Cumulative structure of a Boolean-valued model of set theory

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A Boolean-valued algebraic system of set-theoretic signature $\{=,\in\}$ is a nonempty class X endowed with Boolean-valued interpretations of the signature symbols which are functions $=_X, \in_X \colon X^2 \to B$ taking values in a complete Boolean algebra B and satisfying the analogs of the classical axioms of equality, such as

$$\in_X(x,y) \wedge =_X(y,z) \leqslant \in_X(x,z)$$

(see [1, 3.1]). By means of the operations in B of supremum $a \vee b$, infimum $a \wedge b$, and complement $\neg b$, as well as suprema $\sup A$ and infima $\inf A$ of subsets $A \subset B$, the truth value $[\varphi(\bar{x})]_X \in B$ in X at $\bar{x} = x_1, \ldots, x_n \in X$ is recursively defined for an arbitrary formula φ of the first-order language of signature $\{=,\in\}$. In the case $[\varphi(\bar{x})]_X = 1_B$, the assertion $\varphi(\bar{x})$ is said to be *true in* X, which fact is denoted as $X \vDash \varphi(\bar{x})$.

A simple and natural example of a Boolean-valued algebraic system is the class $\mathbb{V}^S$ of all functions defined on a nonempty set S with the interpretations

$$=_{\mathbb{V}^S}(x,y) = \{s \in S : x(s) = y(s)\},$$
$$\in_{\mathbb{V}^S}(x,y) = \{s \in S : x(s) \in y(s)\},$$

which take values in the Boolean algebra $\mathcal{P}(S)$ of all subsets of S. In this case, the truth value of every formula can be calculated pointwise:

$$[\varphi(x_1,\ldots,x_n)]_{\mathbb{V}^S} = \big\{s \in S : \varphi\big(x_1(s),\ldots,x_n(s)\big)\big\}.$$

A more general function example of a Boolean-valued system is the class $C(Q, V^Q)$ of continuous sections of a bundle V^Q of models of set theory over an extremally disconnected compact space Q (see [2; 3, Ch. 6]).

Let X be a Boolean-valued system with truth algebra B. A *Boolean-valued class* in X is a function $\Phi\colon X \to B$ subject to the relation

$$\Phi(x) \wedge [x = y]_X \leqslant \Phi(y)$$

*The work was supported by the program of fundamental scientific researches of the SB RAS № I.1.2, project № 0314-2019-0005.

for all $x, y \in X$ (see [1, 3.5; 3, 4.6.1]). Such natural agreements as

$$[x \in \Phi]_X = \Phi(x), \quad [x = \Phi]_X = [(\forall\, y)(y \in x \Leftrightarrow y \in \Phi)]_X$$

make it possible to use Boolean-valued classes inside the truth value expressions, analogously to the use of classes in the language of set theory. Say that an *element* $x \in X$ *represents a Boolean-valued class* Φ and write $x \simeq \Phi$ if $X \vDash (x = \Phi)$.

Let B^X be the class of all functions $F\colon \operatorname{dom} F \to B$ on subsets $\operatorname{dom} F \subset X$. The *ascent* of $F \in B^X$ is the Boolean-valued class $F{\uparrow}\colon X \to B$ defined as follows:

$$F{\uparrow}(x) = \bigvee_{y \in \operatorname{dom} F} [x = y]_X \wedge F(y).$$

An element $x \in X$ is the *mixing* of a family $(x_i)_{i \in I} \subset X$ with respect to a partition of unity $(d_i)_{i \in I} \subset B$ whenever $[x = x_i]_X \geqslant d_i$ for all $i \in I$. The symbol $\operatorname{mix} Y$ denotes the totality of various mixings of elements of a subset $Y \subset X$.

A Boolean-valued algebraic system X with truth algebra B is called a *Boolean-valued universe* (see [1, 3.4]) or a *B-valued universe*, if it meets the following five conditions:

(1) $(\forall\, x, y \in X)\big(X \vDash (x = y) \Rightarrow x = y\big)$;
(2) $(\forall\, F \in B^X)(\exists\, x \in X)(x \simeq F{\uparrow})$;
(3) $(\forall\, x \in X)(\exists\, F \in B^X)(x \simeq F{\uparrow})$;
(4) $X \vDash (\forall\, x, y)\big((\forall\, z)(z \in x \Leftrightarrow z \in y) \Rightarrow x = y\big)$;
(5) $X \vDash (\forall\, x)\big((\exists\, y)(y \in x) \Rightarrow (\exists\, y \in x)(\forall\, z \in x)(z \notin y)\big)$.

As is known (see [1, 3]), for every complete Boolean algebra B, there exists a B-valued universe $\mathbb{V}^{(B)}$ which occurs a model of ZFC: if φ is a theorem of ZFC then the assertion $\mathbb{V}^{(B)} \vDash \varphi$ is also a theorem of ZFC.

In the research paper [4] under announcement, we show that every B-valued universe X has the following multilevel structure analogous to the von Neumann cumulative hierarchy:

$X_0 = \varnothing$;
$X_{\alpha+1} = \big\{x \in X : x \simeq F{\uparrow},\ F \in B^{X_\alpha}\big\}$ for every ordinal α;
$X_\alpha = \bigcup_{\beta<\alpha} X_\beta$ for every limit ordinal α;
$X = \bigcup_{\alpha \in \mathrm{Ord}} X_\alpha$, where Ord is the class of ordinals.

Another cumulative structure is obtained if we consider the ascents of constant functions only and add mixings at the limit steps:

$Y_0 = \varnothing$;
$Y_{\alpha+1} = \big\{x \in X : x \simeq (D \times \{b\}){\uparrow},\ D \subset Y_\alpha,\ b \in B\big\}$ for every ordinal α;
$Y_\alpha = \operatorname{mix} \bigcup_{\beta<\alpha} Y_\beta$ for every limit ordinal α;
$X = \bigcup_{\alpha \in \mathrm{Ord}} Y_\alpha$.

Such cumulative hierarchies clarify the structure of Boolean-valued systems and, in particular, make it possible to easily prove the uniqueness of a Boolean-valued universe up to isomorphism.

In addition, [4] contains a general tool for adding ascents to Boolean-valued systems which builds the hierarchy $(X_\alpha)_{\alpha\in\mathrm{Ord}}$ for an arbitrary system X_0 satisfying (4) and enlarges X_0 to a system $X = \bigcup_{\alpha\in\mathrm{Ord}} X_\alpha$ subject to (2)–(4), as well as to (1) and (5) as soon as X_0 meets the corresponding requirements. This makes it possible to construct examples of Boolean-valued systems with unusual properties. By means of the tool, we show in [4] that each of the five conditions (1)–(5) listed in the definition of a Boolean-valued universe, is essential and does not follow from the other conditions.

References

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