* $H \ni h$.

[A] *If* H *is convex set, and the condition* (2aff) *is fulfilled, then there exists a lower envelope* $h \leq F$ *on* D *for* F *with respect to* $H \ni h$.

In our review [2], this Theorem is proved in a much more general form for arbitrary functions $F \in L^1_{\mathrm{loc}}(D)$ without condition $\mathbb{R}\mathbf{1} \in H$. Special cases of this Theorems and corollaries from it have been successfully applied in our articles [5–10] to study the distribution of zero sets of holomorphic functions under restrictions on their growth. Research on the application of this Theorem in complex analysis will be continued. More general abstract forms of our Theorem from [2] and [3] can find applications in other functional spaces far from the space $L^1_{\mathrm{loc}}(D)$ since they are given for projective limits of vector lattices or topological projective limits of Frechet lattices (see [2, Ch. 1; 3] and bibliography in them).

References

1. Hayman W. K., Kennedy P. B. *Subharmonic functions*, vol. **1**. London etc.: Acad. Press, 1976.

2. Khabibullin B. N., Rozit A. P., Khabibullina E. B., *Order versions of the Hahn–Banach theorem and envelopes. II. Applications to the function theory*, Complex Analysis. Mathematical Physics, Itogi Nauki i Tekhniki. Ser. Sovrem. Mat. Pril. Temat. Obz., **162**, VINITI, Moscow, 2019, 93–135 (in Russian); translated in English and are published by Springer Verlag in Journal of Mathematical Sciences.
3. Khabibullin B. N. *Dual representation of superlinear functionals and its applications in function theory. I*, Izv. RAN. Ser. Mat., **65**:4 (2001), 205–224; Izv. Math., 65:4 (2001), 835–852.
4. Khabibullin B. N. *Dual representation of superlinear functionals and its applications in function theory. II*, Izv. RAN. Ser. Mat., 65:5 (2001), 167–190; Izv. Math., **65**:5 (2001), 1017–1039.
5. Khabibullin B. N., Tamindarova N. R. *Uniqueness Theorems for Subharmonic and Holomorphic Functions of Several Variables on a Domain*, Azerbaijan J. Math. **7**:1 (2017), 70–79.
6. Khabibullin B. N., Tamindarova N. R. *Subharmonic test functions and the distribution of zero sets of holomorphic functions*, Lobachevskii J. Math. **38**:1 (2017), 38–43.
7. Khabibullin B. N., Abdullina Z. F., Rozit A. P. *A uniqueness theorem and subharmonic test functions*, Algebra i Analiz, **30**:2 (2018), 318–334; St. Petersburg Math. Journal, **30**:2 (2019)
8. Khabibullin B. N., Rozit A. P. *On the Distribution of Zero Sets of Holomorphic Functions*, Funktsional. Anal. i Prilozhen. **52**:1 (2018), 26–42; Funct. Anal. Appl. **52**:1 (2018), 21–34.
9. Menshikova E. B., Khabibullin B. N. *On the Distribution of Zero Sets of Holomorphic Functions. II*, Funktsional. Anal. i Prilozhen. **53**:1 (2019), 84–87; Funct. Anal. Appl. **53**:1 (2019), 65–68.
10. Khabibullin B. N., Khabibullin F. B. *On the Distribution of Zero Sets of Holomorphic Functions. III. Inversion Theorems*, Funktsional. Anal. i Prilozhen. **53**:2 (2019), 42–58; Funct. Anal. Appl. **53**:2 (2019), 110–123.

Einstein equation on metric Lie groups with vectorial torsion

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Let (M, g) be a (pseudo)Riemannian manifold admitting a metric connection ∇ of the form

$$\nabla_X Y = \nabla^g_X Y + g(X, Y)V - g(V, Y)X,$$

where V is a fixed vector field, X, Y are arbitrary vector fields, ∇^g is the Levi-Civita connection on M. Connection ∇ is one of the three basic connections, described by E. Cartan in [1], and is called a metric connection with vectorial torsion or semi-symmetric connection (up to direction).

Manifolds of constant Ricci curvature or Einstein manifolds are sufficiently known in the case of (pseudo)Riemannian manifolds with a Levi-Civita connection (see for example [2]). They admit several generalizations to the case of manifolds with metric connection with vectorial torsion [3].

Next we will use the following

Definition. A (pseudo)Riemannian manifold (M, g) with metric connection with vectorial torsion will be called a *Einstein manifold* if the Ricci tensor satisfies the Einstein equation

$$r = \Lambda g$$

for some function Λ.

The main result of this paper is

Theorem 1. *Let G be a three or four-dimensional Lie group with a left-invariant Riemannian metric and metric connection ∇ with left-invariant vectorial torsion. If the Einstein equation is satisfied, then either the vector field V is trivial or the Ricci tensor of the connection ∇ is zero.*

References

1. Cartan E. *Sur les variétés à connexion affine et la théorie de la relativité généralisée (deuxième partie)*, Ann. Ecole Norm. Sup. 42 (1925), 17–88.
2. Besse A. *Einstein Manifolds*, Springer-Verlag Berlin Heidelberg, 1987.
3. Chaturvedi B. B., Gupta B. K. *Study on Semi-symmetric Metric Spaces*, Novi Sad J. Math. 44:2 (2014), 183–194.

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Three-dimensional locally homogeneous manifolds with vectorial torsion and zero curvature tensor

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Let (M, g) be a (pseudo)Riemannian manifold. One can define a metric connection ∇ on M by the formula

$$\nabla_X Y = \nabla^g_X Y + g\,(X, Y)\,V - g\,(V, Y)\,X,$$

where V is a fixed vector field, X, Y are arbitrary vector fields, ∇^g is the Levi-Civita connection on M. Connection ∇ is one of the three basic connections, described by E. Cartan in [1], and is called a metric connection with vectorial torsion or semi-symmetric connection (up to direction).

The curvature tensor of a metric connection ∇ with vectorial torsion is determined by the equality

$$R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z.$$

K. Yano in [2] proved that a Riemannian manifold admits a metric connection with vectorial torsion, the curvature tensor of which is zero, if and only if it is conformally flat.

This paper is devoted to obtaining a classification of three-dimensional metric Lie groups with a zero curvature tensor, the vectorial torsion of which is generated by some left-invariant vector field.

References

1. Cartan E. *Sur les variétés à connexion affine et la théorie de la relativité généralisée (deuxième partie)*, Ann. Ecole Norm. Sup. 42 (1925), 17–88.
2. Yano K. *On semi-symmetric metric connection*, Revue Roumame de Math. Pure et Appliquees 15 (1970), 1579–1586.

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Interpolation of Hardy-type spaces and of their intersections

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A short survey of interpolation results for the scale $H^p(\mathbb{T})$, $0 < p \leq \infty$, of Hardy classes on the unit circle will be given. Also, some quite recent interpolation theorem for intersections of "one-parameter" Hardy-type spaces will be exposed. This theorem covers the case of Herdy classes on the two-dimensional torus, the case of co-invariant subspaces for the shift operator on the unit circle, as well as a number of nonclassical situations.

Conformal deformations and (pseudo)Riemannian manifolds with vectorial torsion

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The necessary and sufficient conditions for the (pseudo)Riemannian manifold to be conformally flat are well known as the **Weyl–Schouten theorem**:

Theorem 1. *(Pseudo)Riemannian manifold of dimension n with $n \geq 3$ is conformally flat if and only if the Cotton tensor vanishes for $n = 3$, or the Weyl tensor vanishes for $n > 3$.*

On the other hand, these conditions can be viewed using the metric connection ∇ with vectorial torsion [1], which is determined by the equality

$$\nabla_X Y = \nabla^g_X Y + g\,(X, Y)\,V - g\,(V, Y)\,X,$$

where V is a fixed vector field, X, Y are arbitrary vector fields, ∇^g is the Levi-Civita connection on M, g is a metric tensor.

Yano proved another criterion that a manifold is conformally flat [2]:

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Theorem 2. *(Pseudo)Riemannian manifold admits a metric connection with vectorial torsion, whose curvature tensor is zero, iff it is conformally flat with respect to the Levi-Civita connection.*

This paper is devoted to the study of (pseudo) Riemannian manifolds which are conformally flat with respect to a metric connection ∇ with vector torsion, i.e. for which there is a conformal deformation of the metric such that the new metric tensor has a zero curvature tensor with respect to the connection ∇.

References

1. Cartan E. *Sur les variétés à connexion affine et la théorie de la relativité généralisée (deuxième partie)*, Ann. Ecole Norm. Sup. 42 (1925), 17–88.
2. Yano K. *On semi-symmetric metric connection*, Revue Roumame de Math. Pure et Appliquees 15 (1970), 1579–1586.

Example of a smooth homeomorphism violating the Lusin $\mathcal{N}^{-1}$ condition

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In the context of Solid Mechanics, the fact that the material can neither be created nor lost during the deformation of a solid body is described by the two mathematical conditions that preserve null sets — Lusin $\mathcal{N}$ and $\mathcal{N}^{-1}$ conditions. A function $f\colon \Omega \to \mathbb{R}^N$, $\Omega \subset \mathbb{R}^N$, is said to satisfy the *Lusin $\mathcal{N}$ condition* if, for every $E \subset \Omega$, we have

$$\mathcal{L}_N(E) = 0 \implies \mathcal{L}_N\left(f(E)\right) = 0.$$

Analogously, f fulfils the *Lusin $\mathcal{N}^{-1}$ condition* if, for every $E \subset \Omega$, we have

$$\mathcal{L}_N\left(f(E)\right) = 0 \implies \mathcal{L}_N(E) = 0.$$

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Here, $\mathcal{L}_N$ denotes the Lebesgue measure on $\mathbb{R}^N$.

Of special interest is the question about the sufficient conditions for mappings to possess Lusin's $\mathcal{N}$ and $\mathcal{N}^{-1}$ properties. In 1966 Reshetnyak [6] proved that the $\mathcal{N}$ condition holds for $W^{1,N}$-homeomorphisms. Later, in 1973, the general case of Sobolev $W^{1,p}$-mappings, for $p > N$, was obtained by Marcus and Mizel [3]. The first crucial counterexample of a Sobolev mapping violating Lusin's conditions was described by Cesari [1] in 1944, and was later improved by Malý and Martio [2] in 1995. They present a continuous mapping in the Sobolev space $W^{1,p}([0,1]^N, [0,1]^M)$, $M \in \mathbb{N}$, $p \leq N$, not satisfying the $\mathcal{N}$ condition. The second counterexample came in 1971 in the form of Ponomarev's celebrated construction [4,5]. It provides a Sobolev homeomorphism in $W^{1,p}([0,1]^N, [0,1]^N)$, for $p < N$, that violates the $\mathcal{N}$ condition.

Once Lusin's conditions are fully described in the setting of Sobolev spaces, it is natural to ask whether other analytic requirements could guarantee Lusin's properties. In [7], the question, if the Lusin $\mathcal{N}$ condition can be guaranteed by the C^∞-regularity of a mapping, was posed, and the answer was negative. In this work, we ask the same question regarding the Lusin $\mathcal{N}^{-1}$ condition. The answer is also negative, and we provide the following counterexample.

Example. *There exists a homeomorphism $g \in C^\infty(\mathbb{R}^N, \mathbb{R}^N)$ violating the Lusin $\mathcal{N}^{-1}$ condition such that $g(x) = x$ outside $(0,1)^N$.*

References

1. Cesari L. *Sulle trasformazioni continue*, Ann. Mat. Pura Appl. 21(4) (1942), 157–188.
2. Malý J., Martio O. *Lusin's condition (N) and mappings of the class $W^{1,n}$*, J. Reine Angew. Math. 458 (1995), 19–36.
3. Marcus M., Mizel V. J. *Transformations by functions in Sobolev spaces and lower semicontinuity for parametric variational problems*, Bull. Amer. Math. Soc. 79 (1973), 790–795.
4. Ponomarev S. P. *An example of an* ACTL^p *homeomorphism that is not absolutely continuous in the sense of Banach*, Dokl. Akad. Nauk SSSR 201 (1971), 1053–1054.
5. Ponomarev S. P. *On the N-property of homeomorphisms of the class W_p^1*, Sib. Math. J. 28:2 (1987), 140–148.
6. Reshetnyak Yu. G. *Certain geometric properties of functions and mappings with generalized derivatives*, Sib. Math. J. 7 (1966), 886–919.
7. Roskovec T. *Higher order Sobolev homeomorphisms and mappings and the Lusin (N) condition*, Rev. Mat. Complut. 31:2 (2018), 379–406.

Estimation of the error of calculating the functional containing higher-order derivatives on triangular grid

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We consider folowing functional

$$I(f) = \int\limits_{\Omega} G(x, f, \nabla f, D^2 f)dx, \;\; D^2 f = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j=1}^{n}. \tag{1}$$

It is proved that a functional (1) can be calculated with an error of order $O(h^{4m+1})$ with a triangular mesh of $h \to 0$ if the piecewise polynomial functions of degree $4m+1$ for $m \geq 1$. For $n = 2$ we give an example of the fact that the piecewise quadratic approximation gives the second order of accuracy for calculating the functional

$$I(u) = \iint\limits_{\Omega} (u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2)dxdy$$

for a special kind of triangulation. Consider the points $x_i = \frac{i}{n}$, $y_j = \frac{j}{n}$, $i, j = 0, \ldots, n$. Put $h = \frac{1}{n}$. We introduce the notation $Q_{ij} = (x_i, y_j)$. In each triangle $T_{ij}^1 = \triangle\, Q_{ij} Q_{i+1j} Q_{ij+1}$ replace the function $u(x, y)$ with the quadratic $P_1(x, y) = ax^2 + bxy + cy^2 + dx + ey + f$. Similarly, in each triangle $T_{ij}^2 = \triangle\, Q_{i+1j} Q_{ij+1} Q_{i+1j+1}$ replace the function $u(x, y)$ with the quadratic $P_2(x, y) = ax^2 + bxy + cy^2 + dx + ey + f$. Polynomials are constructed about the values of the function u at the vertices of the triangles and the midpoints of their sides. Then the estimate

$$\iint\limits_{\Omega} (u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2)dxdy - \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \iint\limits_{T_{ij}^1} (((P_1)_{xx})^2 + 2((P_1)_{xy})^2 + ((P_1)_{yy})^2)dxdy -$$

$$- \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \iint\limits_{T_{ij}^2} (((P_2)_{xx})^2 + 2((P_2)_{xy})^2 + ((P_2)_{yy})^2)dxdy = O(h^2)$$

for $h \to 0$.

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On existence of triangulations which are not affine-like to Delaunay triangulation

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Let P be arbitrary finite set of plane $\mathbb{R}^2$. Triangulation T of P is called affine-like to Delaunay triangulation if there exists affine map $L : \mathbb{R}^2 \to \mathbb{R}^2$ such that the triangulation $L(T)$ of set $L(P)$ is Delaunay triangulation. We discuss the question: is there triangulation which is not affine-like to Delaunay triangulation. Note that this question is connected with problems of structure of the set all triangulations of finite set on the plane [1–3].

Consider some triangle $\Delta ABC \subset \mathbb{R}^2$. Let S be it's circumcircle. Now we have three ball segments bounded by sides of triangle ΔABC and corresponding arcs of S. Choose the points D, E, F in each such segments and construct three new triangles ΔABD, ΔBCE, ΔCAF. We prove that the given triangulation of points A, B, C, D, E, F is not affine-like Delaunay triangulation.

References

1. Ajtai M., Chvátal V., Newborn M. M., Szemerédi E. *Crossing-free subgraphs*, Annals Discrete Math. 12 (1982), 9–12.
2. Garcia A., Noy M., Tejel J. *Lower bounds on the number of crossing-free subgraphs of K N*, Comput. Geom. Theory Appl. 16 (2000), 211–221.
3. Hurtado F., Noy M. *Counting triangulations of almost-convex polygons*, Ars Combinatorica 45 (1997), 169–179.

Conformal mapping from the half-plane onto circular-arc polygon with boundary normalization

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Let $\Gamma(t) = \{w : w = \varphi(\tau), t_0 \leqslant \tau \leqslant t\}$ be a piecewise smooth curve consisting of $(n+1)/2$ (n is an odd number) circular arcs $\Gamma_*(t) = \{w : w = \varphi(\tau), t_{\frac{n-1}{2}} \leqslant$

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$\tau \leqslant t\}$, $\Gamma_k = \{w : w = \varphi(\tau), t_{k-1} \leqslant \tau \leqslant t_k\}$, $k = 1, \ldots, \frac{n-1}{2}$; $w = \varphi(\tau)$ is a path, $0 \leqslant t_0 \leqslant \tau \leqslant t \leqslant t_{\frac{n+1}{2}}$. Denote by $A_0 = \varphi(t)$, $A_1, \ldots, A_n$ the vertices of polygon $\Delta(t) = \overline{\mathbb{C}}\backslash\Gamma(t)$, going along the boundary of $\Delta(t)$ in the positive direction, $\varphi(t_k) = A_{\frac{n+1}{2}\pm k}$, $k = 0, \ldots, \frac{n-1}{2}$.

There is a family of functions $w = f(z,t)$, that maps the upper half-plane $\Pi^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ onto family of polygons $\Delta(t)$, $f : \Pi^+ \to \Delta(t)$, $t_2 \leqslant t \leqslant t_{\frac{n+1}{2}}$. Let the family $f(z,t)$ satisfy the boundary normalization $f(0,t) = w_0$, $f(1,t) = w_1$, $f(\infty,t) = w_\infty$, $t_2 \leqslant t \leqslant t_{\frac{n+1}{2}}$, $w_0, w_1, w_\infty \in \partial\Delta(t_2)$. We assume that points w_0, w_1, w_∞ do not coincide with vertices $\Delta(t_2)$. Denote by a_k, $a_k = a_k(t)$, $k = 0, 1, \ldots, n$, $a_0 = \lambda(t)$ the preimages of the vertices A_k and let the angles at these vertices equal $\alpha_k\pi$, $k = 0, 1, \ldots, n$. Let $0 < \alpha_k < 2$, $k = 1, \ldots, \frac{n-1}{2}$. We note that $\alpha_0 = \alpha_{\frac{n+1}{2}} = 2$, $\alpha_k = 2 - \alpha_{n+1-k}$, $k = 1, \ldots, \frac{n-1}{2}$.

A family of mappings $w = f(z,t)$ satisfies the Schwarz differential equation

$$\frac{f'''(z,t)}{f'(z,t)} - \frac{3}{2}\left(\frac{f''(z,t)}{f'(z,t)}\right)^2 = \sum_{k=0}^{n}\left(\frac{L_k}{(z-\widetilde{a}_k(t))^2} + \frac{\widetilde{M}_k(t)}{z-\widetilde{a}_k(t)}\right),$$

where $L_k = \frac{1}{2}(1-\alpha_k^2)$, M_k are the accessory parameters.

This paper solves the problem of finding the parameters $a_k = a_k(t)$, $M_k = M_k(t)$, $k = 0, 1, \ldots, n$, of a family $f = w(z,t)$ with boundary normalization. The works [1, 2] solve this problem for the case when family has hybrid normalization. The method was proposed in the work [3].

Suppose that if t tends to $t_{\frac{n+1}{2}}$, the endpoint $\varphi(t)$ of the curve $\Gamma = \Gamma(t)$ achieve some point of the curve Γ (for example the point $\varphi(t_0)$). Then family $\Delta(t)$ has two kernels, denote by Δ the one which is bounded, another one $\Delta_\infty = \overline{\mathbb{C}}\backslash\big(\Delta \bigcup \Gamma(t_{\frac{n+1}{2}})\big)$ is a kernel relative to infinity. If the points w_0, w_1, w_∞ belonging to the boundary of $\Delta(t)$ are also boundary points of polygon Δ (Δ_∞), then by Caratheodory kernel theorem the family of mappings $w = f(z,t)$ converges to the map f from the upper half-plane onto polygon Δ (correspondingly polygon Δ_∞). Thus by solving the problem of finding parameters of family $w = f(z,t)$, one can find a map from half-plane onto arbitrary polygon.

Theorem. *For all $t_{(n-1)/2} < t < t_{(n+1)/2}$ the parameters $a_k(t)$, $M_k(t)$, $k = 0, 1, \ldots, n$, $a_0(t) = \lambda(t)$, of the mapping f from the upper half-plane Π^+ onto* *closed plane with excluded curve $\Gamma(t)$ satisfy the system of differential equations*

$$\begin{cases} \dot{a}_k(t) = \dfrac{a_k(t)(a_k(t)-1)}{a_k(t)-\lambda(t)}, \quad k=1,\dots,n, \\ \dot{\lambda}(t) = 2\lambda(t) - \lambda(t)(\lambda(t)-1)M_0(t) - 1, \\ \dot{M}_k(t) + M_k(t) + 2L_k \dfrac{\lambda(t)(\lambda(t)-1)}{\big(a_k(t)-\lambda(t)\big)^3} - M_k(t)\dfrac{\lambda(t)(\lambda(t)-1)}{\big(a_k(t)-\lambda(t)\big)^2} = 0,\ k=1,\dots,n. \end{cases}$$

If $\alpha_1, \alpha_n \neq 0, 2$ the system has a unique solution with respect to $x = \sqrt{t - t_{n-1/2}}$

$$\begin{aligned} a_0\big(t(x)\big) &= \widetilde{a}_0(x) = \widetilde{\lambda}(x) = \sigma + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 + \dots, \\ a\big(t(x)\big) &= \widetilde{a}_k(x) = a_{k0} + a_{k1}x + a_{k2}x^2 + a_{k3}x^3 + \dots, && k=1,\dots,n, \\ M_k\big(t(x)\big) &= \widetilde{M}_k(x) = m_{k0} + m_{k1}x + m_{k2}x^2 + m_{k3}x^3 + \dots, && k=2,\dots,n-1, \\ M_k\big(t(x)\big) &= \widetilde{M}_k(x) = \frac{m_{k,-1}}{x} + m_{k0} + m_{k1}x + m_{k2}x^2 + \dots, && k=0,1,n, \end{aligned}$$

on an interval $[t_{(n-1)/2}, t_{(n+1)/2}]$, satisfying the initial conditions

$$a_{11} = \sqrt{q\frac{\alpha_n}{\alpha_1}}, \quad a_{n1} = -\sqrt{q\frac{\alpha_1}{\alpha_n}}, \quad \lambda_1 = a_{11} + a_{n1}, \quad a_{k1} = 0, \quad k=2,\dots,n-1,$$

$$m_{0,-1} = -\frac{\lambda_1}{q},\ m_{k,-1} = 2L_k \frac{a_{k1}^3}{q(a_{k1}^2+q)},\ k=1,n,\ m_{k1}=0,\ k=2,\dots,n-1,$$

$$\lambda_2 = (2\sigma-1)\left(\frac{\lambda_1^2}{q}+1\right) + \frac{q}{2}\sum_{k=1}^{n} m_{k0},$$

where $q = 2\sigma(\sigma-1)$, and parameter λ_3, relating to the curvature of the movable arc of the curve $\Gamma(t)$, satisfy the condition

$$|f(z,\lambda_3,t) - \omega| = r, \quad t_{\frac{n-1}{2}} \leqslant t \leqslant t_{\frac{n+1}{2}},$$

where ω is the center of circle Γ_*, r is the radius of this circle, f is a desired mapping.

References

1. Baibarin B. G. *On a numerical method for determining the parameters of the Schwarz derivative for functions that map the half-plane to circular domains*, Proc. of Tomsk Univ. **189** (1966), 123–136.
2. Kolesnikov I. A. *On the problem of determining parameters in the Schwarz equation*, Issues Anal. **7**(25):2 (2018), 50–62.
3. Kufarev P. P. *On a numerical method for determining parameters in the Schwarz-Christoffel integral*, Dokl. Akad. Nauk SSSR **57**:6 (1947), 535–537.

Unique determination of conformal type for domains

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The main goal of the lecture is to present a new, apparently, very interesting, and yet very difficult trend in the classical geometric topic of the unique determination of convex surfaces by their intrinsic metrics. It consists of three parts. In the first part, we consider results on the problem of unique determination for (generally speaking) nonconvex domains in $\mathbb{R}^n$. One of these results is as follows: If U is a (generally speaking) nonconvex bounded polyhedral domain in $\mathbb{R}^n$ $(n \geq 4)$ whose boundary is an $(n-1)$-dimensional connected manifold of class C^0 without boundary and is representable as a finite union of pairwise nonoverlapping $(n-1)$-dimensional cells is uniquely determined by the relative conformal moduli of its boundary condensers. Results on the unique determination (of polyhedral domains) of isometric type are also obtained. In contrast to the classical case, these results present a new approach in which the notion of the p-modulus of path families is used.

The second part is devoted to the study of the problems of the unique determination of convex polyhedral domains in the three-dimensional Euclidean space by the relative conformal moduli of boundary condensers. The main result of this part is that any convex bounded polyhedral domain in the three-dimensional Euclidean space is uniquely determined by the relative conformal moduli of its boundary condensers.

Finally, in the third part, the main result is that, for $n \leq 3$, any n-connected plane domain U is uniquely determined by the relative conformal moduli of pairs of boundary components.

Some calculations of Orlicz cohomology

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We carry out calculations of Orlicz cohomology for some basic Riemannian manifolds (the real line, the hyperbolic plane, the ball). We also discuss how Orlicz cohomology is related to Poincaré–Sobolev–Orlicz type inequalities.

This is a joint work with Prof. V. Gol′dshtein.

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Subtwistor bundle over four-dimensional sphere

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Let M be a smooth manifold of arbitrary dimension $n \geq 3$ and Ω be a skew-symmetric 2-form on M. A radical of 2-form Ω at the point $x \in M$ is a subspace $\operatorname{rad}\Omega_x$ of the tangent space T_xM formed as

$$\operatorname{rad}\Omega_x = \{v \in T_xM : \Omega_x(v, \cdot) = 0\},$$

where $\Omega(v, \cdot)$ denotes that the second vector argument can be an arbitrary vector from T_xM. Note that a radical can also be well-defined for any bilinear form at the point x. It is easy to see that at the point x there is always a tangent subspace D_x such that the restriction of 2-form Ω_x to D_x is a non-degenerate 2-form, and $\dim D_x$ is always an even number for any n. This tangent subspace is called the work subspace.

An affinor associated with 2-form Ω_x at the point $x \in M$ is an endomorphism Φ_x of the tangent space T_xM that satisfies the following conditions:

1) $\ker \Phi_x = \operatorname{rad}\Omega_x$;

2) $\Phi_x^2|_{D_x} = -\mathrm{id}$, where id is an identity operator in D_x;

3) $\Omega_x \circ \Phi_x = \Omega_x$;

4) For all $v \in T_xM$ we have $\Omega_x(v, \Phi_x v) \geq 0$.

The properties of affinor Φ_x imply that the bilinear form

$$\Omega_x(v, \Phi_x w), \quad v, w \in T_xM,$$

is the inner product in the Work Subspace D_x and is equal to zero when either v or w belongs to $\operatorname{rad}\Omega_x$.

A radical metric for 2-form Ω_x at the point $x \in M$ is a symmetric 2-form β_x such that $\operatorname{rad}\beta_x = D_x$ and the restriction of β_x to $\operatorname{rad}\Omega_x$ is an inner product in $\operatorname{rad}\Omega_x$.

In the work [1] we define the concept of subtwistor structure on a manifold of arbitrary dimension. We define the subtwistor structure with the radical of rank r at the point $x \in M$ as the quadruple $(\Omega_x, D_x, \Phi_x, \beta_x)$ where Ω_x is a skew-symmetric 2-form with the radical of dimension r at the point x, D_x is a fixed work subspace for Ω_x, Φ_x is an affinor associated with Ω_x, and β_x is a radical metric for Ω_x. This subtwistor structure induces the inner product

$$(v, w)_x = \Omega_x(v, \Phi_x w) + \beta(v, w)$$

in T_xM.

A subtwistor bundle with radical of rank r over manifold M is a bundle P over M with the projection $\pi : P \mapsto M$ such that for any point $x \in M$ the fibre $\pi^{-1}(x)$ is a variety of all subtwistor structures with the radical of rank r at the point x as defined above.

We describe the subtwistor bundle over four-dimensional sphere S^4 and show that S^4 admits only subtwistor bundle with the radical of rank 0 or 2. A subtwistor bundle with the radical of rank 0 over S^4 is a two-leaf covering of a classical subtwistor bundle over S^4 and is isomorphic to $\mathbb{H}P^1 \times \mathbb{C}P^3 \times \mathbb{Z}_2$, where $\mathbb{H}P^n$ is a quaternion projective space of dimension n, $\mathbb{C}P^n$ is a complex projective space of dimension n, and $\mathbb{Z}_2$ is the discrete group $\{1, -1\}$. A subtwistor bundle with the radical of rank 2 over S^4 is a manifold isomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^2 \times \mathrm{SM}(2)$, where $\mathrm{SM}(2)$ is a variety of symmetric non-degenerate positive-defined matrices 2×2. Additionaly, the structure of the subtwistor bundle with the radical of rank 0 over S^4 allows to prove that S^4 does not admit an almost complex structure as well as a non-degenerate skew-symmetric 2-form.

References

1. Kornev E. S. *Subcomplex And Sub-kähler Structures*, Siberian Math. J. 57:5 (2016), 830–840.

The generalized polar transform conformally-flat metrics of positive one-dimensional curvature

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Let R be a real number line, R^{n+1} be Euclidean $(n+1)$-dimensional arithmetic space, $M^{n+2} = R^{n+1} \times R$ be a pseudo-Euclidean space, scalar square of vector $w = [x, \zeta] \in M^{n+2}$ in which is $\langle w \rangle^2 = |x|^2 - \zeta^2$, where $|x|^2$ – scalar square of vector $x \in R^{n+1}$. Denote by

$$C^+ = \left\{ [x, \zeta] \in M^{n+2} : |x|^2 - \zeta^2 = 0,\ \zeta > 0 \right\},$$

the upper part of an isotropic cone in M^{n+2}.

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Lemma 1. *Let on a unit sphere $S^n \subset \mathbb{R}^{n+1}$ the conformal-plane metric be given*

$$ds^2 = \frac{dx^2}{f^2(x)}, \quad x \in S^n \subseteq \mathbb{R}^{n+1},$$

where $f(x)$ is a function of class C^1. Then the canonical isometric embedding specified by the formula is defined

$$Z : x \in S^n \to \left[\frac{x}{f(x)}, \frac{1}{f(x)}\right] \in C^+. \tag{1}$$

Image $Z(S^n) = F \subseteq C^+$ is a space-like n-dimensional surface. It identifies conformally-flat metric on the surface F. Suppose that the function $f(x)$ is smooth enough, then the surface F is regular, and at each point $Z(x) \in F$ is defined tangent n-dimensional space $T_x(F)$. There is a single vector $Z^*(x) \in C^+$ such that

$$\langle Z, Z^* \rangle = -1, \quad Z^* \perp T_x(F),$$

where orthogonality is understood with respect to the scalar products in M^{n+2}.

Lemma 2. *Let the function $f(x)$, which gives a conformally flat metric, by homogeneity extended to all space $\mathbb{R}^{n+1}$. Then the vector Z^* is explicitly expressed in f and ∇f in R^{n+1}:*

$$Z^*(x) = \left[-\nabla f + \frac{|\nabla f|^2}{2f} x, \frac{|\nabla f|^2}{2f}\right], \tag{2}$$

where $x \in S^n \subset \mathbb{R}^{n+1}$, ∇f gradient functions f in space $\mathbb{R}^{n+1}$.

Definition 1. If the point $Z \in F$ runs through the surface F, then the point Z^* runs through the dual surface F^*. The corresponding conformal-flat the metric $ds^{*2} = \frac{dy^2}{f^{*2}(y)}$, $y \in S^n$, is called **polar** to the original metric [1–7]. Comparing formulas (1) and (2) we have:

$$\left[-\nabla f + \frac{|\nabla f|^2}{2f} x, \frac{|\nabla f|^2}{2f}\right] \equiv \left[\frac{y}{f^*(y)}, \frac{1}{f^*(y)}\right].$$

From where we get the formulas for the transition to the polar conformal flat metric:

$$f^*(y) = \frac{2f(x)}{|\nabla f|^2}, \quad y = x - 2f(x)\frac{\nabla f}{|\nabla f|^2}. \tag{3}$$

Lemma 3. *Let $f : R^{n+1} \to R$ be an arbitrary homogeneous power of one function by R^{n+1}. Mapping $H_f : S^n \to S^n$ defined by the formula*

$$H_f : x \in S^n \to x - 2f(x)\frac{\nabla f}{|\nabla f|^2} \in S^n, \tag{4}$$

saves the norm of the vector: $|H_f(x)| = |x|$.

Definition 2. The mapping H_f is called the conformal gradient of the function f. If the mapping H_f has the inverse H_f^{-1}, then the polar metric is defined by the function:

$$f^*(y) = \frac{2f(x)}{|\nabla f|^2}\bigg|_{x=H_f^{-1}(y)}.$$

Note. From the definition (1) follows the duality of the metric $ds^2 = \frac{dx^2}{f^2(x)}$, $x \in S^n$ and the metric $ds^{*2} = \frac{dy^2}{f^{*2}(y)}$, $y \in S^n$. Therefore, under an appropriate regularity of the functions $f^*(y)$ there are equalities:

$$\begin{aligned} f^*(y) &= \frac{2f(x)}{|\nabla f(x)|^2}, \quad y = x - 2f(x)\frac{\nabla f(x)}{|\nabla f(x)|^2}, \\ f(x) &= \frac{2f^*(y)}{|\nabla f^*(y)|^2}, \quad x = y - 2f^*(y)\frac{\nabla f^*(y)}{|\nabla f^*(y)|^2}. \end{aligned} \tag{5}$$

Theorem. *In the presence of the corresponding regularity and positivity of the one-dimensional sectional curvature of the metric* $ds^2 = \frac{dx^2}{f^2(x)}$, $x \in S^n$ *are equal:*

$$f^*(y) = \max_{x\in S^n} \frac{\|x-y\|^2}{2f(x)}, \quad f(x) = \max_{y\in S^n} \frac{\|x-y\|^2}{2f^*(y)}, \tag{6}$$

where $\|x - y\|$ *is the chord distance between the points* $x, y \in S^n$*.*

Formulas (6) allow us to abandon the regularity and positivity of one-dimensional sectional curvature in determining the polar transformation of a conformally flat metric and formulate

Definition 2. The generalized polar transform conformally-flat metrics:

$$f^*(y) = \max_{x\in S^n} \frac{\|x-y\|^2}{2f(x)}, \quad f^{**}(x) = \max_{y\in S^n} \frac{\|x-y\|^2}{2f^*(y)}, \tag{6}$$

where $\|x-y\|$ is the chord distance between points $x, y \in S^n$, from the positivity of one-dimensional sectional curvature the equation $f^{**}(x) \equiv f(x)$ follows.

References

1. Slavsky V. V. *Conformally flat metrics bound curve on n-dimensional sphere*, Studies on geometry "in General" and mathematical analysis. Novosibirsk: Science, 1987. V. 9, P. 183–199.
2. Slavskii V. V. *Conformally flat metrics and the geometry of the pseudo-Euclidean space*, Siberian Math. J. 35:3 (1994), 674–682.

3. Balashchenko V. V., Nikonorov Yu. G., Rodionov E. D., Slavsky V. V. *Homogeneous spaces: theory and applications: monograph.* Khanty-Mansiysk: Polygraphist, 2008.
4. Rodionov E. D., Slavsky V. V. *one-Dimensional sectional curvature of Riemannian variations*, Reports Academy of Sciences 387:4 (2002), 454–457.
5. Nikonorov Yu. G., Rodionov E. D., Slavskii V. V. *Geometry of homogeneoues Riemannian manifolds*, J. Math. Sci. 146:2 (2007), 6313–6390.
6. Kurkina M. V., Rodionov E. D., Slavskii V. V. *Conformally Convex Functions and Conformally Flat Metrics of Nonnegative Curve*, Doklady Math. 91:3 (2015), 1–3.
7. Rodionov E. D., Slavsky V. V. *Polar transform of conformally flat metrics*, Siberian Adv. Math. 28:2 (2018), 101–114.

The Boolean valued Ando theorem

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The famous Ando Theorem states that a Banach lattice X of dimension ≥ 3 is isometrically lattice isomorphic to $L^p(\Omega, \Sigma, \mu)$ for some $1 \leq p \in \mathbb{R}$ and measure space (Ω, Σ, μ) or to $c_0(\Gamma)$ for some nonempty set Γ if and only if each closed sublattice of X is the image of a positive contractive projection (see, for example, [1, Theorem 1.b.8]) or [2, Theorem 2.7.13]. Below we present a counterpart of the Ando Theorem for the class of $\mathbb{B}$-cyclic Banach lattices obtained by using the Boolean valued transfer principle for injective Banach lattices, see [2] for details. Unexplained terms can be found in books [3, 4].

Theorem. *Let $\mathbb{B}$ be a complete Boolean algebra and let Q be the Stone representation space of $\mathbb{B}$. The following are equivalent for a $\mathbb{B}$-cyclic Banach lattice X satisfying the condition $\mathbb{B}$-dim ≥ 3:*

(1) *There is a positive contractive projection onto any $\mathbb{B}$-complete closed sublattice in X that commutes with the projections from $\mathbb{B}$.*

(2) *There is a partition of unity $(\pi_\gamma)_{\gamma\in\Gamma\cup\{0\}}$ in $\mathbb{B}$, where Γ is a nonempty set of cardinals, such that $\pi_0 X \simeq_{\pi_0\mathbb{B}} L^p(\Phi)$ for some $\mathbf{1} \leq p \in \Lambda^u$ and an injective Banach lattice $L := L^1(\Phi)$ with $\mathbb{M}(L) \simeq \pi_0\mathbb{B}$ and $\pi_\gamma X \simeq_{\pi_\gamma\mathbb{B}} C_{\#}(Q_\gamma, c_0(\gamma))$ for all $\gamma \in \Gamma$, where Q_γ is a clopen subset of Q corresponding to the projection π_γ.*

References

1. Lindenstrauss J., Tzafriri L. *Classical Banach Spaces. Vol. 2. Function Spaces*, Springer-Verlag, Berlin etc., 1979.
2. Meyer-Nieberg P. *Banach Lattices.* Berlin etc.: Springer, 1991.
3. Kusraev A. G., Kutateladze S. S. *Two applications of Boolean valued analysis*, Siberian Math. J. 60:5 (2019).
4. Kusraev A. G., Kutateladze S. S. *Boolean Valued Analysis: Selected Topics.* Vladikavkaz: Vladikavkaz Scientific Center Press, 2014.

On representation of homogeneous polynomials in vector lattices

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Polynomials over infinite-dimensional spaces, appeared at the end of the 19th century and from the very beginning were associated with the studies on infinite-dimensional holomorphy. Relevant historical notes can be found in the book [1]. Despite the fact that the algebraic and linear-topological study of polynomials has a long history and is well represented in literature, the study of the order properties of polynomials has begun only recently, starting with [2].

Let X and Y be vector spaces, $1 \le n \in \mathbb{N}$. A mapping $P : X \to Y$ is said to be an *n-homogeneous polynomial* if $P(x) = \varphi(x, \dots, x)$ $(x \in X)$ for some multilinear operator $\varphi : X^n \to Y$. In this event φ is called *generating*. If φ is symmetric, then it is a unique for P and is called the *associated* operator.

Assume now that E and F are vector lattices, see [3]. (All vector lattices are assumed to be Archimedean). A homogeneous polynomial $P : E \to Y$ is called *orthogonally additive* whenever $P(x + y) = P(x) + P(y)$ for any pair of disjoint members $x, y \in E$. If $P : C(Q) \to Y$ is an orthogonally additive s-homogeneous polynomial with Q an arbitrary compactum and Y a Banach space, then there exists a unique bounded linear operator $S : C(Q) \to Y$ such that $P(f) = S(f^n)$ for all $f \in C(Q)$, see [4]. Below we present two general representation results of this type. Let Y be a bornological space and E is considered with the bornology of order bounded sets. Denote by $\mathcal{P}^b_o({}^nE; F)$ the space of all bounded orthogonally additive n-homogeneous polynomials from E to Y and $L_b(E, Y) := \mathcal{P}^b_o({}^1E, Y)$.

For every vector lattice E and every natural $n \in \mathbb{N}$ there exists a unique (up to lattice isomorphism) vector lattice $E^{n\odot}$ and an odd order isomorphism $\iota_n : E \to E^{n\odot}$ such that the mapping $x \mapsto x^{n\odot} := \iota_n(x^+) + (-1)^n \iota_n(x^-)$ $(x \in E)$ is an orthogonally additive n-homogeneous polynomial.

Theorem 1 [5]. *Let E be a vector lattice, Y convex bornological space. Then for every bounded orthogonally additive n-homogeneous polynomial $P : E \to Y$ there exists a unique bounded linear operator $S : E^{n\odot} \to Y$ such that*

$$P(x) = S(x^{n\odot}) \quad (x \in E).$$

Moreover the correspondence $P \leftrightarrow S$ is an isomorphism $\mathcal{P}^b_o({}^nE, Y) \simeq L_b(E^{n\odot}, Y)$.

Say that vector lattices $G_1, \ldots, G_n$ *admit product in* a vector lattice F if for any choice of a unital f-algebra multiplication $*$ in the universal completion F^u there exist vector sublattices $F_1, \ldots, F_n$ of F^u such that the set

$$F_1 * \cdots * F_n := \{f_1 * \cdots * f_n : f_k \in F_k, k = 1 \ldots, n\}$$

is contained in F and G_j is lattice isomorphic to F_j for all $j = 1, \ldots, n$. In this case G_k and F_k are identified for all $k = 1, \ldots, n$. Denote

$$K_n(m) := \{(k_1, \ldots, k_m) \in \mathbb{N}^m : \ k_1 + \cdots + k_m = n\}.$$

Theorem 2 [6]. *Let E and F be vector lattices with F Dedekind complete and let P be an n-homogeneous polynomial from E to F. The following conditions are equivalent:*

(1) *P is generated by an order bounded disjointness preserving* (*not necessarily symmetric*) *n-linear operator T from E^n to F.*

(2) *There exist vector lattices $F_1, \ldots, F_n$ admitting product $*$ in F, lattice homomorphisms T_k from E to F_k, $k = 1, \ldots, n$, and a partition of unity $\varrho_1, \ldots, \varrho_n$ in $\mathbb{P}(F)$, such that for every $m \le n$ there exists a disjoint family $\{a_{k_1,\ldots,k_m} : (k_1, \ldots, k_m) \in K_n(m)\}$ in $\mathcal{Z}(F)$ such that for all $x \in E$ the representation holds:*

$$P(x) = \sum_{m=1}^{n} \varrho_m \sum_{(k_1,\ldots,k_m)\in K_n(m)} a_{k_1,\ldots,k_m} \Big(T_1(x)^{k_1\odot} * \ldots * T_m(x)^{k_m\odot} \Big).$$

This is a fundamentally different type of monomial decomposition, since linear operators (lattice homomorphisms) appear instead of variables.

References

1. Dineen S. *Complex Analysis on Infinite Dimensional Spaces*, Springer Berlin, 1999.

2. Grecu B. C., Ryan R. A. *Polynomials on Banach spaces with unconditional bases*, Proc. Amer. Math. Soc. 133(4) (2005), 1083–1091.
3. Aliprantis C. D., Burkinshaw O. *Positive Operators*, Acad. Press Inc., London etc., 1985.
4. Perez-Garcia D., Villanueva I. *Orthogonally additive polynomials on spaces of continuous functions*, J. Math. Anal. Appl. 306(1) (2005), 97–105.
5. Kusraeva Z. A. *Representation of orthogonally additive polynomials*, Sib. Math. J. 52(2) (2011), 248–255.
6. Kusraev A. G., Kusraeva Z. A. *Monomial decomposition of homogeneous polynomials in vector lattices*, Adv. Oper. Theory. 4(2) (2019), 428–446.

Solution of the problem of embedding for three-dimensional geometries of local maximum mobility

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Consider a four-dimensional analytic manifold M, which is locally diffeomorphic to the direct product of the three-dimensional analytic manifold N and one-dimensional analytic manifold L. We construct a function of a pair of points $f : M \times M \to R$ with an open and dense domain of definition $S_f \subset M^2$ using the following formula:

$$f = \chi(g(\pi_1(h), \pi_1(h)), \pi_2(h), \pi_2(h)), \tag{1}$$

where $h : M \to N \times L$ is a local diffeomorphism, $\pi_1 : N \times L \to N$ and $\pi_2 : N \times L \to L$ are projections, $g : N \times N \to R$ is the function of a pair of points with an open and dense domain S_g in N^2, $\chi : R \times L \times L \to R$ is a nondegenerate function. All the functions listed here are analytical.

The embedding method is used to construct a classification of four-dimensional geometries of local maximum mobility. The functions of a pair of points (1) are found by the known functions of a pair of points g of three-dimensional geometries. The main results are published in [1–4].

The list of functions of a pair of points of three-dimensional geometries of local maximum mobility:

$$g(A, B) = \sin z_A \sin z_B[\sin y_A \sin y_B \cos(x_A - x_B) + \cos y_A \cos y_B] + \cos z_A \cos z_B;$$

$$g(A, B) = \operatorname{ch} z_A \operatorname{ch} z_B[\operatorname{sh} y_A \operatorname{sh} y_B \cos(x_A - x_B) - \operatorname{ch} y_A \operatorname{ch} y_B] + \operatorname{sh} z_A \operatorname{sh} z_B;$$

$$g(A,B)=\operatorname{sh} z_A \operatorname{sh} z_B[\operatorname{sh} y_A \operatorname{sh} y_B \cos(x_A-x_B)-\operatorname{ch} y_A \operatorname{ch} y_B]+\operatorname{ch} z_A \operatorname{ch} z_B;$$

$$g(A,B)=\operatorname{sh} z_A \operatorname{sh} z_B[\operatorname{sh} y_A \operatorname{sh} y_B \operatorname{ch}(x_A-x_B)-\operatorname{ch} y_A \operatorname{ch} y_B]+\operatorname{ch} z_A \operatorname{ch} z_B;$$

$$g(A,B)=(x_A-x_B)^2+(y_A-y_B)^2+(z_A-z_B)^2;$$

$$g(A,B)=(x_A-x_B)^2+(y_A-y_B)^2-(z_A-z_B)^2;$$

$$g(A,B)=x_Ay_B-x_By_A+z_A-z_B;$$

$$g(A,B)=\frac{y_A-y_B}{x_A-x_B}+z_A+z_B;$$

$$g(A,B)=\frac{y_A-y_B}{x_A-x_B}e^{z_A+z_B};$$

$$g(A,B)=\operatorname{arctg}\frac{y_A-y_B}{x_A-x_B}+z_A+z_B;$$

$$g(A,B)=((x_A-x_B)^2+(y_A-y_B)^2)e^{2z_A+2z_B};$$

$$g(A,B)=((x_A-x_B)^2-(y_A-y_B)^2)e^{2z_A+2z_B};$$

$$g(A,B)=((x_A-x_B)^2+(y_A-y_B)^2)e^{2\gamma\operatorname{arctg}\frac{y_A-y_B}{x_A-x_B}+2z_A+2z_B};$$

$$g(A,B)=\beta\ln(y_A-y_B)+\varepsilon\ln(x_A-x_B)+z_A+z_B;$$

$$g(A,B)=(x_A-x_B)^2e^{2\frac{y_A-y_B}{x_A-x_B}+2z_A+2z_B},$$

where (x_A,y_A,z_A) are the local coordinates of the point A of the manifold N, $\beta,\gamma=\mathrm{const}$, $\gamma\neq 0$, $\beta\neq 0,\pm 1$, $\varepsilon=\pm 1$.

The list of functions of a pair of points of four-dimensional geometries of local maximum mobility:

$$f(A,B)=\sin w_A \sin w_B[\sin z_A \sin z_B[\sin y_A \sin y_B \cos(x_A-x_B)+\cos y_A \cos y_B]+$$
$$+\cos z_A \cos z_B]+\cos w_A \cos w_B;$$

$$f(A,B)=\operatorname{ch} w_A \operatorname{ch} w_B[\operatorname{ch} z_A \operatorname{ch} z_B[\operatorname{sh} y_A \operatorname{sh} y_B \cos(x_A-x_B)-\operatorname{ch} y_A \operatorname{ch} y_B]+$$
$$+\operatorname{sh} z_A \operatorname{sh} z_B]+\operatorname{sh} w_A \operatorname{sh} w_B;$$

$$f(A,B)=\operatorname{sh} w_A \operatorname{sh} w_B[\operatorname{ch} z_A \operatorname{ch} z_B[\operatorname{sh} y_A \operatorname{sh} y_B \cos(x_A-x_B)-\operatorname{ch} y_A \operatorname{ch} y_B]+$$
$$+\operatorname{sh} z_A \operatorname{sh} z_B]+\operatorname{ch} w_A \operatorname{ch} w_B;$$

$$f(A,B)=\operatorname{ch} w_A \operatorname{ch} w_B[\operatorname{sh} z_A \operatorname{sh} z_B[\operatorname{sh} y_A \operatorname{sh} y_B \cos(x_A-x_B)-\operatorname{ch} y_A \operatorname{ch} y_B]+$$
$$+\operatorname{ch} z_A \operatorname{ch} z_B]-\operatorname{sh} w_A \operatorname{sh} w_B;$$

$$f(A,B) = \operatorname{ch} w_A \operatorname{ch} w_B[\operatorname{sh} z_A \operatorname{sh} z_B[\operatorname{sh} y_A \operatorname{sh} y_B \operatorname{ch}(x_A - x_B) - \operatorname{ch} y_A \operatorname{ch} y_B]+$$
$$+ \operatorname{ch} z_A \operatorname{ch} z_B] - \operatorname{sh} w_A \operatorname{sh} w_B;$$
$$f(A,B) = (x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2 + (w_A - w_B)^2;$$
$$f(A,B) = (x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2 - (w_A - w_B)^2;$$
$$f(A,B) = (x_A - x_B)^2 + (y_A - y_B)^2 - (z_A - z_B)^2 - (w_A - w_B)^2;$$
$$f(A,B) = ((x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2)e^{w_A + w_B};$$
$$f(A,B) = ((x_A - x_B)^2 + (y_A - y_B)^2 - (z_A - z_B)^2)e^{w_A + w_B};$$
$$f(A,B) = x_A y_B - x_B y_A + z_A w_B - z_B w_A,$$

where (x_A, y_A, z_A, w_A) are the local coordinates of the point A of the manifold M.

References

1. Kyrov V. A., Mikhailichenko G. G. *Analytical method of embedding symplectic geometry.* Siberian Electronic Mathematical Reports, 14 (2017), 657–672.
2. Kyrov V. A. *Analytical embedding method for multidimensional pseudo-Euclidean geometries.* Siberian Electronic Mathematical Reports, 15 (2018), 741–758.
3. Kyrov V. A. *Embedding of multidimensional singular extensions of pseudo-Euclidean geometries.* Chelyab. Phys.-Math. J. 4:3 (2018), 408–420.
4. Kyrov V. A. *Analytic embedding of three-dimensional simplicial geometries.* Trudy Instituta Matematiki i Mekhaniki URO RAN, 25:2 (2019), 125–136.

Complex and real Hurwitz numbers: from symmetric groups to new algebras

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Conventional Hurwitz numbers enumerate meromorphic functions on complex curves with prescribed ramification over a finite collection of points in the complex projective line. They can be also expressed in terms of connection coefficients in the centers of the group algebras of symmetric groups. Simple Hurwitz numbers (those enumerating ramified coverings with all of the ramification points but one being simple) are closely related to the Kadomtsev–Petviashvili integrable hierarchy of partial differential equations of mathematical physics.

The talk will be devoted to the real version of Hurwitz numbers, which is much less studied than the complex one. These real Hurwitz numbers enumerate *real* meromorphic functions on real algebraic curves, with prescribed ramification type over finitely many *real* points. In the real case, the group algebras of the symmetric groups must be replaced by certain other algebras, which we call the algebras of *transitions*. Multiplication in these algebras is more complicated, but, nevertheless, it allows one to write out partial differential equations for the generating functions for real Hurwitz numbers. However, currently it is unclear, whether these partial differential equations are related to any integrable systems. Many other natural questions also are open.

The talk will be based on the preprint [1].

References

1. Kazarian M., Lando S., Natanzon S. *On framed simple purely real Hurwitz numbers*, arXiv:1809.04340, 2018.

Linear ordering of classes of $W^{1,2}$-extension domains

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Let for any open set $G \in \mathbb{R}^n$ be defined some space $\mathcal{F}(G)$ of functions $f : G \to \mathbb{R}$. We will use the usual abbreviation $\mathcal{F} = \mathcal{F}(\mathbb{R}^n)$. The function $f_{ext} \in \mathcal{F}$ is called an extension of the function $f \in \mathcal{F}(G)$, if $f_{ext}|_G = f$. The set G is called the $\mathcal{F}$-extension set for function $f \in \mathcal{F}(G)$ if there is an extension $f_{ext} \in \mathcal{F}$. The set of open $\mathcal{F}$-extended sets is denoted by $\mathrm{Ext}(\mathcal{F})$.

Examples known to the author show that if the domain G belongs to $\mathrm{Ext}(W^{1,p})$ or Ext $(L^{1,p})$, then G belongs to $\mathrm{Ext}(W^{k,p})$ or $\mathrm{Ext}(L^{k,p})$ for all natural $k > 1$. ($W^{k,p}$, $L^{k,p}$ - Sobolev spaces): [1–3].

Notations. Let α be a multi-index, $k \in \mathbb{N}$.

$H^k(G)$ is the space of functions $f \in L_{2,\,loc}(G)$ that have weak partial derivatives of all orders from 1 to k, belonging to $L_2(G)$, $W^{k,2}(G) = H^k(G) \cap L_2(G)$.

The inner product and seminorm in $H^k(G)$.

$$\langle u, v\rangle_{H^k(G)} = \int\limits_G \sum_{1\le|\alpha|\le k} D^\alpha u \cdot D^\alpha v \, dx, \qquad \|f\|_{H^k(G)} = \sqrt{\langle f, f\rangle_{H^k(G)}}.$$

Definition. Let a domain $G \subset \mathbb{R}^n$ belong to the class $\mathrm{Ext}(H^1)$.

$E_f = \{F \in H^1 : F|_G = f\}$ is called the set of extensions of the function f.

Theorem. *If a bounded domain $G \in \mathbb{R}^n$ belongs to the class Ext(H^1), then $G \in$Ext(H^k) for $k \in \mathbb{N}$.*

Scheme of proof.

1) Let $f \in H^1(G)$. $\mathcal{E}_f$ is convex closed set in a Hilbert space $\mathcal{H}^1(\mathbb{R}^n)$, therefore $\mathcal{E}_f$ contains a unique minimal element h, which we call the minimal extension:

$$\|h\|^2_{\mathcal{H}^1} = \min\left\{ \int\limits_{\mathbb{R}^n} |\nabla F|^2 dx, \quad F \in \mathcal{E}_f \right\}.$$

Function h is a harmonic on the open set $\Omega = \mathbb{R}^n \setminus \overline{G}$.

2) Let $G \in\mathrm{Ext}(H^1)$, $f \in H^1(G)$ and $F \in H^1$ be an extension of the function f, moreover $F|_\Omega$ is a harmonic function. Such an extension is called harmonic.

The harmonic extension is unique and coincides with the minimal extension of h.

3) Let $G \in\mathrm{Ext}(H^1$ and $f \in H^2(G)$, then $f \in H^1(G)$, therefore there is minimal extension h. In this case, $h \in H^2$, that is, $G \in \mathrm{Ext}(H^2)$.

4) If $G \in\mathrm{Ext}(H^k)$, then $G \in\mathrm{Ext}(H^{k+1})$.

References

1. Jones P. W. *Quasiconformal mappings and extendability of functions in Sobolev spaces*, Acta Math. 147 (1981), 71–88.
2. Shvartsman P. *On Sobolev extension domains in Rn*, J. Func. Anal. 258 (2010), 2205–2245.
3. Shvartsman P., Zobin N. *On planar Sobolev L^m_p-extension domains*, Advances in Mathematics 287 (2016).

Reduced modules in the geometric theory of functions

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The reduced module concept based on the properties of the modules of families of curves, which first appeared in the work of Teichmüller [1], was firmly included in the list of important metric characteristics of flat and spatial domains and found numerous applications in the geometric theory of functions. Analysis of the condenser capacitance behavior in the degeneration of one of its plates led to another the reduced module definition, which directly connected with the concept of the inner (conformal) radius of a domain at a point and which has found applications in geometric theory of functions and mathematical physics (see [2–4]). Based on the same connections, a new progress to the concept of a reduced module was proposed by Dubinin [5], who developed a comprehensive approach to the definition of this concept, which made it possible to significantly expand the scope of its possible applications.

In our talk we present various options for defining and applying the concept of a reduced modules. In particular, the definition of Dubinin [5] extends to the case of reduced p-modules of open spatial sets with respect to system of interior points with weight functions corresponding to these points. The basis for this is the connection established by the author between the reduced p-module with respect to a fixed point and its p-harmonic inner radius. (see [7, 8]).

Let $B \subset \overline{E^n}$ be an open set, $B \neq \overline{E^n}$, $1 < p \leq n$. In this set, we consider the set of interior points $X = \{x_l\}$ and the system of functions $\Psi_p = \left\{\psi_l^{(p)}(r)\right\}$, $l = 1, 2, \ldots, m$, where $\psi_l^{(p)}(r) = \begin{cases} r^{\beta_l}, & p = n \\ \nu_l r, & 1 < p < n \end{cases}$, and ν_l, β_l are arbitrary positive numbers. We choose $r > 0$ so small that the balls $V(x_l, \psi_l^{(p)}(r))$ do not intersect in pairs and lie entirely in the set B. We define the p-module of the condenser $(B, X, \Psi_p(r)) = \left(\overline{E^n} \backslash B, \bigcup\limits_{l=1}^{m} \overline{V(x_l, \psi_l^{(p)}(r))}\right)$ by formula

$$mod_p(B, X, \Psi_p(r)) = \left(\frac{n\omega_n}{Cap_p(B, X, \Psi_p(r))}\right)^{\frac{1}{p-1}},$$

where $Cap_p(B, X, \Psi_p(r))$ is p-capacity of condenser [6].

Using the inequality for p-modules of a system of disjoint condensers separating the boundary components of the condenser in question, it can be shown that there exists

$$M_p(B,X,\Psi_p)=\lim_{r\to 0}\begin{cases}(mod_n(B,X,\Psi_n(r))+\beta\ln r), & p=n,\\ (mod_p(B,X,\Psi_p(r))-\frac{\nu}{\gamma}r^{-\gamma}), & 1<p<n,\end{cases}$$

which is called the generalized reduced p-module of the set B with respect to the system of points X and the system of functions Ψ_p. Here $\beta=\left(\sum_{l=1}^{m}\beta_l{}^{1-n}\right)^{\frac{1}{1-n}}$, $\nu=\left(\sum_{l=1}^{m}\nu_l^{\gamma(p-1)}\right)^{\frac{1}{1-p}}$, $\gamma=\frac{n-p}{p-1}$.

Theorem 1. *If the set B consists of m disjoint domains B_l with a regular boundary and $x_l\in B_l$, $l=1,2,...,m$, then we have the equality*

$$M_p(B,X,\Psi_p)=\begin{cases}\beta^n\sum_{l=1}^{m}\beta_l^{-n}h_n(x_l,B_l), & p=n,\\ \nu^p\sum_{l=1}^{m}\nu_l^{\gamma p}h_p(x_l,B_l), & 1<p<n,\end{cases}$$

where $h_p(x,B_l)$ is the Roben p-function of the domain B_l [8].

Theorem 2. *If, under the conditions of the previous theorem, the set B belongs to the domain $D\subset\overline{E^n}$, then*

$$M_p(D,X,\Psi_p)\geq\begin{cases}\beta^n\sum_{l=1}^{m}\beta_l^{-n}h_n(x_l,B_l), & p=n,\\ \nu^p\sum_{l=1}^{m}\nu_l^{\gamma p}h_p(x_l,B_l), & 1<p<n.\end{cases}$$

These theorems allow one to obtain some results on extreme decomposition of spatial domains, similar to the results established in [9].

References

1. Teichmüller O. *Untersuchungen über konforme und quasiconforme Abbildung*, Deutshe Math. 3 (1938), 621–678.
2. Polya G., Szegö G. *Isoperimetric Inequalities in Mathematical Physics*, Princeton Univ. Press, Princeton, N.J., 1951.
3. Hayman W. K. *Multivalent functions, Second edition*, Cambridge Tracts in Mathematics, Cambridge Tracts University Press, 1994.
4. Mityuk I. P. *Generalized reduced module and some of its applications*, Izv. Vyssh. Uchebn. Zaved. Mat. 2 (1964), 110–119 [in Russian].
5. Dubinin V. N. *Condenser capacities and symmetrization in geometric function theory*, Birkhäuser Basel, 2014.
6. Shlyk V. A. *On the equality between p-capacity and p-modulus*, Sibirsk. Mat. Zh. 34:6 (1993), 216–221 [in Russian].

7. Levitskii B. *Reduced p-modulus and the interior p-harmonic radius*, Dokl. Akad. Nauk SSSR 316:4 (1991), 812–815 [in Russian].
8. Levitskii B. E. *Reduced p-modulus, p-harmonic radius and p-harmonic Green's mappings*, Problemy Analiza - Issues of Analysis: scientific journal 7(25):2 (2018), 82–97.
9. Kalmykov S., Prilepkina E. *On the p-harmonic Robin radius in the Euclidian Space*, J. Math. Sci. 225:6, 969–979.

Analytical solution of Newton's aerodynamic problem without the assumption of rotational symmetry

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The aerodynamic problem on the form of the convex body having minimal resistance while moving in a rare media was proposed and solved by Sir Isaac Newton for surfaces in revolution. At the very end of the 20th century, it appears that the rejection of the hypothesis of axial symmetry allows reducing resistance: non-axially symmetric bodies with less resistance than symmetric ones of the same length and cross-section were found. The exact form of the best shape of bodies with minimal resistance was still unknown. It was considered as a challenge for experts in optimal control theory. We will present a new result, in which the body shape in the class of bodies with a vertical symmetry plane is analytically derived, and its local optimality is proved. The resistance obtained agrees well with the numerical computations performed by Lachand-Robertand, Oudet and Wachmuth, which suggests its asymptotic optimality among all convex bodies.

This is a joint work with professor M. I. Zelikin.

References

1. Lokusievskiy L. V., Zelikin M. I. *The analytical solution of Newton's aerodynamic problem without the assumption of axial symmetry*, arXiv:1905.02028.

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Boundary value problems for elliptic equations on model Riemannian manifolds

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The intrinsic relationship between the classical problems of function theory and the theory of partial differential equations, on the one hand, and the geometry of Riemannian manifolds, on the other hand, is well known. In particular, the following statements of problems have developed historically in this field of mathematics.

1. Find conditions on a Riemannian manifolds guaranteeing that every solution of a given class is trivial on this manifolds (theorems of the Liouville type).

2. Find conditions on a Riemannian manifolds ensuring the unique solvability of boundary value problems on this manifold.

One source of this topic is the classification theory of noncompact Riemannian surfaces. A distinctive property of two-dimensional surfaces of parabolic type is that they satisfy the Liouville theorem, which states that every positive superharmonic function on a given surface is identically constant. This property served as a basis for the extension of the concepts of parabolicity and hyperbolicity to arbitrary Riemannian manifolds. Namely, manifolds on which every lower-bounded superharmonic function is a constant are called manifolds of parabolic type.

The problems on the existence of nontrivial harmonic and superharmonic functions naturally lead to the theorems of the Liouville type. However, the class of Riemannian manifolds containing fairly ample sets of bounded harmonic functions is very broad. In this connection, the solvability of the Dirichlet problem is of great interest. Note that even the very statement of the Dirichlet problem is of great a solution of the equation from the boundary data at ¡¡infinity¿¿ encounters serious difficulties on an arbitrary noncompact Riemannian manifolds. However, the geometric compactification of a manifold allows one to do this in the ¡¡classical¿¿ setting in some cases. One class of Riemannian manifolds on which the statement of boundary value problems has a natural geometric interpretation is the set of model Riemannian manifolds. In recent years, a number of papers studying the behavior of solutions of various elliptic equations and inequations on such manifolds have been published.

The present paper deals with noncompact Riemannian manifolds representable in the form $M_g = B \cup D$, where B is a precompact set with nonempty interior and D is isometric to the Cartesian product $[r_0, +\infty) \times S$ (where $r_0 > 0$ and S is a closed Riemannian manifold) with the metric

$$ds^2 = dr^2 + g^2(r)d\theta^2.$$

Here $g(r)$ is a positive smooth function on the interval $[r_0, +\infty)$, and $d\theta^2$ is a metric on S. The following assertions are true [1].

Theorem 1. *The manifold M_g is parabolic if and only if*

$$K = \int_{r_0}^{\infty} g^{1-n}(t)dt = \infty.$$

Let us introduce the notation

$$J = \int_{r_0}^{\infty} g^{1-n}(t)(\int_{r_0}^{t} g^{n-3}(z)dz)dt,$$

where $n = dim M_g$.

Theorem 2. *The following statements hold on the Riemannian manifolds M_g.*
1. If $J = \infty$, then each positive harmonic function om M_g is identically constant.
2. If $J < \infty$, then for any functions $F_1(\theta) \in C(S)$ and $F_2(\theta) \in C(S)$ there exists a unique harmonic function on D such that

$$u(r_0, \theta) = F_1(\theta)$$

and

$$\lim_{r\to\infty} u(r, \theta) = F_2(\theta).$$

3. If $J < \infty$, then for any function $F_2(\theta) \in C(S)$ there exists a unique harmonic function on M_g such that

$$\lim_{r\to\infty} u(r, \theta) = F_2(\theta).$$

We also note that the passage to the limit in the boundary conditions at "infinity" in assertions 2 and 3 of Theorem 2 is understood in the sense of uniform convergence. Namely, the following equality is implied:

$$\lim_{r\to\infty} ||u(r, \theta) - F_2(\theta)||_{C(M_g \setminus B(r))} = 0.$$

The following statement holds.

Theorem 3. *If $J < \infty$ and $g''(r) > 0$, then for any functions $F_1(\theta) \in C^\infty(S)$ and $F_2(\theta) \in C^\infty(S)$ there exists a unique harmonic function on D such that*

$$u(r_0, \theta) = F_1(\theta)$$

and

$$\lim_{r\to\infty} ||u(r, \theta) - F_2(\theta)||_{C^1(M_g \setminus B(r))} = 0.$$

To prove Theorem 3, we will need next statement.

Lemma 1. *Let $w_k(\theta)$ be the eigenfunction of the Laplace operator $-\Delta_\theta$ corresponding to the k-th eigenvalue λ_k of the compact set S with the normalization condition*

$$\int_S w_k^2(\theta)d\theta = 1.$$

Then for every natural number m there exist positive constants $C > 0$ and $p > 0$ such that

$$||w_k||_{C^m(S)} < Ck^p.$$

References

1. Losev A. G., Mazepa E. A. *On the asymptotic behavior of solutions of some elliptic-type equations on noncompact Riemannian manifolds*, Russian Math. (Iz. VUZ) 43:6 (1999), 39–47.

Connections for non-holonomic distributions

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There were several approaches to define a privileged connection on non-Riemannian manifolds. Tanaka-Webster connection for CR-manifolds, Tanno connection for contact manifolds, Wagner connection for general distribution on a manifold. We construct the holonomy flag — generalization of the holonomy algebra in Riemannian case. Calculate it for each mentioned connection for the 3-dimensional sub-Riemannian Lie group. And give description for all connections for which holonomy flag vanishes.

References

1. Malkovich E. G. *On a Holonomy Flag of Non-holonomic Distributions*, J. Dynamical and Control Systems 24:3 (2018), 355–370.

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Sub–Riemannian geometry in image processing and modelling of human visual system

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In this talk, we explain the role of sub–Riemannian (SR) geometry in digital image processing and modelling of human visual system. In recently proposed mathematical models of human vision (J. Petitot, G. Citti, A. Sarti), it was shown that SR geodesics appear as natural curves caused by a mechanism of the primary visual cortex V1 of a human brain for completion of contours that are partially corrupted or hidden from observation. We extend the model by including data adaptivity via a suitable external cost in the SR metric and show the advantages of such extension: 1) it leads to a powerful method of extraction the information from digital images; 2) it provides a refined model of V1 that takes into account a presence of visual stimulus.

We start from explanation of basic concepts of SR geometry and then show how they provide brain inspired methods in digital image processing. We discuss how considering of SR structures on 2D and 3D images (or more precisely on their lift to the extended space of positions and directions) helps to detect some features, e.g. salient curves. We consider several particular examples: tracking of blood vessels in planar and spherical images of human retina, tracking of neural fibers in MRI images of human brain. Afterwards we show how a proper choice of the external cost based on a response of simple cells to the visual stimulus provides a model for geometrical optical illusions.

The talk is based on joint works [1–5]

References

1. Bekkers E., Duits R., Mashtakov A., Sanguinetti G. *A PDE approach to data-driven sub-Riemannian geodesics in SE(2)*, SIAM J. Imaging Sci. 8:4 (2015), 2740–2770.

2. Duits R., Ghosh A., Dela Haije T., Mashtakov A. *On sub-Riemannian geodesics in SE(3) whose spatial projections do not have cusps*, J. Dynamical and Control Systems 22:4 (2016), 771–805.

3. Mashtakov A. P., Popov A. Yu. *Extremal controls in the sub-Riemannian problem on the group of motions of Euclidean space*, Regular and Chaotic Dynamics 22:8 (2017), 952–957.

*This work is supported by the Russian Science Foundation under grant 17-11-01387.

4. Mashtakov A., Duits R., Sachkov Yu., Bekkers E., Beschastnyi I. *Tracking of lines in spherical images via sub-Riemannian geodesics on SO(3)*, J. Math. Imaging and Vision 58:2 (2017), 239–264.
5. Franceschiello B., Mashtakov A., Citti G., Sarti A. *Geometrical optical illusion via sub-Riemannian geodesics in the roto-translation Group*, Diff. Geom. Appl. 65 (2019), 55–77.

Jacobian group for cyclic coverings of a graph

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The notion of the Jacobian group of a graph, which is also known as the Picard group, the critical group, and the dollar or sandpile group, was independently introduced by many authors. We define Jacobian of a graph as the maximal Abelian group generated by the flows obeying two Kirchhoff's laws. This notion arises as a discrete version of the Jacobian in the classical theory of Riemann surfaces. It also admits a natural interpretation in various areas of physics, coding theory, and financial mathematics. The Jacobian group is an important algebraic invariant of a finite graph. In particular, its order coincides with the number of spanning trees of the graph, which is well known for some simplest graphs, such as the wheel, fan, prism, ladder, and Möbius ladder. At the same time, the structure of the Jacobian is known only in some particular cases. The class of circulant graphs is fairly large and includes the cyclic graphs, complete graphs, Möbius ladder, antiprisms, and other graphs. The purpose of this report is to determine the structure of the Jacobian for circulant graphs, the generalized Petersen graph, I-,Y-,H- graphs and some others. We also present new formulas for the number of spanning trees and investigate arithmetical properties of these numbers.

Enumeration of rooted spanning forests in circulant graphs

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We develop a new method to produce explicit formulas for the number $f_G(n)$ of rooted spanning forests in the circulant graph $G = C_n(s_1, s_2, \ldots, s_k)$ and $G = C_{2n}(s_1, s_2, \ldots, s_k, n)$. These formulas are expressed through Chebyshev polynomials. We prove that in both cases the number of rooted spanning forests can be represented in the form $f_G(n) = p\,a(n)^2$, where $a(n)$ is an integer sequence and p is a prescribed natural number. Finally, we find an asymptotic formula for $f_G(n)$ through the Mahler measure of the associated Laurent polynomial $2k + 1 - \sum_{i=1}^{k}(z^{s_i} + z^{-s_i})$.

We note that similar results for the number of spanning trees in circulant graphs were obtained earlier in the paper [1].

References

1. Mednykh A., Mednykh I. *The number of spanning trees in circulant graphs, its arithmetic properties and asymptotic*, Discrete Math, 342 (2019), 1772–1781.

Nattall's decomposition of a three-sheeted torus

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The type II Pade-Hermite approximations play an important role in the modern approximation theory. For a given functions f_j, $1 \le j \le m$, holomorphic in a neighborhood of infinity, their type II Pade-Hermite approximants are rational functions

$$\frac{Q_{nj}(\tau)}{P_n(\tau)}, \quad 1 \le j \le m,$$

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such that the polynomials $Q_{nj}(\tau)$ and $P_n(\tau)$, $\deg P_n \le mn$, satisfy the condition:

$$P_n(\tau)f_j(\tau) - Q_{nj}(\tau) = O(\tau^{-(n+1)}),\ \tau \to \infty, \quad 1 \le j \le m.$$

Assume that all f_j can be extended from infinity along any path lying outside a fixed compact $\mathcal{E}$. An important problem is to find maximal domains of convergence of the Pade-Hermite approximants to these functions.

In the case of one function ($m = 1$), the problem was solved by Stahl [1,2]. He proved that the maximal convergence domain is the exterior of some compact $\mathcal{K}$ which is described with the help of orthogonal critical trajectories of a quadratic differential, connected with Green's function for the exterior of $\mathcal{K}$.

For several functions the question is open. Nattall [3] conjectured that, in the case, the asymptotics of the Pade-Hermite approximants is closely connected with an $(m+1)$-sheeted Riemann surface S over the Riemann sphere; its branch-points are over the set $\mathcal{E}$. On the Riemann surface we construct an Abelian integral $\mathcal{N}$ with prescribed logarithmic singularities at the points lying over infinity. With the help of the Abelian integral $\mathcal{N}$, we can separate S into $(m + 1)$ parts; we call this division the Nattall decomposition, and its parts are called the Nattall sheets.

It is of interest to investigate the problem when the set $\mathcal{E}$ is finite. In this connection, Aptekarev and Tulyakov [4] considered the Riemann surface S of the function

$$\omega = \sqrt[3]{(\tau - a_1)(\tau - a_2)(\tau - a_3)} \tag{1}$$

corresponding to the set $\mathcal{E}$ consisting of three points, a_1, a_2, and a_3. They describe the Nattall decomposition when the triangle with vertices a_1, a_2, and a_3 is close to a regular one.

We investigate the problem for arbitrary location of the points a_1, a_2, and a_3. The Riemann surface S of the function (1) is of genus 1, i.e. it is a complex torus. With the help of the Weierstrass elliptic functions, on its universal covering we construct the Abelian integral $\mathcal{N}$ and describe the Nattall decomposition. The curves separating the Nattall's sheets from each other are trajectories of a quadratic differential $\omega = Q(z)dz^2$ uniquely defined by $\mathcal{N}$. We find all cases when the critical points of ω lie on the boundaries of Nattall's sheets. This occurs if and only if either a_1, a_2, and a_3 are on the same straight line or they form an isosceles triangle $\Delta(a_1, a_2, a_3)$ with vertex angle $\ge \pi/3$.

Moreover, we describe the differential-topological structure of trajectories of ω depending on location of the points a_1, a_2, and a_3 both on the universal covering and on the base S over the τ-plane. We also investigate connectedness of Nattall' sheets and show that in some cases all three the sheets can be disconnected.

Then we give a full classification of Nattall decomposition for the surface S corresponding to (1). There are the following possible cases, defining the structure of Nattall's sheets:

- the triangle $\Delta(a_1, a_2, a_3)$ is isosceles with vertex angle $> \pi/3$;
- the triangle $\Delta(a_1, a_2, a_3)$ is isosceles with vertex angle $< \pi/3$;
- the triangle $\Delta(a_1, a_2, a_3)$ is regular;
- a_1, a_2, and a_3 are on the same straight line and none of the points is the middle of the segment formed by the other two points;
- a_1, a_2, and a_3 are on the same straight line and one of them is the middle of the segment formed by the other two points;
- the triangle $\Delta(a_1, a_2, a_3)$ is non-degenerate and the points a_1, a_2, and a_3 are in a general position.

Numerical calculations, performed with the help of the Wolfram Mathematica package, give a possibility to better understand transformation of the structure of Nattall's sheets when we change the parameters a_1, a_2, and a_3 and transfer from one of the possible cases, described above, to another.

We also establish some properties of the Weierstrass elliptic functions which are of independence interest. They concern geometric characteristics of conformal mappings, realized by the functions and related to them, and some estimations.

References

1. Stahl H. *The structure of extremal domains associated with an analytic function*, Complex Variables Theory Appl. 4:4 (1985), 339–354.
2. Stahl H. *Orthogonal polynomials with complex-valued weight function. I, II*, Constr. Approx. 2:1 (1986), 225–240, 241–251.
3. Nuttall J. *Asymptotics of diagonal Hermite-Padé polynomials*, J. Approx. Theory 42:4 (1984), 299–386.
4. Aptekarev A. I., Tulyakov D. N. *Nuttall's Abelian integral on the Riemann surface of the cube root of a polynomial of degree 3*, Izvestiya: Mathematics 80:6 (2016), 997–1034.

Surfaces with L^2 second fundamental form: isothermal coordinates, Jordan-Brouwer theorem, extension of Sobolev functions

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We deal with immersions $\mathbf{\Phi} : \mathbb{R}^2 \to \mathbb{R}^3$ with the square integrable second fundamental form. For the state of art in the domain, see [1–6] and the references therein. With applications to the hydroelastic wave theory in mind, we will consider special types of immersions having the periodic structure. The results obtained also hold true for the immersions of $2D$ tori in $\mathbb{R}^3$.

Let periods $\mathbf{l}_1, \mathbf{l}_2 \in \mathbb{R}^2$ be linearly independent vectors, and Γ be a corresponding lattice in $\mathbb{R}^2$. Denote by Π_Γ the fundamental cell of the lattice Γ. The class $\mathcal{S}$ of immersions $\mathbf{\Phi} : \mathbb{R}^2 \to \mathbb{R}^3$ consists of all immersions satisfying the following conditions. The mapping $\mathbf{\Phi}$ admits the representation

$$\mathbf{\Phi}(X) = (X \cdot \mathbf{l}_1')\,\mathbf{i} + (X \cdot \mathbf{l}_2')\,\mathbf{j} + \Phi_0(X), \quad X \in \mathbb{R}^2, \quad \mathbf{i} = (1,0,0), \quad \mathbf{j} = (0,1,0).$$

Here $\Phi_0 : \mathbb{R}^2 \to \mathbb{R}^3$ is a Lipschitz Γ- periodic mapping, the vectors $\mathbf{l}_i'$ form a lattice dual to Γ. Furthermore, the first fundamental form $\mathbf{g} = (g_{ij})$ of the immersion Φ is bounded and has the bounded inverse, and the second fundamental form of Φ is square integrable over period. This means that

$$\|\mathbf{g}\|_{L^\infty(\Pi_\Gamma)} + \|\mathbf{g}^{-1}\|_{L^\infty(\Pi_\Gamma)} < \infty, \quad \int_{\Pi_\Gamma} |\nabla \mathbf{n}(X)|^2\, dX < \infty$$

Here $\mathbf{n}(X) = |\partial_{X_1}\Phi \times \partial_{X_2}\Phi|^{-1}\partial_{X_1}\Phi \times \partial_{X_2}\Phi$ is the unit normal vector to $S = \Phi(\mathbb{R}^2)$. The following theorem, cf. [2, 6], constitutes the existence of isothermal coordinates on a surface of the class $\mathcal{S}$.

Theorem 1. *Let $\mathbf{\Phi}$ be an immersion of the class $\mathcal{S}$ with the lattice of periods Γ. Then there exist a lattice Υ with the periods $\mathbf{s}_1 = (\mu, 0)$, $\mathbf{s}_2 = (\nu, \mu^{-1})$, $\mu > 0$, and bi-Lipshitz homeomorphism $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ such that the immersion $\mathbf{\Psi} = \mathbf{\Phi} \circ \varphi$ admits the representation*

$$\mathbf{\Psi}(X) = (\mathbf{s}_1' \cdot X)\,\mathbf{i} + (\mathbf{s}_2' \cdot X)\,\mathbf{j} + \mathbf{\Psi}_0(X).$$

Here the vectors $(\mathbf{s}_1', \mathbf{s}_2')$ are dual to $(\mathbf{s}_1, \mathbf{s}_2)$, and $\mathbf{\Psi}_0$ is the Lipschitz Υ-periodic mapping. The immersion $\mathbf{\Psi} : \mathbb{R}^2 \to \mathbb{R}^3$ is conformal, i.e.,

$$\partial_j \mathbf{\Psi}(X) = e^{f(X)}\mathbf{e}_j(X), \quad \mathbf{e}_j \cdot \mathbf{e}_i = \delta_{ji},$$

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The Υ- periodic vectors $\mathbf{e}_i$, the logarithm f of the conformal factor, and the parameters of the lattice Υ admit the estimates

$$\|\mathbf{e}_i\|_{W^{1,2}(\Pi_\Upsilon)} + \|f\|_{L^\infty(\Pi_\Upsilon)} + \|f\|_{W^{1,2}(\Pi_\Upsilon)} + |\mu| + |\mu^{-1}| + |\nu| \le c.$$

Here the constant c depend on Φ and Γ. The vector fields $\mathbf{e}_i(X)$ are orthogonal to $\mathbf{n} := \mathbf{n}(\varphi^{-1}(X))$. The immersion $\mathbf{\Psi}$ is unique with the accuracy up to the shift of the independent variable.

The proof is based on the Chern moving frame method and on the following version of the Hadamard Theorem for Lipschitz mappings.

Theorem 2. *Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be a Lipschitz mapping satisfying the following condition*

$$\|D\varphi(X)\| + \|(D\varphi(X))^{-1}\| \le M < \infty, \quad \det D\varphi(X) > 0 \ \ a.e. \ in \ \ \mathbb{R}^2.$$

Here $D\varphi$ is the Jacobi matrix of the mapping φ, the constant M is independent of X. We assume that $D\varphi$ is Γ- periodic. Then φ takes homeomorphically $\mathbb{R}^2$ onto $\mathbb{R}^2$. There is a lattice $\mathcal{C}$ with linearly independent periods $\mathbf{c}_i$ such that the inverse φ^{-1} admits the representation

$$\varphi^{-1}(Y) = \mathbf{l}_1(\mathbf{c}_1' \cdot Y) + \mathbf{l}_2(\mathbf{c}_2' \cdot Y) + \varphi_0(Y).$$

Here φ_0 is the Lipschitz $\mathcal{C}$-periodic mapping.

For applications it is important that an immersion forms the boundary of some physical domain in the ambient space $\mathbb{R}^3$. In our case this results from the following version of the Jordan-Brouwer Theorem

Theorem 3. *Assume that, under the assumptions of Theorem 1,*

$$2\pi^2 \text{ area } \Sigma + \int_{\Pi_\Upsilon} |\nabla \mathbf{n}|^2 \, dX \le 8\pi - \delta, \quad \delta > 0, \quad \Sigma = \mathbf{\Psi}(\Pi_\Upsilon).$$

Then the surface $S = \mathbf{\Psi}(\mathbb{R}^2)$ splits the space $\mathbb{R}^3$ into two connected components, i.e., $\mathbb{R}^3 \setminus S = G^- \cup G^+$. The component G^+ (G^-) contains the half space $\{x_3 > N\}$ ($\{x_3 < -N\}$) for sufficiently large N.

Finally, we consider the question of well-posedness of the Neumann problem in domain G^- bounded by the surface $S \in \mathcal{S}$,

$$\text{div } (\nabla u - \mathbf{h}) = 0 \ \text{ in } \ G^-, \quad \nabla u \cdot \mathbf{n} - \mathbf{h} \cdot \mathbf{n} = 0 \ \text{ on } \ S,$$

$$u(x + n\mathbf{i} + m\mathbf{j}) = u(x), \quad m, n \in \mathbb{Z}, \ \mathbf{i} = (1, 0, 0), \mathbf{j} = (0, 1, 0).$$

Here $\mathbf{h} \in L^\infty(\mathbb{R}^3)$ is 1-periodic in x_1, x_2 and vanishes as $x_3 \to -\infty$. Denote by $\mathcal{W}^{1,2}$ the class of all 1-periodic in x_1, x_2 locally integrable functions $u : G^- \to \mathbb{R}$ with $\nabla u \in L^2(G^-_\Pi)$ such that the average value of u over some fixed ball $B_0 \subset G^-$ equals zero. Here G^-_Π is the intersection of G^- with the slab of periods $\Pi = \{0 \le x_1, x_2 \le 1\}$. Supplemented with the norm $\|\nabla u\|_{L^2(G^-_\Pi)}$ the class $\mathcal{W}^{1,2}$ becomes the Hilbert space. We prove that the Neumann problem has a unique weak solution in $\mathcal{W}^{1,2}$. This result is based on the following extension theorem which is of independent interest.

Theorem 4. *Under the assumptions of Theorem 3, for every $u \in \mathcal{W}^{1,2}$ and for every $\delta > 0$, there is a function $u_e : \mathbb{R}^3 \to \mathbb{R}$ with the following properties. The function u_e is 1-periodic in x_1, x_2 and vanishes for all sufficiently large x_3. The function u_e belongs to $\mathcal{W}^{1,2}$ and satisfies the inequalities*

$$\|\nabla u_e - \nabla u\|_{L^2(G^-_\Pi)} \le \delta, \quad \|\nabla u\|_{L^2(\Pi)} \le c(\delta).$$

References

1. Bauer M., Kuwert E. *Existence of minimizing Willmore surfaces of prescribed genus*, Int. Math. Res. Not. 10 (2003), 553–576.
2. Helein F. *Harmonic maps, conservation laws and moving frames*, Cambridge University Press, Cambridge (2002).
3. Simon L. *Existence of surfaces minimizing the Willmore functional*, Communications in analysis and geometry 1:2 (1993), 281–326.
4. Taimanov I. A. *The two-dimensional Dirac operator and the theory of surfaces.* (Russian) Uspekhi Mat. Nauk 61:1(367) (2006), 85–164; translation in Russian Math. Surveys 61:1 (2006), 79–159.
5. Toro T. *Geometric conditions and existence of bi-Lipschitz parametrisations*, Duke Math. J. 77:1 (1995), 193–227.
6. Riveire T. *Weak immersions of surfaces with L^2 bounded second fundamental form*, Lecture Notes written by N. Loose, PCMI Graduate summer school (2013).

On deviation of ergodic averages

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Let $(\Omega, \mathfrak{F}, \lambda)$ be a probability measure space, and let T be its automorphism, i. e., an invertible map $T : \Omega \to \Omega$ such that the sets $T^{-1}A$ and TA lie in $\mathfrak{F}$ and $\lambda(A) = \lambda(T^{-1}A)$ for all $A \in \mathfrak{F}$. Given $f \in L_1(\Omega, \mathbb{C})$, $\omega \in \Omega$, $n \geq 1$, we set

$$S_n f(\omega) = \sum_{k=0}^{n-1} f(T^k\omega), \quad A_n f(\omega) = \frac{1}{n} S_n f(\omega).$$

The individual Birkhoff ergodic theorem asserts that the limit $f^* = \lim\limits_{n\to\infty} A_n f$ exists a. e. and $\int_\Omega f^* d\lambda = \int_\Omega f d\lambda$. If the map T is ergodic, then $f^* \equiv \int_\Omega f d\lambda$ a. e.

It is well known that, for a real-valued function $f \in L_1^0(\Omega, \mathbb{R})$, the sequence $S_n f(\omega)$ infinitely often changes sign a. e. and, moreover, approaches zero arbitrarily closely a. e. [1]. For complex-valued functions $f \in L_1^0(\Omega, \mathbb{C})$, this is not necessarily the case (see [2] for example). In context of study of such asymptotical properties of ergodic averages we formulate the main result (see [3]).

For each function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$, monotonically decreasing to zero at infinity, we consider the following $\mathfrak{F}$-measurable sets:

$$E_\varphi(f, T) = \{\omega \in \Omega : \ A_n f(\omega) = o(\varphi(n)) \ \text{ as } \ n \to \infty\},$$
$$F_\varphi(f, T) = \{\omega \in \Omega : \ A_n f(\omega) = \mathcal{O}(\varphi(n)) \ \text{ as } \ n \to \infty\}.$$

Theorem 1. *Suppose that $f \in L_1^0(\Omega, \mathbb{C})$, $f \not\equiv 0$, and T is an ergodic automorphism of $(\Omega, \mathfrak{F}, \lambda)$. Then, for **any** $\varphi(n)$ of positive numbers decreasing to zero, the sets $E_\varphi(f, T)$, $F_\varphi(f, T)$ are invariant* $(\mathrm{mod}\,\lambda)$ *with respect to T, and consequently have measure* 1 *or* 0.

References

1. Schmidt K. *On reccurence*, Z. Wahrscheinlichkeitstheor. Verw. Geb. 68:1 (1984), 75–95.
2. Yoccoz J.-C. *Petits diviseurs en dimension 1*, Astérisque 231 (Soc. Math. France, Paris, 1995).
3. Kachurovskii A. G., Podvigin I. V. *Measuring the rate of convergence in the Birkhoff ergodic theorem*, Math. Notes 106:1 (2019), 52–62.

Measurement of the arcs on certain classes of curves

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It is known that the length of a segment of a straight line can measured by the successive construction of a standard segment and its binary parts [1, p. 371-378]. Circular arcs can be measured in the same way if the measure is required to be invariant with respect to the group of similarities. What other curves can be measured in the same way?

The author proved the existence of a measure on the moment curve that is invariant with respect to the group of unimodular affine transformations and coinciding with the process of measuring arcs.

The moment curve (also named enika) is a curve in n-dimensional affine space, given in some affine frame by parameterization

$$(t, t^2, \ldots, t^n), \quad t \in R.$$

It is established that the moment curve, on which the equality of arcs is defined as equivalence with respect to the group of unimodular affine transformations, is a model of a straight line in absolute geometry according to Hilbert's axioms. The author shows both internal and external geometric characteristics of the middle of the arc and affine equality of the arcs.

Measuring the arcs of curves by construction of the standard arc and its binary parts can be extended to the orbits of one-parameter transformation groups. In this case, the measure must be invariant with respect to a wider group of transformations, which includes in addition to transformations from one-parameter transformation group transformations that map one orbit to another.

References

1. Atanasian L. S., Bazilev V. T. *Geometry: in 2 p.*, P. 2, Press: KNORUS, Moscow, 2015.

On a class of functionals on a weighted Sobolev space of first order on the real line

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Let $I := (a,b) \subset \mathbb{R}$, $\mathfrak{M}(I)$ be the space of all Lebesgue measurable on I functions $f : I \mapsto [-\infty, \infty]$, $\mathfrak{M}^+(I) := \{f \in \mathfrak{M}(I) : f \geq 0\}$, $0 < p, q \leq \infty$, $\rho, v \in \mathfrak{M}^+(I)$. Denote by $W^1_{p,q}(I)$ the space of all functions $u \in L^1_{\text{loc}}(I)$, whose distributional derivative Du belongs to $L^1_{\text{loc}}(I)$ and

$$\|u\|_{W^1_{p,q}(I)} := \|\rho D u\|_{L^p(I)} + \|vu\|_{L^q(I)} < \infty.$$

Denote by $\overset{\circ}{W}{}^1_{p,q}(I)$ the closure in $W^1_{p,q}(I)$ of subspace

$$\overset{\circ\circ}{W}{}^1_{p,q}(I) := \{f \in AC_{\text{loc}}(I) \cap W^1_{p,q}(I) : \operatorname{supp} f \text{ is compact in } I\}.$$

Let $g \in \mathfrak{M}(I)$, $X \in \{\overset{\circ\circ}{W}{}^1_{p,p}(I), \overset{\circ}{W}{}^1_{p,p}(I), W^1_{p,p}(I)\}$. For $p \in (1, \infty)$, v^p, ρ^p, $\rho^{-p'} \in L^1_{\text{loc}}(I)$, $\|v\|_1 > 0$ the description of element Λ_g of the space X^*, having the form $\Lambda_g(f) = \int_I f(x)g(x)\,dx$, $f \in X$, is given in papers [3, 4]. The main tool is the scheme Oinarov – Otelbaev, which was set out in the papers [1, 2].

Put $\Lambda'_g(f) := \int_I g(x)(Df)(x)\,dx$, $f \in X$. We give the answer on the question: under what conditions on the function $g \in \mathfrak{M}(I)$ the map Λ'_g is a continuous linear functional on the space X; and obtain estimates on the norm in X^* of the functional Λ'_g.

References

1. Oinarov R. *On weighted norm inequalities with three weights*, J. London Math. Soc. 48 (1993), 103–116.
2. Oinarov R. *Boundedness of integral operators in weighted Sobolev spaces*, Izv. Math. 78:4 (2014), 836–853.
3. Prokhorov D. V., Stepanov V. D., Ushakova E. P. *On associate spaces of weighted Sobolev space on the real line*, Math. Nachr. 290 (2017), 890–912.
4. Prokhorov D. V., Stepanov V. D., Ushakova E. P. *Spaces associated with weighted Sobolev spaces on the real line*, Dokl. Math. 98:1 (2018), 373–376.

Some functional properties of $L^{p,\omega}(\mathcal{D})$ on the Riemann surface

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Let $\mathcal{R}$ be a Riemann surface from [1]. The projection operation $W \to \operatorname{pr} W = w$ for $W \in \mathcal{R}$ induces a two-dimensional Lebesgue measure σ on $\mathcal{R}$ (for more details see [2]). Let $\widetilde{\omega}$ be A_p Muckenhoupt weight on $\mathbb{C}$, $p \in (1, +\infty)$ and $\omega(W) = \widetilde{\omega}(w)$ if $\operatorname{pr} W = w \neq \infty$, $\omega(W) = 1$ if $\operatorname{pr} W = w = \infty$. Given an open set $\mathcal{D}$ on $\mathcal{R}$, by $L^{p,\omega}(\mathcal{D})$ we denote the class of functions $u : \mathcal{D} \to [-\infty, +\infty]$ for which $\|u\|_{p,\omega}(\mathcal{D}) = \big(\int\limits_{\mathcal{D}} |u|^p \, \omega \, d\sigma\big)^{\frac{1}{p}} < \infty$. Let $f = (f_1, f_2)$ be a vector function on an open set $\mathcal{D} \subset \mathcal{R}$. By definition, $f \in L^{p,\omega}(\mathcal{D})$ whenever $f_1, f_2 \in L^{p,\omega}(\mathcal{D})$. For such functions, one can obtain analogues of the well-known Clarkson's, Hölder's and Minkowski's inequalities.

Theorem 1. Let vector functions $f, g \in L^{p,\omega}(\mathcal{D})$.
If $2 \le p < \infty$ then

$$\int\limits_{\mathcal{D}} \Big|\frac{f+g}{2}\Big|^p \omega \, d\sigma + \int\limits_{\mathcal{D}} \Big|\frac{f-g}{2}\Big|^p \omega \, d\sigma \le \frac{1}{2}\Big(\int\limits_{\mathcal{D}} |f|^p \, \omega \, d\sigma + \int\limits_{\mathcal{D}} |g|^p \, \omega \, d\sigma\Big).$$

If $1 < p \le 2$ then

$$\Big(\int\limits_{\mathcal{D}} \Big|\frac{f+g}{2}\Big|^p \omega \, d\sigma\Big)^{\frac{1}{p-1}} + \Big(\int\limits_{\mathcal{D}} \Big|\frac{f-g}{2}\Big|^p \omega \, d\sigma\Big)^{\frac{1}{p-1}} \le \Big(\frac{1}{2}\int\limits_{\mathcal{D}} |f|^p \, \omega \, d\sigma + \frac{1}{2}\int\limits_{\mathcal{D}} |g|^p \, \omega \, d\sigma\Big)^{\frac{1}{p-1}},$$

$$\int\limits_{\mathcal{D}} |f| \cdot |g| \, d\sigma \le \Big(\int\limits_{\mathcal{D}} |f|^p \, \omega \, d\sigma\Big)^{\frac{1}{p}} \cdot \Big(\int\limits_{\mathcal{D}} |g|^q \, \omega^{1-q} \, d\sigma\Big)^{\frac{1}{q}},$$

$$\|f+g\|_{p,\omega}(\mathcal{D}) \le \|f\|_{p,\omega}(\mathcal{D}) + \|g\|_{p,\omega}(\mathcal{D}).$$

References

1. Dubinin V. *Circular symmetrization of condensers on Riemann surfaces*, Sb. Math., **206**:1 (2015), 61–86.
2. Pugach P., Shlyk V. *Moduli, capacity, BV-functions on the Riemann surfaces*, Lobachevskii J. Math. **38**:2 (2017), 338–351.

Legendre transformation of conformal convex functions

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Minkowski duality and Legendre transformation of convex functions play an important role in the theory of convex subsets of Euclidean space. In this paper we consider conformally flat Riemannian metrics of positive one-dimensional curvature defined on Euclidean space and corresponding conformal-convex functions. For this class of functions an analog of Legendre transform is defined and studied in detail

Introduction.

Let R^n be an Euclidean n-dimensional space, M^{n+2} is a pseudo-Euclidean space, with scalar square of vector $\langle w\rangle^2 = |x|^2 - 2ac$, where $w = [x, a, c] \in M^{n+2}$ and $|x|^2$ – scalar square of vector $x \in R^n$. Let as denote by

$$C^+ = \left\{[x, a, c] \in M^{n+2} : |x|^2 - 2ac = 0,\ a > 0, c > 0\right\},$$

part of an isotropic cone in M^{n+2}. Then for Arbitrary conformally-flat metric we have an imbedding [1] $ds^2 = \frac{dx^2}{f^2(x)}$

$$Z\left(x\right) = \frac{1}{f(x)}\left[-x, \frac{1}{\sqrt{2}}, \frac{|x|^2}{\sqrt{2}}\right] \in C^+, \tag{1}$$

and paired with him conjugate imbedding:

$$Z^*\left(x\right) = \left[\frac{2f\nabla f - x\,\|\nabla f|^2}{2f}, \frac{|\nabla f|^2}{2\sqrt{2}f}, \frac{\left(2f\nabla f - x\,|\nabla f|^2\right)^2}{2\sqrt{2}f\,|\nabla f|^2}\right] \in C^+. \tag{2}$$

The following equality are true:

$$\langle Z\left(x\right), Z^*\left(x\right)\rangle = -1, \quad \langle dZ\left(x\right), Z^*\left(x\right)\rangle = 0,$$
$$\langle dZ\left(x\right), dZ\left(x\right)\rangle = \frac{(dx, dx)}{f(x)^2}, \quad \langle dZ^*\left(x\right), dZ^*\left(x\right)\rangle = \frac{(Adx, Adx)\,,}{f(x)^2},$$

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where the symmetric matrix A corresponds to the quadratic form (one-dimensional sectional curvature of the metric):

$$(A\xi, \xi) = f(x)d^2f(\xi, \xi) - \frac{1}{2}|\nabla f|^2 |\xi|^2. \tag{3}$$

The mapping Z is an isometrical imbedding of the metric $ds^2 = \frac{dx^2}{f^2(x)}$, $x \in R^n$, into the isotropic cone C^+, a the mapping Z^* is an isometrical imbedding of the dual metric $ds^{*2} = \frac{dy^2}{f^{*2}(y)}$, $y \in R^n$, into the isotropic cone C^+. From formulas (1) and (2) we have equality:

$$\left[\frac{2f\nabla f - x|\nabla f|^2}{2f}, \frac{|\nabla f|^2}{2\sqrt{2}f}, \frac{\left(2f\nabla f - x|\nabla f|^2\right)^2}{2\sqrt{2}f|\nabla f|^2}\right] \equiv \frac{1}{f^*(y)}\left[-y, \frac{1}{\sqrt{2}}, \frac{|y|^2}{\sqrt{2}}\right],$$

from where we get the formulas for the dual metric in parametric form:

$$f^*(y) = \frac{2f(x)}{|\nabla f|^2}, \quad y = x - 2f(x)\frac{\nabla f}{|\nabla f|^2}. \tag{4}$$

Definition. Let us call the function $g(x) = \sqrt{f(x)}$ conformal convex if the matrix A is positive-definite. This condition is equivalent to the fulfillment of the three-point inequality

$$g(x) \le g(x_1)\frac{|x_2 - x|}{|x_2 - x_1|} + g(x_2)\frac{|x - x_1|}{|x_2 - x_1|}, \tag{5}$$

for any triple of points x, x_1, $x_2 \in R^n$ [2]. For conformal-convex functions, a number of remarkable relations are performed:

$$g^*(y) = \max_x \frac{|x-y|}{\sqrt{2}g(x)}, \quad g(x) = \max_y \frac{|x-y|}{\sqrt{2}g^*(y)}, \quad |x-y| \le \sqrt{2}g(x)g^*(y), \tag{6}$$

equality in the last inequality is achieved if x, y are connected by the relation (4).

Example. *Let $g(x) = \frac{1}{5}|x-2| + \frac{3}{10}|x+1| + \frac{2}{5}|x+2|$. Then the dual function $g^*(y)$ explicitly equals:*

$$g^*(y) = \frac{|y-9|}{22\sqrt{2}} + \frac{18\sqrt{2}|y+\frac{7}{9}|}{55} + \frac{3|y+3|}{10\sqrt{2}},$$

where graphs of function $g(x)$ and dual function $g^*(y)$ is shown in Fig. 2.

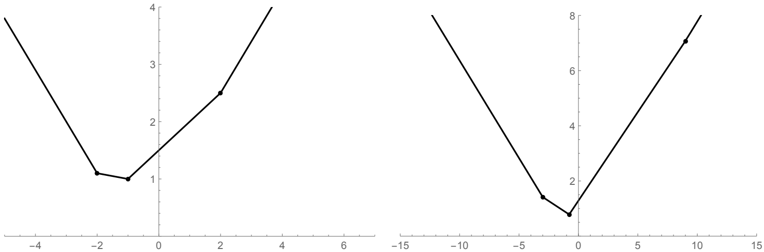

Figure 2: Graph of function $g(x)$ and dual function $g^*(y)$.

The relations (6) have a nontrivial form:

$$\frac{|y-9|}{22\sqrt{2}} + \frac{18\sqrt{2}|y+\frac{7}{9}|}{55} + \frac{3|y+3|}{10\sqrt{2}} \equiv$$
$$\equiv \max_x \frac{|y-x|}{\sqrt{2}\left(\frac{1}{5}|x-2| + \frac{3}{10}|x+1| + \frac{2}{5}|x+2|\right)},$$
$$\frac{1}{5}|x-2| + \frac{3}{10}|x+1| + \frac{2}{5}|x+2| \equiv$$
$$\equiv \max_y \frac{|y-x|}{\sqrt{2}\left(\frac{|y-9|}{22\sqrt{2}} + \frac{18\sqrt{2}|y+\frac{7}{9}|}{55} + \frac{3|y+3|}{10\sqrt{2}}\right)},$$

$$\frac{|y-x|}{\sqrt{2}} \le \left[\frac{|y-9|}{22\sqrt{2}} + \frac{18\sqrt{2}|y+\frac{7}{9}|}{55} + \frac{3|y+3|}{10\sqrt{2}}\right]\left[\frac{1}{5}|x-2| + \frac{3}{10}|x+1| + \frac{2}{5}|x+2|\right].$$

References

1. Slavsky V. V. *Conformally flat metrics bounded curvature on n-dimensional sphere*, Studies on geometry "in General" and mathematical analysis. Novosibirsk: Science Vol. 9, 1987. 183–199.
2. Kurkina M. V., Rodionov E. D., Slavskii V. V. *Conformally Convex Functions and Conformally Flat Metrics of Nonnegative Curve*, Doklady Math. 91:3 (2015), 1–3.

On p-extremal mappings

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Consider a simply connected domain G on the plane R^2, bounded by a closed Jordan curve Γ, as well as four different consecutive boundary points $a_1, a_2, a_3, a_4 \in \Gamma$. Suppose for definiteness that the numbering of points is consistent with the positive orientation of the boundary. A domain G with four marked boundary points will be a quadrilateral and denote it by $G_\star$. Closed boundary arcs $F_0 = \Gamma_{a_1 a_4}$, $F_1 = \Gamma_{a_2 a_3}$, $E_0 = \Gamma_{a_1 a_2}$ and $E_1 = \Gamma_{a_3 a_4}$ are called "sides" of $G_\star$.

Given a pair of "opposite sides" F_0 and F_1, define the class of admissible functions as

$$D(F_0, F_1) = \left\{ u \in ACL(G) \cap C(D \cup F_0 \cup F1) \middle| \; u|_{F_0} = 0, u|_{F_1} = 1 \right\},$$

and define the corresponding p-capacity for $1 \leq p < \infty$

$$C_p = cap_p(F_0, F_1) = \inf_{u \in D(F_0, F_1)} \int\limits_G |\nabla u|^p \, dxdy.$$

When $1 < p < \infty$ there is a unique p-extremal function $u \in \overline{D(F_0, F_1)}$ such that

$$\|u \mid L^1_p(G)\|^p = cap_p(F_0, F_1).$$

Let $1/p + 1/p' = 1$ and v_0 - p'-extremal function for pairs of "sides" E_0 and E_1, $v = C_p \cdot v_0$.

Consider the p-extremal mapping $\Phi = (u, v)$ and some its properties.

1. The mapping $\Phi : G \to P$ is a diffeomorphism, $P = (0, 1) \times (0, C_p)$.

2. Coordinate functions of p-extremal mapping $\Phi(x, y) = (u(x, y), v(x, y))$ satisfy system of equations

$$\begin{cases} \dfrac{\partial v}{\partial y} = |\nabla u|^{p-2} \dfrac{\partial u}{\partial x}, \\ \dfrac{\partial v}{\partial x} = -|\nabla u|^{p-2} \dfrac{\partial u}{\partial y}, \end{cases}$$

which for $p = 2$ turns into a Cauchy-Riemann system.

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3. Gradients of coordinate functions are orthogonal, $|\nabla v| = |\nabla u|^{p-1}$.
4. Jacobi matrix p-extremal mappings $\Phi : G \to P$ has the form

$$J(\Phi) = \begin{pmatrix} |\nabla u| & 0 \\ 0 & |\nabla u|^{p-1} \end{pmatrix} \circ \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}.$$

Denote the class of all p-extremal homeomorphisms of $\Phi : G \to R^2$ by the symbol $H_p(G)$.

Let $G_1, G_2 \subset R^2$ be bounded simply connected domains with Jordan boundaries. We say that a homeomorphism $L : G_1 \to G_2$ is p-extremal, if $L = \Phi_2^{-1} \circ \Phi_1$, where $\Phi_1 \in H_p(G_1), \Phi_2 \in H_p(G_2)$. The class of the corresponding p-extremal homeomorphisms we denote by $H_p(G_1, G_2)$. When $p = 2$ we obtain the class of conformal mappings.

For p-extremal homeomorphisms true analogue of the Riemann theorem on conformal equivalence simply connected domains.

Theorem. *For all $1 < p < \infty$ for any bounded simply connected domains $G_1, G_2 \subset R^2$ with Jordan boundaries there exists homeomorphism $L : G_1 \to G_2$ belonging to class $H_p(G_1, G_2)$.*

The similar result is true for doubly-connected domains.

References

1. Romanov A. S. *Capacity relations in a flat quadrilateral*, Siberian Math. J. 49:4 (2008), 709–717.
2. Romanov A. S. *On a class of extremal mappings*, Siberian Electronic Mathematical Reports, to appear.

Critical points of the Godbillon-Vey functional

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Let a smooth manifold M^3 be equipped with a vector field T and a 1-form ω such that $\omega(T)=1$. We construct a 3-form $\eta\wedge d\eta$ analogous to the Godbillon–Vey invariant [1] of a foliation (i.e., integrable plane field $\ker\omega$) and explore critical points of the functional $\mathrm{gv}=\int_M \eta\wedge d\eta$. It is known that the Godbillon–Vey class is not rigid: there is a one parameter family of foliations on a 3-sphere with the Godbillon–Vey number taking all values in an interval [5]. Now, let g be an adapted Riemannian metric on M, i.e., T becomes the unit vector field normal to the (generally, non-integrable) distribution $\ker\omega$. The unit normal N and the binormal B of T-curves, the torsion τ are defined on $U=\{x\in M: k(x)\neq 0\}$, where k is the curvature of T-curves. Then, one can put $\eta=k\,N^\flat$ and get $\mathrm{gv}=\int_M k^2(\tau-h_{B,N})\,\mathrm{d}V_g$ (see [2] for integrable $\ker\omega$), where h is the nonsymmetric second fundamental form of $\ker\omega$. We consider two types of variations of gv: 1) $\ker\omega$ varies starting from integrable distributions; 2) a metric g varies, while a non-integrable plane field $\ker\omega$ is fixed. We find examples of critical points for gv, e.g. for geodesic vector fields T and among Reeb foliations and twisted products [3,4]. The results can be extended to a manifold M^{2q+1} equipped with a q-form ω and a multivector field $\mathbf{T}=T_1\wedge\ldots\wedge T_q$ such that $\omega(\mathbf{T})=1$.

References

1. Godbillon C., Vey J. *Un invariant des feuilletages de codimension 1*, C. R. Acad. Sci. Paris, Comptes Rendus, sér. A, 273 (1971), 92–95.
2. Reinhart B. L., Wood J. W. *A metric formula for the Godbillon–Vey invariant for foliations*, Proc. AMS 38:2 (1973), 427–430.
3. Rovenski V., Walczak P. *A Godbillon–Vey type invariant for a 3-dimensional manifold with a plane field*, DGA 66 (2019), 212–230.
4. Rovenski V., Walczak P. *Variations of the Godbillon-Vey invariant of foliated 3-manifolds*, Complex Analysis and Operator Theory, 2018.
5. Thurston W. *Noncobordant foliations of S^3*, BAMS 78:4 (1972), 511–514.

*To the 80th anniversary of Academician Yu.G.Reshetnyak.

W. Blaschke's invariants application to image processing

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The earch of new mathematical approaches to the multichannel images analysis and processing is one of the most relevant problems of digital images processing. The relevance of these researches is caused by need of optimization for the available methods, increases in efficiency and quality of the received results.

In work the new approach to digital RGB image processing, based on the W. Blaschke's three-web theory is offered. Invariants of three-channel images for the widest "topological" group of transformations are defined and investigated.

Introduction

Consider a three-channel RGB-image. Suppose that RGB-image is a set of three non-negative functions $u_i(x, y)$, $i = 1, 2, 3$ in a two-dimensional domain D. Families of lines for these functions can be written as $L_i = \{(x, y) : u_i(x, y) = const$, $i = 1, 2, 3$.

Call these three families of lines an image's topographical grid (or three-webs) [1]. The three-webs function is any function $W(u_1, u_2, u_3)$ nonidentically equal to a constant such that $W(u_1(x, y), u_2(x, y), u_3(x, y)) \equiv 0$ in domain D.

W. Blaschke suggested to consider "topological" differential geometry. He studied differential-geometrical (the local!) properties of various objects invariant to topological transformations. Thus use of the classical differential geometry forces to limit set of transformations by functions differentiable sufficient times or analytical.

Connection form, element of surface and curvature as RGB-images invariants

Assume that at the point P_0 the following condition is satisfied

$$u_1(P_0) = u_2(P_0) = u_3(P_0) = 0.$$

Decompose the web function in a series $\{u_1, u_2, u_3\}$:

$$W = W_1\, u_1 + W_2\, u_2 + W_3\, u_3 + \frac{1}{2} W_1\, u_1^2 + W_{12}\, u_1\, u_2 + \dots$$

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Here W_i, W_{ij} are partial derivatives of the first and second order with respect to u_i. In the paper [1] connection form γ, element of surface Ω and curvature κ are defined as three-web invariants:

$$\Omega = W_1 W_2 \, du_1 \wedge du_2 = \ldots, \quad \gamma = -\sum_{i=1,2,3} \frac{\partial}{\partial u_i} \ln(W_i)\, du^i + d\ln(W_1 W_2 W_3),$$
$$\kappa = A_{23} + A_{31} + A_{12},$$

where A_{rs}:

$$A_{rs} = \frac{1}{W_r W_s} \frac{\partial^2}{\partial u_r \partial u_s} \ln \frac{W_r}{W_s} = \frac{W_{rrs}}{W_r^2 W_s} - \frac{W_{rss}}{W_r W_s^2} + \frac{W_{rs}}{W_r W_s} \left(\frac{W_{ss}}{W_s^2} - \frac{W_{rr}}{W_r^2} \right).$$

These invariants are effective RGB-images characteristics which can be used in various tasks. In work [2] the technique of invariants calculation based on transition to a discrete grid of points (image pixels) is offered.

Integral-topographical characteristics of RGB-images

Let‘s pass from three-webs (level lines) to Lebesgue sets of functions and their integrated characteristics. A main objective of such transition from contours to integral-geometrical characteristics of Lebesgue sets is that these images characteristics are easily calculated and are widely used in various applications of digital processing of images.

In the work [2] the important concepts for digital image were defined – Lebesgue sets:

$$M_1(c_1) = \{(x, y) : u_1(x, y) \geq c_1\}$$
$$M_2(c_2) = \{(x, y) : u_2(x, y) \geq c_2\}$$
$$M_3(c_3) = \{(x, y) : u_3(x, y) \geq c_3\}$$

where $u_1(x, y)$, $u_2(x, y)$, $u_3(x, y)$ are functions of images brightness, which accepts values in the range of $[0, 255]$. To them there correspond various numerical characteristics. The area of sets $M_1(c_1)$, $M_2(c_2)$, $M_2(c_3)$ is an example of such characteristics, where $c_1, c_2, c_3 \in [0, 255]$.

Let u be one of the functions u_1, u_2, u_3. Designate through $F_1(c)$ and $F_2(c)$ the following integrals:

$$F_1(c, u) = \iint\limits_{M(c)} |\nabla u| \, dxdy, \quad F_2(c, u) = \iint\limits_{M(c)} |\nabla u| \, \kappa(x, y) \, dxdy,$$

where $|\nabla u|$ is a gradient function and $\kappa(x,y)$ is a curvature of level lines for function $u(x,y)$:

$$\kappa(x,y) = \frac{\frac{\partial^2 u}{\partial y^2}\left(\frac{\partial u}{\partial x}\right)^2 - 2\frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial y}\right)\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial x^2}\left(\frac{\partial u}{\partial y}\right)^2}{\left(\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right)^{3/2}}.$$

For border length $L(c)$ and border curvature $\kappa(c)$ for smooth regular function are defined by the following integral-geometrical ratios:

$$L(c) = \frac{dF_1}{dc}, \quad \kappa(c) = \frac{dF_2}{dc}.$$

Let's call subintegral values

$$\lambda_1(x,y) = |\nabla u|, \quad \lambda_2(x,y) = |\nabla u|\,\kappa(x,y) = \frac{\frac{\partial^2 u}{\partial y^2}\left(\frac{\partial u}{\partial x}\right)^2 - 2\frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial y}\right)\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial x^2}\left(\frac{\partial u}{\partial y}\right)^2}{\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2}$$

as density of lengths and density of curvature for image contour lines. We defined them for smooth functions, in the obvious way they are defined also for grid functions.

Generally, when function $u(x,y)$ is irregular or grid, we call functions $F_1(c)$, $F_2(c)$ irregularity of the first and second order for contour grid lines.

Conclusion

The approach to the images analysis based on the three-webs theory is the perspective direction in the field of digital image processing. Problems of special points identification for three-channel images, search of three-web integral characteristics, determination of local correlation for two images are requiring attention, in our opinion. The presented methods of three-web invariant calculation and integral-topographical characteristics definition can find application in the wide class of images processing, analysis and classification problems.

References

1. Blaschke W. *Einfuhrung in die Geometrie der Waben*, Russian transl.: GITTL, Moskva, 1959.
2. Samarina O. V., Slavsky V. V. *W. Blaschke's theory application in digital image processing*, J. Math. Sci. Appl. 1:2 (2013), 17–23.

On quasiconformal connection of the Lagrangean and Eulerian coordinates

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The study considers the application methods of spatial quasiconformal mapping while solving the problems on continuum strain within the continuous medium. The deformation process with the transition of the continuous medium initial status into the current status is modeled via the Lagrangean and Eulerian coordinates connection. When taking the axiom of the quasiconformal connection, we come to conclusion that any of the known definitions of spatial quasiconformal mapping is able to save the property of the continuous medium initial status under its deformation. Quasiconformity of the Lagrangean and Eulerian coordinates connection is determined for the classical models of the fluids' motion when based on conclusion that standard deviator of strain-rate tensor represents the velocity of change in volume form of an infinitely small particle [1], theorem 6. The class of the dynamic systems, which give rise to quasiconformal isotopy, was considered in the context of the same approach as the only solution of the Cauchy problem for the definite class of vector-fields [2, 3]. According to the continuous medium models with the hypothetical quasiconformal connection of the Lagrangean and Eulerian coordinates, smoothness of vector field generating quasiconformal isotopy does not always allow us to formulate sufficient conditions for existence and uniqueness of the Cauchy problem solution in terms of vector-field modules continuity. These models suppose correctness and uniqueness of the Cauchy problem solution for vector-fields in terms of isotopy classes generated by vector-fields [4]. The existence and uniqueness theorems for the Cauchy problem in terms of quasiconformal isotopy properties generated by vector-fields are recognized to be one of the quasiconformal mapping application methods for the kinematic connection renewal problem of the Lagrangean and Eulerian coordinates for non-smooth vector-fields. Another use of the quasiconformal connections of the Lagrangean and Eulerian coordinates is referred to the compactness properties of the quasiconformal mapping families. In this regard, we study the model of the limiting transition for the viscous incompressible fluid into the ideal fluid at vanishing viscosity. This model is based on Stokes' law that postulates proportionality of deviator of stress tensor to deviator of strain-rate tensor, where coefficient of proportionality appears to be viscosity coefficient. It follows from this property that the uniform boundedness of deviator of strain-rate tensor in certain norm causes the boundedness of maximum shear stress – to – coefficient of viscosity ratio. The estimation, which defines the compactness property for the quasiconformal isotopy family of the definite class, follows from this fact; and we obtain the sequence of viscous incompressible fluid, which tends to the ideal fluid at vanishing viscosity.

References

1. Monakhov V. N., Seleznev V. A. *Dynamical systems determining quasiconformal mappings*, Soviet Math. Dokl. 45:1 (1992), 189–192.
2. Monakhov V. N., Seleznev V. A. *On certain properties of quaziconformal dynamical systems in the large*, Soviet Math. Dokl. 45:3 (1992), 541–544.
3. Seleznev V. A. *Some problems quasiconformal izotopies*, Siberian Math. J. 38:2 (1997), 372–382.
4. Seleznev V. A. *On the only initial task for ordinary differential equations*, Dinamika Sploshn. Sredy 97 (1990), 107–113. [in Russian]

Invariants and 3D Navier–Stokes equations: local and global solutions

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I consider local solutions of the 3D Navier-Stokes equations and its properties such as an existence of global and smooth solution, uniform boundedness (see [1, 2]).The basic role is assigned to a special invariant class of solenoidal vector fields and three parameters that are invariant with respect to the scaling procedure. Since in spaces of even dimension the scaling procedure is a conformal mapping on Heisenberg group then an application of invariant parameters can be considered as the application of conformal invariants. All this gives the possibility to prove the sufficient and necessary conditions for existence of a global regular solution. It is the main result and one among some new statements. With some compliments the rest improves well-known classical results.

Another aspect close to conformal structure is connected with integral identities for solenoidal fields belonging to S. Ju. Dobrokhotov and A. I. Shafarevich. Applying new integral identities it can see an influence of space dimension on an existence of global regular solutions.

References

1. Semenov V. I. *Some New integral identities for solenoidal fields and applications*, Mathematics 2(1) (2014), 29–36.
2. Semenov V. I. *The 3D Navier–Stokes Equations: Invariants, local and global Solutions*, Axioms 8(2),41 (2019), 1–50.

On fixed points and coincidence points of mappings in (q_1, q_2)-quasimetric spaces

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We study the existence of fixed points of contraction mappings and coincidence points of set-valued mappings in $(q_1, q_2)-$quasimetric spaces. In order to pass on to the main results, let us recall some definitions.

Definition 1. A function $\rho_X : X \times X \to \mathbb{R}_+$, such that $\rho_X(x, y) = 0 \iff x = y$, is called a (q_1, q_2)-quasimetric for $q_1, q_2 \geq 1$, if the generalized (q_1, q_2)-triangle inequality holds:

$$\rho_X(x, z) \leq q_1 \rho_X(x, y) + q_2 \rho_X(y, z) \quad \forall x, y, z \in X.$$

The pair (X, ρ_X) is called a (q_1, q_2)-quasimetric space. $(1, 1)$-quasimetric space is called a quasimetric space.

Definition 2. For $\beta > 0$ the mapping $\Phi : X \to X$ is called β-Lipschitz if

$$\rho_X(\Phi(x), \Phi(y)) \leq \beta \rho_X(x, y) \quad \forall x, y \in X.$$

If $\beta < 1$ then the mapping Φ is said to be a contraction mapping.

Definition 3. A sequence $\{x_i\}_{i=1}^{\infty} \subset X$ such that

$$\rho_X(x_j, x_i) < \varepsilon \quad \forall i, j \in \mathbb{N} : \quad i > j > N,$$

is called a Cauchy sequence.

Definition 4. A sequence $\{x_i\}_{i=1}^{\infty} \subset X$ is said to converge to a point $x_0 \in X$ in a quasimetric space (X, ρ_X) if $\lim_{i \to \infty} \rho_X(x_0, x_i) = 0$.

Definition 5. A quasimetric space (X, ρ_X) is said to be complete if any Cauchy sequence converges in this space.

For a mapping $\Phi : X \to Y$ denote $gph\Phi = \{(x, y) \in X \times Y : y = \Phi(x)\}$.

Definition 6. The mapping $\Phi : X \to Y$ is said to be closed if for all sequences $\{x_i\} \subset X$ and $\{y_i\} \subset Y$, such that they converge to points x_0 and y_0 respectively and $(x_i, y_i) \in gph(\Phi)$ for all i, we have $(x_0, y_0) \in gph(\Phi)$.

Let us consider the following fixed point theorem from [1].

Theorem 1. *A closed contraction mapping acting from a complete (q_1, q_2)-quasimetric space to itself has a unique fixed point.*

It can be seen that the theorem above is similar to the Banach fixed point theorem but additionally requires the graph of the mapping to be closed as (q_1, q_2)-quasimetric spaces do not necessarily satisfy the T_2 axiom and hence the graph of a contraction mapping may not be closed. A natural question arises whether the assumption of the mapping being closed is essential in Theorem 1.

The example below illustrates and gives an affirmative answer to the question above.

Example 1. Set $X = \{0, 1, 2, 3, ...\}$. Let us define a $(1,1)$-quasimetric $\rho : X \times X \to \mathbb{R}_+$,

$$\rho(k,n) = \begin{cases} \frac{1}{2^{n-1}} - \frac{1}{2^{k-1}}, & \text{if} \quad k > n. \\ \frac{1}{2^n}, & \text{if} \quad k < n. \\ 0, & \text{if} \quad k = n. \end{cases}$$

Let us define a $\frac{1}{2}-$Lipschitz mapping $\Phi : X \to X$, $\Phi(n) = n + 1$ for all $n \in X$. It is obvious, that the mapping Φ does not have fixed points. It can be easily verified that mapping Φ is not closed.

Let us now pass on to set-valued mappings and state the sufficient conditions for existence of coincidence points. Again we start with definitions.

Definition 7. A point ξ is said to be a coincidence point of set-valued mappings Φ, Ψ if $\Phi(\xi) \cap \Psi(\xi) \neq \emptyset$.

Definition 8. Let $\alpha > 0$. The set-valued mapping $\Psi : X \rightrightarrows Y$ is called $\alpha-$covering if

$$\bigcup_{y \in \Psi(x)} B_Y(y, \alpha r) \subseteq \Psi(B_X(x,r)) \quad \forall r \geq 0, \quad \forall x \in X.$$

Definition 9. For $U, V \subset X$ the function $h^+_X(U,V) = \sup\limits_{u \in U} dist(u, V)$ is called the Hausdorff deviation, where $dist(U,V) = inf\{\rho_X(x_1, x_2), x_1 \in U, x_2 \in V\}$.

Definition 10. For $U, V \subset X$ the function $h_X(U,V) = \max\{h^+_X(U,V), h^+_X(V,U)\}$ is called the Hausdorff distance.

Definition 11. The set-valued mapping $\Phi : X \rightrightarrows Y$ is called $\beta-$ Lipschitz if

$$h(\Phi(x_1), \Phi(x_2)) \leq \beta \rho_X(x_1, x_2) \quad \forall x_1, x_2 \in X.$$

Let α, β be non-negative. The ordered pair of set-valued mappings (Ψ, Φ) belong to $\mathcal{F}_{\alpha,\beta}$ if

1) The set-valued mapping Ψ is $\alpha-$covering and its graph is closed;
2) The set-valued mapping Φ is $\beta-$ Lipschitz;
3) At least one of the graphs $gph(\Psi)$ or $gph(\Phi)$ is complete.
For $x \in X$ and $r > 0$ set

$$M(x,r) = \{y \in \Phi(x) : dist(\Psi(x), y) < r\},$$
$$S(\theta, n) = \frac{1-\theta^n}{1-\theta}, \quad \text{where} \quad \theta \geq 0.$$

Theorem 2 (on existence of coincidence points)**.** *Let an ordered pair of set-valued mappings $(\Psi, \Phi) \in \mathcal{F}_{\alpha,\beta}$ be given. Let $m_0 = \min\{j \in \mathbb{N} : q_2\beta^j < 1\}$. Then for all $x_0 \in X, r_0 > dist(\Psi(x_0), \Phi(x_0)), y_1 \in M(x_0, r_0)$ there exist $\xi \in X, \eta \in Y$ such that $\eta \in \Psi(\xi) \cap \Phi(\xi)$, and the following inequalities hold:*

$$\varliminf_{\lambda \to \xi} \rho(x_0, \lambda) \leq \frac{q_1^2 \alpha^{m_0 - 1} S(q_2 \frac{\beta}{\alpha}, m_0 - 1) + q_1 (q_2 \beta)^{m_0 - 1}}{\alpha_0^m - q_2 \beta_0^m} r_0,$$

$$\varliminf_{\kappa \to \eta} \rho(y_1, \kappa) \leq \beta \frac{q_1^2 \alpha^{m_0 - 1} S(q_2 \frac{\beta}{\alpha}, m_0 - 1) + q_1 (q_2 \beta)^{m_0 - 1}}{\alpha_0^m - q_2 \beta_0^m} r_0.$$

The above theorem is a supplementation to an analogous theorem from [1] with an additional estimate on $y_1 \in M(x_0, r_0)$.

References

1. Arutyunov A. V., Greshnov A. V. *Theory of (q_1, q_2)-quasimetric spaces and coincidence points*, Doklady Mathematics. 94:1 (2016), 434–437.

Harmonic maps and Yang–Mills fields

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Theorem of Atiyah and Donaldson asserts that there is a 1-1 correspondence between the moduli space of G-instantons on $\mathbb{R}^4$ and based holomorphic maps of the Riemann sphere to the loop space ΩG of the compact Lie group G. Generalizing this theorem, we have proposed the harmonic spheres conjecture establishing a 1-1 correspondence between the moduli space of Yang–Mills G-fields on $\mathbb{R}^4$ and based harmonic maps of the Riemann sphere to ΩG. In our talk we shall discuss this conjecture and propose a method of its proof using the adiabatic limit construction.

Wiener-Hopf equation in measures

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Let $\mathbb{R}$ be the set of all real numbers, $\mathbb{R}_+$ be the set of all nonnegative numbers and $\mathbb{R}_- := \mathbb{R} \setminus \mathbb{R}_+$. Denote by $\mathcal{B}$ the σ-algebra of all Borel sets of the line $\mathbb{R}$. Put $\mathcal{B}(\mathbb{R}_\pm) := \{A \in \mathcal{B} \colon A \subseteq \mathbb{R}_\pm\}$. The classical Wiener-Hopf equation is

$$z(x) = \int_0^\infty k(x-y)z(y)\,dy + g(x), \qquad x \in \mathbb{R}_+, \tag{1}$$

where $z(x)$, $x \in \mathbb{R}_+$, is unknown, and $k(x)$, $x \in \mathbb{R}$, and $g(x)$, $x \in \mathbb{R}_+$, are known functions. Equation (1) may be written as

$$z(x) = \int_{-\infty}^x z(x-y)k(y)\,dy + g(x), \qquad x \in \mathbb{R}_+. \tag{2}$$

Consider the following equation in measures

$$z(A) = \int_{\mathbb{R}} z[(A-y) \cap \mathbb{R}_+]\,F(dy) + g(A), \qquad A \in \mathcal{B}(\mathbb{R}_+), \tag{3}$$

where z is an unknown measure on $\mathcal{B}(\mathbb{R}_+)$, F is a probability distribution on $\mathcal{B}$ and g is a known finite measure on $\mathcal{B}(\mathbb{R}_+)$. Denote by F^{n*} the nth-fold convolution of F:

$$F^{1*} := F, \quad F^{(n+1)*} := F^{n*} * F, \quad n \geq 1,$$

and by $F^{0*} := \delta_0$ the atomic measure of unit mass at 0. A solution to equation (3) is given in terms of the Wiener-Hopf factorization

$$\delta_0 - F = (\delta_0 - F_-) * (\delta_0 - F_+),$$

where $F_\pm$ are specific probability distributions concentrated on $\mathbb{R}_\pm$ respectively. Let $U_\pm := \sum_{k=0}^\infty F_\pm^{k*}$ be the renewal measures generated by $F_\pm$ respectively.

Theorem 1. *Let F be a nonarithmetic probability distribution on $\mathbb{R}$ and let g be a finite measure defined on $\mathcal{B}(\mathbb{R}_+)$. Then the measure*

$$z(A) := U_+ * \left[(U_- * g)|_{\mathbb{R}_+}\right](A), \qquad A \in \mathcal{B}(\mathbb{R}_+), \tag{4}$$

is a solution to equation (3), which coincides with the solution obtained by successive approximations, i.e. $z(A) = \lim_{n\to\infty} z^{(n)}(A)$, where

$$z^{(0)}(A) = g(A), \quad z^{(n)}(A) = \int_{\mathbb{R}} z^{(n-1)}[(A-y) \cap \mathbb{R}_+]\,F(dy) + g(A), \qquad A \in \mathcal{B}(\mathbb{R}_+).$$

Theorem 2. *Let F be a nonarithmetic probability distribution on $\mathbb{R}$ such that*

$$\mu := \int_{-\infty}^{\infty} x\, F(dx) \in (0, +\infty],$$

and let g be a finite measure defined on $\mathcal{B}(\mathbb{R}_+)$. Then the solution z defined by (4) of equation (3) satisfies the following relation:

$$z((x, x+h]) \to \frac{hg(\mathbb{R}_+)}{\mu} - \frac{h}{\mu_+} \int_{-\infty}^{0+} g((-\infty, -y))\, U_-(dy), \qquad x \to \infty; \tag{5}$$

here $h > 0$ is a fixed number and $\mu_+ := \int_0^\infty x\, F_+(dx)$.

Remark 1. Suppose that F is concentrated on $\mathbb{R}_+$ and $g = F$. Then equation (3) coincides with the renewal equation $z = z*F+F$ and Blackwell's renewal theorem states that $\mu > 0$ implies $z((x, x+h]) \to h/\mu$ as $x \to \infty$, which in turn coincides with (5) since $F(\mathbb{R}_+) = 1$, $U_- = \delta_0$ and $F((-\infty, 0)) = 0$.

Remark 2. What is the connection between equations (2) and (3)? Suppose that g is absolutely continuous and that so is F with density $k(x)$. Then the solution z of equation (3) given by (4) is absolutely continuous. In fact, if one of two measures is absolutely continuous, then so is their convolution. Hence $U_- * g$, its restriction to $\mathbb{R}_+$ and the solution given by $z = U_+ * \left[(U_- * g)|_{\mathbb{R}_+}\right]$ are all absolutely continuous measures. Moreover, the integral in (3) is also an absolutely continuous measure, being the difference $z - g$ of absolutely continuous measures. Take $A = [0, x]$ in (3). After some manipulations we arrive at

$$\int_0^x \left[z'(u) - \int_{-\infty}^{u} z'(u-y)k(y)\, dy - g'(u) \right] du \equiv 0.$$

Differentiating this identity, we obtain the equation for the density z' of the solution to (3):

$$z'(x) = \int_{-\infty}^{x} z'(x-y)k(y)\, dy + g'(x).$$

It follows that in the absolutely continuous case equation (3) in measures is equivalent to equation (2) — or, which is the same, to the classical Wiener-Hopf equation (1) — for the corresponding densities.

References

1. Sgibnev M. S. *Wiener-Hopf equation in measures with probabilistic kernel*, Siberian Electronic Mathematical Reports, 16 (2019), 609–617.

The Reshetnyak formula for the momentum ray transform

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The ray transform integrates symmetric tensor fields over lines. The family of oriented straight lines in $\mathbb{R}^n$ is parameterized by points of the manifold

$$T\mathbb{S}^{n-1} = \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \mid |\xi| = 1, \langle x,\xi\rangle = 0\} \subset \mathbb{R}^n \times \mathbb{R}^n$$

that is the tangent bundle of the unit sphere $\mathbb{S}^{n-1}$. Let $\mathcal{S}(T\mathbb{S}^{n-1})$ be the Schwartz space of smooth functions $u(x,\xi)$ on $T\mathbb{S}^{n-1}$ rapidly decaying in x together with all derivatives.

Let $S^m\mathbb{R}^n$ be the vector space of rank m symmetric tensors on $\mathbb{R}^n$ and let $\mathcal{S}(\mathbb{R}^n; S^m\mathbb{R}^n)$ be the Schwartz space of $S^m\mathbb{R}^n$-valued functions that are called *rank m smooth fast decaying symmetric tensor fields* on $\mathbb{R}^n$. The *momentum ray transforms*

$$I^k : \mathcal{S}(\mathbb{R}^n; S^m\mathbb{R}^n) \to \mathcal{S}(T\mathbb{S}^{n-1}) \quad (k = 0, 1, \dots, m)$$

are defined by

$$(I^k f)(x,\xi) = \int\limits_{-\infty}^{\infty} t^k \langle f(x+t\xi), \xi^m\rangle\, dt = \int\limits_{-\infty}^{\infty} t^k f_{i_1\dots i_m}(x+t\xi)\, \xi^{i_1}\dots\xi^{i_m}\, dt.$$

A tensor field $f \in \mathcal{S}(\mathbb{R}^n; S^m\mathbb{R}^n)$ is uniquely determined by the data $(I^0f, I^1f, \dots, I^mf)$, we present the inversion algorithm.

Let $H^s_t(\mathbb{R}^n)$ be the generalized Sobolev space introduced in [1]. The Reshetnyak formula expresses the norm $\|f\|_{H^s_t(\mathbb{R}^n;S^m\mathbb{R}^n)}$ of a symmetric tensor field f through some adapted norm of the data $(I^0f, I^1f, \dots, I^mf)$. For example, the Reshetnyak formula in the case of $m = 1$ looks as follows:

$$\begin{aligned} \|f\|^2_{H^s_t(\mathbb{R}^n;S^1\mathbb{R}^n)} = a_n\Big[\sum_{i=1}^{n} \|\Xi_i(I^0f)\|^2_{H^{s+1/2}_{t+1/2}(T\mathbb{S}^{n-1})} + \|I^0f\|^2_{H^{s+1/2}_{t+1/2}(T\mathbb{S}^{n-1})} \\ + \|I^1f\|^2_{H^{s+3/2}_{t+3/2}(T\mathbb{S}^{n-1})} + 2\Re\big(i\,Z(I^0f), I^1f\big)_{H^{s+1}_{t+1}(T\mathbb{S}^{n-1})}\Big]. \qquad (1) \end{aligned}$$

Here Ξ_i are first order differential operators on $T\mathbb{S}^{n-1}$ defined by

$$\Xi_i = \frac{\partial}{\partial\xi^i} - x_i\xi^p\frac{\partial}{\partial x^p} - \xi_i\xi^p\frac{\partial}{\partial\xi^p}$$

and Z is the first order pseudodifferential operator on $T\mathbb{S}^{n-1}$ defined, with the help of the Fourier transform, by

$$\widehat{Z\varphi}(y,\xi) = \frac{y^i}{|y|}(\Xi_i\hat{\varphi})(y,\xi).$$

Formula (1) contains 4 terms on the right-hand side. We also present the corresponding formula for $m = 2$ which contains 19 terms on the right-hand side. In principle, the Reshetnyak formula can be written for an arbitrary m, but the bulkiness of the formula grows rapidly with m. For example, the Reshetnyak formula for $m = 3$ contains more than 250 terms on the right-hand side, and we cannot write the formula in a compact form.

The Reshetnyak formula gives the best stability estimate for the inverse problem of recovering a rank m symmetric tensor field f from the data $(I^0 f, \dots, I^m)$ and allows to extend the map

$$f \mapsto (I^0 f, \dots, I^m)$$

to a linear continuous operator on the space $H^s_t(\mathbb{R}^n; S^m\mathbb{R}^n)$ for every $s \in \mathbb{R}$ and $t > -n/2$.

This is the joint work with Venky Krishnan, Ramesh Manna, and Suman Kumar (TIFR Centre for Applicable Mathematics, Bangalore, India).

References

1. Sharafutdinov V. *The Reshetnyak formula and Natterer stability estimates in tensor tomography*, Inverse Problems 33:2 (2017), 025002.

Logarithmic coefficients of the inverse or univalent functions

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Let $\mathcal{S}$ be the class of analytic and univalent functions in the unit disk $|z| < 1$, that have a series of the form $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$. Let F be the inverse of the function $f \in \mathcal{S}$ with the series expansion $F(w) = f^{-1}(w) = w + \sum\limits_{n=2}^{\infty} A_n w^n$ for $|w| < 1/4$. The logarithmic inverse coefficients Γ_n of F are defined by the formula $\log(F(w)/w) = 2\sum_{n=1}^{\infty}\Gamma_n(F)w^n$. In this talk, we will first discuss about the logarithmic coefficients bound for certain univalent functions, then we will determine the sharp bound for the absolute value of $\Gamma_n(F)$ when f belongs to $\mathcal{S}$ and for all $n \geq 1$. This result motivates us to carry forward similar problems for some of its important geometric subclasses. In some cases, we have managed to solve this question completely but in some other cases it is difficult to handle for $n \geq 4$. For example, in the case of convex functions f, we show that the logarithmic inverse coefficients $\Gamma_n(F)$ of F satisfy the inequality

$$|\Gamma_n(F)| \leq \frac{1}{2n} \text{ for } n \geq 1, 2, 3$$

and the estimates are sharp for the function $l(z) = z/(1-z)$. Although this cannot be true for $n \geq 10$, it is not clear whether this inequality could still be true for $4 \leq n \leq 9$.

This talk is based on the following articles.

References

1. Ponnusamy S., Sharma N. L., Wirths K.-J. *Logarithmic coefficient of the inverse of univalent functions*, Results Math. **73**:160 (2018).
2. Ponnusamy S., Sharma N. L., Wirths K.-J. *Logarithmic coefficient problems in families related to starlike and convex functions*, J. Aust. Math. Soc. (2019), 1–20.

On Gibbs equilibrium surfaces

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It is known from the dates of Young and Laplace that Laplace formula for equilibrium surfaces

$$2H = P_2 - P_1,$$

may be deduced as from macroscopic considerations as from the microscopic ones.

The advantages of the microscopic theory lie in the possibility to take into account the flexural energy of the intermediate layer, the so-called Willmore energy.

We propose a model, which takes into account also the energy needed for the formation of the intermediate layer described by the mathematical surface or Gibbs surface.

In order to find equilibrium surface we use as in the classical case the variational principle.

The work of the forces forming intermediate layer is equal to

$$2H - Kl_p.$$

Here K is the Gaussian curvature of the surface of separation.

The functionals whose first variation is determined by the mean curvature are well known. We propose the functionals whose first variation yield Gauss curvature of the admissible surfaces. We use theory of quasiconformal mappings connected with half geodesic parameterizations of surfaces in order to find such functionals over the class of the convex Liouville surfaces.

In the general case the conjugated Beltrami equation is non-linear one and degenerating on the unknown set with unknown velocity.

Using functionals of the Gaussian curvature, we study axisymmetrical problem for liquid drops pending from the plane.

The equation we get has the following form

$$\mu \cdot \Delta_S H + 2\mu \cdot H \left(H^2 - K\right) + 2H + l_p K = \lambda + \frac{1}{\sigma}\Gamma \cdot \rho.$$

Here Δ_S – Laplace – Beltrami operator corresponding to the metrical tensor of the surface, H – mean curvature, K – Gaussian curvature of equilibrium surface and l_p – the thickness of the intermediate layer. The right hand expression corresponds to the gravitational forces. The first two terms on the left correspond to the forces responsible for the formation of the intermediate layer.

Besides we get the formulas connecting the thickness of intermediate layer and contact angle.

When $\mu = l_p = 0$ we get classical result.

In the absence of gravitational forces the surfaces we get can be considered as the generalizations of the minimal surfaces including those indicated in the monograph of Nitszhe on minimal surfaces.

On the closure of the set of finitely supported smooth functions in anisotropic weighted Hölder spaces and de Rham complex over it

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We give a description of the closure of the set of finitely supported smooth functions ín anisotropic Hölder spaces weighted at the infinity with respect to the space variables over the strip $\mathbb{R}^n \times [0, T]$ with a finite time $T > 0$ and the dimension $n \geq 2$, cf. [1, §1.3] for the usual isotropic Hölder spaces over bounded open sets and compacts or [2] for the isotropic weighted Hölder spaces over the strip $\mathbb{R}^n$. We use the functions $(1 + |x|^2)^{\gamma/2}$, $\gamma > 0$, as the weights.

We also consider the de Rham differentials as bounded linear operators over the closure and the scale anisotropic weighted Hölder spaces itself. We indicate the range of weights such that the operators have the Fredholm property, cf. [3] for the Laplace operator in $\mathbb{R}^n$, [2] for the de Rham complex over isotropic weighted Hölder spaces and [4] for anisotropic weighted Hölder spaces corresponding to a very narrow range of the weight-indexes γ.

References

1. Tarkhanov N. *The Cauchy Problem for Solutions of Elliptic Equations*, Akademie-Verlag, Berlin, 1995.
2. Sidorova (Gagelgans) K. V., Shlapunov A. A. Mathematical Notes, 105:4 (2019), 604–617.
3. McOwen R. *Behavior of the Laplacian on weighted Sobolev spaces*, Comm. Pure Appl. Math. 32 (1979), 783–795.
4. Shlapunov A., Tarkhanov N. *An Open Mapping Theorem for the Navier-Stokes Equations*, Advances and Applications in Fluid Mechanics 21:2 (2018), 127-246.

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Weighted Sobolev space and capacity with Muckenhoupt weight

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Let Ω be an open subset of the real Euclidean n-dimensional space $\mathbb{R}^n$, $n \geq 2$, $1 < p < \infty$, $m = 1, 2, \ldots$ Below let w be the A_p-Muckenhoupt weight [1]. Then w is a p-admissible weight in the Heinonen-Kilpeläinen-Martio sense from [2].

Let $D^\alpha u$ be a distributional derivative of order $|\alpha| = \sum\limits_{j=1}^{m} \alpha_j$, $\alpha = (\alpha_1, \ldots, \alpha_m) \geq 0$. Here we set $\tilde{\nabla}_m u = \{D^\alpha u : 0 \leq |\alpha| \leq m\}$.

The Sobolev space $W^m_{p,w}(\Omega)$ is defined as

$$\left\{ u : \|u\|_{W^m_{p,w}(\Omega)} = \left(\int\limits_\Omega \Big(\sum_{0 \leq |\alpha| \leq m} (D^\alpha u)^2 \Big)^{p/2} w\, dx \right)^{1/p} < \infty \right\},$$

where dx is the element of the n-dimensional Lebesgue measure. The space $\overset{\circ}{W}{}^m_{p,w}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^m_{p,w}(\Omega)$.

Suppose that K is a compact subset of Ω. Let $W(K, \Omega) = \{u \in C_0^\infty(\Omega) : u = 1$ in some neighbourhood of $K\}$ and define

$$\mathrm{Cap}^m_{p,w}(K, \Omega) = \inf_{u \in W(K,\Omega)} \int_\Omega |\widetilde{\nabla}_m u|^p w\, dx.$$

For an arbitrary $E \subset \Omega$ we set $\mathrm{Cap}^m_{p,w}(E, \Omega) = \sup \mathrm{Cap}^m_{p,w}(K, \Omega)$ where supremum is taken over all compacts $K \subset E$. The number $\mathrm{Cap}^m_{p,w}(E, \Omega)$ is called the (m, p, w)-capacity of the condenser (E, Ω).

A set E in $\mathbb{R}^n$ is said to be of (m, p, w)-capacity zero if $\mathrm{Cap}^m_{p,w}(E, R^n) = 0$. In this case we write $\mathrm{Cap}^m_{p,w}(E) = 0$.

Let u_0 be an extremal for $\mathrm{Cap}^m_{p,w}(K, \Omega)$, i. e. such that $u_0 = \lim\limits_{j\to\infty} u_j$ in $W^m_{p,w}(\Omega)$ where $u_j \in W(K, \Omega)$ and $\int_\Omega |\widetilde{\nabla}_m u_0|^p w\, dx. = \mathrm{Cap}^m_{p,w}(K, \Omega)$ for a compact subset K of Ω.

Theorem 1. *Let $u \in W^m_{p,w}(\Omega)$. Then there is a sequence $u_j \in C^\infty(\Omega)$ such that $u = \lim\limits_{j\to\infty} u_j$ in $W^m_{p,w}(\Omega)$.*

Theorem 2. *A function $u \in W^m_{p,w}(\Omega)$ is extremal for $\mathrm{Cap}^m_{p,w}(K, \Omega)$ if and only if $\int\limits_\Omega |\widetilde{\nabla}_m u|^{p-2} \widetilde{\nabla}_m u \cdot \widetilde{\nabla}_m v\, w\, dx = 0$ for every $v \in C_0^\infty(\Omega)$ and $v = 0$ in some neighbourhood of compact K.*

Theorem 3. *Let E be a relatively closed subset of Ω. Then $\overset{\circ}{W}{}_w^{m,p}(\Omega) = \overset{\circ}{W}{}_w^{m,p}(\Omega \setminus E)$ if and only if E has a (m,p,w)-capacity zero.*

Theorem 4. *Suppose that E is a relatively closed subset of Ω. If E is of (m,p,w)-capacity zero or E is a $NC_{p,w}$-set* [3] *then $W_w^{m,p}(\Omega) = W_w^{m,p}(\Omega \setminus E)$.*

Theorem 5. *$W_w^{m,p}(\Omega) = \overset{\circ}{W}{}_w^{m,p}(\Omega)$ if and only if $\mathbb{R}^n \setminus \Omega$ has a (m,p,w)-capacity zero.*

Remark 1. *For $m = 1$ theorems 3–5 follow from the results of* [2, 4].

Remark 2. *The theorems 1–4 can be extended to the weighted spaces similar to W_p^m, L_p^m in* [5].

Remark 3. *The theorem 3 can be extended to the spaces similar to $H^{1,p}(\Omega, \mu)$ in* [2].

References

1. Muckenhoupt B. *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207–226.
2. Heinonen J., Kilpeläinen T., Martio O. *Nonlinear potential theory of degenerate elliptic equations*, New York: Dover Publications, 2006.
3. Dymchenko Yu. V., Shlyk V. A. *Sufficiency of broken lines in the modulus method and removable sets*, Sib. Mat. Zh. **51**:6 (2010), 1028—1042.
4. Kilpeläinen T. *Weighted Sobolev spaces and capacity*, Ann. Acad. Sci. Fenn. Ser. A. I. Math. **11** (1994), 95–113.
5. Mazya V. G. *Sobolev spaces*, L., 1985.

Types of growth points of a dendrite

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Let $\mathcal{S} = \{S_1, S_2, \dots S_m\}$ be a self-similar linear zipper [1] in $\mathbb{R}$, whose attractor K is $[0, 1]$.

We consider the question:

Can we add a similarity map S_* of the complex plane $\mathbb{C}$ to the system $\mathcal{S}$ in such a way, that the attractor K' of the resulting system $\mathcal{S}'$ would be a dendrite?

A point $c \in [0, 1]$ is called a *dendrite growth point*, if there is such $h > 0$, that the attractor K' of a system $\mathcal{S}' = \{S_1, S_2, \dots S_m, S_*\}$, where $S_*(z) = ihz + c$, is a dendrite.

We prove that there are 3 types of dendrite growth points, depending on their address $i_1 i_2 \dots i_n \dots$ in K:

(1) The point c has a periodic address $c = \pi(\overline{i_1 i_2 \dots i_n})$. Then c is a growth point if $c \neq 0, 1$ and $\mathcal{S}_{i_1 i_2 \dots i_n} = S_* S_A^k S_*^{-1}$ where S_A is some element of the semigroup generated by $\mathcal{S}$ which fixes the point 0. In this case K' is a p. c. f. dendrite [2].

(2) c has a preperiodic address $c = \pi(j_1 j_2 \dots j_k i_1 \overline{i_1 i_2 \dots i_n})$ then for sufficiently small h, depending on $j_1 j_2 \dots j_k$ only, K' is a p. c. f. dendrite.

(3) c has aperiodic address such that there is initial word $i_1 i_2 \dots i_n$ such that for any $k > 0$,

$$i_{k+1} i_{k+2} \dots i_{k+n} \neq i_1 i_2 \dots i_n.$$

Then there is $h > 0$ depending on $i_1 i_2 \dots i_n$ only for which K' is a non-p. c. f. dendrite.

References

1. Aseev V. V., Tetenov A. V., Kravchenko A. S. *On Self-Similar Jordan Curves on the Plane*, Sib. Math. J. **44**:3 (2003), 379–386.
2. Singh P., Tetenov A. *A Self-Similar Dendrite with One-Point Intersection and Infinite Post-Critical Set*, Adv. Theory Nonlinear Anal. Appl. **2** (2018) 81–84.

Invariant para-Sasakian structures on Lie groups

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In this paper, we consider left-invariant paracontact Sasaki structures on Lie groups, which are obtained by central extensions of almost para-Kähler Lie groups. The topic of paracontact structures is currently quite in great request, there are several different approaches to the definition of the concept of paracontact and para-Sasakian structures [1–3]. In this work, the para-Sasakian structures are defined according to the same scheme as the usual Sasaki structures in the case of contact structures [4].

Let M be a smooth $(2n+1)$-dimensional manifold. A 1-form η on M is called a *contact form* if $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. Then the pair (M, η) is called a *contact* manifold. The contact form η on M defines a rank $2n$ distribution $D = \{X \in TM \,|\, \eta(X) = 0\}$, which is called the *contact*. The contact manifold M has a vector field ξ, called the *Reeb field*, which is defined by the properties: $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for all vector fields X on M.

Definition 1. A paracontact structure on the manifold M is a triple (η, ξ, ϕ), where η is a contact form, ξ – the Reeb vector field and ϕ is an affinor on M with the property $\phi^2 = I - \eta \otimes \xi$. In addition, the affinor ϕ acts on contact distribution D as a para-complex structure.

Let M be the paracontact manifold. Consider the direct product $M \times \mathbb{R}$. A vector field on the $M \times \mathbb{R}$ can be represented as a pair $(X, f\partial_t)$, where X is a tangent to M, t is the coordinate on $\mathbb{R}$ and f is a function on the $M \times \mathbb{R}$. We define an almost para-complex structure J on the direct product $M \times \mathbb{R}$ as follows: $J(X, f\partial_t) = (\phi(X) - f\xi, -\eta(X)\partial_t)$.

Definition 2. Paracontact structure (η, ξ, ϕ) is called normal if the almost para-complex structure J on $M \times \mathbb{R}$ is integrable.

Definition 3. Let (M, η, ξ, ϕ) be the paracontact manifold. Then the paracontact metric structure is quadruple (η, ξ, ϕ, g), where g is the pseudo-Riemannian metric on M for which the following properties hold:

1. $d\eta(X, Y) = g(\phi X, Y)$;
2. $g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y)$.

Definition 4. A para-Sasakian manifold is a normal paracontact metric manifold.

If the manifold under consideration is a Lie group G, it is natural to consider the left-invariant contact structure. In this case, the contact form η, Reeb vector field ξ, affinor ϕ and associated metric g are defined by its values in the unit e, that is, on the Lie algebra $\mathfrak{g}$. We will call the $\mathfrak{g}$ a *contact Lie algebra* if it is defined in a contact form $\eta \in \mathfrak{g}^*$ and the vector $\xi \in \mathfrak{g}$, such that $\eta \wedge (d\eta)^n \neq 0$, $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$. Note that $d\eta(x, y) = -\eta([x, y])$. In a similar sense, it is considered a *contact metric* Lie algebra, *symplectic* Lie algebra, *para-Kähler* Lie algebra, *para-Sasakian* Lie algebra, etc.

Contact Lie algebra can be obtained as a result of the *central extension* of the symplectic Lie algebra $\mathfrak{h}$. Recall this procedure. If $(\mathfrak{h}, \omega)$ is the symplectic Lie algebra, then the central extension $\mathfrak{g} = \mathfrak{h} \times_\omega \mathbb{R}$ is a Lie algebra in which the Lie brackets are defined as follows:

$$[X, \xi]_{\mathfrak{g}} = 0, \qquad [X, Y]_{\mathfrak{g}} = [X, Y]_{\mathfrak{h}} + \omega(X, Y)\xi$$

for any $X, Y \in \mathfrak{h}$, where $\xi = \partial_t$ is the unit vector in $\mathbb{R}$.

On the Lie algebra $\mathfrak{g} = \mathfrak{h} \times_\omega \mathbb{R}$ contact form given by the form $\eta = \xi^*$, and $\xi = \partial_t$ is the Reeb field. If $x = X + \lambda\xi$ and $y = Y + \mu\xi$, where $X, Y \in \mathfrak{h}$, $\lambda, \mu \in \mathbb{R}$, then: $d\eta(x, y) = -\eta([x, y]) = -\xi^*\left([X, Y]_{\mathfrak{h}} + \omega(X, Y)\xi\right) = -\omega(X, Y)$.

To define the affinor ϕ on $\mathfrak{g} = \mathfrak{h} \times_\omega \mathbb{R}$ we can use an almost para-complex structure J on $\mathfrak{h}$ as follows: if $x = X + \lambda\xi$, where $X \in \mathfrak{h}$, then $\phi(x) = JX$. If this almost para-complex structure J on $\mathfrak{h}$ is also compatible with the ω, that is, it has the property of $\omega(JX, JY) = -\omega(X, Y)$, we will get the paracontact metric structure (η, ξ, ϕ, g) on $\mathfrak{g} = \mathfrak{h} \times_\omega \mathbb{R}$, where $g(X, Y) = d\eta(\phi X, Y) + \eta(X)\eta(Y)$. Let $h(X, Y) = \omega(X, JY)$ be a associated (pseudo) Riemannian metric on the symplectic Lie algebra $(\mathfrak{h}, \omega)$. Then for $x = X + \lambda\xi$ and $y = Y + \mu\xi$, we have: $g(x, y) = -\omega(JX, Y) + \lambda\mu = h(X, Y) + \lambda\mu$.

Theorem 1. *The paracontact metric structure (η, ξ, ϕ, g) on the central expansion $\mathfrak{g} = \mathfrak{h} \times_\omega \mathbb{R}$ is para-Sasakian if and only if the symplectic algebra $(\mathfrak{h}, \omega, J)$ is a para-Kähler.*

Recall that the curvature tensor is defined by the formula $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. The Ricci tensor *Ric* is a convolution of the curvature tensor R with respect to the first and fourth (upper) indices. For the pseudo-Riemannian metric g the Ricci tensor $Ric(X, Y)$ can also be determined by the formula: $Ric(X, Y) = \sum_i \varepsilon_i g(R(e_i, Y)Z, e_i)$, where $\{e_i\}$ is an orthonormal basis and $\varepsilon_i = g(e_i, e_i)$.

Theorem 2. *Let (ω, J, h) be a para-Kähler structure on the Lie algebra $\mathfrak{h}$ and (η, ξ, ϕ, g) be the corresponding para-Sasakian structure on the central expansion $\mathfrak{g} = \mathfrak{h} \times_\omega \mathbb{R}$. Then the curvature tensor R on $\mathfrak{g}$ is expressed in terms of the curvature tensor $R_{\mathfrak{h}}$ on $\mathfrak{h}$, the symplectic form ω and the para-complex structure*

J on $\mathfrak{h}$ as follows:

$$R(X,Y)Z = R_{\mathfrak{h}}(X,Y)Z + \frac{1}{4}\omega(Y,Z)JX - \frac{1}{4}\omega(X,Z)JY - \frac{1}{2}\omega(X,Y)JZ,$$

$$R(X,Y)\xi = 0, \qquad R(X,\xi)Z = \frac{1}{4}g(X,Z)\xi, \qquad R(X,\xi)\xi = -\frac{1}{4}X,$$

where $X, Y \in \mathfrak{h}$.

Theorem 3. *Let (ω, J, h) be a para-Kähler structure on the $2n$-dimensional Lie algebra $\mathfrak{h}$ and (η, ξ, ϕ, g) be the corresponding para-Sasakian structure on the central expansion $\mathfrak{g} = \mathfrak{h} \times_\omega \mathbb{R}$. Then the Ricci tensor Ric of $\mathfrak{g}$ is expressed in terms of the Ricci tensor $Ric_{\mathfrak{h}}$ on $\mathfrak{h}$ as follows:*

$$Ric(Y,Z) = Ric_{\mathfrak{h}}(Y,Z) + \frac{1}{2}h(Y,Z),$$
$$R(Y,\xi) = 0, \qquad R(\xi,\xi) = -\frac{n}{2}.$$

where $X, Y \in \mathfrak{h}$.

References

1. Bejan C. L., Eken S., Kılıç E. *Legendre Curves on Generalized Paracontact Metric Manifolds*, Bull. Malays. Math. Sci. Soc. 42 (2019), 185–199.
2. Prakasha D. G. *Para-Sasakian manifolds and *-Ricci solitons*, arXiv:1801.01727.
3. Alekseevsky D. V., Cortes V., Galaev A., Leistner T. *Cones over pseudo-Riemannian manifolds and their holonomy*, J. Reine Angew. Math 635 (2009), 23–69.
4. Blair D. E. *Contact Manifolds in Riemannian Geometry*, Springer Verlag, 1976.

Sharp estimates of sublinear and bilinear integral operators

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Denote $\mathfrak{M}$ the set of all Lebesgue measurable functions on $\mathbb{R}_+ := [0;\infty)$, and let $\mathfrak{M}^+ \subset \mathfrak{M}$ be the subset of all nonnegative functions.

For $0 < p \le \infty$ and $v \in \mathfrak{M}^+$ we define weighted Lebesgue space

$$L_v^p := \left\{ f \in \mathfrak{M} : \|f\|_{p,v} := \left(\int_0^\infty |f(x)|^p v(x) dx \right)^{\frac{1}{p}} < \infty \right\},$$

$$L_v^\infty := \left\{ f \in \mathfrak{M} : \|f\|_{\infty,v} := \operatorname*{ess\,sup}_{x \ge 0} v(x)|f(x)| < \infty \right\}.$$

Suppose that $u, v, w \in \mathfrak{M}^+$, $0 < q, p, r < \infty$, and a kernel $k : [0,\infty)^2 \to [0,\infty)$ is a Borel function satisfying the following Oinarov condition: $k(x,y) = 0$ for $0 \le x < y$ and $k(x,y) \ge 0$ for $0 \le y \le x$ and

$$D^{-1}((k(x,z) + k(z,y)) \le k(x,y) \le D((k(x,z) + k(z,y)),\ x \ge z \ge y \ge 0$$

with a constant $D \ge 1$ independent on x, z, y.

We consider the weighted inequality

$$\|Rf\|_{r,u} \le C\|f\|_{p,v},\ f \in \mathfrak{M}^+, \tag{1}$$

where the sublinear integral operator R has one of the following form:

$$Tf(x) := \left(\int_x^\infty k_1(y,x) w(y) \left(\int_0^y k_2(y,z) f(z) dz \right)^q dy \right)^{\frac{1}{q}},$$

$$\mathcal{T}f(x) := \left(\int_0^x k_1(x,y) w(y) \left(\int_y^\infty k_2(z,y) f(z) dz \right)^q dy \right)^{\frac{1}{q}},$$

$$Sf(x) := \left(\int_x^\infty k_1(y,x) w(y) \left(\int_y^\infty k_2(z,y) f(z) dz \right)^q dy \right)^{\frac{1}{q}},$$

$$\mathcal{S}f(x) := \left(\int_0^x k_1(x,y) w(y) \left(\int_0^y k_2(y,z) f(z) dz \right)^q dy \right)^{\frac{1}{q}}$$

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and give sharp two-sided estimates for a least possible constant C.

Analogous problem is solved for the inequality (1) restricted to the cone of monotone functions or, when operator R has a bilinear form including the case of Hardy-Steklov operators or multidimensional Hardy-type transform.

A talk is based on joint publications [1–10].

References

1. Prokhorov D. V., Stepanov V. D. *On weighted Hardy inequalities in mixed norms*, Proceedings Steklov Inst. Math. 283 (2013), 149–164.
2. Prokhorov D. V., Stepanov V. D. *Weighted inequalities for quasilinear integral operators on the semi-axis and applications to Lorentz spaces*, Sbornik: Mathematics 207:8 (2016), 1159–1186.
3. Stepanov V. D., Shambilova G. E. *Boundedness of quasilinear integral operators on the cone of monotone functions*, Siberian Math. J. 57:5 (2016), 884–904.
4. Stepanov V. D., Shambilova G. E. *On the Boundedness of Quasilinear Integral Operators of Iterated Type with Oinarov's Kernels on the Cone of Monotone Functions*, Eurasian Math. J. 8:2 (2017), 47–73.
5. Stepanov V. D., Shambilova G. E. *Reduction of weighted bilinear inequalities with integration operators on the cone of nondecreasing functions*, Siberian Math. J. 59:3 (2018), 505–522.
6. Stepanov V. D., Shambilova G. E. *On weighted iterated Hardy-type operators*, Anal. Math. 44:2 (2018), 273–283.
7. Stepanov V. D., Shambilova G. E. *Iterated integral operators on the cone of monotone functions*, Math. Notes 104:3 (2018), 443–453.
8. Jain P., Kanjilal S., Stepanov V. D., Ushakova E. P. *Bilinear Hardy-Steklov operators*, Math. Notes 104:6 (2018), 823–832.
9. Stepanov V. D., Shambilova G. E. *On Iterated and Bilinear integral Hardy-type operators*, Math. Inequal. Appl. (2018), to appear.
10. Stepanov V. D., Shambilova G. E. *Multidimensional bilinear Hardy inequalities*, Doklady Math. 100:1 (2019).

Variational field theory in Sobolev spaces

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In our paper we show that classical theory of Weierstrass–Hilbert can be strengthen. Precisely, for every field of extremals we show that if an extremal is an element of the field then it gives minimum in the class of Sobolev functions with the same boundary data and with the graphs in the set covered by the field. This result remains valid if one of the extremals is singular. In the case there is a field containing more then one singular extremal we can claim that each such extremal gives problem for which there are no solution in the class of admissible Lipschitz functions.

On families of subarcs in non-PCF dendrites

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Post-critically finite (PCF) self-similar sets occupy significant position in the theory of self-similar sets. They have a very clear structure which allows to build productive models of analysis and differential equations. Such sets can also have very attractive geometric features: as it was proved by C. Bandt in [1], the set of dimensions of minimal subarcs of a PCF set is finite. This is also true for any postcritically finite self-similar dendrite K, and the set of cut points of such dendrite may be represented as a countable union of images of arcs γ_k, $k = 1, \ldots, n$ which are the components of attractor of some graph-directed IFS [2].

In this connection, much less is known about non-postcritically finite self-similar dendrites. Nevertheless, it turns out that in non-PCF dendrites which satisfy one-point intersection property the set of dimensions of subarcs may be also finite. We construct a sufficiently wide family of non-postcritically finite systems of contraction similarities $\mathcal{S} = \{S_0, S_1, S_2, S_3\}$, whose attractors K are dendrites, lying in a triangle $\Delta \subset \mathbb{R}^2$ with the vertices $(0,0), (1,0), (1/2, \sqrt{3}/2)$ and for which the following properties hold:

1. All subarcs $\gamma_{xy} \subset K, x \neq y$ and the set of cut points of the dendrite K have the same Hausdorff dimension s.

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2. The set of s-dimensional measures ℓ_{Ox} of paths connecting the point $O = \mathrm{Fix}(S_0)$ with the points $x \in K \cap [0,1]$ lying on the base of the triangle Δ is a self-similar Cantor discontinuum.

Since none of the arcs in the family can be represented as the attractor of a finite graph-directed IFS, the main difficulty is to show that these arcs have finite positive s-dimensional measure. We propose a method allowing to prove these positivity and finiteness properties for a wide class of infinitely generated self-similar arcs.

References

1. Bandt C., Stahnke J. *Self-similar sets 6. Interior distance on deterministic fractals*, preprint, Greifswald, 1990.
2. Samuel M., Tetenov A. V., Vaulin D. A. *Self-similar dendrites generated by polygonal systems in the plane*, Sib. El. Math. Rep. 14 (2017), 737–751.

Convexity in complex analysis and tropical geometry

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In relation to hypersurfaces in $\mathbb{R}^n$ the notion of k-convexity is determined by a number k equal to the number of nonnegative principal curvatures of the hypersurface. The number k is found from the Hessian quadratic form for the equation which defines the hypersurface. The complex analog of this notion is k-pseudoconvexity related to the properties of the Levi form for the hypersurface.

Following a classical theorem of Aumann, compact convex sets in a linear space can be characterized using a homological condition on sections by linear subspaces of a fixed dimension. Gromov intuited from this that such homological conditions for non-compact sets should be considered in the frame of the Lefschetz section property for smooth compact varieties. Using this idea Henriques introduced the following

Definition. A subset $X \in \mathbb{R}^n$ is called *k-convex* if the homomorphisms of reduced homology groups

$$i_\pi : \widetilde{H}_{k-1}(\pi \cap X) \to \widetilde{H}_{k-1}(X)$$

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are injective for each k-plane $\pi \subseteq \mathbb{R}^n$.

In the talk we consider questions of k-pseudoconvexity for complements of complex algebraic sets and synchronous questions of k-convexity for complements of amoebas of these sets. Recall that *an amoeba* of an algebraic set $V \subset (\mathbb{C} \setminus 0)^n$ is its image $\mathcal{A}_V$ in $\mathbb{R}^n$ under the logarithmic mapping

$$(z_1, \ldots, z_n) \to (\log|z_1|, \ldots, \log|z_n|).$$

In the case when V is a complete intersection (i.e. has dimension $n-k$ and is defined by k equations) the following holds true

Theorem (Bushueva-Tsikh). *The complement of V is $(k-1)$-pseudoconvex. The complement of the amoeba $\mathcal{A}_V$ is $(k-1)$-convex.*

Also, in the talk some results on relation between an amoeba $\mathcal{A}_V$ and a tropical variant of an algebraic set V in the spirit of the recent work [4] will be announced.

References

1. Aumann G., Annals of Mathematics 37:2 (1936), 443–447.
2. Gromov M., Rend. Sem. Mat. Fis. Milano 61 (1991), 9–123 (1994).
3. Henriques A., Adv. Geom, 4:1 (2004), 61–73.
4. Adiprasito K., Babaee F., Mathematische Annalen 373:1-2 (2019), 237–251.

Korevaar-Schoen energy on strongly rectifiable spaces

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We extend Korevaar-Schoen's theory of metric valued Sobolev maps to cover the case of the source space being an $\mathrm{RCD}(K, N)$ space. In this situation it appears that no version of the "subpartition lemma" holds: to obtain both existence of the limit of the approximated energies and the lower semicontinuity of the limit energy we shall rely on the fact that such spaces are "strongly rectifiable" a notion which is first-order in nature (as opposed to measure-contraction-like properties, which are of second order). When the target space is CAT(0) we can also identify the energy density as the Hilbert- Schmidt norm of the differential, in line with the smooth situation.

*based on the joint work with Nicola Gigli.

The convergence order of minimal and almost minimal cubature formulas

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On the unit cube $Q = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid 0 \le x_j < 1,\ j = 1, \ldots, n\}$, $n \ge 2$, we examine the cubature formulas of the form

$$\int\limits_Q \varphi(x)\,dx \approx \sum_{j=1}^{N} c_j \varphi(x^{(j)}), \qquad \sum_{j=1}^{N} c_j = 1. \tag{1}$$

Here $x^{(j)}$ are nodes of the formula, lying in Q, and c_j are its nonzero weights. It is assumed that any integrand $\varphi(x)$ is periodic with period 1 with respect to each of its variables, i.e. $\varphi(x + \gamma) = \varphi(x)$ for every vector γ in $\mathbb{Z}^n$. Examples of periodic functions are given by trigonometric polynomials. By definition, a trigonometric polynomial $\varphi(x)$ is a linear combination of exponents of the following form:

$$\varphi(x) = \sum_{\beta \in \mathbb{Z}^n} c[\beta] e^{-i2\pi\beta x}. \tag{2}$$

Here $\beta = (\beta_1, \ldots, \beta_n)$ is a multi-index, $\beta x = \sum\limits_{j=1}^{n} \beta_j x_j$, and only finitely many coefficients $c[\beta]$ are nonzero. The degree d of the polynomial (2) is defined by the equality $d = \max\{|\beta_1| + |\beta_2| + \ldots |\beta_n| \mid c[\beta] \ne 0\}$.

Let a cubature formula (1) be exact at all trigonometric polynomials up to degree d inclusive. If in addition there exists a trigonometric polynomial of degree $d + 1$ at which the formula is not exact then the formula (1) is said to have trigonometric degree d. An example of a cubature formula of degree d is given by the direct product of the quadrature formulas for rectangles constructed for the segments $0 \le x_j \le 1$, $j = 1, 2, \ldots, n$. Each of the factors in the direct product has equal weights and nodes distributed uniformly with meshsize h, where $1/h = d + 1$ is a natural number. The left end $x_j = 0$ of the segment belongs to the set of nodes, whereas the right end $x_j = 1$ is not a node. The corresponding direct product is defined by the relation

$$\int\limits_Q \varphi(x)\,dx \approx h^n \sum_{h\gamma \in Q} \varphi(h\gamma), \tag{3}$$

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where $\gamma \in \mathbb{Z}^n$. The meshsize h and the number of nodes N in the direct product are connected by the relation $Nh^n = 1$, i.e. $N = (d+1)^n$. The weights of the cubature formula are equal and their sum is equal to one. A number of investigations has aimed to construct cubature formulas of a given trigonometric degree d having less nodes than in (3). Among all cubature formulas of fixed trigonometric degree d *minimal and almost minimal cubature formulas* are distinguished. To give appropriate definition, we consider an extremal problem.

Let L be a finite-dimensional subspace in the space of continuous functions $C(\overline{Q})$, and let $\mathbf{\Sigma} = \mathbf{\Sigma}(L)$ be the set of all cubature formulas of the form (1), which are exact for every function in L. Given a cubature formula σ from $\mathbf{\Sigma}(L)$ with fixed number of nodes N, we define the value of a functional $\mathbf{N}$ by the equality $\mathbf{N}(\sigma) = N$. The problem is first to find the infimum $N_{\min}(L) = \inf\limits_{\sigma \in \mathbf{\Sigma}(L)} \mathbf{N}(\sigma)$, and, second, specify the cubature formula $\sigma_{\min}$ such that $\sigma_{\min} = \arg \inf\limits_{\sigma \in \mathbf{\Sigma}(L)} \mathbf{N}(\sigma)$.

As a finite-dimensional subspace L we can consider the space $\mathcal{T}_d$ of all trigonometric polynomials up to degree d, $L = \mathcal{T}_d$. There exist cubature formulas that are exact for all functions from $\mathcal{T}_d$. Moreover, the value of $N_{\min}(\mathcal{T}_d) = N_{\min}(d)$ allows the following two-sided estimate: $N_0(d) \le N_{\min}(d) \le (d+1)^n$. Here $N_0(d)$ is the so-called Möller's lower bound. If d is even then $N_0(d) = \dim \mathcal{T}_{d/2} = \sum_{s=0}^{d} 2^s C_n^s C_{d/2}^s$. Similar relationship holds in the case of an odd degree d.

A cubature formula σ of degree d is said to be minimal, if the number of its nodes coincides with Möller's lower bound, i.e. $\mathbf{N}(\sigma) = N_0(d)$. For some values of d and n, the exact equality holds $N_{\min}(d) = N_0(d)$, and the minimal cubature formula is simultaneously the solution to the formulated above extremal problem. At the same time, there exist values of d and n such that $N_{\min}(d) > N_0(d)$. In this case, a cubature formula with fixed number of nodes N is said to be almost minimal if N is equal or close to $N_{\min}(d)$.

Note that for $n = 1$ the above-mentioned extremal problem degenerates. In this case $N_{\min}(d) = N_0(d) = d + 1$, and the minimal quadrature formula coincides with the rectangular rule. For $n = 2$ and each odd d the problem is completely solved, and the minimal cubature formula has nodes twice smaller than the cubature formula of rectangles with the same degree d.

For an arbitrary dimension n, the above-formulated extremal problem is very difficult. For this reason, the process of its solution has been historically stretched, evolving gradually from the lower values of n to large. For $n \ge 3$ instead of minimal cubature formulas a number of authors have suggested take advantage of special series of cubature formulas, accurate on subspace $\mathcal{T}_d$ and having nodes in any number times smaller than the cubature formula of rectangles of the same degree d. The construction of series replacing the minimal cubature formulas is usually carried out by the number-theoretic methods. In the process of searching for series consisting of almost minimal formulas, in particular, cubature formulas of trigonometric degree d were obtained with number of nodes N, satisfying the inequality

$$N \leq \frac{(d+1)^n}{2^{n-1}} < (d+1)^n, \quad n \geq 2.$$

A cubature formula with such properties is said to be a formula of high trigonometric degree. The number of nodes in the cubature formula of high trigonometric degree d at least in 2^{n-1} times less than the cubature formula of rectangles with the same d. When $n = 3$ this is four times less nodes; when $n = 4$ this is eight times less nodes, etc. It is due to this property that the minimal and almost minimal cubature formulas are extremely attractive for practice. With an unlimited increase in the number of nodes N trigonometric degree $d = d(N)$ of minimal or almost minimal cubature formula also increases indefinitely. For the formula of high trigonometric degree the growth of $d = d(N)$ is limited by the lower bound

$$d(N) \geq 2^{1-1/n} N^{1/n}.$$

Theorem 1. *Let the number of nodes of the cubature formula of trigonometric degree d be such that $n^{n/2} < N < \frac{(d+1)^n}{2^{n-1}}$, $n \geq 4$. Moreover, the sums $\sum\limits_{j=1}^{N} |c_j|$ are uniformly bounded with respect to N, i.e. $K = \sup\limits_{N\geq 1} \sum\limits_{j=1}^{N} |c_j| < \infty$. Then for any function φ from the periodic Sobolev space $\widetilde{H}^s(Q)$, $s > n/2+1$, and for any real q such that $\frac{n}{2} < q < s-1$ the following error estimate is true*

$$\Big| \int\limits_Q \varphi(x)\, dx - \sum_{j=1}^{N} c_j \varphi(x^{(j)}) \Big| \leq M(n,s,q) K N^{-\alpha} \|\varphi \mid \widetilde{H}^s(Q)\|.$$

The positive constant $M(n,s,q)$ does not depend on φ and on the cubature formula (1). The order of convergence α is defined by $\alpha = \frac{2}{n}(s-q-1) > 0$.

The rate of decreasing error of the formula to zero depends on the value of power α. The maximum of power α in the conditions of Theorem 1 is achieved by $q = \frac{n}{2}$. Unfortunately, the transition in the error estimate to the limit as $q \to \frac{n}{2}$ does not lead to a meaningful result. In this case, the constant $M(n,s,q)$ increases indefinitely. For the same reason, the error estimate degenerates when $q \to s-1$.

Twisted simplicial groups and homology of categories

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The James construction $J(X)$ of a topological space X is underlying space of the free (topological) monoid generated by X. For a connected CW complex X, the space $J(X)$ has the same homotopy type as the loop space of the suspension of X. Our main motivation is to give a twisted version of the James construction.

We investigate simplicial complexes with twisted structure on vertices and small categories with twisted structure on objects. Then we introduce a twisted construction of simplicial groups for such simplicial complexes and small categories by varying faces and degeneracies from twisted data in the free product construction of simplicial groups, which gives a new construction of simplicial groups. The homotopy type of the resulting twisted simplicial groups is different from the untwisted case because of variation on faces and degeneracies. The main result determines the homotopy type of the twisted simplicial groups.

The talk is based on the joint work with Jingyan Li and Jie Wu [1].

References

1. Li J. Y., Vershinin V. V., Wu J. *Twisted simplicial groups and twisted homology of categories*, Homology Homotopy Appl. 19:2 (2017), 111–130.

Convergence of spline interpolation processes

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Assume that the values $f_i = f(x_i)$ of some function f are known at the knots of the partition

$$\Delta : a = x_0 < x_1 < \ldots < x_N = b$$

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and that a spline s of degree $2n-1$ interpolates $\{f_i\}$. Additional conditions are needed to define the interpolation spline unambiguously. In practice, there are many recipes for what boundary conditions should be used depending on the known additional information about f. The best results are achieved by using the values of the lowest $n-1$ derivatives at the endpoints of $[a, b]$, namely,

$$s^{(\nu)}(a) = f^{(\nu)}(a), \quad s^{(\nu)}(b) = f^{(\nu)}(b), \qquad \nu = 1, \ldots, n-1.$$

A spline with boundary conditions of this type is called complete. We will consider interpolation using complete splines.

We are interested in the problem of convergence of interpolation processes: Will the sequence of the derivatives of splines $s^{(k)}$ converge to the corresponding derivative $f^{(k)}$ for any function $f \in C^k[a, b]$, $0 \leq k \leq 2n-1$? If there is no convergence in the general case, then which limitations should be imposed on the sequence of meshes $\{\Delta\}$ in order to attain the convergence?

This problem was posed by Schoenberg at the first conference on spline approximation held in 1963 at Oberwolfach. Regarding minimum smoothness requirements for an interpolated function, we should note that, for $k < n-1$, in the complete spline problem some of the derivatives required to specify boundary conditions may not exist. In this case the values of the derivatives which are lacking will be replaced by some numerical values.

In 1975, de Boor conjectured [1] that convergence without restrictions is possible only for the two middle derivatives of orders $n-1$ and n, while for other derivatives and the splines themselves convergence requires restrictions on the sequence of meshes $\{\Delta\}$. He only proved this conjecture for lower derivatives $k = 0, \ldots, n-2$. Later, this author constructed an example of divergence for $k = 0, \ldots, n-2$ and $k = n+1, \ldots, 2n-1$.

De Boor's conjecture on the unconditional convergence of the nth derivative was proved by Shadrin in [2], while for the $(n-1)$st derivative it was proved by this author in [3] and [4] in the periodic case and in [5] for complete splines.

The history of the problem of convergence of spline interpolation processes, the main steps in its investigation, the results and related aspects are described in greater detail in the survey [6].

The convergence of $s^{(k)}$ to $f^{(k)}$ is understood as convergence in the uniform norm, that is, $\|s^{(k)} - f^{(k)}\|_{L_\infty} \to 0$ as $|\Delta| = \max_i(x_{i+1} - x_i) \to 0$.

As a rule, the value of the error function or its derivative is estimated in terms of the norm of the inverse of the matrix of a system used to determine some spline parameters. A good estimate for this norm allows us to speak about the convergence of the interpolation process for the function or the corresponding derivative. On the other hand, the divergence of the interpolation process means bad conditionality of the related matrix. An interesting question arises: Can we use the convergence of the interpolation process to tell us about a matrix having good conditionality and find an estimate for the norm of its inverse regardless of the mesh?

The author has previously proposed an approach to constructing interpolation splines by finding the coefficients of the expansions of the derivatives of the required splines in B-splines. Systems of linear equations with respect to the above parameters were written down, and the properties of the matrices of systems of this type were investigated (see [7]).

If the matrix $\mathbf{A}_k$, $0 \le k \le 2n-1$, is a matrix of the system of linear equations with respect to the coefficients of the expansions of kth derivative of the interpolating spline s in B-splines of the order $2n-k$, then our estimates for the deviations of derivatives of the interpolation splines from the corresponding derivatives of the functions under interpolation show that the convergence of the interpolation process for the kth derivative, with the smoothness assumptions $f \in C^k[a,b]$ guarantees that the max-norm of the inverse matrix to $\mathbf{A}_k$ is bounded.

And conversely [8], if for some sequence of meshes $\{\Delta\}$ the norms $\|\mathbf{A}_k^{-1}\|$ can be bounded by a mesh-independent constant, then $s^{(k)}$ converges to $f^{(k)}$ for $f \in C^k[a,b]$, and it is simultaneously equivalent to the convergence of $s^{(2n-k-1)}$ to $f^{(2n-k-1)}$ for $f \in C^{2n-k-1}[a,b]$ on the same mesh sequence $\{\Delta\}$.

We consider the problem of the convergence of interpolation processes in terms of the boundedness of the norms of the projectors that, to a given derivative of a function under interpolation, assign the corresponding derivative of an interpolation spline of degree $2n-1$.

References

1. de Boor C., *On bounding spline interpolation*, J. Approx. Theory 14:3 (1975), 191–203.
2. Shadrin A. Yu., *The L_∞-norm of the L_2-spline projector is bounded independently of the knot sequence: A proof of de Boor's conjecture*, Acta Math. 187:1 (2001), 59–137.
3. Volkov Yu. S., *Unconditional convergence of one more middle derivative for odd degree spline interpolation*, Dokl. Math. 71:2 (2005), 250–252.
4. Volkov Yu. S., *Inverse of cyclic band matrices and the convergence of interpolation processes for derivatives of periodic interpolation splines*, Numer. Anal. Appl. 3:3 (2010), 199–207.
5. Volkov Yu. S., *Convergence analysis of an interpolation process for the derivatives of a complete spline*, J. Math. Sci. 187:1 (2012), 101–114.
6. Volkov Yu. S., Subbotin Yu. N., *Fifty years of Schoenberg's problem on the convergence of spline interpolation*, Proc. Steklov Inst. Math. 288:1 suppl. (2015), 222–237.

7. Volkov Yu. S., *Totaly positive matrices in the methods of constructing interpolation splines of odd degree*, Siberian Adv. Math. 15:4 (2005), 96–125.

8. Volkov Yu. S., *Convergence of spline interpolation processes and conditionality of systems of equations for spline construction*, Sb. Math. 210:4 (2019), 550–564.

Conformal invariants: a look at computation and theory

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Conformal invariants (such as harmonic measure, extremal length, capacity of condensers and hyperbolic metric) are key tools of geometric function theory in the plane. But their application is limited because of lack of explicit formulas. In the cases where formulas exist, they may be challenging to use for paper and pen calculations. In the first part of the talk we report about recent progress in numerical computation of several conformal invariants [1–4] in the planar case. Experiments based on this work have led to a number of conjectures, which will be also discussed. In the second part of the talk we discuss efforts, due to many authors, to generalize the hyperbolic metric to $\mathbb{R}^n$, $n \geq 3$. Metrics sharing some (but not all) properties of the hyperbolic metric are called hyperbolic type metrics. Some of these hyperbolic type metrics were introduced by F. W. Gehring, J. Ferrand, A. Beardon, P. Seittenranta, P. Hästö: the quasihyperbolic, modulus, Apollonian, Möbius and generalized hyperbolic metrics, resp. Here we discuss a new hyperbolic type metric [5] and give an application to quasiconformal maps.

References

1. Dautova D., Nasyrov S., Vuorinen M. *Conformal module of the exterior of two rectilinear slits*, arXiv:1908.02459.

2. Nasser M., Vuorinen M. *Numerical computation of the capacity of generalized condensers*, arXiv:1908.03866.

3. Nasser M., Vuorinen M. *Computation of conformal invariants*, arXiv:1908.04533.

4. Kalmoun E. M., Nasser M., Vuorinen M., *Numerical computation of Mityuk's function and radius for circular/radial slit domains*, arXiv:1908.03874.

5. Fujimura M., Mocanu M., Vuorinen M. *Barrlund's distance function and quasiconformal maps.*, arXiv:1903.12475.

The quaternionic Monge–Ampère operator and closed positive currents on the Heisenberg group

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I will discuss the generalization of some of our previous results in the pluripotential theory on the quaternionic space $\mathbb{H}^n$ to the $(4n+1)$-dimensional Heisenberg group. We introduce notions of a plurisubharmonic function, the quaternionic Monge-Ampère operator, differential operators d_0 and d_1 and a closed positive current on the Heisenberg group. The quaternionic Monge-Ampère operator is the coefficient of $d_0 d_1 u \wedge \cdots \wedge d_0 d_1 u$. The Chern-Levine-Nirenberg type estimate, the existence of quaternionic Monge-Ampère measure for continuous quaternionic plurisubharmonic functions and the minimum principle for the quaternionic Monge-Ampère operator are established. Unlike the tangential Cauchy-Riemann operator $\overline{\partial}_b$ on the Heisenberg group which behaves badly as $\partial_b \overline{\partial}_b \neq -\overline{\partial}_b \partial_b$, the quaternionic counterpart d_0 and d_1 satisfy $d_0 d_1 = -d_1 d_0$. This is the main reason that we have a better theory for the quaternionic Monge-Ampère operator than the natural CR generalization of the complex Monge-Ampère operator: $\partial_b \overline{\partial}_b u \wedge \cdots \wedge \partial_b \overline{\partial}_b u$.

References

1. Wang W. *On the optimal control method in quaternionic analysis*, Bull. Sci. Math. 135:8 (2011), 988–1010.
2. Wan D., Wang W. *Viscosity solutions to quaternionic Monge-Ampère equations*, Nonlinear Anal. 140 (2016), 69–81.
3. Wan D., Wang W. *On quaternionic Monge-Ampère operator, closed positive currents and Lelong-Jensen type formula on the quaternionic space*, Bull. Sci. Math. 141:4 (2017), 267–311.
4. Wang W. *The linear algebra in the quaternionic pluripotential theory*, Linear Alg. Appl. 562 (2019), 223–241.
5. Wang W. *The quaternionic Monge-Ampère operator and closed positive currents on the Heisenberg group*, preprint, 2019.

Smooth parameterizations in dynamics, analysis, and Diophantine geometry

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Smooth parameterization consists of a subdivision of a mathematical object under consideration into simple pieces, and then parametric representation of each piece, while keeping control of high order derivatives. Main examples for this talk are C^k or analytic parameterizations of semi-algebraic and o-minimal sets.

We provide an overview of some results, open and recently solved problems on smooth parameterizations, and their applications in several apparently rather separated domains: Smooth Dynamics, Diophantine Geometry, and Analysis. This includes a short report on a remarkable progress, recently achieved in this (large) direction by two independent groups (G. Binyamini, D. Novikov, on one side, and R. Cluckers, J. Pila, A. Wilkie, on the other).

The structure of the results, open problems, and conjectures in each of these domains shows in many cases a remarkable similarity, which we plan to stress.

Finally, we plan to consider a special case of smooth parameterization: "doubling coverings" (or "conformal invariant Whitney coverings"), and "Doubling chains". We present some new results (joint with O. Friedland and R. Cluckers) on the complexity bounds for doubling coverings, doubling chains, and on the resulting bounds in Kobayashi metric and Doubling inequalities. We believe that some connections with quasi-conformal geometry may exist, and plan to shortly discuss this direction.

Gromov–Hausdorff closed spaces and approximate capture

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This work focuses on the possibility of transporting some structures from one metric compact space to another 'nearby' metric compact space. Our approach combines methods of both geometry and game theory. Game problems inevitably require some specific structures, e. g. a set of admissible trajectories or strategies, besides geometry occasionally considers similar objects. This study was initially motivated by one particular game problem, but it also essentially uses features of Gromov–Hausdorff distance and related notions. On this way, pursuit–evasion games theory turns out to be mutual benefit to geometry of geodesic spaces.

For the sake of brevity we consider a sufficient simple game problem called Lion and Man. It typically deals with an approximate capture, i. e., α-capture, because the pursuer typically is interested in reaching an α-neighbourhood of the evader. We design a way for transporting pursuers' winning strategies which keeps good enough radius of capture. This approach uses the idea of guidance, but it is applied for couples of spaces; in addition, we evolve some methods of Krasovskii and Subbotin to deal with general compact metric spaces, e. g., we do not need a smooth structure.

Let us formulate the main result omitting details and impose the required definitions in special sections.

Theorem 1. *Let two compact geodesic spaces be such that the Gromov–Hausdorff distance between them is not greater than ε. In Lion and Man game, existence of α-capture by a time T in one space implies existence of $\big(\alpha+(20T+8)\sqrt{\varepsilon}\big)$-capture by the time T in the other space.*

Notice that this result justifies replacing a space by finite graphs: for example, we can find ε-closed finite graph for a suitable ε and check 2α-capture in this graph.

Gromov-Hausdorff distance. Let us introduce the definition borrowed from the book [1, Def. 7.3.10.];

Definition 1. *Let X and Y be metric spaces. The Gromov–Hausdorff distance between them, denoted by $d_{\mathrm{GH}}(X,Y)$, is defined by the following relation. For an $r>0$, we have $d_{\mathrm{GH}}(X,Y)<r$ iff there exist a metric space Z and its subspaces X' and Y' that are isometric to X and Y, respectively, and such that the Hausdorff distance between X' and Y' (as subsets of Z) is less than r.*

*This study was supported by the Russian Science Foundation (project no. 17-11-01093).

In other words, this $d_{\mathrm{GH}}(X, Y)$ is the infimum of positive r for which the above-mentioned Z, X', and Y' exist.

Pursuit–evasion game. We consider Lion and Man game, which is a two-person pursuit-evasion game with equal players' admissible trajectories. More precisely, we assume that both players move in a metric space (X, ρ) and denote by $L(\cdot)$ and $M(\cdot)$ the trajectories of Lion and Man respectively. Both $L(\cdot)$ and $M(\cdot)$ must be 1-Lipschitz functions of time to the space X. Moreover we assume that the pursuer uses non-anticipative strategies against arbitrary possible movements of the evader, which means that the evader might know pursuer's strategy and can choose his trajectory using this information. By non-anticipative strategies we mean the following. Let $1\mathrm{Lip}(X)$ denotes the set of all 1-Lipschitz curves from $\mathbb{R}_+$ to X, i. e. the set of admissible trajectories.

Definition 2. *A map $\mathfrak{s}_{L_0} : 1\mathrm{Lip}(X) \to 1\mathrm{Lip}(X)$ is called Lion's non-anticipative strategy (with Lion's initial position L_0) if it satisfies the equality $\mathfrak{s}_{L_0}(M)(0) = L_0$ for all $M \in 1\mathrm{Lip}(X)$ and the following implication holds true: for admissible trajectories $M_1, M_2 \in 1\mathrm{Lip}(X)$ and a number $\tau \geq 0$,*

$$\begin{array}{lll} \textit{if} & M_1(t) = M_2(t) & \forall t \in [0, \tau], \\ \textit{then} & \mathfrak{s}_{L_0}(M_1)(t) = \mathfrak{s}_{L_0}(M_2)(t) & \forall t \in [0, \tau]. \end{array}$$

Connection. Let us notice that transporting of pursuer's winning strategy from one space (say X) to another (say Y) means that each evader's admissible trajectory in Y corresponds to pursuer's one; so we should match it with a path in X, find a pursuer's response in X, and return it into Y. To realise this approach we design the special construction related with Gromov–Hausdorff distance and distortions of functions. More details and proofs are presented in the author's work [2].

References

1. Burago D. et al. *A course in metric geometry*, American Math. Soc., 2001. V. 33.
2. Yufereva O. *Robustness of capture radius with respect to Gromov–Hausdorff distance in Lion and Man game*, preprint arXiv:1806.03667 (2018).

cc-uniformity and hyperspaces on canonical Engel group

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Let us consider the standard euclidean vector space $\mathbb{R}^4$. We suppose that every point $u \in \mathbb{R}^4$ can be represented as $u = xe_1 + ye_2 + te_3 + ze_4$ where e_1, e_2, e_3, e_4 are standart Euclidean orts. Engel canonical group $\mathbb{E}_{\alpha,\beta}$ (see, for example, [1]) is defined in such $\mathbb{R}^4$ using the table of commutators

$$[e_1, e_2] = \alpha e_3, \quad [e_1, e_3] = \beta e_4, \quad \alpha, \beta > 0; \ [e_1, e_4] = [e_2, e_3] = [e_2, e_4] = [e_3, e_4] = 0.$$

Carnot-Carathéodory metric d_{cc} of $\mathbb{E}_{\alpha,\beta}$ (induced by e_1, e_2) is

$$d_{cc}(u, v) = \inf_{\gamma}\{l(\gamma) \mid \gamma \text{ — horizontal path connected points } u, v \in \mathbb{E}_{\alpha,\beta}\},$$

where $l(\gamma)$ is the length of γ defined via Riemannian scalar product (see, for example, [2]). Domain $\mathcal{D} \subset (\mathbb{E}_{\alpha,\beta}, d_{cc})$ is called cc-uniform if for every pair of points $u, v \in \mathcal{D}$ there exists a horizontal path $\gamma : l(\gamma) \to \mathcal{D}$, $\gamma \subset \mathcal{D}$, connecting points v, u, such that

$$\begin{cases} l(\gamma) \le ad_{cc}(u, v), \quad a = const, \\ \min\{l(\gamma_{u,w}), l(\gamma_{v,w})\} \le bd_{cc}(w, \partial\mathcal{D}), \quad b = const, \end{cases}$$

where $w \in \gamma$ is an arbitrary point, $\gamma_{u,w}$ is a part of γ from u to w, $\gamma_{v,w}$ is a part of γ from v to w, and constants a, b do not depend on u, v, w. We proved

Theorem. 1^0 *Hyperspace* $\{(x, y, t, z) \in \mathbb{E}_{\alpha,\beta} \mid t > 0\}$ *is cc-uniform domain,*
2^0 *hyperspace* $\{(x, y, t, z) \in \mathbb{E}_{\alpha,\beta} \mid z > 0\}$ *is not cc-uniform domain.*

Note that 2^0 can be proved using results of the work [3], but our method is different.

References

1. Greshnov A. V., Tryamkin M. V. *Exact values of constants in the generalized triangle inequality for some* $(1, q_2)$*-quasimetrics on canonical Carnot groups*, Math. Notes 98:4 (2015), 694–698.
2. Greshnov A. V. *Uniform domains and NTA-domains in Carnot groups*, Siberian Math. J. 42:5 (2001), 851–864.
3. Monti R., Morbidelli D. *Regular domains in homogeneous groups*, TAMS 357:8 (2005), 2975–3011.

SRA-free spaces and their applications

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Let $k \in \mathbb{N}$ and $0 < \alpha < 1$. We say that a metric space X is free of k-point SRA(α)-subspaces if for every k-point subset $Y \subset X$ there exist $x, z, y \in Y$ such that

$$d(x, y) > \max\{d(x, z) + \alpha\, d(z, y), \alpha\, d(x, z) + d(z, y)\}.$$

We say that a metric space is an SRA(α)-free space if it is free of k-point SRA(α)-subspaces for some $k \in \mathbb{N}$.

In the talk I will discuss examples of such spaces which are: finite dimensional Alexandrov spaces of non-negative curvature, complete Berwald spaces of non-negative flag curvature, Cayley Graphs of virtually abelian groups and doubling metric spaces of non-positive Busemann curvature with extendable geodesics. And also arbitrary big balls in complete, locally compact CAT(k)-spaces ($k \in \mathbb{R}$) with locally extendable geodesics, finite-dimensional Alexandrov spaces of curvature $\geq k$ with $k \in \mathbb{R}$ and complete Finsler manifolds satisfying the doubling condition.

I also will talk about applications of such spaces to (a) non-embeddability of snowflakes into SRA(α)-free spaces, (b) rectifiability of bounded self-contracted curves in SRA(α)-free spaces and (c) bi-Lipschitz embeddability of SRA(α)-free spaces into Euclidean spaces.

The talk is based on series of papers [1, 2, 3].

References

1. Zolotov V. *Sets with small angles in self-contracted curves*, preprint, arXiv:1804.00234 (2018).
2. Lebedeva N., Ohta S., Zolotov V. *Self-contracted curves in spaces with weak lower curvature bound*, preprint, arXiv:1902.01594 (2019).
3. Zolotov V. *Bi-Lipschitz embeddings of* SRA-*free spaces into Euclidean spaces*, preprint, arXiv:1906.02477 (2019).

О структуре множества гиперболических прямоугольных многогранников

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Исследование прямоугольных многогранников в пространстве Лобачевского восходит к работе Погорелова [1], где рассматривался вопрос их существования. Метод построения замкнутых, как ориентируемых, так и неориентируемых, трехмерных гиперболических многообразий из ограниченных гиперболических многогранников, описан в [2]. Эти же многогранники играют важную роль в торической топологии, поскольку для строящихся из них многообразий имеет место когомологическая жесткость [3].

Вопрос об описании начального множества ограниченных прямоугольных гиперболических многогранников исследовался в [4].

В докладе будут представлены некоторые известные результаты о структуре множества ограниченных прямоугольных гиперболических многогранников и новые результаты о структуре множества идеальных (со всеми вершинами на абсолюте) прямоугольных гиперболических многогранников, полученные в [5].

References

1. Погорелов А. В. *О правильном разбиении пространства Лобачевского*, Матем. заметки 1:1 (1967), 3–8.
2. Веснин А. Ю. *Прямоугольные многогранники и трехмерные гиперболические многообразия*, Успехи математических наук 72(2) (2017), 147–190.
3. Бухштабер В. М., Ероховец Н. Ю., Масуда М., Панов Т. Е., Пак С. *Когомологическая жесткость многообразий, задаваемых трехмерными многогранниками*, Успехи математических наук 72(2) (2017), 3–66.
4. Inoue T. *Exploring the list of smallest right-angled hyperbolic polyhedra*, Experimental Mathematics, Published online April 11, 2019. DOI:10.1080/10586458.2019.1593897.
5. Веснин А. Ю., Иванов А. А. *Идеальные прямоугольные многогранника в пространстве Лобачевского*, Препринт, 2019, 21 стр.

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