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BOOLEAN-VALUED UNIVERSE AS AN ALGEBRAIC SYSTEM. II: INTENSIONAL HIERARCHIES

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Abstract: Under study are the notions of transitivity, regularity, and σ-regularity for Boolean-valued algebraic systems of set-theoretic signature. The notion of a universe over an arbitrary extensional Boolean-valued system is introduced. Some description is proposed for the structure of the universe by means of various hierarchies. The results are used for proving the uniqueness of a Boolean-valued universe up to a unique isomorphism.

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The article continues [1] and presents the second part of the study of Boolean-valued algebraic systems of set-theoretic signature.

In the first part, we developed the apparatus of partial elements of a Boolean-valued system, studied the predicative Boolean-valued classes that admit quantification, and proposed an interpretation of the mixing principle in terms of the joins of antichains of partial elements. We also established the relationship between cyclicity and the attainment of a maximum by extensional Boolean-valued functions.

We enumerate the sections in the second part starting from 7, thus continuing the enumeration of [1]. In § 7, we study the notion of transitive Boolean-valued subsystem and prove an analog of Lévy's Lemma on the absoluteness of bounded formulas for transitive models. In § 8, the syntax of truth values is extended by quantifiers over predicative Boolean-valued classes. In § 9, we study the notion of regular Boolean-valued system and describe the Boolean algebras B for which the classes of regular and σ-regular B-systems coincide. In § 10, we introduce the notion of Boolean-valued universe over an arbitrary extensional Boolean-valued system and establish a close interrelation between such a universe and an intensional hierarchy, a Boolean-valued analog of the von Neumann cumulative hierarchy. In § 11, we characterize the classical Boolean-valued universe $\mathbb{V}^{(B)}$ as an algebraic system, prove the uniqueness of such a system up to a unique isomorphism, and, for every complete Boolean algebra B, demonstrate the logical independence of the conditions involved in the definition of B-valued universe. In § 12, we describe the structure of the Boolean-valued universe by means of various hierarchies.

When referring to any subsections of Sections 1–6, we implicitly presume the article [1]: For example, a reference to 2.8 means [1, 2.8].

Throughout the sequel, B is an arbitrary complete Boolean algebra, $\mathbb{N} = \{1, 2, \dots\}$ is the set of naturals, and $\omega = \{0, 1, 2, \dots\}$ is the least infinite ordinal. The class of all ordinals is denoted by Ord; and the class of all limit ordinals, by Lim Ord. Moreover, we use the notation

$$\mathrm{Ord}^\bullet := \mathrm{Ord} \cup \{\infty\}, \quad \mathrm{Lim\,Ord}^\bullet := \mathrm{Lim\,Ord} \cup \{\infty\},$$

where $\alpha < \infty$ for all $\alpha \in \mathrm{Ord}$.

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§ 7. Transitivity

A set or a class Y is called *transitive* if

$$(\forall x, y)(x \in y \in Y \;\Rightarrow\; x \in Y).$$

Transitive classes are traditionally used in set theory as models of fragments and modifications of set theory itself. Lévy's Lemma on the absoluteness of bounded formulas is a useful tool in working with such models. In this section, we study the notion of transitive Boolean subsystem and prove the corresponding analog of Lévy's Lemma.

7.1. If Y is a subsystem of a B-system X (see 2.8) then the notation ${}^{\%}Y$ can be understood in two ways: as the set $(Y \times B)/{\sim}$ of partial elements of Y (see 3.1) or as the subclass $\{x|_b : x \in Y,\ b \in B\} \subset {}^{\%}X$. The choice is immaterial in this case since the correspondence

$$\sim_Y(y, b) \;\mapsto\; \sim_X(y, b), \quad y \in Y,\ b \in B,$$

is a natural embedding of ${}^{\%}Y = (Y \times B)/{\sim}$ into ${}^{\%}X = (X \times B)/{\sim}$, and we may always assume that ${}^{\%}Y \subset {}^{\%}X$.

An analogous ambiguity appears in interpreting the symbol $P{\uparrow}$ for $P \subset {}^{\%}Y$ (see 3.5). Within the system X or Y, the Boolean-valued class $P{\uparrow}$ is the function $P{\uparrow}_X : X \to B$ or $P{\uparrow}_Y : Y \to B$ respectively. But also in this case the choice of an interpretation is immaterial since $P{\uparrow}_X$ and $P{\uparrow}_Y$ coincide on Y; therefore, the validity of the inclusion $[y \in P{\uparrow}]$ for $y \in Y$ does not depend on the system in which it is calculated. Indeed, if $y \in Y$ and $P = \{y_i|_{b_i} : i \in I\}$, where $y_i \in Y$ and $b_i \in B$ then

$$\begin{aligned} [y \in P{\uparrow}_Y]_Y &= P{\uparrow}_Y(y) = \bigvee_{i \in I} [y = y_i|_{b_i}]_Y = \bigvee_{i \in I} [y = y_i]_Y \wedge b_i \\ &= \bigvee_{i \in I} [y = y_i]_X \wedge b_i = \bigvee_{i \in I} [y = y_i|_{b_i}]_X = P{\uparrow}_X(y) = [y \in P{\uparrow}_X]_X. \end{aligned}$$

For this reason, we will simply write $[y \in P{\uparrow}]$ instead of $[y \in P{\uparrow}_Y]_Y$ or $[y \in P{\uparrow}_X]_X$ and sometimes add the indices X and Y just to specify the system where the calculation is carried out.

Slightly more accuracy is needed in dealing with the formula $y \simeq P{\uparrow}$ for $y \in Y$ and $P \subset {}^{\%}Y$ (see 3.19). The relation $y \simeq P{\uparrow}_X$ in X implies the analogous relation $y \simeq P{\uparrow}_Y$ in Y but, as is shown in 7.3, the converse holds for all $y \in Y$ and $P \subset {}^{\%}Y$ if and only if the subsystem $Y \subset X$ is transitive.

7.2. Let X be an arbitrary B-system. Call a subclass $Y \subset X$ *transitive* in X if the ascent $Y{\uparrow}$ is transitive inside X:

$$X \vDash (\forall x, y)(x \in y \in Y{\uparrow} \;\Rightarrow\; x \in Y{\uparrow}).$$

We say that Y is a *transitive subsystem* of X and write $Y \preccurlyeq X$ if Y is a nonempty transitive subclass of X endowed with the interpretations induced from X (see 2.8).

The following properties of a subclass $Y \subset X$ are equivalent:

(a) *Y is transitive in X;*

(b) $(\forall y \in Y)\ X \vDash (y \subset Y{\uparrow})$;

(c) $(\forall x \in X)(\forall y \in Y)\ [x \in y] \leqslant [x \in Y{\uparrow}]$;

(d) $(\forall x \in X)(\forall y \in Y)\ [x \in y] = \bigvee_{z \in Y} [x = z] \wedge [z \in y]$.

◁ The implications (a)⇒(b)⇔(c)⇐(d) are obvious; (b)⇒(a) stems from Lemma 3.18(d).

(c)⇒(d): By (c) and 2.4, if $x \in X$ and $y \in Y$ then

$$\bigvee_{z \in Y} [x = z] \wedge [z \in y] \leqslant [x \in y] \leqslant [x \in Y{\uparrow}] \wedge [x \in y] = \bigvee_{z \in Y} [x = z] \wedge [x \in y] = \bigvee_{z \in Y} [x = z] \wedge [z \in y]. \;▷$$

7.3. *Let Y be a subsystem of a B-system X. The following are equivalent:*
(a) $X \vDash (y = P{\uparrow}_X) \Leftrightarrow Y \vDash (y = P{\uparrow}_Y)$ *for every element* $y \in Y$ *and an arbitrary class* $P \subset {}^{\%}Y$*;*
(b) $X \vDash (y = y{\Downarrow}_Y{\uparrow}_X)$ *for all* $y \in Y$*;*
(c) $Y \preccurlyeq X$.

$\lhd$ (a)$\Rightarrow$(b): By Lemma 4.3, if $y \in Y$ then $Y \vDash (y = y{\Downarrow}_Y{\uparrow}_Y)$, and so $X \vDash (y = y{\Downarrow}_Y{\uparrow}_X)$ due to (a).
The implication (b)$\Rightarrow$(c) stems from 7.2(b) and Lemma 3.17(b).
(c)$\Rightarrow$(a): Let $Y \preccurlyeq X$, $y \in Y$, and $P \subset {}^{\%}Y$. From 7.2(b) and Lemma 3.17(b) it follows that $X \vDash (y \subset Y{\uparrow}_X,\ P{\uparrow}_X \subset Y{\uparrow}_X)$, and so

$$X \vDash (y = y \cap Y{\uparrow}_X,\ P{\uparrow}_X = P{\uparrow}_X \cap Y{\uparrow}_X).$$

Therefore, from Lemma 3.18(d) we deduce that

$$\begin{gathered} X \vDash (y = P{\uparrow}_X) \Leftrightarrow X \vDash (y \cap Y{\uparrow}_X = P{\uparrow}_X \cap Y{\uparrow}_X) \Leftrightarrow X \vDash (\forall z \in Y{\uparrow}_X)(z \in y \Leftrightarrow z \in P{\uparrow}_X) \\ \Leftrightarrow (\forall z \in Y)\ X \vDash (z \in y \Leftrightarrow z \in P{\uparrow}_X) \Leftrightarrow (\forall z \in Y)\ Y \vDash (z \in y \Leftrightarrow z \in P{\uparrow}_Y) \\ \Leftrightarrow Y \vDash (\forall z)(z \in y \Leftrightarrow z \in P{\uparrow}_Y) \Leftrightarrow Y \vDash (y = P{\uparrow}_Y). \ \rhd \end{gathered}$$

7.4. Let φ be a formula of signature $\{=, \in\}$, and let U be a variable, a term, or a class symbol. The *relativization* of φ to U is the formula obtained from φ by replacing each quantifier $(\forall x)$ and $(\exists x)$, where x is an arbitrary variable, by the corresponding quantifier $(\forall x \in U)$ and $(\exists x \in U)$ (see [2, 12.6]). The relativization of φ to U is denoted by $U \vDash \varphi$ or, in more detail, $U \vDash \varphi(\bar{x})$, where $\bar{x} = x_1, \dots, x_n$ is the list containing all free variables of φ. The choice of the notation for relativization is based on the fact that, for every $\bar{x} \in U$, the assertion $U \vDash \varphi(\bar{x})$ is equivalent to the validity of $\varphi(\bar{x})$ in the two-valued algebraic system $(U, =_U, \in_U)$ with the standard interpretations of equality and containment.

7.5. Refer to a formula of signature $\{=, \in\}$ as *syntactically bounded* if it can be constructed from the atomic formulas $x = y$ and $x \in y$ by disjunction, negation, and some quantifiers of the form $(\exists x \in y)$, where x and y are variables. A formula φ is said to be *bounded* if there exists a syntactically bounded formula ψ such that $\vdash (\varphi \Leftrightarrow \psi)$; the latter means that the equivalence $\varphi \Leftrightarrow \psi$ is provable within the theory of predicates, i.e., $\varphi \Leftrightarrow \psi$ is a tautology. Bounded formulas are also referred to as formulas of class Σ_0, Π_0, or Δ_0. As is easy to see, if each occurrence of a quantifier in φ has the form $(\exists x \in y)$ or $(\forall x \in y)$ then φ is a bounded formula. As an example of a bounded formula serves the relativization $U \vDash \varphi$ of any formula φ to a variable U.

The following assertion is a consequence of the classical lemma by A. Lévy on the absoluteness of bounded formulas for transitive models [3, Lemma 34; 2, Lemma 12.9].

Lemma. *Let $\varphi(\bar{y})$ be a bounded formula with free variables $\bar{y} = y_1, \dots, y_n$. If X is a B-system and $Y \preccurlyeq X$ then*

$$(\forall \bar{y} \in Y)\ [\varphi(\bar{y})]_X = [\varphi(\bar{y})]_Y.$$

$\lhd$ Since Lévy's Lemma is proven within the theory of predicates without special axioms, it is a tautology. By 3.9, the conclusion of the lemma for the Boolean-valued class $Y{\uparrow}_X$, transitive in X, is valid inside X:

$$X \vDash (\forall \bar{y} \in Y{\uparrow}_X)\big(\varphi(\bar{y}) \Leftrightarrow Y{\uparrow}_X \vDash \varphi(\bar{y})\big);$$

and, in particular,

$$(\forall \bar{y} \in Y)\ [\varphi(\bar{y})]_X = [Y{\uparrow}_X \vDash \varphi(\bar{y})]_X,$$

where $Y{\uparrow}_X \vDash \varphi(\bar{y})$ is the relativization of $\varphi(\bar{y})$ to $Y{\uparrow}_X$. The equality

$$[Y{\uparrow}_X \vDash \varphi(\bar{y})]_X = [\varphi(\bar{y})]_Y$$

is easily proven by induction on the complexity of a syntactically bounded formula φ. $\rhd$

7.6. In what follows, we will need a stronger version of Lévy's Lemma that involves Boolean-valued classes.

Theorem. *Let φ be a bounded formula with free variables $y_1, \dots, y_n$, $z_1, \dots, z_m$. If X is a B-system and $Y \preccurlyeq X$ then*

$$\begin{aligned}&[\varphi(y_1, \dots, y_n, \Phi_1, \dots, \Phi_m)]_X = [\varphi(y_1, \dots, y_n, \Phi_1|_Y, \dots, \Phi_m|_Y)]_Y \\ &\text{for all } y_1, \dots, y_n \in Y,\ \Phi_1, \dots, \Phi_m \Subset X,\ \Phi_1, \dots, \Phi_m \leqslant Y{\uparrow}_X.\end{aligned} \tag{12}$$

◁ Refer to φ as an *absolute formula* if φ satisfies (12) for every partition of the list of free variables into the two groups $y_1, \dots, y_n$ and $z_1, \dots, z_m$. (Note that, owing to the presence of arbitrary Boolean-valued classes, condition (12) is in general an infinite assertion; see § 1.) It follows from 3.9 that, for proving the theorem, it suffices to prove the absoluteness of syntactically bounded formulas. To this end, we use the induction on the complexity of the formula.

Show that the atomic formulas are absolute. Assume that $y \in Y$, $\Phi, \Psi \Subset X$, and $\Phi, \Psi \leqslant Y{\uparrow}_X$. Then $X \vDash (y \subset Y{\uparrow}_X)$ due to the transitivity of the subsystem Y and, moreover, $X \vDash (\Phi, \Psi \subset Y{\uparrow}_X)$ by Lemma 3.17(a). Therefore, using Lemma 3.18, we have

$$[y \in \Phi]_X = \Phi(y) = \Phi|_Y(y) = [y \in \Phi|_Y]_Y;$$

$$\begin{aligned}[y = \Phi]_X &= [(\forall z)(z \in y \Leftrightarrow z \in \Phi)]_X = [(\forall z \in Y{\uparrow}_X)(z \in y \Leftrightarrow z \in \Phi)]_X = \bigwedge_{z \in Y} [z \in y]_X \Leftrightarrow_B [z \in \Phi]_X \\ &= \bigwedge_{z \in Y} [z \in y]_Y \Leftrightarrow_B [z \in \Phi|_Y]_Y = [(\forall z)(z \in y \Leftrightarrow z \in \Phi|_Y)]_Y = [y = \Phi|_Y]_Y;\end{aligned}$$

$$\begin{aligned}[\Phi \in y]_X &= [(\exists z)(z = \Phi \wedge z \in y)]_X = [(\exists z \in Y{\uparrow}_X)(z = \Phi \wedge z \in y)]_X = \bigvee_{z \in Y} [z = \Phi]_X \wedge [z \in y]_X \\ &= \bigvee_{z \in Y} [z = \Phi|_Y]_Y \wedge [z \in y]_Y = [(\exists z)(z = \Phi|_Y \wedge z \in y)]_Y = [\Phi|_Y \in y]_Y;\end{aligned}$$

$$\begin{aligned}[\Phi = \Psi]_X &= [(\forall z)(z \in \Phi \Leftrightarrow z \in \Psi)]_X = [(\forall z \in Y{\uparrow}_X)(z \in \Phi \Leftrightarrow z \in \Psi)]_X = \bigwedge_{z \in Y} [z \in \Phi]_X \Leftrightarrow_B [z \in \Psi]_X \\ &= \bigwedge_{z \in Y} [y \in \Phi|_Y]_Y \Leftrightarrow_B [z \in \Psi|_Y]_Y = [(\forall z)(z \in \Phi|_Y \Leftrightarrow z \in \Psi|_Y)]_Y = [\Phi|_Y = \Psi|_Y]_Y;\end{aligned}$$

$$\begin{aligned}[\Phi \in \Psi]_X &= [(\exists z)(z = \Phi \wedge z \in \Psi)]_X = [(\exists z \in Y{\uparrow}_X)(z = \Phi \wedge z \in \Psi)]_X = \bigvee_{z \in Y} [z = \Phi]_X \wedge [z \in \Psi]_X \\ &= \bigvee_{z \in Y} [z = \Phi|_Y]_Y \wedge [z \in \Psi|_Y]_Y = [(\exists z)(z = \Phi|_Y \wedge z \in \Psi|_Y)]_Y = [\Phi|_Y \in \Psi|_Y]_Y.\end{aligned}$$

As is easy to see, the absoluteness of formulas is preserved by negation and disjunction. It remains to show that the absoluteness of $\varphi(x, y, \bar{y}, \bar{z})$ implies the absoluteness of $\psi(y, \bar{y}, \bar{z}) := (\exists x \in y)\, \varphi(x, y, \bar{y}, \bar{z})$. Indeed, if $y, \bar{y} = y_1, \dots, y_n \in Y$, $\Phi, \overline{\Phi} = \Phi_1, \dots, \Phi_m \Subset X$, and $\Phi, \Phi_1, \dots, \Phi_m \leqslant Y{\uparrow}_X$; then, in view of the relations $X \vDash (y \subset Y{\uparrow}_X)$, $X \vDash (\Phi \subset Y{\uparrow}_X)$ and the absoluteness of φ, we have

$$\begin{aligned}[\psi(y, \bar{y}, \overline{\Phi})]_X &= \big[(\exists x)\big(x \in y \wedge \varphi(x, y, \bar{y}, \overline{\Phi})\big)\big]_X = \big[(\exists x \in Y{\uparrow}_X)\big(x \in y \wedge \varphi(x, y, \bar{y}, \overline{\Phi})\big)\big]_X \\ &= \bigvee_{x \in Y} [x \in y]_X \wedge [\varphi(x, y, \bar{y}, \overline{\Phi})]_X = \bigvee_{x \in Y} [x \in y]_Y \wedge [\varphi(x, y, \bar{y}, \overline{\Phi}|_Y)]_Y \\ &= \big[(\exists x)\big(x \in y \wedge \varphi(x, y, \bar{y}, \overline{\Phi}|_Y)\big)\big]_Y = [\psi(y, \bar{y}, \overline{\Phi}|_Y)]_Y;\end{aligned}$$

$$\begin{aligned}[\psi(\Phi, \bar{y}, \overline{\Phi})]_X &= \big[(\exists x)\big(x \in \Phi \wedge \varphi(x, \Phi, \bar{y}, \overline{\Phi})\big)\big]_X = \big[(\exists x \in Y{\uparrow}_X)\big(x \in \Phi \wedge \varphi(x, \Phi, \bar{y}, \overline{\Phi})\big)\big]_X \\ &= \bigvee_{x \in Y} [x \in \Phi]_X \wedge [\varphi(x, \Phi, \bar{y}, \overline{\Phi})]_X = \bigvee_{x \in Y} [x \in \Phi|_Y]_Y \wedge [\varphi(x, \Phi|_Y, \bar{y}, \overline{\Phi}|_Y)]_Y \\ &= \big[(\exists x)\big(x \in \Phi|_Y \wedge \varphi(x, \Phi|_Y, \bar{y}, \overline{\Phi}|_Y)\big)\big]_Y = [\psi(\Phi|_Y, \bar{y}, \overline{\Phi}|_Y)]_Y.\end{aligned}$$ ▷

7.7. Corollary. *If a B-system X is extensional and $Y \preccurlyeq X$ then Y is extensional.*

◁ By Definition 2.7, the extensionality of X means that $X \vDash (\forall\, x, y)\, \varphi(x, y)$, where

$$\varphi(x,y) \;:=\; (\forall\, z \in x)(z \in y) \wedge (\forall\, z \in y)(z \in x) \Rightarrow x = y.$$

Thus, if X is extensional then $X \vDash (\forall\, x, y \in Y{\uparrow}_X)\, \varphi(x,y)$, whence, by Theorem 7.6 and the boundedness of the formula $(\forall\, x, y \in u)\, \varphi(x,y)$, we have $Y \vDash \big(\forall\, x, y \in (Y{\uparrow}_X)|_Y\big)\, \varphi(x,y)$, i.e., $Y \vDash (\forall\, x, y)\, \varphi(x,y)$, which is equivalent to the extensionality of Y. ▷

7.8. Corollary. *Let $\varphi(y_1,\dots,y_n,z_1,\dots,z_m)$ be an arbitrary formula. If X is a B-system and $Y \preccurlyeq X$ then*

$$[Y{\uparrow}_X \vDash \varphi(y_1,\dots,y_n,\Phi_1,\dots,\Phi_m)]_X = [\varphi(y_1,\dots,y_n,\Phi_1|_Y,\dots,\Phi_m|_Y)]_Y$$

for all $y_1,\dots,y_n \in Y$, $\Phi_1,\dots,\Phi_m \Subset X$, $\Phi_1,\dots,\Phi_m \leqslant Y{\uparrow}_X$, where $Y{\uparrow}_X \vDash \varphi$ is the relativization of φ to $Y{\uparrow}_X$.

◁ Since the formula $x \vDash \varphi(\bar y, \bar z)$ is bounded, Theorem 7.6 implies

$$[Y{\uparrow}_X \vDash \varphi(\bar y, \overline{\Phi})]_X = [(Y{\uparrow}_X)|_Y \vDash \varphi(\bar y, \overline{\Phi}|_Y)]_Y = [Y{\uparrow}_Y \vDash \varphi(\bar y, \overline{\Phi}|_Y)]_Y = [\varphi(\bar y, \overline{\Phi}|_Y)]_Y. \;▷$$

7.9. Let I be a nonempty set or class. Call a family of B-systems $(X_i)_{i\in I}$ *directed* if for all $i, j \in I$ there is $k \in I$ such that X_i and X_j are subsystems of X_k. As is easy to see, on the union $X := \bigcup_{i\in I} X_i$ of such a family, there is a unique pair of functions $=_X, \in_X \colon X^2 \to B$ turning X into a B-system that contains all X_i's as subsystems. With this circumstance in mind, when considering a directed family of B-systems, agree to tacitly assume the union of the family to be a B-system.

7.10. Lemma. (a) *Let Y and Z be subsystems of a B-system X. If $Z \preccurlyeq X$ and $Z \subset Y$ then $Z \preccurlyeq Y$.*

(b) *If X, Y, and Z are B-systems and $Z \preccurlyeq Y \preccurlyeq X$ then $Z \preccurlyeq X$.*

(c) *Let $(X_i)_{i\in I}$ be a nonempty family of subsystems of a B-system X. If $X_i \preccurlyeq X$ for all $i \in I$ then $\bigcup_{i\in I} X_i \preccurlyeq X$.*

(d) *Let I be a nonempty directed ordered set or class, let $(X_i)_{i\in I}$ be a family of B-systems, and let $X_i \preccurlyeq X_j$ for $i \leqslant j$. Then $X_i \preccurlyeq \bigcup_{j\in I} X_j$ for all $i \in I$.*

(e) *Let $(X_\alpha)_{\alpha\in \mathrm{Ord}}$ be a family of B-systems, let $X_\alpha \preccurlyeq X_{\alpha+1}$ for $\alpha \in \mathrm{Ord}$, and let $\bigcup_{\beta<\alpha} X_\beta \preccurlyeq X_\alpha$ for $\alpha \in \mathrm{Lim\,Ord}$. Then $X_\gamma \preccurlyeq X_\beta \preccurlyeq \bigcup_{\alpha\in\mathrm{Ord}} X_\alpha$ for all $\gamma \leqslant \beta \in \mathrm{Ord}$.*

◁ (a) If $Z \subset Y \subset X$, $Z \preccurlyeq X$, $y \in Y$, and $z \in Z$, then $[y \in z] \leqslant [y \in Z{\uparrow}_X]_X = [y \in Z{\uparrow}_Y]_Y$.

(b) If $Z \preccurlyeq Y \preccurlyeq X$ then, using 7.2(d), for all $x \in X$ and $z \in Z$, we have

$$[x \in z] = \bigvee_{y\in Y} [x = y] \wedge [y \in z] \leqslant \bigvee_{y\in Y} [x = y] \wedge [y \in Z{\uparrow}] \leqslant [x \in Z{\uparrow}].$$

(c) If $x \in X$ and $y \in Y := \bigcup_{i\in I} X_i$ then $y \in X_i$ for some $i \in I$. Since $X_i \preccurlyeq X$, this implies that $[x \in y] \leqslant [x \in X_i{\uparrow}]_X \leqslant [x \in Y{\uparrow}]_X$.

(d) If $i \in I$, $x \in X := \bigcup_{j\in I} X_j$, and $y \in X_i$; then $x \in X_j$ for some $j \in I$, $i \leqslant j$. Since $X_i \preccurlyeq X_j$, it follows that $[x \in y] \leqslant [x \in X_i{\uparrow}]_{X_j} = [x \in X_i{\uparrow}]_X$.

(e) Given $0 \neq \alpha \in \mathrm{Ord}$, put $\mathscr{X}_\alpha := \bigcup_{\beta<\alpha} X_\beta$ and observe that $\mathscr{X}_\alpha \preccurlyeq X_\alpha$. Indeed, for $\alpha \in \mathrm{Lim\,Ord}$, this relation explicitly occurs in the hypothesis; and if $\alpha = \alpha_0 + 1$ then, by the obvious monotonicity of the family $(X_\alpha)_{\alpha\in\mathrm{Ord}}$, we have $\mathscr{X}_\alpha = \bigcup_{\beta<\alpha} X_\beta = \bigcup_{\beta\leqslant\alpha_0} X_\beta = X_{\alpha_0} \preccurlyeq X_{\alpha_0+1} = X_\alpha$.

Show by induction on α that $X_\beta \preccurlyeq X_\alpha$ for $\beta < \alpha$. Consider an arbitrary ordinal α, suppose that

$$X_\gamma \preccurlyeq X_\beta \;\text{ for }\; \gamma < \beta < \alpha, \tag{13}$$

and establish the relation $X_\beta \preccurlyeq X_\alpha$ for all $\beta < \alpha$. If $\beta < \alpha$ then, by (13) and (d), we have $X_\beta \preccurlyeq \mathscr{X}_\alpha$, whence, owing to $\mathscr{X}_\alpha \preccurlyeq X_\alpha$ and using (b), we have $X_\beta \preccurlyeq X_\alpha$.

It remains to observe that the above together with (d) implies $X_\beta \preccurlyeq \bigcup_{\alpha\in\mathrm{Ord}} X_\alpha$ for all $\beta \in \mathrm{Ord}$. ▷

§ 8. Quantification over Boolean-Valued Classes

In this section, the language of truth values is extended by quantifiers over predicative Boolean-valued classes, an analog of Lévy's Lemma is proven for formulas of the extended language, and the maximum principle is established for Boolean-valued classes.

8.1. Let φ be an arbitrary formula and let X be a B-system. Agree to write as $(\forall\,\Phi \Subset X)\,\varphi(\Phi)$ and $(\exists\,\Phi \Subset X)\,\varphi(\Phi)$ the assertions that $\varphi(\Phi)$ holds for every or, respectively, for some predicative Boolean-valued class $\Phi \Subset X$. Since predicative classes admit quantification (see 4.6), the assertions are not infinite and each of them can be written down by a single formula:

$$(\forall\,\Phi \Subset X)\,\varphi(\Phi) \;\Leftrightarrow\; (\forall\,P \subset {}^{\%}X)\,\varphi(P\!\uparrow), \quad (\exists\,\Phi \Subset X)\,\varphi(\Phi) \;\Leftrightarrow\; (\exists\,P \subset {}^{\%}X)\,\varphi(P\!\uparrow).$$

The expressions $\bigwedge_{\Phi \Subset X} \tau(\Phi)$ and $\bigvee_{\Phi \Subset X} \tau(\Phi)$ for a term $\tau(x)$ are defined similarly:

$$\bigwedge_{\Phi \Subset X} \tau(\Phi) \,=\, \bigwedge_{P \subset {}^{\%}X} \tau(P\!\uparrow), \qquad \bigvee_{\Phi \Subset X} \tau(\Phi) \,=\, \bigvee_{P \subset {}^{\%}X} \tau(P\!\uparrow).$$

8.2. Suppose that φ is a formula, X is a B-system, and $\Phi \Subset X$. Henceforth, in using the expression $[\varphi(\Phi,\dots)]_X$ or $X \vDash \varphi(\Phi,\dots)$, we mean that φ can contain several free variables, some of which are possibly replaced by symbols of Boolean-valued classes; i.e., the notation $\varphi(\Phi,\dots)$ serves as an abbreviation for $\varphi(\Phi, y_1, \dots, y_m, \Psi_1, \dots, \Psi_n)$, where $y_i \in X$ and $\Psi_j \Subset X$ are arbitrary preassigned elements and Boolean-valued classes.

Extend the syntax of Boolean-valued truth values by quantifiers over predicative Boolean-valued classes:

$$[(\boldsymbol{\forall}\,\Phi)\,\varphi(\Phi,\dots)]_X := \bigwedge_{\Phi \Subset X} [\varphi(\Phi,\dots)]_X, \quad [(\boldsymbol{\exists}\,\Phi)\,\varphi(\Phi,\dots)]_X := \bigvee_{\Phi \Subset X} [\varphi(\Phi,\dots)]_X.$$

As is easy to see, if X is extensional and satisfies the ascent principle then the quantifiers over classes in X are tantamount to the conventional quantifiers: $[(\boldsymbol{\forall}\,\Phi)\,\varphi(\Phi,\dots)]_X = [(\forall\,x)\,\varphi(x,\dots)]_X$, $[(\boldsymbol{\exists}\,\Phi)\,\varphi(\Phi,\dots)]_X = [(\exists\,x)\,\varphi(x,\dots)]_X$.

8.3. The following assertion shows that, in any B-system, the Boolean-valued classes satisfy an analog of the maximum principle (see 6.1).

If φ is a formula and X is a B-system then the function $\Phi \Subset X \mapsto [\varphi(\Phi,\dots)] \in B$ attains a maximum:

$$(\exists\,\Psi \Subset X)\ [\varphi(\Psi,\dots)] = [(\boldsymbol{\exists}\,\Phi)\,\varphi(\Phi,\dots)].$$

In particular,

$$X \vDash (\boldsymbol{\forall}\,\Phi)\,\varphi(\Phi,\dots) \;\Leftrightarrow\; (\forall\,\Phi \Subset X)\ X \vDash \varphi(\Phi,\dots),$$
$$X \vDash (\boldsymbol{\exists}\,\Phi)\,\varphi(\Phi,\dots) \;\Leftrightarrow\; (\exists\,\Phi \Subset X)\ X \vDash \varphi(\Phi,\dots).$$

$\lhd$ Put $b := [(\boldsymbol{\exists}\,\Phi)\,\varphi(\Phi,\dots)] = \bigvee_{\Phi \Subset X} [\varphi(\Phi,\dots)]$. By the exhaustion principle [4, 2.1.10(1)], there exist an antichain $(d_i)_{i\in I} \subset B$ and a family of predicative classes $\Phi_i \Subset X$ $(i \in I)$ such that $\bigvee_{i\in I} d_i = b$ and $d_i \leqslant [\varphi(\Phi_i,\dots)]$ for all $i \in I$. Define the predicative class $\Phi\colon X \to B$ by putting $\Phi(x) := \bigvee_{i\in I} \Phi_i(x) \wedge d_i$ for $x \in X$. Then, for all $i \in I$ and $x \in X$, we have $\Phi_i(x) \wedge d_i = \Phi(x) \wedge d_i$, whence

$$[\Phi_i = \Phi] = [(\forall\,x)(x \in \Phi_i \Leftrightarrow x \in \Phi)] = \bigwedge_{x \in X} \big(\Phi_i(x) \Leftrightarrow_B \Phi(x)\big) \geqslant d_i$$

and so, recalling 3.9, we conclude that

$$b = \bigvee_{i\in I} d_i \leqslant \bigvee_{i\in I} [\varphi(\Phi_i,\dots)] \wedge [\Phi_i = \Phi] \leqslant [\varphi(\Phi,\dots)]. \ \rhd$$

8.4. Lemma. *Suppose that φ is a formula, X is a B-system, and $\Psi \Subset X$. Then*

$$[(\forall\,\Phi \subset \Psi)\,\varphi(\Phi,\dots)] = [(\forall\,\Phi)\,\varphi(\Phi\cap\Psi,\dots)] = \bigwedge_{\Phi\Subset X:\ \Phi\leqslant\Psi} [\varphi(\Phi,\dots)].$$

In particular, the following properties of the Boolean-valued class Ψ are equivalent:

(a) $X \vDash (\forall\,\Phi\subset\Psi)\,\varphi(\Phi,\dots)$;

(b) $X \vDash (\forall\,\Phi)\,\varphi(\Phi\cap\Psi,\dots)$;

(c) $(\forall\,\Phi \Subset X)\ \Phi\leqslant\Psi \Rightarrow X\vDash\varphi(\Phi,\dots)$.

$\triangleleft$ Given a predicative class $\Phi \Subset X$, define $\Phi_0 \Subset X$ by putting $\Phi_0(x) := \Phi(x)\wedge\Psi(x)$ for all $x \in X$. Then $\Phi_0 \leqslant \Psi$ and $[\Phi_0 \subset \Psi] = [\Phi_0 = \Phi\cap\Psi] = 1_B$.

Since $[\Phi_0 \subset \Psi \Rightarrow \varphi(\Phi_0,\dots)] = [\varphi(\Phi_0,\dots)] = [\varphi(\Phi\cap\Psi,\dots)]$; therefore,

$$\bigwedge_{\Phi\Subset X}[\Phi\subset\Psi\Rightarrow\varphi(\Phi,\dots)] \leqslant \bigwedge_{\Phi\Subset X}[\varphi(\Phi\cap\Psi,\dots)].$$

Furthermore, if $\Phi\leqslant\Psi$ then $[\Phi\cap\Psi=\Phi]=1_B$ and $[\varphi(\Phi\cap\Psi,\dots)] = [\varphi(\Phi,\dots)]$, and so

$$\bigwedge_{\Phi\Subset X}[\varphi(\Phi\cap\Psi,\dots)] \leqslant \bigwedge_{\Phi\Subset X:\ \Phi\leqslant\Psi}[\varphi(\Phi\cap\Psi,\dots)] = \bigwedge_{\Phi\Subset X:\ \Phi\leqslant\Psi}[\varphi(\Phi,\dots)].$$

Finally, the relations

$$\big[\varphi(\Phi\cap\Psi,\dots)\Rightarrow\big(\Phi\subset\Psi\Rightarrow\varphi(\Phi,\dots)\big)\big] = \big[\big(\varphi(\Phi\cap\Psi,\dots)\wedge\Phi\cap\Psi=\Phi\big)\Rightarrow\varphi(\Phi,\dots)\big] = 1_B$$

imply that

$$[\varphi(\Phi_0,\dots)] = [\varphi(\Phi\cap\Psi,\dots)] \leqslant [\Phi\subset\Psi\Rightarrow\varphi(\Phi,\dots)],$$

and so

$$\bigwedge_{\Phi\Subset X:\ \Phi\leqslant\Psi}[\varphi(\Phi,\dots)] \leqslant \bigwedge_{\Phi\Subset X}[\Phi\subset\Psi\Rightarrow\varphi(\Phi,\dots)].\ \triangleright$$

8.5. Lemma. *Let φ be a bounded formula. If X is a B-system, $Y \preccurlyeq X$, $\bar y = y_1,\dots,y_m \in Y$, $\Psi,\overline{\Psi} = \Psi_1,\dots,\Psi_n \Subset X$, and $\Psi,\overline{\Psi} \leqslant Y{\uparrow}_X$ then*

$$[(\forall\,\Phi\subset\Psi)\,\varphi(\Phi,\bar y,\overline{\Psi})]_X = [(\forall\,\Phi\subset\Psi|_Y)\,\varphi(\Phi,\bar y,\overline{\Psi}|_Y)]_Y.$$

$\triangleleft$ If $\Phi\Subset Y$ and $\Phi\leqslant\Psi|_Y$ then $\Phi{\uparrow}_X\leqslant\Psi$ by Lemma 3.7(b) and, moreover, $(\Phi{\uparrow}_X)|_Y=\Phi$. Therefore, using Theorem 7.6 and Lemma 8.4, we have

$$[(\forall\,\Phi\subset\Psi)\,\varphi(\Phi,\bar y,\overline{\Psi})]_X = \bigwedge_{\Phi\Subset X:\ \Phi\leqslant\Psi}[\varphi(\Phi,\bar y,\overline{\Psi})]_X \leqslant \bigwedge_{\Phi\Subset Y:\ \Phi\leqslant\Psi|_Y}[\varphi(\Phi{\uparrow}_X,\bar y,\overline{\Psi})]_X$$

$$= \bigwedge_{\Phi\Subset Y:\ \Phi\leqslant\Psi|_Y}\big[\varphi\big((\Phi{\uparrow}_X)|_Y,\bar y,\overline{\Psi}|_Y\big)\big]_Y = \bigwedge_{\Phi\Subset Y:\ \Phi\leqslant\Psi|_Y}[\varphi(\Phi,\bar y,\overline{\Psi}|_Y)]_Y = [(\forall\,\Phi\subset\Psi|_Y)\,\varphi(\Phi,\bar y,\overline{\Psi}|_Y)]_Y.$$

The reverse inequality is also guaranteed by Theorem 7.6 and Lemma 8.4:

$$[(\forall\,\Phi\subset\Psi|_Y)\,\varphi(\Phi,\bar y,\overline{\Psi}|_Y)]_Y = \bigwedge_{\Phi\Subset Y:\ \Phi\leqslant\Psi|_Y}[\varphi(\Phi,\bar y,\overline{\Psi}|_Y)]_Y \leqslant \bigwedge_{\Phi\Subset X:\ \Phi|_Y\leqslant\Psi|_Y}[\varphi(\Phi|_Y,\bar y,\overline{\Psi}|_Y)]_Y$$

$$= \bigwedge_{\Phi\Subset X:\ \Phi|_Y\leqslant\Psi|_Y}[\varphi(\Phi,\bar y,\overline{\Psi})]_X \leqslant \bigwedge_{\Phi\Subset X:\ \Phi\leqslant\Psi}[\varphi(\Phi,\bar y,\overline{\Psi})]_X = [(\forall\,\Phi\subset\Psi)\,\varphi(\Phi,\bar y,\overline{\Psi})]_X.\ \triangleright$$

§ 9. Regularity

The axiom of regularity (foundation) has the form

$$\rho \;:=\; (\forall\, x)\, \mu(x),$$

where

$$\mu(x) \;:=\; \bigl((\exists\, y)(y \in x) \Rightarrow (\exists\, y \in x)(\forall\, z \in x)(z \notin y)\bigr) \tag{14}$$

or, which is the same,

$$\mu(x) \;:=\; \bigl(x \neq \varnothing \Rightarrow (\exists\, y \in x)(y \cap x = \varnothing)\bigr).$$

If $\mathbb{V}$ is the class of all sets and $(\mathbb{V}_\alpha)_{\alpha\in\mathrm{Ord}}$ is the von Neumann cumulative hierarchy defined by the recursive rule

$$\begin{aligned} &\mathbb{V}_0 = \varnothing; \\ &\mathbb{V}_{\alpha+1} = \mathscr{P}(\mathbb{V}_\alpha), \quad \alpha \in \mathrm{Ord}; \\ &\mathbb{V}_\alpha = \bigcup_{\beta<\alpha} \mathbb{V}_\beta, \qquad \alpha \in \mathrm{Lim\,Ord}; \end{aligned} \tag{15}$$

then, in the theory obtained from ZFC by excluding the axiom of regularity ρ, the equality $\mathbb{V} = \bigcup_{\alpha\in\mathrm{Ord}} \mathbb{V}_\alpha$ is equivalent to ρ (see [2, § 6]). Moreover, in this theory, the axiom of regularity is equivalent to σ-regularity, that is the absence of a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_{n+1} \in x_n$ for all $n \in \mathbb{N}$.

In the present section, we introduce and study the notions of regular and σ-regular Boolean-valued system and describe the Boolean algebras B for which the classes of regular and σ-regular B-systems coincide.

9.1. Let X be a B-system and let $\Psi \Subset X$. We say that the *Boolean-valued class* Ψ *is regular in* X if

$$X \vDash (\forall\, \Phi \subset \Psi)\, \mu(\Phi)$$

(see (14)) and that the *B-system X is regular outside a subclass* $Y \subset X$ if the complement $\neg(Y{\uparrow})$ of the Boolean-valued class $Y{\uparrow}$ is regular in X:

$$X \vDash (\forall\, \Phi)\bigl(\Phi \cap Y{\uparrow} = \varnothing \;\Rightarrow\; \mu(\Phi)\bigr).$$

Call a system X *regular* if the greatest Boolean-valued class $X{\uparrow}$ is regular in X:

$$X \vDash (\forall\, \Phi)\, \mu(\Phi).$$

The regularity of a system X is obviously related to the validity of the axiom of regularity ρ in X:
(a) *if X is predicative then the regularity of X implies the validity $X \vDash \rho$;*
(b) *if X is intensional then the validity $X \vDash \rho$ implies the regularity of X;*
(c) *if X satisfies the ascent principle then the regularity of X is equivalent to the validity $X \vDash \rho$.*

9.2. Lemma. *The following properties of a B-system X are equivalent:*
(a) *X is not regular;*
(b) *there exists a set $P \subset {}^{\%}X$ such that*

$$(\exists\, p \in P)\;\; \mathrm{dom}\, p \neq 0_B,$$

$$(\forall\, p \in P)\;\; \bigvee_{q\in P} [q \in p] \geqslant \mathrm{dom}\, p;$$

(c) *there exists a sequence of sets $P_n \subset {}^{\%}X$ $(n \in \mathbb{N})$ such that*

P_n *is an antichain*;

$$\bigvee_{p\in P_n} \mathrm{dom}\, p \;=\; \bigvee_{p\in P_1} \mathrm{dom}\, p \neq 0_B;$$

$$(\forall\, p \in P_n)(\forall\, q \in P_{n+1})(\mathrm{dom}\, p \wedge \mathrm{dom}\, q \neq 0_B \;\Rightarrow\; \mathrm{dom}\, q = [q \in p]);$$

$$(\forall\, p \in P_n)\;\; \bigvee_{q\in P_{n+1}} [q \in p] = \mathrm{dom}\, p.$$

◁ (a)⇒(b): If X is not regular; then, by 8.3 and Theorem 4.5(a), there exists a set $P \subset {}^{\%}X$ for which $X \not\models \mu(P\!\uparrow)$. This means that

$$b := [P\!\uparrow \neq \varnothing] \wedge [(\forall\, y \in P\!\uparrow)(\exists\, z \in P\!\uparrow)(z \in y)] \neq 0_B.$$

Show that the set $\{p|_b : p \in P\}$ satisfies (b). Indeed, with account taken of (3),

$$\bigvee_{p\in P} \operatorname{dom} p|_b = \bigvee_{p\in P} \operatorname{dom} p \wedge b = [P\!\uparrow \neq \varnothing] \wedge b = b \neq 0_B.$$

Moreover, by Lemma 3.18,

$$b \leqslant [(\forall\, y \in P\!\uparrow)(\exists\, z \in P\!\uparrow)(z \in y)] = \bigwedge_{p\in P} \neg \operatorname{dom} p \vee \bigvee_{q\in P} [q \in p].$$

Therefore, for all $p \in P$,

$$b \leqslant \neg \operatorname{dom} p \vee \bigvee_{q\in P} [q \in p],$$

and so

$$\bigvee_{q\in P} [q \in p|_b] = \bigvee_{q\in P} [q \in p] \wedge \operatorname{dom} p \wedge b = \Big(\neg \operatorname{dom} p \vee \bigvee_{q\in P} [q \in p]\Big) \wedge \operatorname{dom} p \wedge b \geqslant \operatorname{dom} p \wedge b = \operatorname{dom} p|_b.$$

(b)⇒(c): Suppose that $P \subset {}^{\%}X$ satisfies (b). By the exhaustion principle, for each $p \in {}^{\%}P$, the equality $\bigvee_{q\in P} [q \in p] = \operatorname{dom} p$ implies the existence of an antichain $D(p) \subset {}^{\%}P$ such that $\operatorname{dom} q = [q \in p]$ for all $q \in D(p)$ and $\bigvee_{q\in D(p)} [q \in p] = \operatorname{dom} p$. For the same reason, there exists an antichain $P_1 \subset {}^{\%}P$ such that

$$\bigvee_{p\in P_1} \operatorname{dom} p = \bigvee_{p\in P} \operatorname{dom} p \neq 0_B.$$

Define the sets $P_n \subset {}^{\%}P$ $(n \in \mathbb{N})$ by putting

$$P_{n+1} := \bigcup_{p\in P_n} D(p), \quad n \in \mathbb{N}.$$

An elementary check shows that P_n's satisfy all conditions in (c).

(c)⇒(a): Suppose that a sequence $(P_n)_{n\in\mathbb{N}}$ satisfies (c). Put $P := \bigcup_{n\in\mathbb{N}} P_n$ and show that $X \not\models \mu(P\!\uparrow)$. Indeed, by (3),

$$[P\!\uparrow \neq \varnothing] = \bigvee_{p\in P} \operatorname{dom} p \geqslant \bigvee_{p\in P_1} \operatorname{dom} p \neq 0_B;$$

but, by Lemma 3.18,

$$[(\exists\, y \in P\!\uparrow)(\forall\, z \in P\!\uparrow)(z \notin y)] = [(\exists\, y \in P\!\uparrow)\neg(\exists\, z \in P\!\uparrow)(z \in y)]$$

$$= \bigvee_{p\in P} \operatorname{dom} p \wedge \neg\Big(\bigvee_{q\in P} \operatorname{dom} q \wedge [q \in p]\Big) = \bigvee_{n\in\mathbb{N}} \bigvee_{p\in P_n} \operatorname{dom} p \wedge \neg\Big(\bigvee_{m\in\mathbb{N}} \bigvee_{q\in P_m} \operatorname{dom} q \wedge [q \in p]\Big)$$

$$\leqslant \bigvee_{n\in\mathbb{N}} \bigvee_{p\in P_n} \operatorname{dom} p \wedge \neg\Big(\bigvee_{q\in P_{n+1}} [q \in p]\Big) = \bigvee_{n\in\mathbb{N}} \bigvee_{p\in P_n} \operatorname{dom} p \wedge \neg \operatorname{dom} p = 0_B. \ \triangleright$$

9.3. The following stems from Lemma 8.5 owing to the boundedness of $\mu(x)$:

Corollary. *Suppose that X is a B-system, $Y \preccurlyeq X$, $\Psi \Subset X$, and $\Psi \leqslant Y\!\uparrow_X$. The class Ψ is regular in X if and only if the class $\Psi|_Y$ is regular in Y.*

9.4. Lemma. *Let X and Y be B-systems and let $Z \subset Y \preccurlyeq X$. If Y is regular outside Z then the assertion on the regularity of the difference $Y{\uparrow} \setminus Z{\uparrow}$ is valid inside X:*

$$X \vDash (\forall\, \Phi \subset Y{\uparrow} \setminus Z{\uparrow})\, \mu(\Phi).$$

$\triangleleft$ Put $\Psi := Y{\uparrow}_X \wedge \neg(Z{\uparrow}_X)$. The regularity of Y outside Z means that the class $\neg(Z{\uparrow}_Y)$ is regular in Y, which, in view of Corollary 9.3 and the relations $\Psi \leqslant Y{\uparrow}_X$, $\Psi|_Y = \neg(Z{\uparrow}_Y)$, implies the regularity of the class Ψ in X. It remains to observe that $X \vDash (\Psi = Y{\uparrow} \setminus Z{\uparrow})$. $\triangleright$

9.5. Lemma. *Let X and Y be B-systems and let $Z \subset Y \preccurlyeq X$. If Y is regular outside Z and X is regular outside Y then X is regular outside Z.*

$\triangleleft$ Validate $X \vDash (\forall\, \Phi)\big(\Phi \cap Z{\uparrow} = \varnothing \Rightarrow \mu(\Phi)\big)$ by "arguing inside X" (see 3.10).

Let $\Phi \cap Z{\uparrow} = \varnothing$ and $\Phi \neq \varnothing$. Find $y \in \Phi$ such that $y \cap \Phi = \varnothing$. If $\Phi \cap Y{\uparrow} = \varnothing$ then a desired y exists by the regularity of $\neg Y{\uparrow}$. Now, let $\Phi \cap Y{\uparrow} \neq \varnothing$. Since the class $Y{\uparrow} \setminus Z{\uparrow}$ is regular (see Lemma 9.4) and $\varnothing \neq \Phi \cap Y{\uparrow} \subset Y{\uparrow} \setminus Z{\uparrow}$, there exists $y \in \Phi \cap Y{\uparrow}$ such that $y \cap \Phi \cap Y{\uparrow} = \varnothing$. Then $y \cap \Phi = y \cap \Phi \cap Y{\uparrow} = \varnothing$ since, by the transitivity of $Y{\uparrow}$, from $y \in Y{\uparrow}$ it follows that $y \subset Y{\uparrow}$. $\triangleright$

9.6. Corollary. *Suppose that X and Y are B-systems and $Y \preccurlyeq X$. If Y is regular and X is regular outside Y then X is regular.*

9.7. Lemma. *If a system X is regular then every subsystem $Y \subset X$ is regular.*

$\triangleleft$ Consider arbitrary families $(y_i)_{i \in I} \subset Y$ and $(b_i)_{i \in I} \subset B$ and show that $Y \vDash \mu(P{\uparrow}_Y)$, where $P := \{y_i|_{b_i} : i \in I\}$. Indeed, using the regularity of X and Lemma 3.18, we conclude that

$$\begin{aligned}
[P{\uparrow}_Y \neq \varnothing]_Y = \bigvee_{i \in I} b_i = [P{\uparrow}_X \neq \varnothing]_X \leqslant [(\exists\, y \in P{\uparrow}_X)(\forall\, z \in P{\uparrow}_X)(z \notin y)]_X \\
= \bigvee_{i \in I} \bigwedge_{j \in I} ([y_j \notin y_i]_X \wedge b_i \wedge b_j) \vee \neg b_j = \bigvee_{i \in I} \bigwedge_{j \in I} ([y_j \notin y_i]_Y \wedge b_i \wedge b_j) \vee \neg b_j \\
= [(\exists\, y \in P{\uparrow}_Y)(\forall\, z \in P{\uparrow}_Y)(z \notin y)]_Y.
\end{aligned}$$ $\triangleright$

9.8. Let X be a B-system and let $\Psi \Subset X$. Say that the *Boolean-valued class Ψ is σ-regular in X* if

$$\bigwedge_{n \in \mathbb{N}} [x_n \in \Psi] \wedge [x_{n+1} \in x_n] = 0_B$$

for every sequence $(x_n)_{n \in \mathbb{N}} \subset X$. We say that the *$B$-system X is σ-regular outside a subclass $Y \subset X$* if the complement $\neg(Y{\uparrow})$ to the Boolean-valued class $Y{\uparrow}$ is σ-regular in X; i.e.,

$$\bigwedge_{n \in \mathbb{N}} [x_n \notin Y{\uparrow}] \wedge [x_{n+1} \in x_n] = 0_B$$

for every $(x_n)_{n \in \mathbb{N}} \subset X$. Call the system X *σ-regular* if the greatest class $X{\uparrow}$ is σ-regular in X; i.e.,

$$\bigwedge_{n \in \mathbb{N}} [x_{n+1} \in x_n] = 0_B$$

for every $(x_n)_{n \in \mathbb{N}} \subset X$.

9.9. Theorem. *Let X be a B-system.*

(a) *If a class $\Psi \Subset X$ is regular in X then Ψ is σ-regular in X.*

(b) *If $Y \subset X$ and the difference $X \backslash Y$ is precyclic then the regularity and the σ-regularity of X outside Y are equivalent.*

$\triangleleft$ (a): Suppose that Ψ is regular in X. Consider an arbitrary sequence $(x_n)_{n \in \mathbb{N}} \subset X$ and show that

$$b := \bigwedge_{n \in \mathbb{N}} [x_n \in \Psi] \wedge [x_{n+1} \in x_n] = 0_B.$$

Put $\Phi := \{x_n : n \in \mathbb{N}\}\!\uparrow$. Then $[\![\Phi \neq \varnothing]\!] = 1_B$; and, moreover, by Lemma 3.18

$$[\![\Phi \subset \Psi]\!] = [\![(\forall\, x \in \Phi)(x \in \Psi)]\!] = \bigwedge_{n\in\mathbb{N}} [\![x_n \in \Psi]\!] \geqslant b,$$

which, in view of the regularity of Ψ in X, implies that $[\![(\exists\, y \in \Phi)(\forall\, z \in \Phi)(z \notin y)]\!] \geqslant b$. On the other hand, from Lemma 3.18 we conclude that

$$[\![(\exists\, y \in \Phi)(\forall\, z \in \Phi)(z \notin y)]\!] = \bigvee_{n\in\mathbb{N}} \bigwedge_{m\in\mathbb{N}} [\![x_m \notin x_n]\!] = \neg \bigwedge_{n\in\mathbb{N}} \bigvee_{m\in\mathbb{N}} [\![x_m \in x_n]\!] \leqslant \neg \bigwedge_{n\in\mathbb{N}} [\![x_{n+1} \in x_n]\!] \leqslant \neg b.$$

(b): Suppose that the difference $Z := X \backslash Y \neq \varnothing$ is precyclic, and that X is σ-regular outside Y. Consider an arbitrary class $\Phi \Subset X$ satisfying $\Phi \leqslant \neg(Y\!\uparrow)$ and show that $X \vDash \mu(\Phi)$; i.e.,

$$b := [\![\Phi \neq \varnothing \wedge (\forall\, y \in \Phi)(y \cap \Phi \neq \varnothing)]\!] = 0_B.$$

Observe first that, for every Boolean-valued class $\Psi \Subset X$,

$$(\exists\, x \in X)\ [\![(\exists\, y \in \Phi)\, \Psi(y)]\!] = [\![x \in \Phi]\!] \wedge [\![\Psi(x)]\!]. \tag{16}$$

Indeed, by Lemma 3.17(d)

$$X \vDash (\Phi \subset X\!\uparrow \backslash\, Y\!\uparrow \subset Z\!\uparrow),$$

which, by Lemmas 3.18 and 5.11, implies the equalities

$$[\![(\exists\, y \in \Phi)\, \Psi(y)]\!] = [\![(\exists\, y \in Z\!\uparrow)(\Phi(y) \wedge \Psi(y))]\!] = [\![(\exists\, y \in (\operatorname{mix} Z)\!\uparrow)(\Phi(y) \wedge \Psi(y))]\!] = \bigvee_{y \in \operatorname{mix} Z} \Phi(y) \wedge \Psi(y).$$

Moreover, owing to Theorem 6.5, the function $\Phi \wedge \Psi$ attains its maximum on the cyclic class $\operatorname{mix} Z$.

By (16), there is $x_1 \in X$ such that $[\![x_1 \in \Phi]\!] = [\![\Phi \neq \varnothing]\!] \geqslant b$. From $[\![(\forall\, y \in \Phi)(y \cap \Phi \neq \varnothing)]\!] \geqslant b$ it follows that $[\![x_1 \cap \Phi \neq \varnothing]\!] \geqslant b$, i.e., $[\![(\exists\, y \in \Phi)(y \in x_1)]\!] \geqslant b$; and so, by (16), there exists $x_2 \in X$ such that $[\![x_2 \in \Phi]\!] \wedge [\![x_2 \in x_1]\!] \geqslant b$. "Iterating" these arguments (and, strictly speaking, applying recursion and the axiom of choice), we obtain a sequence $(x_n)_{n\in\mathbb{N}}$ of elements in X satisfying $[\![x_n \notin Y\!\uparrow]\!] \wedge [\![x_{n+1} \in x_n]\!] \geqslant [\![x_n \in \Phi]\!] \wedge [\![x_{n+1} \in x_n]\!] \geqslant b$ for all $n \in \mathbb{N}$. Owing to the σ-regularity of X outside Y, this implies that $b = 0$. ▷

9.10. We finish the section by describing the complete Boolean algebras B for which the notions of regular and σ-regular B-systems coincide.

Say that an element $c \in B$ is *refined* (*strictly refined*) *from* $D \subset B$ and write $c \preccurlyeq D$ ($c \prec D$), if $c \leqslant d$ ($c < d$) for some element $d \in D$. Say that a set $C \subset B$ is *refined* (*strictly refined*) *from* D and write $C \preccurlyeq D$ ($C \prec D$), if $c \preccurlyeq D$ ($c \prec D$) for all $c \in C$. An element c or a set C is said to be *refined from a sequence* $(D_n)_{n\in\mathbb{N}}$ of subsets of B if $c \preccurlyeq D_n$ ($C \preccurlyeq D_n$) for all $n \in \mathbb{N}$.

A set $C \subset B$ is called a *cover* of a Boolean algebra B if $\vee C = 1_B$. A *partition* of a Boolean algebra is a partition of unity; i.e., a cover that is an antichain. Recall that, by the exhaustion principle, from each cover we can refine a partition.

Theorem [5, § 19; 6]**.** *The following properties of a complete Boolean algebra B are equivalent:*

(a) *for every set I and every family $\big(b_i^n\big)_{i\in I}^{n\in\mathbb{N}}$ of elements in B we have*

$$\bigwedge_{n\in\mathbb{N}} \bigvee_{i\in I} b_i^n = \bigvee_{i\in I^{\mathbb{N}}} \bigwedge_{n\in\mathbb{N}} b_{i(n)}^n;$$

(b) *from each sequence of covers B we can refine a cover;*

(c) *from each sequence of partitions B we can refine a partition;*

(d) *for every sequence of partitions $(D_n)_{n\in\mathbb{N}}$ of B and every nonzero $a \in B$, there exists a sequence of $d_n \in D_n$ ($n \in \mathbb{N}$) such that*

$$a \wedge \bigwedge_{n\in\mathbb{N}} d_n \neq 0_B.$$

A Boolean algebra B satisfying any of the equivalent conditions (a)–(d) is called ω*-distributive* or (ω, ∞)*-distributive.*

Every atomic complete Boolean algebra is ω-distributive. The completion of the quotient Boolean algebra $\mathscr{P}(\mathbb{N})/\mathscr{P}_{\mathrm{fin}}(\mathbb{N})$ is an atomless ω-distributive Boolean algebra (see [6, Corollary to Lemma 5; 7, Example 9]). A classical example of a Boolean algebra that is not ω-distributive is given by the Boolean algebra of cosets of Lebesgue measurable subsets of $\mathbb{R}$.

9.11. Consider a sequence $(C_n)_{n\in\mathbb{N}}$ of subsets of B. Call $(C_n)_{n\in\mathbb{N}}$ a *refinement* in B if

(a) $\vee C_n = \vee C_1 \neq 0_B$ for all $n \in \mathbb{N}$;

(b) $\bigwedge_{n\in\mathbb{N}} c_n = 0_B$ for every sequence of $c_n \in C_n$ $(n \in \mathbb{N})$.

Say that a refinement $(C_n)_{n\in\mathbb{N}}$ is *partitioning* if, for all $n \in \mathbb{N}$,

(c) $0_B \notin C_n$;

(d) C_n is an antichain;

(e) $C_{n+1} \prec C_n$.

9.12. The following is readily verified:

Lemma. *Let $(C_n)_{n\in\mathbb{N}}$ be a partitioning refinement in B and let $X := \bigcup_{n\in\mathbb{N}} C_n$.*

(a) *For each $x \in X$, there exists a unique finite sequence of elements $c_1 \in C_1, c_2 \in C_2, \dots, c_n \in C_n$ such that $c_1 > c_2 > \dots > c_n = x$.*

(b) *If $n \neq m$ then $C_n \cap C_m = \varnothing$.*

(c) *Given $x \in X$, denote by $h(x)$ the only $n \in \mathbb{N}$ for which $x \in C_n$. If $(x_i)_{i\in I} \subset X$ and $\bigvee_{i\in I} h(x_i) = \infty$ then $\bigwedge_{i\in I} x_i = 0_B$.*

9.13. Theorem. *The following properties of a complete Boolean algebra B are equivalent:*

(a) *B admits a partitioning refinement;*

(b) *B admits a refinement;*

(c) *B is not ω-distributive.*

$\triangleleft$ The implication (a)⇒(b) is trivial. The implication (b)⇒(c) is easily proven by Theorem 9.10(d). Show that (c)⇒(a).

If B is not ω-distributive; then, by Theorem 9.10(d), there exist a sequence of partitions $(D_n)_{n\in\mathbb{N}}$ and a nonzero $a \in B$ such that

$$a \wedge \bigwedge_{n\in\mathbb{N}} d_n = 0_B \tag{17}$$

for every sequence of $d_n \in D_n$ $(n \in \mathbb{N})$. By (17), it is impossible to refine from $(D_n)_{n\in\mathbb{N}}$ any nonzero element $b \leqslant a$, and so for each $b \leqslant a$ we can consider the natural

$$m(b) := \min\{n \in \mathbb{N} : b \not\preccurlyeq D_n\}.$$

Since $\vee D_{m(b)} = 1_B$ and $b \not\preccurlyeq D_{m(b)}$, there exists an element $d \in D_{m(b)}$ such that $0_B < b \wedge d < b$. Thus, the set

$$P(b) := \{b \wedge d : d \in D_{m(b)}\} \setminus \{0_B\}$$

possesses the following properties:

$$P(b) \text{ is an antichain}, \quad \vee P(b) = b, \quad 0_B < c < b \text{ for all } c \in P(b).$$

Define $C_n \subset B$ $(n \in \mathbb{N})$ recursively by putting

$$C_1 := P(a);$$
$$C_{n+1} := \bigcup_{b\in C_n} P(b), \quad n \in \mathbb{N},$$

and demonstrate that the sequence $(C_n)_{n\in\mathbb{N}}$ is a partitioning refinement. Conditions 9.11(a),(c)–(e) are obvious. It remains to justify 9.11(b).

Show by induction on $n \in \mathbb{N}$ that $m(b) \geqslant n$ for all $b \in C_n$. The induction base $n = 1$ is trivial. Suppose that $m(b) \geqslant n$ for $b \in C_n$, consider an arbitrary $c \in C_{n+1}$, and prove that $m(c) \geqslant n + 1$. By the definition of C_{n+1}, we have the representation $c = b \wedge d$ for some $b \in C_n$ and $d \in D_{m(b)}$. Since b is refined from $D_1, \dots, D_{m(b)-1}$; by the inequality $c \leqslant b$, the same holds for c, and so $m(c) \geqslant m(b) \geqslant n$. Moreover, from $c \not\preccurlyeq D_{m(c)}$ and $c \leqslant d \in D_{m(b)}$ it follows that $m(c) \neq m(b)$, and so $m(c) \geqslant n+1$.

Now, let $c_n \in C_n$ for all $n \in \mathbb{N}$. By the above, $m(c_{n+1}) > n$, whence $c_{n+1} \preccurlyeq D_n$, and so $c_{n+1} \leqslant d_n$ for some sequence of $d_n \in D_n$ $(n \in \mathbb{N})$. Applying (17), we conclude that

$$\bigwedge_{n\in\mathbb{N}} c_n = c_1 \wedge \bigwedge_{n\in\mathbb{N}} c_{n+1} \leqslant a \wedge \bigwedge_{n\in\mathbb{N}} d_n = 0_B. \ \rhd$$

9.14. Theorem. *The classes of regular and σ-regular B-systems coincide if and only if the Boolean algebra B is ω-distributive.*

$\lhd$ Necessity: If B is not ω-distributive; then, by Theorem 9.13, there exists a partitioning refinement $(C_n)_{n\in\mathbb{N}}$ in B. Put $X := \bigcup_{n\in\mathbb{N}} C_n$. As in Lemma 9.12(c), given $x \in X$, denote by $h(x)$ the only $n \in \mathbb{N}$ for which $x \in C_n$. Turn X into a B-system by putting for $x, y \in X$

$$[x = y] := \begin{cases} 1_B & \text{if } x = y, \\ 0_B & \text{otherwise;} \end{cases} \qquad [x \in y] := \begin{cases} x & \text{if } h(x) = h(y) + 1, \\ 0_B & \text{otherwise.} \end{cases}$$

Establish the σ-regularity of X. To this end, consider an arbitrary sequence $(x_n)_{n\in\mathbb{N}} \subset X$ and show that $\bigwedge_{n\in\mathbb{N}} [x_{n+1} \in x_n] = 0_B$. This relation is obvious if $[x_{n+1} \in x_n] = 0_B$ for some $n \in \mathbb{N}$. Otherwise, $h(x_{n+1}) = h(x_n) + 1$ and $[x_{n+1} \in x_n] = x_{n+1}$ for all $n \in \mathbb{N}$. Then $\bigvee_{n\in\mathbb{N}} h(x_{n+1}) = \infty$, whence, by Lemma 9.12(c), we have

$$\bigwedge_{n\in\mathbb{N}} [x_{n+1} \in x_n] = \bigwedge_{n\in\mathbb{N}} x_{n+1} = 0_B.$$

The system X is not regular since

$$P := \{x|_x : x \in X\} \subset {}^{\%}X$$

satisfies (b) of Lemma 9.2. Indeed, for all $n \in \mathbb{N}$ and $x \in C_n$,

$$\begin{aligned} \bigvee_{y\in X} [y|_y \in x|_x] &= x \wedge \bigvee_{y\in X} [y \in x] \wedge y \geqslant x \wedge \bigvee_{y\in C_{n+1}} [y \in x] \wedge y \\ &= x \wedge \bigvee_{y\in C_{n+1}} y = x \wedge \vee C_{n+1} = x \wedge \vee C_n = x = \operatorname{dom} x|_x. \end{aligned}$$

Sufficiency: Suppose that X is a B-system which is σ-regular and not regular. Consider the sets $P_n \subset {}^{\%}X$ $(n \in \mathbb{N})$ satisfying (c) of Lemma 9.2, put

$$C_n := \{\operatorname{dom} p : p \in P_n\}, \quad n \in \mathbb{N}, \tag{18}$$

and show that the sequence $(C_n)_{n\in\mathbb{N}}$ is a refinement in B (see Theorem 9.13). Indeed, for all $n \in \mathbb{N}$,

$$\vee C_n = \bigvee_{p\in P_n} \operatorname{dom} p = \bigvee_{p\in P_1} \operatorname{dom} p = \vee C_1 \neq 0_B.$$

Let $c_n \in C_n$ for all $n \in \mathbb{N}$. Show that $\bigwedge_{n\in\mathbb{N}} c_n = 0_B$. The last relation is obvious if $(\exists\, n \in \mathbb{N})\ c_n \wedge c_{n+1} = 0_B$. Now, let $c_n \wedge c_{n+1} \neq 0_B$ for all $n \in \mathbb{N}$. By (18), for each $n \in \mathbb{N}$ there exists $p_n \in P_n$ such that $c_n = \operatorname{dom} p_n$. Since $\operatorname{dom} p_n \wedge \operatorname{dom} p_{n+1} = c_n \wedge c_{n+1} \neq 0_B$; therefore, 9.2(c) implies that $c_{n+1} = \operatorname{dom} p_{n+1} = [p_{n+1} \in p_n]$. Thus

$$\bigwedge_{n\in\mathbb{N}} c_n \leqslant \bigwedge_{n\in\mathbb{N}} c_{n+1} = \bigwedge_{n\in\mathbb{N}} [p_{n+1} \in p_n] = 0_B$$

owing to the σ-regularity of X. $\rhd$

§ 10. Intensional Hierarchy

The von Neumann cumulative hierarchy $(V_\alpha)_{\alpha\in\mathrm{Ord}}$ over a set or a class V_0 is defined by the transitive recursion

$$\begin{aligned} V_{\alpha+1} &= V_\alpha \cup \mathscr{P}(V_\alpha), \quad && \alpha\in\mathrm{Ord};\\ V_\alpha &= \bigcup_{\beta<\alpha} V_\beta, && \alpha\in\mathrm{Lim\,Ord}. \end{aligned} \tag{19}$$

In this section, we define an intensional hierarchy that serves as an analog for hierarchy (19) for Boolean-valued systems, introduce the notion of Boolean-valued universe over an arbitrary extensional Boolean-valued system, and establish a close relationship of such a universe with the corresponding intensional hierarchy.

10.1. We begin with a characterization of the superstructure that is a Boolean-valued analog of the discrete step $V_{\alpha+1} = V_\alpha \cup \mathscr{P}(V_\alpha)$ of (19).

Let X be an extensional B-system, and let Y be a subclass of X. Introduce the following notions:

$$\begin{aligned} X \textit{ is intensional over } Y &\Leftrightarrow (\forall P \subset {}^{\%}Y)(\exists x\in X)(x \simeq P{\uparrow})\\ &\Leftrightarrow (\forall \Phi \Subset X:\ \Phi \leqslant Y{\uparrow})(\exists x\in X)(x\simeq \Phi);\\ X \textit{ is predicative over } Y &\Leftrightarrow (\forall x\in X\backslash Y)(\exists P\subset {}^{\%}Y)(x\simeq P{\uparrow})\\ &\Leftrightarrow X = Y\cup \mathscr{P}_X(Y);\\ X \textit{ is separated over } Y &\Leftrightarrow (\forall x\in X)(\forall z\in X\backslash Y)(x\simeq z \Rightarrow x=z)\\ &\Leftrightarrow (\forall x_1,x_2\in X)(x_1\simeq x_2,\ x_1\neq x_2 \Rightarrow x_1,x_2\in Y). \end{aligned}$$

We say that an extensional B-system X is a *superstructure* over a subsystem Y if

(a) $Y \preccurlyeq X$;

(b) X is intensional over Y;

(c) X is predicative over Y;

(d) X is separated over Y.

Refer to an extensional B-system X as a *superstructure over a copy* of a B-system Z if X is a superstructure over a subsystem isomorphic to Z.

10.2. Lemma. *For every extensional Boolean-valued system Z, there exists a superstructure over a copy of Z.*

◁ Consider an arbitrary extensional B-system Z, its isomorphic copy $Y := \{\varnothing\}\times Z$ and put (see 4.2)

$$\mathscr{Y} := \{P\subset {}^{\%}Y :\ P \text{ is saturated},\ \neg(\exists y\in Y)(y\simeq P{\uparrow}_Y)\}.$$

Note that $Y\cap\mathscr{Y} = \varnothing$. Indeed, by Definition 3.1, every subset $P\subset {}^{\%}Y$ consists of pairs, whereas every element of the product $Y = \{\varnothing\}\times Z$ in Kuratowski's approach has the form $\big\{\{\varnothing\},\{\varnothing, z\}\big\}$ and so contains the element $\{\varnothing\}$ which is not a pair.

Put $X := Y\cup\mathscr{Y}$ and extend the interpretations $=_Y, \in_Y$ onto X^2 by putting

$$\begin{aligned} =_X(y,z) &:= =_Y(y,z), & \in_X(y,z) &:= \in_Y(y,z),\\ =_X(P,Q) &:= [P{\uparrow}_Y = Q{\uparrow}_Y]_Y, & \in_X(P,Q) &:= [P{\uparrow}_Y \in Q{\uparrow}_Y]_Y,\\ =_X(P,y) &:= [P{\uparrow}_Y = y]_Y, & \in_X(P,y) &:= [P{\uparrow}_Y \in y]_Y,\\ =_X(y,P) &:= [y = P{\uparrow}_Y]_Y, & \in_X(y,P) &:= [y\in P{\uparrow}_Y]_Y \end{aligned}$$

for all $y,z\in Y$ and $P,Q\in\mathscr{Y}$.

The fact that $(X, =_X, \in_X)$ is a B-system is established by an elementary check of the conditions of Definition 2.1. In most cases the "syntactic sugar" 3.8 and the validity of the propositional axioms and the equality axioms (see 3.9) are enough for this check. Clarify only five cases, in three of which the extensionality of Y is used, and the other two employ the obvious inequality $[\varphi(x)]_Y \leqslant [(\exists x)\, \varphi(x)]_Y$.

If $x, y, z \in Y$ and $P, Q \in \mathscr{Y}$ then

$$
\begin{aligned}
&=_X(x,P) \wedge =_X(P,z) = [x = P{\uparrow_Y} \wedge P{\uparrow_Y} = z]_Y = [(\forall y)(y \in x \Leftrightarrow y \in P{\uparrow_Y}) \wedge (\forall y)(y \in P{\uparrow_Y} \Leftrightarrow y \in z)]_Y \\
&\qquad \leqslant [(\forall y)(y \in x \Leftrightarrow y \in z)]_Y \leqslant [x = z]_Y = =_X(x,z);
\end{aligned}
$$

$$
\begin{aligned}
&\in_X(P,y) \wedge =_X(P,z) = [P{\uparrow_Y} \in y \wedge P{\uparrow_Y} = z]_Y = [(\exists x)(P{\uparrow_Y} = x \wedge x \in y) \wedge P{\uparrow_Y} = z]_Y \\
&\qquad = \big[(\exists x)\big((\forall u)(u \in P{\uparrow_Y} \Leftrightarrow u \in x) \wedge x \in y\big) \wedge (\forall u)(u \in P{\uparrow_Y} \Leftrightarrow u \in z)\big]_Y \\
&\qquad \leqslant \big[(\exists x)\big((\forall u)(u \in z \Leftrightarrow u \in x) \wedge x \in y\big)\big]_Y \\
&\qquad \leqslant [(\exists x)(z = x \wedge x \in y)]_Y \leqslant [z \in y]_Y = \in_X(z,y);
\end{aligned}
$$

$$
\begin{aligned}
&\in_X(P,Q) \wedge =_X(P,z) = [P{\uparrow_Y} \in Q{\uparrow_Y} \wedge P{\uparrow_Y} = z]_Y = [(\exists x)(P{\uparrow_Y} = x \wedge x \in Q{\uparrow_Y}) \wedge P{\uparrow_Y} = z]_Y \\
&\qquad = \big[(\exists x)\big((\forall y)(y \in P{\uparrow_Y} \Leftrightarrow y \in x) \wedge x \in Q{\uparrow_Y}\big) \wedge (\forall y)(y \in P{\uparrow_Y} \Leftrightarrow y \in z)\big]_Y \\
&\qquad \leqslant \big[(\exists x)\big((\forall y)(y \in z \Leftrightarrow y \in x) \wedge x \in Q{\uparrow_Y}\big)\big]_Y \\
&\qquad \leqslant [(\exists x)(z = x \wedge x \in Q{\uparrow_Y})]_Y \leqslant [z \in Q{\uparrow_Y}]_Y = \in_X(z,Q);
\end{aligned}
$$

$$
\begin{aligned}
&\in_X(x,y) \wedge =_X(x,P) = [x \in y \wedge x = P{\uparrow_Y}]_Y \\
&\qquad \leqslant [(\exists x)(P{\uparrow_Y} = x \wedge x \in y)]_Y = [P{\uparrow_Y} \in y]_Y = \in_X(P,y);
\end{aligned}
$$

$$
\begin{aligned}
&\in_X(x,P) \wedge =_X(x,Q) = [x \in P{\uparrow_Y} \wedge x = Q{\uparrow_Y}]_Y \\
&\qquad \leqslant [(\exists x)(Q{\uparrow_Y} = x \wedge x \in P{\uparrow_Y})]_Y = [Q{\uparrow_Y} \in P{\uparrow_Y}]_Y = \in_X(Q,P).
\end{aligned}
$$

Show that X is extensional. Indeed, if $u, v \in X$ and $[\cdot \in u]_X = [\cdot \in v]_X$; then, in each of the three cases

$$u, v \in Y; \quad u \in Y,\ v \in \mathscr{Y}; \quad u, v \in \mathscr{Y},$$

we have

$$
[u = v]_X = \left\{
\begin{aligned}
&[u = v]_Y &&= \bigwedge_{z \in Y} [z \in u]_Y \Leftrightarrow_B [z \in v]_Y \\
&[u = v{\uparrow_Y}]_Y &&= \bigwedge_{z \in Y} [z \in u]_Y \Leftrightarrow_B [z \in v{\uparrow_Y}]_Y \\
&[u{\uparrow_Y} = v{\uparrow_Y}]_Y &&= \bigwedge_{z \in Y} [z \in u{\uparrow_Y}]_Y \Leftrightarrow_B [z \in v{\uparrow_Y}]_Y
\end{aligned}
\right\} = \bigwedge_{z \in Y} [z \in u]_X \Leftrightarrow_B [z \in v]_X = 1_B.
$$

10.1(a): Establish the transitivity of the subsystem $Y \subset X$ by checking 7.2(c). Suppose that $u \in X$ and $v \in Y$. If $u \in Y$ then $[u \in v]_X \leqslant 1_B = [u \in Y{\uparrow_X}]_X$, and if $u \in \mathscr{Y}$ then

$$
\begin{aligned}
[u \in v]_X &= [u{\uparrow_Y} \in v]_Y = [(\exists y)(u{\uparrow_Y} = y \wedge y \in v)]_Y \\
&\leqslant [(\exists y)(u{\uparrow_Y} = y)]_Y = \bigvee_{y \in Y} [u{\uparrow_Y} = y]_Y = \bigvee_{y \in Y} [u = y]_X = [u \in Y{\uparrow_X}]_X.
\end{aligned}
$$

The following relation in X will be useful below:

$$P \simeq P{\uparrow_X} \ \text{ for all } P \in \mathscr{Y}. \tag{20}$$

Let $P = \{y_i|_{b_i} : i \in I\} \in \mathscr{Y}$. Demonstrate that $[x \in P]_X = [x \in P{\uparrow_X}]_X$ for all $x \in X$. Indeed, if $y \in Y$ then $[y \in P]_X = [y \in P{\uparrow_Y}]_Y = [y \in P{\uparrow_X}]_X$ (see 7.1, 7.6), and if $Q \in \mathscr{Y}$; then by Lemma 3.16

$$[Q \in P]_X = [Q{\uparrow_Y} \in P{\uparrow_Y}]_Y = \bigvee_{p \in P} [Q{\uparrow_Y} = p]_Y = \bigvee_{i \in I} [Q{\uparrow_Y} = y_i]_Y \wedge b_i = \bigvee_{i \in I} [Q = y_i]_X \wedge b_i = [Q \in P{\uparrow_X}]_X.$$

10.1(b): Consider a subset $P \subset {}^{\%}Y$ and show that $x \simeq P{\uparrow}_X$ for some $x \in X$.

Let $\overline{P} := P{\uparrow}_Y{\Downarrow}_Y \subset {}^{\%}Y$ be the saturated hull of P in Y (see 4.4). Since $P \subset \overline{P} \subset P{\uparrow}_X{\Downarrow}_X$; therefore, $P{\uparrow}_X \leqslant \overline{P}{\uparrow}_X \leqslant P{\uparrow}_X{\Downarrow}_X{\uparrow}_X$, and so $P{\uparrow}_X = \overline{P}{\uparrow}_X$ by the equality $P{\uparrow}_X{\Downarrow}_X{\uparrow}_X = P{\uparrow}_X$ (see Lemma 4.3). If $y \simeq \overline{P}{\uparrow}_Y$ in Y for some $y \in Y$; then $y \simeq \overline{P}{\uparrow}_X$ in X by 7.3, and so $x := y \simeq \overline{P}{\uparrow}_X = P{\uparrow}_X$. If $\neg(\exists\, y \in Y)(y \simeq \overline{P}{\uparrow}_Y)$ then $\overline{P} \in \mathscr{Y}$ and, in this case, $x := \overline{P} \simeq \overline{P}{\uparrow}_X = P{\uparrow}_X$ owing to (20).

10.1(c): This is a direct consequence of (20).

10.1(d): Suppose that $y \in Y$ and $P \in \mathscr{Y}$. By the definition of $\mathscr{Y}$, we have $y \not\simeq P{\uparrow}_Y$ in Y; whence from 7.3 it follows that $y \not\simeq P{\uparrow}_X$ in X and so $y \not\simeq P$ by (20).

Now, let $P, Q \in \mathscr{Y}$, $P \simeq Q$. Relation (20) implies $P{\uparrow}_X = Q{\uparrow}_X$. Then $P{\uparrow}_Y = (P{\uparrow}_X)|_Y = (Q{\uparrow}_X)|_Y = P{\uparrow}_Y$. Since P and Q are saturated subsets of ${}^{\%}Y$; by Lemma 4.2 we have $P = P{\uparrow}_Y{\Downarrow}_Y = Q{\uparrow}_Y{\Downarrow}_Y = Q$. ▷

10.3. Lemma. (a) *If a B-system X is a superstructure over a subsystem $Y \subset X$, $f\colon X \leftrightarrow_B X$, and $f|_Y = \mathrm{id}_Y$; then $f = \mathrm{id}_X$.*

(b) *If B-systems X and X' are superstructures over subsystems $Y \subset X$ and $Y' \subset X'$ then each isomorphism $f\colon Y \leftrightarrow_B Y'$ extends to a unique isomorphism $\bar{f}\colon X \leftrightarrow_B X'$. In particular, a superstructure over a copy of a Boolean-valued system is unique up to isomorphism.*

◁ (a): Observe first of all that the extensionality of X and 10.1(c) imply the relation

$$(\forall\, x_1, x_2 \in X\backslash Y)\ \ [x_1 = x_2]_X = \bigwedge_{y\in Y} [y \in x_1 \Leftrightarrow y \in x_2]_X. \tag{21}$$

Indeed, for every $x \in X\backslash Y$, there is a subset $P \subset {}^{\%}Y$ such that $X \vDash (x = P{\uparrow}_X)$, which in view of the validity $X \vDash (P{\uparrow}_X \subset Y{\uparrow}_X)$ (see Lemma 3.17(b)) implies $X \vDash (x \subset Y{\uparrow}_X)$. Consequently,

$$X \vDash \big(x_1 = x_2 \Leftrightarrow (\forall\, y \in Y{\uparrow}_X)(y \in x_1 \Leftrightarrow y \in x_2)\big)$$

for all $x_1, x_2 \in X\backslash Y$. It remains to refer to Lemma 3.18.

Suppose that X, Y, and f satisfy the conditions of (a). Consider an arbitrary $x \in X\backslash Y$ and show that $f(x) = x$. Reckoning with (21) and the equality $f(y) = y$ for $y \in Y$, we conclude that

$$[f(x) = x]_X = \bigwedge_{y\in Y} [y \in f(x) \Leftrightarrow y \in x]_X = \bigwedge_{y\in Y} [f(y) \in f(x)]_X \Leftrightarrow_B [y \in x]_X = \bigwedge_{y\in Y} [y \in x]_X \Leftrightarrow_B [y \in x]_X = 1_B.$$

Thus, $f(x) \simeq x$, which by 10.1(d) implies that $f(x) = x$ since $x \notin Y$.

(b): The uniqueness of an extension $\bar{f}$ follows from (a). Prove the existence. Consider arbitrary extensional B-systems X and X' that are superstructures over subsystems $Y \subset X$ and $Y' \subset X'$ and an isomorphism $f\colon Y \leftrightarrow_B Y'$. By 10.1(c), there exists a family of subsets $P_x \subset {}^{\%}Y$ such that $x \simeq P_x{\uparrow}_X$ for all $x \in X\backslash Y$. Given $x \in X\backslash Y$, put $P'_x := f^{\%}(P_x) \subset {}^{\%}Y'$ (see 3.15). By 10.1(b), there is a function $g\colon X\backslash Y \to X'$ satisfying $g(x) \simeq P'_x{\uparrow}_{X'}$ for all $x \in X\backslash Y$. Define $\bar{f}\colon X \to X'$ by putting

$$\bar{f}(x) := \begin{cases} f(x) & \text{if } x \in Y,\\ g(x) & \text{if } x \in X\backslash Y,\end{cases}$$

and show that $\bar{f}\colon X \leftrightarrow_B X'$. For convenience, put $x' := \bar{f}(x) \in X'$ for $x \in X$. Therefore, $f\colon y \mapsto y'$ is an isomorphism from Y onto Y', and, for all $x \in X\backslash Y$,

$$\begin{aligned} &P_x \subset {}^{\%}Y, \quad P'_x = f^{\%}(P_x) \subset {}^{\%}Y',\\ &x \simeq P_x{\uparrow}_X, \quad x' \simeq P'_x{\uparrow}_{X'}.\end{aligned}$$

Show that $\bar{f}$ preserves the truth values of atomic formulas. Let $\varphi(x_1, x_2, y_1, y_2)$ be any of the (bounded) formulas $x_1 = x_2$, $x_1 \in x_2$, $x_1 = y_1$, $x_1 \in y_1$, $y_1 \in x_1$, $y_1 = y_2$, or $y_1 \in y_2$. Then, by Lemma 3.15 and Theorem 7.6, for all $x_1, x_2 \in X\backslash Y$ and $y_1, y_2 \in Y$, we have

$$\begin{aligned} &[\varphi(x_1, x_2, y_1, y_2)]_X = [\varphi(P_{x_1}{\uparrow}_X, P_{x_2}{\uparrow}_X, y_1, y_2)]_X = [\varphi(P_{x_1}{\uparrow}_Y, P_{x_2}{\uparrow}_Y, y_1, y_2)]_Y\\ &= [\varphi(P'_{x_1}{\uparrow}_{Y'}, P'_{x_2}{\uparrow}_{Y'}, y'_1, y'_2)]_{Y'} = [\varphi(P'_{x_1}{\uparrow}_{X'}, P'_{x_2}{\uparrow}_{X'}, y'_1, y'_2)]_{X'} = [\varphi(x'_1, x'_2, y'_1, y'_2)]_{X'}.\end{aligned} \tag{22}$$

Show that $\bar{f}\colon X \to X'$ is injective. The injectivity of $\bar{f}$ on Y is guaranteed by the injectivity of the isomorphism $f\colon Y \to Y'$. If at least one of the elements $x_1, x_2 \in X$ does not belong to Y; then since $[x_1 = x_2]_X = [x'_1 = x'_2]_{X'}$ (see (22)) from $x'_1 = x'_2$ it follows that $x_1 \simeq x_2$, and so $x_1 = x_2$ by 10.1(d).

Finally, prove the surjectivity of $\bar{f}\colon X \to X'$. Consider an arbitrary $z \in X'$ and show that $z = x'$ for some $x \in X$. If $z \in Y'$ then the desired $x \in X$ exists by the surjectivity of the isomorphism $f\colon Y \to Y'$. Let $z \in X' \setminus Y'$. By 10.1(c), there is a subset $P' \subset {}^{\%}Y'$ for which $z \simeq P'{\uparrow}_{X'}$. Put $P := (f^{\%})^{-1}(P') \subset Y$. By 10.1(b), there exists $x \in X$ satisfying $x \simeq P{\uparrow}_X$. If $x \in Y$; then from 7.3 and Lemma 3.15 we infer that

$$x \simeq P{\uparrow}_X \Rightarrow x \simeq P{\uparrow}_Y \Rightarrow x' \simeq P'{\uparrow}_{Y'} \Rightarrow x' \simeq P'{\uparrow}_{X'} \Rightarrow x' \simeq z,$$

which leads to a contradiction to 10.1(d). Consequently, $x \notin Y$. So, owing to 7.3 and Lemma 3.15,

$$x \simeq P{\uparrow}_X \;\Rightarrow\; P_x{\uparrow}_X = P{\uparrow}_X \;\Rightarrow\; P_x{\uparrow}_Y = P{\uparrow}_Y \;\Rightarrow\; P'_x{\uparrow}_{Y'} = P'{\uparrow}_{Y'} \;\Rightarrow\; P'_x{\uparrow}_{X'} = P'{\uparrow}_{X'} \;\Rightarrow\; x' \simeq z,$$

which, by 10.1(d), implies that $x' = z$. ▹

10.4. Say that a family of extensional B-systems $(X_\alpha)_{\alpha \in \mathrm{Ord}^\bullet}$ is an *intensional hierarchy* or, more exactly, a *B-valued intensional hierarchy over X_0* if

$$X_{\alpha+1} \text{ is a superstructure over } X_\alpha, \quad \alpha \in \mathrm{Ord};$$

$$X_\alpha = \bigcup_{\beta<\alpha} X_\beta, \quad \alpha \in \mathrm{Lim}\,\mathrm{Ord}^\bullet.$$

In this case, $X_\beta \preccurlyeq X_\alpha$ for all $\beta \leqslant \alpha \in \mathrm{Ord}^\bullet$ (see Lemma 7.10(e)).

10.5. Lemma. *For every extensional B-system Y, there exists an intensional hierarchy $(X_\alpha)_{\alpha \in \mathrm{Ord}^\bullet}$ such that $X_0 \leftrightarrow_B Y$.*

◃ The desired hierarchy can be easily built on using Lemmas 7.10 and 10.2 together with the construction of the direct limit (see, for example, [8, III.1.11]). Indeed, define the family of B-systems Y_α ($\alpha \in \mathrm{Ord}^\bullet$) and isomorphic embeddings $f^\alpha_\beta\colon Y_\beta \to Y_\alpha$ ($\beta \leqslant \alpha \in \mathrm{Ord}^\bullet$) satisfying the relations $f^\alpha_\alpha = \mathrm{id}_{Y_\alpha}$ and $f^\alpha_\gamma = f^\alpha_\beta \circ f^\beta_\gamma$ ($\gamma \leqslant \beta \leqslant \alpha \in \mathrm{Ord}^\bullet$), by means of the following recursive procedure:

$Y_0 := Y$, $f^0_0 := \mathrm{id}_{Y_0}$;
for $\alpha \in \mathrm{Ord}$
  $Y_{\alpha+1}$ is a superstructure over a copy of the B-system Y_α (see 10.1, 10.2),
  $f^{\alpha+1}_\alpha$ is the corresponding isomorphism of Y_α onto a subsystem of $Y_{\alpha+1}$,
  $f^{\alpha+1}_{\alpha+1} := \mathrm{id}_{Y_{\alpha+1}}$, $f^{\alpha+1}_\beta := f^{\alpha+1}_\alpha \circ f^\alpha_\beta$ for $\beta < \alpha+1$;
for $\alpha \in \mathrm{Lim}\,\mathrm{Ord}^\bullet$
  $Y_\alpha := \big(\bigcup_{\beta<\alpha} \{\beta\} \times Y_\beta\big)/{\sim}$, where, for $\beta, \gamma < \alpha$, $\mu := \max\{\beta, \gamma\}$, $x \in Y_\beta$, $y \in Y_\gamma$,
    $(\beta, x) \sim (\gamma, y) \Leftrightarrow f^\mu_\beta(x) = f^\mu_\gamma(y)$,
    $=_{Y_\alpha}\!\big({\sim}(\beta,x), {\sim}(\gamma,y)\big) := =_{Y_\mu}\!\big(f^\mu_\beta(x), f^\mu_\gamma(y)\big)$, $\in_{Y_\alpha}\!\big({\sim}(\beta,x), {\sim}(\gamma,y)\big) := \in_{Y_\mu}\!\big(f^\mu_\beta(x), f^\mu_\gamma(y)\big)$,
  $f^\alpha_\alpha := \mathrm{id}_{Y_\alpha}$, $f^\alpha_\beta(x) := {\sim}(\beta, x)$ for $\beta < \alpha$, $x \in Y_\beta$.

From 7.3, 7.10 and 10.1 it follows that the family of B-systems $X_\alpha := f^\infty_\alpha(Y_\alpha)$ ($\alpha \in \mathrm{Ord}^\bullet$) is the desired intensional hierarchy. ▹

10.6. Lemma. *Let $(X_\alpha)_{\alpha \in \mathrm{Ord}^\bullet}$ and $(Y_\alpha)_{\alpha \in \mathrm{Ord}^\bullet}$ be B-valued intensional hierarchies.*
(a) *If $f\colon X_\infty \leftrightarrow_B Y_\infty$ and $f|_{X_0}\colon X_0 \leftrightarrow_B Y_0$ then $f|_{X_\alpha}\colon X_\alpha \leftrightarrow_B Y_\alpha$ for all $\alpha \in \mathrm{Ord}^\bullet$.*
(b) *If $f, g\colon X_\infty \leftrightarrow_B Y_\infty$ and $f|_{X_0} = g|_{X_0}$ then $f|_{X_\alpha} = g|_{X_\alpha}$ for all $\alpha \in \mathrm{Ord}^\bullet$.*
(c) *Every isomorphism $f_0\colon X_0 \leftrightarrow_B Y_0$ extends to a unique isomorphism $f\colon X_\infty \leftrightarrow_B Y_\infty$.*

◃ (a) and (b): In employing induction on $\alpha \in \mathrm{Ord}^\bullet$, the base and limit steps are trivial, and the step $\alpha \mapsto \alpha+1$ is easy to justify for item (a) with the use of Lemma 3.15 and 10.1(b),(c), and for item (b), with the use of (a) and Lemma 10.3(a).

(c) Using transfinite recursion and basing on Lemma 10.3(b), it is easy to construct a family of isomorphisms $f_\alpha\colon X_\alpha \leftrightarrow_B Y_\alpha$ ($\alpha \in \mathrm{Ord}^\bullet$) such that $f_\alpha \subset f_\beta$ for $\alpha \leqslant \beta$. Then $f := f_\infty$ is a desired isomorphism. The uniqueness of the extension follows from (b). ▹

10.7. We say that a B-system X is a *Boolean-valued* (B*-valued*) *universe over* X_0 if the following conditions are fulfilled:

(a) $X_0 \preccurlyeq X$;

(b) X is extensional;

(c) X is intensional;

(d) the elements of $X \backslash X_0$ are predicative;

(e) X is separated over X_0;

(f) X is regular outside X_0.

By Corollary 6.9, conditions (b) and (c) imply the cyclicity of the class $\mathscr{P}_X(X)$ of all predicative elements in X, which in view of (d) implies that the difference $X \backslash X_0$ is precyclic, and hence, by Theorem 9.9(b), condition (f) is equivalent to the σ-regularity of X outside X_0.

As is easy to see, a Boolean-valued universe X over X_0 is a Boolean-valued universe over every subsystem Y satisfying the relations $X_0 \subset Y \preccurlyeq X$.

10.8. Theorem. (a) *If* $(X_\alpha)_{\alpha \in \mathrm{Ord}^\bullet}$ *is an intensional hierarchy then* X_∞ *is a Boolean-valued universe over* X_0.

(b) *If* X *is a Boolean-valued universe over* X_0 *then there exists a unique intensional hierarchy* $(X_\alpha)_{\alpha \in \mathrm{Ord}^\bullet}$ *such that* $X = X_\infty$. *Moreover,*

$$\begin{aligned} X_{\alpha+1} &= X_\alpha \cup \mathscr{P}_X(X_\alpha), && \alpha \in \mathrm{Ord}; \\ X_\alpha &= \bigcup_{\alpha<\beta} X_\beta, && \alpha \in \mathrm{Lim\,Ord}^\bullet. \end{aligned} \tag{23}$$

$\lhd$ (a): Verify 10.7(a)–(f) for $X = X_\infty$.

10.7(a): By Lemma 7.10(e), the system X_0, as well as each of the systems X_α, is a transitive subsystem of X_∞. Note that, for this reason, the fulfillment of the relation $x \simeq P\!\uparrow$ in one of the systems X_α is equivalent to its fulfillment in X_∞ (see 7.3).

Condition 10.7(b) is included in the definition of intensional hierarchy.

10.7(c): Let P be a subset of ${}^{\%}X_\infty = \bigcup_{\alpha \in \mathrm{Ord}} {}^{\%}X_\alpha$. Choosing for each element $p \in P$ an ordinal $\alpha(p)$ satisfying the condition $p \in {}^{\%}X_{\alpha(p)}$, we conclude that $P \subset {}^{\%}X_\alpha$, where $\alpha := \vee\{\alpha(p) : p \in P\}$, and so $P\!\uparrow \simeq x$ for some $x \in X_{\alpha+1}$ by 10.1(b).

10.7(d): If $x \in X_\infty \backslash X_0$ then $x \in X_{\alpha+1}$ for some $\alpha \in \mathrm{Ord}$, and so $(\exists\, P \subset {}^{\%}X_\alpha)(x \simeq P\!\uparrow)$ by 10.1(c).

10.7(e): Suppose that $x, y \in X_\infty$, $x \simeq y$, and $x \neq y$. Put $\alpha := \min\{\beta \in \mathrm{Ord} : x, y \in X_\beta\}$ and show that $\alpha = 0$. Indeed, α cannot be a limit ordinal; since, in this case $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$, and so there is an ordinal $\beta < \alpha$ satisfying $x, y \in X_\beta$. If $\alpha = \beta + 1$ then $x, y \in X_\beta$ by 10.1(d).

10.7(f): Prove that all systems X_α (including X_∞) are regular outside X_0 by inducting on $\alpha \in \mathrm{Ord}^\bullet$.

The case of $\alpha = 0$ is trivial: the system X_0 is obviously regular outside X_0.

Suppose that X_α is regular outside X_0. By Lemma 9.5, for proving the regularity of $X_{\alpha+1}$ outside X_0, it suffices to demonstrate the regularity of $X_{\alpha+1}$ outside X_α. By Lemma 3.17(b), 10.1(c) implies that

$$X_{\alpha+1} \vDash (\forall\, y)(y \in X_\alpha\!\uparrow \,\vee\, y \subset X_\alpha\!\uparrow). \tag{24}$$

Consider an arbitrary Boolean-valued class $\Phi \Subset X_{\alpha+1}$ and prove the validity

$$X_{\alpha+1} \vDash \big(\Phi \cap X_\alpha\!\uparrow = \varnothing \,\wedge\, \Phi \neq \varnothing \;\Rightarrow\; (\exists\, y \in \Phi)(y \cap \Phi = \varnothing)\big)$$

by "arguing inside $X_{\alpha+1}$" (see 3.10). Let $\Phi \cap X_\alpha\!\uparrow = \varnothing$ and $\Phi \neq \varnothing$. Take $y \in \Phi$ and show that $y \cap \Phi = \varnothing$. Indeed, from $\Phi \cap X_\alpha\!\uparrow = \varnothing$ it follows that $y \notin X_\alpha\!\uparrow$, and so $y \subset X_\alpha\!\uparrow$ by (24) and, therefore, $y \cap \Phi \subset X_\alpha\!\uparrow \cap \Phi = \varnothing$.

Suppose now that $\alpha \in \mathrm{Lim\,Ord}^\bullet$, and let X_β be regular outside X_0 for all $\beta < \alpha$. By Lemma 8.4, for proving the regularity of X_α outside X_0, it suffices to consider a Boolean-valued class $\Phi \Subset X_\alpha$ satisfying $\Phi \leqslant \neg(X_0\!\uparrow_{X_\alpha})$ and show that $X_\alpha \vDash \mu(\Phi)$ or, which is equivalent,

$$(\forall\, x \in X_\alpha)\; X_\alpha \vDash \big(x \in \Phi \Rightarrow (\exists\, y \in \Phi)(y \cap \Phi = \varnothing)\big).$$

If $x \in X_\alpha = \bigcup_{\beta<\alpha} X_\beta$ then $x \in X_\beta$ for some ordinal $\beta < \alpha$. Prove the validity

$$X_\alpha \vDash \big(x \in \Phi \Rightarrow (\exists\, y \in \Phi)(y \cap \Phi = \varnothing)\big)$$

by "arguing inside X_α." Relying upon the relations $\Phi \cap X_0{\uparrow} = \varnothing$, $x \in X_\beta{\uparrow}$, and $x \in \Phi$, demonstrate that $(\exists\, y \in \Phi)(y \cap \Phi = \varnothing)$. Since the class $X_\beta{\uparrow} \setminus X_0{\uparrow}$ is regular (see Lemma 9.4) and $\varnothing \neq \Phi \cap X_\beta{\uparrow} \subset X_\beta{\uparrow} \setminus X_0{\uparrow}$, there exists $y \in \Phi \cap X_\beta{\uparrow}$ such that $y \cap \Phi \cap X_\beta{\uparrow} = \varnothing$. By the transitivity of $X_\beta{\uparrow}$, from $y \in X_\beta{\uparrow}$ it follows that $y \subset X_\beta{\uparrow}$. Thus, $y \cap \Phi = y \cap \Phi \cap X_\beta{\uparrow} = \varnothing$.

(b): Consider the family of subsystems $X_\alpha \subset X$ ($\alpha \in \mathrm{Ord}^\bullet$) constructed with the use of transfinite recursion by (23) starting from the given subsystem $X_0 \subset X$.

Induct on $\alpha \in \mathrm{Ord}^\bullet$ to show that $X_\alpha \preccurlyeq X$. The case of $\alpha = 0$ is contained in 10.7(a). Suppose that $X_\alpha \preccurlyeq X$, consider arbitrary $x \in X$, $y \in X_{\alpha+1}$, and establish that $[x \in y] \leqslant [x \in X_{\alpha+1}{\uparrow}]$. If $y \in X_\alpha$ then $[x \in y] \leqslant [x \in X_\alpha{\uparrow}] \leqslant [x \in X_{\alpha+1}{\uparrow}]$; and if $y \in \mathscr{P}_X(X_\alpha)$ then $y \simeq P{\uparrow}$, $P \subset {}^{\%}X_\alpha$ and so by Lemma 3.17(b) $[x \in y] = [x \in P{\uparrow}] \leqslant [x \in X_\alpha{\uparrow}] \leqslant [x \in X_{\alpha+1}{\uparrow}]$. If $\alpha \in \mathrm{Lim\,Ord}^\bullet$ and $X_\beta \preccurlyeq X$ for all $\beta < \alpha$, then $X_\alpha = \bigcup_{\alpha<\beta} X_\beta \preccurlyeq X$ by Lemma 7.10(c).

By Lemma 7.10(a), the above implies that $X_\alpha \preccurlyeq X_{\alpha+1}$ for all $\alpha \in \mathrm{Ord}$, which corresponds to 10.1(a) in the definition of superstructure. Conditions 10.1(b),(c) are ensured by the definition of $X_{\alpha+1}$, and 10.1(d) follows from 10.7(e). Thus, the family $(X_\alpha)_{\alpha\in\mathrm{Ord}^\bullet}$ is an intensional hierarchy.

For proving the equality $X = X_\infty$, we will need several auxiliary facts. Show that

$$(\forall\, x \in X)(\exists\, y \in X_\infty)\ [x = y] = [x \subset X_\infty{\uparrow}]. \tag{25}$$

In the case of $x \in X_0$, (25) is obvious. Let $x \notin X_0$. By 10.7(d) the element $x \in X$ is predicative, and so by Lemma 4.10 there is a set $P \subset {}^{\%}X_\infty$ such that $[x \subset X_\infty{\uparrow}] = [x = P{\uparrow}]$. By (a), the system X_∞ satisfies 10.7(c); therefore, the Boolean-valued class $P{\uparrow}$ is represented by some $y \in X_\infty$, which is desired.

Show also that

$$(\forall\, x \in X)\ [x \in X_\infty{\uparrow}] = [x \subset X_\infty{\uparrow}]. \tag{26}$$

The inequality "$\leqslant$" is guaranteed by the transitivity of the subsystem $X_\infty \subset X$. On the other hand, by (25), for every $x \in X$ there is $y \in X_\infty$ satisfying $[x = y] = [x \subset X_\infty{\uparrow}]$, and so

$$[x \subset X_\infty{\uparrow}] = [x = y] \wedge [y \in X_\infty{\uparrow}] \leqslant [x \in X_\infty{\uparrow}].$$

Passing to the proof of the equality $X = X_\infty$, assume on the contrary that there exists $x \in X$ not belonging to X_∞. Put $b := [x \notin X_\infty{\uparrow}] = [x \not\subset X_\infty{\uparrow}]$ (see (26)). Note that $b \neq 0_B$. Indeed, if $b = 0_B$ then $X \vDash (x \subset X_\infty{\uparrow})$ and then, by (25), there exists $y \in X_\infty$ such that $[x = y] = [x \subset X_\infty{\uparrow}] = 1_B$, i.e., $x \simeq y$, which, with account taken of 10.7(e), implies that $x = y$, and so $x \in X_\infty$. Let $\Phi := \neg(X_\infty{\uparrow})$ be the complement to $X_\infty{\uparrow}$ inside X. Since X is regular outside X_0 (see 10.7(f)), $\Phi \leqslant \neg(X_0{\uparrow})$ and $[\Phi \neq \varnothing] \geqslant b$ imply that $[(\exists\, y \in \Phi)(y \cap \Phi = \varnothing)] \geqslant b$; therefore, $[y \in \Phi\ \wedge\ y \cap \Phi = \varnothing] \neq 0_B$ for some $y \in X$. On the other hand, by (26),

$$[y \in \Phi\ \wedge\ y \cap \Phi = \varnothing] = [y \notin X_\infty{\uparrow}\ \wedge\ y \subset X_\infty{\uparrow}] = 0_B.$$

The uniqueness of the intensional hierarchy in (b) stems from Lemma 10.6(a). Indeed, if $(X_\alpha)_{\alpha\in\mathrm{Ord}^\bullet}$ and $(Y_\alpha)_{\alpha\in\mathrm{Ord}^\bullet}$ are intensional hierarchies such that $X_0 = Y_0$ and $X_\infty = Y_\infty$ then, for all $\alpha \in \mathrm{Ord}^\bullet$, we have $(\mathrm{id}_{X_\infty})|_{X_\alpha}\colon X_\alpha \leftrightarrow_B Y_\alpha$, i.e., $X_\alpha = Y_\alpha$. ▷

10.9. The assertion below follows from Lemmas 10.5, 10.6(c), and Theorem 10.8.

Theorem. (a) *For every extensional B-system Y, there exists a B-valued universe X over X_0 such that $X_0 \leftrightarrow_B Y$; moreover, such an X is unique up to isomorphism.*

(b) *If X and Y are B-valued universes over X_0 and Y_0 then each isomorphism $f_0\colon X_0 \leftrightarrow_B Y_0$ extends to a unique isomorphism $f\colon X \leftrightarrow_B Y$.*

Applying item (a) to some extensional B-system Y and considering a B-valued universe X over a copy X_0 of the system Y, agree to identify X_0 with Y and refer to X as a *Boolean-valued universe over Y*.

§ 11. The Boolean-Valued Universe

In the present section, we formulate the defining properties of the classical Boolean-valued universe $\mathbb{V}^{(B)}$ as an algebraic system, prove the existence of such a system and its uniqueness up to a unique isomorphism. These well-known facts (see [4, 9]) are reproduced here as direct consequences of the general properties of a Boolean-valued universe over an arbitrary extensional system which are established in § 10. The new results presented in the section include examples of Boolean-valued systems with unusual combinations of properties. The examples show that, for each complete Boolean algebra B, none of the conditions in the definition of a B-valued universe follows from the others.

11.1. A Boolean-valued system X is called a *Boolean-valued* (more exactly, *B-valued*) *universe* (see [9, 3.4]) if X satisfies the following conditions:

(a) X is extensional: $X \vDash (\forall\, x, y)\big((\forall\, z)(z \in x \Leftrightarrow z \in y) \;\Rightarrow\; x = y\big)$;
(b) X is intensional: $(\forall\, P \subset {}^{\%}\!X)(\exists\, x \in X)(x \simeq P{\uparrow})$;
(c) X is predicative: $(\forall\, x \in X)(\exists\, P \subset {}^{\%}\!X)(x \simeq P{\uparrow})$;
(d) X is separated: $(\forall\, x, y \in X)\big(X \vDash (x = y) \;\Rightarrow\; x = y\big)$;
(e) X is regular: $X \vDash (\forall\, \Phi)\big((\exists\, y)(y \in \Phi) \Rightarrow (\exists\, y \in \Phi)(\forall\, z \in \Phi)(z \notin y)\big)$.

Recall that, by (b) and (c), the regularity, (e), of X is equivalent to the validity in X of the axiom of regularity (see 9.1).

11.2. Lemma. *Let Y be an arbitrary extensional B-system. If a B-system X is a Boolean-valued universe over Y then*

(a) *X is extensional;*
(b) *X is intensional;*
(c) *X is predicative if and only if Y is predicative;*
(d) *X is separated if and only if Y is separated;*
(e) *X is regular if and only if Y is regular.*

◃ Assertions (a) and (b) are explicitly contained in Definition 10.7; (c) follows from 7.3(a) and 10.7(a); (d) is guaranteed by condition 10.7(e); (e) stems from Lemma 9.7, Corollary 9.6, and Condition 10.7(f). ▹

11.3. Lemma 11.2 implies that the notion of B-valued universe coincides with the notion of Boolean-valued universe over a predicative separated regular B-system. The simplest of these B-systems is a singleton regular B-system. (It would be even easier to speak of a universe over $\varnothing$ but an algebraic system cannot be empty.)

Corollary. *The following properties of a B-system X are equivalent:*

(a) *X is a Boolean-valued universe;*
(b) *X is a Boolean-valued universe over $\{y\}$, where $y \in X$ is such that $X \vDash (y = \varnothing)$;*
(c) *X is a Boolean-valued universe over a singleton B-system $Y = \{y\}$ such that $Y \vDash (y \notin y)$.*

11.4. The next assertion, given (without proof) in [9, 3.4], follows from Theorem 10.9 and Corollary 11.3.

Theorem.[1)] (a) *For every complete Boolean algebra B, there exists a B-valued universe unique up to isomorphism.*

(b) *For arbitrary B-valued universes X and Y, there exists a unique isomorphism $f \colon X \leftrightarrow_B Y$.*

11.5. The Boolean-valued universe, characterized up to isomorphism, is denoted by $\mathbb{V}^{(B)}$; and the corresponding truth values $[\varphi]_{\mathbb{V}^{(B)}}$ are written down as $[\![\varphi]\!]$ (see [4, 9]).

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1) The author is indebted to Professor Robert M. Solovay for a fruitful discussion of the scheme of a proof of this theorem.

The Boolean-valued universe is a model of ZFC (see [4, 4.4]). More exactly, there are classes $\mathbb{V}^{(B)}$, $[\![\cdot = \cdot]\!]$, and $[\![\cdot \in \cdot]\!]$, defined with parameter B, such that

$$\begin{aligned}&\text{ZFC}, B \text{ is a complete Boolean algebra} \vdash \\ &\qquad (\mathbb{V}^{(B)}, [\![\cdot = \cdot]\!], [\![\cdot \in \cdot]\!]) \text{ is a } B\text{-valued universe}, \\ &\qquad (\mathbb{V}^{(B)}, [\![\cdot = \cdot]\!], [\![\cdot \in \cdot]\!]) \vDash \text{ZFC}.\end{aligned}$$

Examples 11.6–11.10 below show that, for every complete Boolean algebra B, each of the five conditions (a)–(e) listed in Definition 11.1 of the Boolean-valued universe is essential, i.e., none of them follows from the other four conditions. The main tools here are Theorem 10.9(a) and Lemma 11.2.

11.6. Example. For every complete Boolean algebra B, there exists a B-valued system that is intensional, predicative, separated, and regular but not extensional.

◁ Extend ZFC by the definitions of the constants $2_1 := \big\{\{\varnothing\}\big\}$ and $2_2 := \big\{\varnothing, \{\varnothing\}\big\}$. Consider the Boolean-valued universe $\mathbb{V}^{(B)}$ and the elements $\varnothing^\wedge, \{\varnothing\}^\wedge, 2_1^\wedge, 2_2^\wedge \in \mathbb{V}^{(B)}$ representing the ascents $\varnothing\!\uparrow$, $\{\varnothing^\wedge\}\!\uparrow$, $\big\{\{\varnothing\}^\wedge\big\}\!\uparrow$, and $\big\{\varnothing^\wedge, \{\varnothing\}^\wedge\big\}\!\uparrow$ respectively. As is easy to see, $\mathbb{V}^{(B)} \vDash (2_1^\wedge = 2_1)$ and $\mathbb{V}^{(B)} \vDash (2_2^\wedge = 2_2)$ (see 2.5).

Show that the subsystem

$$X := \big\{x \in \mathbb{V}^{(B)} : \mathbb{V}^{(B)} \vDash (x \not\subset \{\varnothing\})\big\} \subset \mathbb{V}^{(B)}$$

possesses the desired properties.

By 2.3(a), the tautologies $2_1, 2_2 \not\subset \{\varnothing\}$ and $2_1 \neq 2_2$ imply that $2_1^\wedge, 2_2^\wedge \in X$ and $[2_1^\wedge = 2_2^\wedge]_X = [\![2_1^\wedge = 2_2^\wedge]\!] = 0_B$. On the other hand, for all $x \in X$,

$$[x \in 2_1^\wedge]_X = [\![x \in 2_1^\wedge]\!] = [\![x \in 2_1 \wedge x \not\subset \{\varnothing\}]\!] = 0_B = [\![x \in 2_2 \wedge x \not\subset \{\varnothing\}]\!] = [\![x \in 2_2^\wedge]\!] = [x \in 2_2^\wedge]_X. \tag{27}$$

Thus, $X \vDash (2_1^\wedge \neq 2_2^\wedge)$ and $X \vDash (\forall\, x)(x \in 2_1^\wedge \Leftrightarrow x \in 2_2^\wedge)$, and so the system X is not extensional.

For proving the intensionality of X, for an arbitrary subset $P \subset {}^{\%}X$, consider some element $u \in \mathbb{V}^{(B)}$ representing the ascent $P\!\uparrow$, put $x := u|_b \sqcup 2_1^\wedge|_{\neg b}$ (see 5.3), where $b := \bigvee_{p \in P} \operatorname{dom} p$, and show that $x \in X$ and $x \simeq P\!\uparrow_X$. By Lemma 3.18(a), we have

$$[\![P\!\uparrow \not\subset \{\varnothing\}]\!] = [\![(\exists\, y \in P\!\uparrow)(y \notin \{\varnothing\})]\!] = \bigvee_{p \in P} [\![p \notin \{\varnothing\}]\!] \geqslant \bigvee_{p \in P} [\![p \not\subset \{\varnothing\}]\!] = \bigvee_{p \in P} \operatorname{dom} p = b,$$

which implies that

$$[\![x \not\subset \{\varnothing\}]\!] = ([\![u \not\subset \{\varnothing\}]\!] \wedge b) \vee ([\![2_1 \not\subset \{\varnothing\}]\!] \wedge \neg b) = ([\![P\!\uparrow \not\subset \{\varnothing\}]\!] \wedge b) \vee \neg b = 1_B;$$

i.e., $x \in X$. Moreover, for all $y \in X$, using the equality $[\![y \in 2_1^\wedge]\!] = 0_B$ (see (27)), we have

$$\begin{aligned}[y \in x]_X &= [\![y \in x]\!] = ([\![y \in u]\!] \wedge b) \vee ([\![y \in 2_1^\wedge]\!] \wedge \neg b) = [\![y \in P\!\uparrow]\!] \wedge b \\ &= [\![y \in P\!\uparrow \wedge P\!\uparrow \neq \varnothing]\!] = [\![y \in P\!\uparrow]\!] = P\!\uparrow(y) = P\!\uparrow_X(y).\end{aligned}$$

The predicativity and separatedness of X are immediate from the predicativity and separatedness of $\mathbb{V}^{(B)}$ because $x\!\Downarrow_X \subset x\!\Downarrow_{\mathbb{V}^{(B)}}$ and $[x = y]_X = [\![x = y]\!]$ for all $x, y \in X$. The regularity of X follows from that of $\mathbb{V}^{(B)}$ by Lemma 9.7. ▷

11.7. Example. For every complete Boolean algebra B, there exists a B-valued system that is extensional, predicative, separated, and regular but not intensional.

◁ The extensionality, predicativity, separatedness, and regularity of a singleton system $\{x\}$ with interpretations $[x = x] = 1_B$ and $[x \in x] = 0_B$ are obvious. Since $[x \in x] = 0_B \neq 1_B = \{x\}\!\uparrow(x)$, the Boolean-valued class $\{x\}\!\uparrow$ is not represented by the (unique) element x; and so the system under consideration is not intensional. ▷

11.8. Example. For every complete Boolean algebra B, there exists a B-valued system that is extensional, intensional, separated, and regular but not predicative.

◃ Let Z be an extensional, separated, regular B-valued system whose underlying class is a proper class different from the class of all sets. (As Z we can take, for example, the isomorphic copy $\{\varnothing\} \times \mathbb{V}^{(B)}$ of the Boolean-valued universe $\mathbb{V}^{(B)}$.)

Consider an arbitrary set ∞ not belonging to Z, put $Y := Z \cup \{\infty\}$, and extend the Boolean-valued interpretations of Z onto Y by putting

$$\begin{aligned}
&[\infty = \infty]_Y = 1_B,\\
&[\infty \in \infty]_Y = [z = \infty]_Y = [\infty = z]_Y = [\infty \in z]_Y = 0_B,\\
&[z \in \infty]_Y = 1_B \ \text{ for all } z \in Z.
\end{aligned}$$

An elementary check shows that Y is a B-system.

For proving that Y is extensional, consider arbitrary $x, y \in Y$ and show that

$$[(\forall\, z)(z \in x \Leftrightarrow z \in y)]_Y \leqslant [x = y]_Y.$$

The case of $x, y \in Z$ amounts to the extensionality of Z, the case of $x = y = \infty$ is trivial, and in the case of $x \in Z$ and $y = \infty$ the desired inequality is guaranteed by the regularity of Z since the validity $Z \vDash \mu(\{x\}{\uparrow})$ implies $Z \vDash (x \notin x)$, and so

$$[(\forall\, z)(z \in x \Leftrightarrow z \in \infty)]_Y \leqslant [x \in x \Leftrightarrow x \in \infty]_Y = [x \in x]_Z = 0_B = [x = \infty]_Y.$$

As is easy to see, Y is separated and $Z \preccurlyeq Y$. Since

$$Y \vDash (\forall\, \Phi)(\Phi \cap Z{\uparrow} = \varnothing \Rightarrow \Phi \subset \{\infty\}),$$

the system Y is regular outside Z. By Corollary 9.6, the regularity of Z implies the regularity of Y. The system Y is not predicative since the saturated descent $\infty{\Downarrow}$ contains the proper class $\infty{\downarrow} = Z$.

Lemma 11.2 implies that the Boolean-valued universe over Y satisfies the requirements listed in the statement. ▹

11.9. Example. For every complete Boolean algebra B, there exists a B-valued system that is extensional, intensional, predicative, and regular but not separated.

◃ By Lemma 11.2, all the properties listed are possessed by the Boolean-valued universe over a two-element B-system Y with interpretations $=_Y : Y^2 \to \{1_B\}$ and $\in_Y : Y^2 \to \{0_B\}$. ▹

11.10. Example. For every complete Boolean algebra B, there exists a B-valued system that is extensional, intensional, predicative, and separated but not regular.

◃ Since a singleton set $Y = \{y\}$ endowed with the interpretations $[y = y]_Y = [y \in y]_Y = 1_B$ is an extensional, predicative, separated, and nonregular B-system, by Lemma 11.2, the Boolean-valued universe over Y possesses the required properties. ▹

§ 12. Hierarchies in the Boolean-Valued Universe

In this section, we propose descriptions of the structure of the Boolean-valued universe $\mathbb{V}^{(B)}$ by means of four hierarchies, one of which reproduces the intensional hierarchy, the second serves as the descent of the von Neumann hierarchy, and the other two are generated by some ascents of a special kind and mixings.

In accordance with Agreement 3.20, for every subset $P \subset {}^{\%}\mathbb{V}^{(B)}$, the Boolean-valued class $P{\uparrow} \Subset \mathbb{V}^{(B)}$ is identified with the element of $\mathbb{V}^{(B)}$ which represents the class.

12.1. The hierarchy in the theorem below corresponds to the classical construction of the (unseparated) Boolean-valued universe [4, 4.1.2; 6, (14.15)] and, without the zero term, coincides with the intensional hierarchy (23) over a singleton regular B-system.

Theorem. *Using transfinite recursion, define the family of subsets* $\mathbb{V}^{(B)}_\alpha \subset \mathbb{V}^{(B)}$ $(\alpha \in \mathrm{Ord})$ *by putting*

$$\begin{aligned} &\mathbb{V}^{(B)}_0 = \varnothing; \\ &\mathbb{V}^{(B)}_{\alpha+1} = \mathscr{P}_{\mathbb{V}^{(B)}}\big(\mathbb{V}^{(B)}_\alpha\big), \quad \alpha \in \mathrm{Ord}; \\ &\mathbb{V}^{(B)}_\alpha = \bigcup_{\beta<\alpha} \mathbb{V}^{(B)}_\beta, \qquad \alpha \in \mathrm{Lim\,Ord}. \end{aligned} \tag{28}$$

Then

$$\mathbb{V}^{(B)} = \bigcup_{\alpha\in\mathrm{Ord}} \mathbb{V}^{(B)}_\alpha. \tag{29}$$

Moreover, the family $(X_\alpha)_{\alpha\in\mathrm{Ord}^\bullet}$ *defined by the rule*

$$X_\alpha = \begin{cases} \mathbb{V}^{(B)}_{\alpha+1} & \text{for } \alpha < \omega; \\ \mathbb{V}^{(B)}_\alpha & \text{for } \omega \leqslant \alpha < \infty; \\ \mathbb{V}^{(B)} & \text{for } \alpha = \infty \end{cases}$$

is an intensional hierarchy over $X_0 = \mathbb{V}^{(B)}_1 = \{\varnothing\!\uparrow\}$. *In particular,* $\mathbb{V}^{(B)}_\beta$ *is a transitive subset of* $\mathbb{V}^{(B)}_\alpha$ *for all* $\beta \leqslant \alpha \in \mathrm{Ord}^\bullet$, *where* $\mathbb{V}^{(B)}_\infty := \mathbb{V}^{(B)}$.

◁ Show by induction on $\alpha\in\mathrm{Ord}$ that $\mathbb{V}^{(B)}_\beta \subset \mathbb{V}^{(B)}_\alpha$ for $\beta<\alpha$. Let $\mathbb{V}^{(B)}_\gamma \subset \mathbb{V}^{(B)}_\beta$ for all $\gamma < \beta < \alpha$. Consider arbitrary $\beta < \alpha$ and $x \in \mathbb{V}^{(B)}_\beta$ and show that $x \in \mathbb{V}^{(B)}_\alpha$. The cases of $\alpha = 0$ and $\alpha \in \mathrm{Lim\,Ord}$ are trivial. Let $\alpha = \alpha_0 + 1$. Definition (28) implies that $x \in \mathscr{P}_{\mathbb{V}^{(B)}}\big(\mathbb{V}^{(B)}_\gamma\big)$ for some ordinal $\gamma < \beta \leqslant \alpha_0 < \alpha$. By the induction hypothesis, $\mathbb{V}^{(B)}_\gamma \subset \mathbb{V}^{(B)}_{\alpha_0}$, and so $x \in \mathscr{P}_{\mathbb{V}^{(B)}}\big(\mathbb{V}^{(B)}_\gamma\big) \subset \mathscr{P}_{\mathbb{V}^{(B)}}\big(\mathbb{V}^{(B)}_{\alpha_0}\big) = \mathbb{V}^{(B)}_\alpha$.

The inclusions $X_\alpha \subset X_{\alpha+1}$ imply that $X_{\alpha+1} = X_\alpha \cup \mathscr{P}_{X_\infty}(X_\alpha)$ for all $\alpha \in \mathrm{Ord}$. It remains to use Theorem 10.8(b). ▷

12.2. Corollary [4, 4.1.3]. *Let* C *be a subclass of* $\mathbb{V}^{(B)}$. *If* $P\!\uparrow \in C$ *for all* $P \subset {}^{\%}C$ *then* $C = \mathbb{V}^{(B)}$.

◁ Suppose that $C \neq \mathbb{V}^{(B)}$. By Theorem 12.1, there exists a least ordinal α for which there is an element $x \in \mathbb{V}^{(B)}_\alpha \setminus C$. As we see from (28), $\alpha \neq 0$ and $\alpha \notin \mathrm{Lim\,Ord}$. On the other hand, if $\alpha = \beta + 1$ then $x = P\!\uparrow$ for some subset $P \subset {}^{\%}\mathbb{V}^{(B)}_\beta$. Hence, $\mathbb{V}^{(B)}_\beta \subset C$ implies $x \in C$. ▷

12.3. Since the Boolean-valued universe $\mathbb{V}^{(B)}$ is a model of ZFC, the equality

$$\mathbb{V} = \bigcup_{\alpha\in\mathrm{Ord}} \mathbb{V}_\alpha$$

is valid in $\mathbb{V}^{(B)}$; i.e., inside $\mathbb{V}^{(B)}$, the class of all sets $\mathbb{V}$ coincides with the union of the classical von Neumann cumulative hierarchy $(\mathbb{V}_\alpha)_{\alpha\in\mathrm{Ord}}$ (see (15)).

Define the *descent of the hierarchy* $(\mathbb{V}_\alpha)_{\alpha\in\mathrm{Ord}}$ from $\mathbb{V}^{(B)}$ by assigning to each ordinal α the descent $\mathscr{V}_\alpha\!\downarrow$ of the element $\mathscr{V}_\alpha \in \mathbb{V}^{(B)}$ that is equal inside $\mathbb{V}^{(B)}$ to the corresponding term $\mathbb{V}_{\alpha^\wedge}$ of the von Neumann hierarchy (see [4, 4.4.10]):

$$\mathbb{V}^{(B)} \models (\mathscr{V}_\alpha = \mathbb{V}_{\alpha^\wedge}).$$

It would be natural to expect that hierarchy (28), which is a Boolean-valued analog of hierarchy (15), would coincide with the descent of the latter: $\big(\mathbb{V}^{(B)}_\alpha\big)_{\alpha\in\mathrm{Ord}} = (\mathscr{V}_\alpha\!\downarrow)_{\alpha\in\mathrm{Ord}}$. However, this fails for any infinite Boolean algebra B since not all subsets $\mathbb{V}^{(B)}_\alpha \subset \mathbb{V}^{(B)}$ are cyclic. Indeed, if $(d_n)_{n\in\omega}$ is a partition of unity constituted by nonzero elements $d_n \in B$ then the join $\bigsqcup_{n\in\omega} n^\wedge|_{d_n}$ belongs to $\mathscr{V}_\omega\!\downarrow$ but does not belong to $\mathbb{V}^{(B)}_\omega = \bigcup_{n\in\omega} \mathbb{V}^{(B)}_n$.

The following assertion shows that, for turning (28) into the descent of the von Neumann hierarchy, it suffices to add mixings at the limit steps.

Theorem. *Using transfinite induction, define the family of subsets* $\mathbb{U}_\alpha^{(B)} \subset \mathbb{V}^{(B)}$ $(\alpha \in \mathrm{Ord})$ *by putting*

$$\mathbb{U}_0^{(B)} = \varnothing;$$
$$\mathbb{U}_{\alpha+1}^{(B)} = \mathscr{P}_{\mathbb{V}^{(B)}}\big(\mathbb{U}_\alpha^{(B)}\big), \quad \alpha \in \mathrm{Ord};$$
$$\mathbb{U}_\alpha^{(B)} = \mathrm{mix} \bigcup_{\beta<\alpha} \mathbb{U}_\beta^{(B)}, \quad \alpha \in \mathrm{Lim\,Ord}.$$

Then $\big(\mathbb{U}_\alpha^{(B)}\big)_{\alpha\in\mathrm{Ord}}$ *is the descent of the von Neumann hierarchy* $(\mathbb{V}_\alpha)_{\alpha\in\mathrm{Ord}}$ *from* $\mathbb{V}^{(B)}$*; i.e.,*

$$\mathbb{U}_\alpha^{(B)} = \mathscr{V}_\alpha{\downarrow} \ \text{ for all } \alpha \in \mathrm{Ord},$$

where $\mathscr{V}_\alpha$ *are the elements of* $\mathbb{V}^{(B)}$ *satisfying* $\mathbb{V}^{(B)} \models (\mathscr{V}_\alpha = \mathbb{V}_{\alpha^\wedge})$.

◁ Prove that $\mathbb{U}_\alpha^{(B)} = \mathscr{V}_\alpha{\downarrow}$ by induction on $\alpha \in \mathrm{Ord}$.

The induction base $\alpha = 0$ is trivial: $\mathbb{U}_0^{(B)} = \varnothing = \varnothing{\uparrow}{\downarrow} = \mathscr{V}_0{\downarrow}$.

If $\mathbb{U}_\alpha^{(B)} = \mathscr{V}_\alpha{\downarrow}$; then, using the relation

$$\mathbb{V}^{(B)} \models \big(\mathscr{P}(\mathbb{U}_\alpha^{(B)}{\uparrow}) = \mathscr{P}(\mathscr{V}_\alpha{\downarrow}{\uparrow}) = \mathscr{P}(\mathscr{V}_\alpha) = \mathscr{P}(\mathbb{V}_{\alpha^\wedge}) = \mathbb{V}_{\alpha^\wedge+1} = \mathbb{V}_{(\alpha+1)^\wedge} = \mathscr{V}_{\alpha+1}\big)$$

and employing Lemma 3.17(b),(c) and Corollary 6.4, we conclude that, for all $x \in \mathbb{V}^{(B)}$,

$$x \in \mathbb{U}_{\alpha+1}^{(B)} \Leftrightarrow x \in \mathscr{P}_{\mathbb{V}^{(B)}}\big(\mathbb{U}_\alpha^{(B)}\big) \Leftrightarrow \big(\exists\, P \subset {}^{\%}\mathbb{U}_\alpha^{(B)}\big)(x = P{\uparrow}) \Leftrightarrow \mathbb{V}^{(B)} \models \big(x \subset \mathbb{U}_\alpha^{(B)}{\uparrow}\big)$$
$$\Leftrightarrow \mathbb{V}^{(B)} \models \big(x \in \mathscr{P}(\mathbb{U}_\alpha^{(B)}{\uparrow})\big) \Leftrightarrow \mathbb{V}^{(B)} \models (x \in \mathscr{V}_{\alpha+1}) \Leftrightarrow x \in \mathscr{V}_{\alpha+1}{\downarrow}.$$

If $\alpha \in \mathrm{Lim\,Ord}$ and $\mathbb{U}_\beta^{(B)} = \mathscr{V}_\beta{\downarrow}$ for all $\beta < \alpha$ then, by the equality $\alpha^\wedge = \{\beta^\wedge : \beta \in \alpha\}{\uparrow}$, Lemma 3.18(a), 5.10, and Corollary 6.4, for all $x \in \mathbb{V}^{(B)}$, we have

$$x \in \mathbb{U}_\alpha^{(B)} \Leftrightarrow x \in \mathrm{mix} \bigcup_{\beta<\alpha} \mathbb{U}_\beta^{(B)} \Leftrightarrow \vee\Big\{[\![x = y]\!] : y \in \bigcup_{\beta<\alpha} \mathbb{U}_\beta^{(B)}\Big\} = 1_B$$
$$\Leftrightarrow \bigvee_{\beta<\alpha} \bigvee_{y\in\mathbb{U}_\beta^{(B)}} [\![x = y]\!] = 1_B \Leftrightarrow \bigvee_{\beta<\alpha} [\![x \in \mathbb{U}_\beta^{(B)}{\uparrow}]\!] = 1_B \Leftrightarrow \bigvee_{\beta<\alpha} [\![x \in \mathscr{V}_\beta{\downarrow}{\uparrow}]\!] = 1_B$$
$$\Leftrightarrow \bigvee_{\beta<\alpha} [\![x \in \mathscr{V}_\beta]\!] = 1_B \Leftrightarrow \bigvee_{\beta\in\alpha} [\![x \in \mathbb{V}_{\beta^\wedge}]\!] = 1_B \Leftrightarrow \mathbb{V}^{(B)} \models (\exists\, \beta \in \alpha^\wedge)(x \in \mathbb{V}_\beta)$$
$$\Leftrightarrow \mathbb{V}^{(B)} \models \Big(x \in \bigcup_{\beta<\alpha^\wedge} \mathbb{V}_\beta\Big) \Leftrightarrow \mathbb{V}^{(B)} \models (x \in \mathbb{V}_{\alpha^\wedge}) \Leftrightarrow \mathbb{V}^{(B)} \models (x \in \mathscr{V}_\alpha) \Leftrightarrow x \in \mathscr{V}_\alpha{\downarrow}. \ ▷$$

12.4. The construction of the hierarchy (28) involves the ascents $P{\uparrow}$ of arbitrary sets P of partial elements:

$$\mathbb{V}_{\alpha+1}^{(B)} = \mathscr{P}_{\mathbb{V}^{(B)}}\big(\mathbb{V}_\alpha^{(B)}\big) = \big\{P{\uparrow} : P \subset {}^{\%}\mathbb{V}_\alpha^{(B)}\big\}.$$

On the other hand, by Corollary 5.15(b), each element in the Boolean-valued universe is represented as the ascent $(Y|_b){\uparrow}$ of the set $Y|_b$ of partial elements with the same domain b. In this connection, it seems natural to conjecture that the Boolean-valued universe can be constructed into the hierarchy of ascents of the form $(Y|_b){\uparrow}$. Nevertheless, the fact below refutes this conjecture.

Theorem. *Suppose that a separated extensional B-system X satisfies the ascent principle. Using transfinite induction, define the family of subsets* $Y_\alpha \subset X$ $(\alpha \in \mathrm{Ord})$ *by putting*

$$Y_\alpha = \Big\{(Y|_b){\uparrow} : Y \subset \bigcup_{\beta<\alpha} Y_\beta,\ b \in B\Big\}, \quad \alpha \in \mathrm{Ord}.$$

If the Boolean algebra B is infinite then $\bigcup_{\alpha\in\mathrm{Ord}} Y_\alpha \neq X$.

◁ Define recursively the sequence $(y_n)_{n\in\omega} \subset X$ by putting

$$y_0 := \varnothing\!\uparrow, \quad y_{n+1} := \{y_n\}\!\uparrow, \quad n \in \omega.$$

Fix an arbitrary antichain $(d_n)_{n\in\omega}$ of nonzero elements in B and consider the ascents

$$x_i := \{y_n|_{d_{n+i}} : n \in \omega\}\!\uparrow, \quad i \in \omega. \tag{30}$$

Since $x_i = \big(\bigcup_{n\in\omega}\{y_n\}|_{d_{n+i}}\big)\!\uparrow$ and $\{y_n\}\!\uparrow = y_{n+1}$; by Lemma 6.8,[2] we have

$$x_i = \operatorname{ext}\bigsqcup_{n\in\omega} y_{n+1}|_{d_{n+i}}, \quad i \in \omega. \tag{31}$$

For completing the proof, it suffices to show that $x_i \notin Y_\alpha$ for all $\alpha \in \mathrm{Ord}$, $i \in \omega$. Induct transfinitely on α. Consider $\alpha \in \mathrm{Ord}$, suppose that $x_i \notin Y_\beta$ for all $\beta < \alpha$, $i \in \omega$, and assume that $x_i \in Y_\alpha$ for some $i \in \omega$. Then $x_i = (Y|_b)\!\uparrow$, where $Y \subset \bigcup_{\beta<\alpha} Y_\beta$, $b \in B$. Moreover, $[y \in x_i] = [x_i \neq \varnothing] = \bigvee_{n\in\omega} d_{n+i} = b$ for all $y \in Y$ (see Corollary 5.15(b)). Since $[x_i \neq \varnothing] \neq 0_B$, the set Y is nonempty, and so there exist $\beta < \alpha$ and $y \in Y_\beta$ such that $[y \in x_i] = b$. Reckoning with (30), we have $\bigvee_{n\in\omega} d_{n+i} = [y \in x_i] = \bigvee_{n\in\omega}[y = y_n] \wedge d_{n+i}$, which, by Lemma 5.1(a), implies that $d_{n+i} = [y = y_n] \wedge d_{n+i}$, i.e., $[y = y_n] \geqslant d_{n+i}$ for all $n \in \omega$, and so $y|_b = \bigsqcup_{n\in\omega} y_n|_{d_{n+i}}$. Using the equality $y_0 = \varnothing\!\uparrow$ and (31), we conclude that

$$\operatorname{ext} y|_b = \operatorname{ext}\bigsqcup_{n\in\omega} y_{n+1}|_{d_{(n+1)+i}} = \operatorname{ext}\bigsqcup_{n\in\omega} y_{n+1}|_{d_{n+(i+1)}} = x_{i+1}. \tag{32}$$

Since $y \in Y_\beta$, there are $Z \subset \bigcup_{\gamma<\beta} Y_\gamma$ and $c \in B$ such that $y = (Z|_c)\!\uparrow$. Then, for all $x \in X$,

$$[x \in \operatorname{ext} y|_b] = [x \in y|_b] = [x \in y] \wedge b = [x \in (Z|_c)\!\uparrow] \wedge b = \bigvee_{z\in Z}[x = z|_c] \wedge b = \bigvee_{z\in Z}[x = z|_{c\wedge b}] = [x \in (Z|_{c\wedge b})\!\uparrow];$$

which, according to (32), implies $x_{i+1} = \operatorname{ext} y|_b = (Z|_{c\wedge b})\!\uparrow \in Y_\beta$ contrary to the induction hypothesis. ▷

12.5. Lemma. *If $(Z_i)_{i\in I}$ is a family of subsets of $\mathbb{V}^{(B)}$, $(d_i)_{i\in I} \subset B$ is a partition of unity, and $(b_i)_{i\in I} \subset B$, then there exist $Y \subset \operatorname{mix}\bigcup_{i\in I} Z_i$ and $b \in B$ such that $\bigsqcup_{i\in I} (Z_i|_{b_i})\!\uparrow\big|_{d_i} = (Y|_b)\!\uparrow$.*

◁ Put $Z := \bigcup_{i\in I} Z_i \subset \mathbb{V}^{(B)}$ and $P := \bigcup_{i\in I} Z_i|_{b_i\wedge d_i} \subset {}^{\%}Z$. By Lemma 6.8, we have $\bigsqcup_{i\in I} (Z_i|_{b_i})\!\uparrow\big|_{d_i} = P\!\uparrow$; and, by Corollary 5.15(a), there exist $Y \subset \operatorname{mix} Z$ and $b \in B$ such that $P\!\uparrow = (Y|_b)\!\uparrow$. ▷

12.6. By Theorem 12.4, for constructing the Boolean-valued universe, the ascents with constant domain are not enough. The results below show that the situation will change if we add mixings at the limit steps of the hierarchy.

Theorem. *Using transfinite recursion, define the family of subsets $Y_\alpha \subset \mathbb{V}^{(B)}$ ($\alpha \in \mathrm{Ord}$) by putting*

$$\begin{aligned} &Y_0 = \varnothing;\\ &Y_{\alpha+1} = \{(Y|_b)\!\uparrow : Y \subset Y_\alpha,\ b \in B\}, && \alpha \in \mathrm{Ord};\\ &Y_\alpha = \operatorname{mix}\bigcup_{\beta<\alpha} Y_\beta, && \alpha \in \mathrm{Lim\,Ord}. \end{aligned} \tag{33}$$

Then

(a) $Y_\alpha \subset Y_\beta$ *for* $\alpha \leqslant \beta$;
(b) Y_α *are cyclic for all* $\alpha \in \mathrm{Ord}$;
(c) $\mathbb{V}^{(B)}_\alpha \subset Y_{\alpha+1}$ *for all* $\alpha \in \mathrm{Ord}$;
(d) $\mathbb{V}^{(B)} = \bigcup_{\alpha\in\mathrm{Ord}} Y_\alpha$.

[2] There is a misprint in the statement of Lemma 6.8 in [1]: $\bigsqcup_{i\in I} x_i$ should be replaced with $\bigsqcup_{i\in I} x_i|_{d_i}$.

◁ (a): Prove that $Y_\alpha \subset Y_\beta$ for all $\alpha \leqslant \beta$ by induction on β. Let $\beta \in \mathrm{Ord}$, and let

$$Y_{\alpha_1} \subset Y_{\alpha_2} \text{ for all } \alpha_1 \leqslant \alpha_2 < \beta. \tag{34}$$

Consider an arbitrary ordinal $\alpha \leqslant \beta$ and prove that $Y_\alpha \subset Y_\beta$. This inclusion is obvious if $\alpha = 0$, $\alpha = \beta$, or $\beta \in \mathrm{Lim\,Ord}$. Therefore, we will assume that $0 < \alpha < \beta$ and $\beta = \beta_0 + 1$ for some $\beta_0 \in \mathrm{Ord}$. Fix an arbitrary element $x \in Y_\alpha$ and show that $x \in Y_\beta$.

If $\alpha = \alpha_0 + 1$ for some $\alpha_0 \in \mathrm{Ord}$ then $x = (Y|_b)\!\uparrow$, where $Y \subset Y_{\alpha_0}$, $b \in B$. Since $\alpha_0 < \beta_0 < \beta$, by (34), we have $Y_{\alpha_0} \subset Y_{\beta_0}$, and so $Y \subset Y_{\beta_0}$ and, therefore, $x = (Y|_b)\!\uparrow \in Y_{\beta_0+1} = Y_\beta$.

Now, let $\alpha \in \mathrm{Lim\,Ord}$. Then $Y_\alpha = \mathrm{mix} \bigcup_{\gamma<\alpha} Y_\gamma$, and so $x = \bigsqcup_{i\in I} x_i|_{d_i}$, where $(d_i)_{i\in I} \subset B$ is a partition of unity, $x_i \in Y_{\gamma_i}$, $\gamma_i < \alpha$ $(i \in I)$. Since α is a limit ordinal, for all $i \in I$ we have $\gamma_i + 1 < \alpha$ and, in particular, $\gamma_i < \gamma_i + 1 < \beta$, which, owing to (34), implies that $Y_{\gamma_i} \subset Y_{\gamma_i+1}$. Thus, $x_i \in Y_{\gamma_i+1}$, and so $x_i = (Z_i|_{b_i})\!\uparrow$ for some $Z_i \subset Y_{\gamma_i}$, $b_i \in B$ $(i \in I)$, and so $x = \bigsqcup_{i\in I} (Z_i|_{b_i})\!\uparrow\!\big|_{d_i}$. By Lemma 12.5, there exist $Y \subset \mathrm{mix} \bigcup_{i\in I} Z_i$ and $b \in B$, such that $x = (Y|_b)\!\uparrow$. Note that

$$\mathrm{mix} \bigcup_{i\in I} Z_i \subset \mathrm{mix} \bigcup_{i\in I} Y_{\gamma_i} \subset \mathrm{mix} \bigcup_{\gamma<\alpha} Y_\gamma = Y_\alpha;$$

and, moreover, by (34), $\alpha \leqslant \beta_0 < \beta$ implies $Y_\alpha \subset Y_{\beta_0}$. Hence, $Y \subset Y_{\beta_0}$, and so $x = (Y|_b)\!\uparrow \in Y_{\beta_0+1} = Y_\beta$.

(b): Applying induction, consider an arbitrary ordinal α, suppose that $\mathrm{mix}\, Y_\beta = Y_\beta$ for all $\beta < \alpha$, and establish the equality $\mathrm{mix}\, Y_\alpha = Y_\alpha$. In the case of $\alpha = 0$ or $\alpha \in \mathrm{Lim\,Ord}$, the equality is obvious. Assume that $\alpha = \beta + 1$ for some $\beta \in \mathrm{Ord}$. Consider an arbitrary element $x \in \mathrm{mix}\, Y_\alpha$ and show that $x \in Y_\alpha$. Since

$$x \in \mathrm{mix}\, Y_{\beta+1} = \mathrm{mix}\{(Z|_b)\!\uparrow : Z \subset Y_\beta,\ b \in B\},$$

there are a partition of unity $(d_i)_{i\in I} \subset B$ and families $Z_i \subset Y_\beta$, $b_i \in B$ $(i \in I)$ such that $x = \bigsqcup_{i\in I} (Z_i|_{b_i})\!\uparrow\!\big|_{d_i}$. Lemma 12.5 implies the representation $x = (Y|_b)\!\uparrow$ for some $Y \subset \mathrm{mix} \bigcup_{i\in I} Z_i$ and $b \in B$. By the induction hypothesis, $\mathrm{mix} \bigcup_{i\in I} Z_i \subset \mathrm{mix}\, Y_\beta = Y_\beta$, and so $Y \subset Y_\beta$; thus, $x = (Y|_b)\!\uparrow \in Y_{\beta+1} = Y_\alpha$.

(c): Again proceeding by induction, fix an ordinal α, suppose that $\mathbb{V}_\beta^{(B)} \subset Y_{\beta+1}$ for all $\beta < \alpha$, consider an arbitrary element $x \in \mathbb{V}_\alpha^{(B)}$, and show that $x \in Y_{\alpha+1}$. From (28) it follows that $x = P\!\uparrow$ for some subset $P \subset {}^{\%}\mathbb{V}_\beta^{(B)}$, where $\beta < \alpha$. By Corollary 5.15(a), there are $Y \subset \mathrm{mix}\, \mathbb{V}_\beta^{(B)}$ and $b \in B$ such that $x = (Y|_b)\!\uparrow$. By the induction hypothesis, $\mathbb{V}_\beta^{(B)} \subset Y_{\beta+1}$. Moreover, (a) implies that $Y_{\beta+1} \subset Y_\alpha$. Thus, $\mathbb{V}_\beta^{(B)} \subset Y_\alpha$, which, in view of (b), implies $Y \subset \mathrm{mix}\, \mathbb{V}_\beta^{(B)} \subset \mathrm{mix}\, Y_\alpha = Y_\alpha$, and so $x = (Y|_b)\!\uparrow \in Y_{\alpha+1}$.

(d): This follows from (c) and (29). ▷

12.7. The following assertion shows that equality 12.6(d) remains valid if, at the discrete steps of hierarchy (33), we confine ourselves to ascents of the sets of everywhere defined elements.

Theorem. *Using transfinite recursion, define the family of subsets $Z_\alpha \subset \mathbb{V}^{(B)}$ $(\alpha \in \mathrm{Ord})$ by putting*

$$\begin{aligned} &Z_0 = \varnothing; \\ &Z_{\alpha+1} = \{Z\!\uparrow\, :\, Z \subset Z_\alpha\}, \quad && \alpha \in \mathrm{Ord}; \\ &Z_\alpha = \mathrm{mix} \bigcup_{\beta<\alpha} Z_\beta, && \alpha \in \mathrm{Lim\,Ord}. \end{aligned}$$

Then

$$\mathbb{V}^{(B)} = \bigcup_{\alpha\in\mathrm{Ord}} Z_\alpha.$$

◁ Define the function $\delta\colon \mathrm{Ord} \to \mathrm{Ord}$ by the recursive formula

$$\delta(\alpha) = \vee\{\delta(\beta) : \beta < \alpha\} + \omega, \quad \alpha \in \mathrm{Ord}.$$

By Theorem 12.6(d), it suffices to prove that

$$(\forall\,\alpha \in \mathrm{Ord})(Y_\alpha \subset Z_{\delta(\alpha)}),$$

where Y_α are defined in (33). Inducting on $\alpha \in \mathrm{Ord}$, assume that $Y_\beta \subset Z_{\delta(\beta)}$ for all $\beta < \alpha$, and establish the inclusion $Y_\alpha \subset Z_{\delta(\alpha)}$.

Let $\alpha = \beta + 1$. The obvious monotonicity of δ implies that

$$\delta(\alpha) = \vee\{\delta(\gamma) : \gamma < \alpha\} + \omega = \vee\{\delta(\gamma) : \gamma \leqslant \beta\} + \omega = \delta(\beta) + \omega.$$

Consider an arbitrary $x \in Y_\alpha = Y_{\beta+1}$. By (33), there are $Y \subset Y_\beta$ and $b \in B$ such that $x = (Y|_b)\!\uparrow$. Then, for all $z \in \mathbb{V}^{(B)}$,

$$\begin{aligned}[\![z \in x]\!] = [\![z \in (Y|_b)\!\uparrow]\!] = \bigvee_{y\in Y} [\![z = y|_b]\!] = \bigvee_{y\in Y} [\![z = y]\!] \wedge b = [\![z \in Y\!\uparrow]\!] \wedge b \\ = [\![z \in (Y\!\uparrow)|_b]\!] = [\![z \in (Y\!\uparrow)|_b]\!] \vee 0_B = [\![z \in (Y\!\uparrow)|_b]\!] \vee [\![z \in (\varnothing\!\uparrow)|_{\neg b}]\!],\end{aligned}$$

which, by Lemma 5.6(d), implies that $x = (Y\!\uparrow)|_b \sqcup \varnothing\!\uparrow\!|_{\neg b}$. The inclusions $Y, \varnothing \subset Y_\beta \subset Z_{\delta(\beta)}$ imply $Y\!\uparrow, \varnothing\!\uparrow \in Z_{\delta(\beta)+1}$, and so

$$x = (Y\!\uparrow)|_b \sqcup \varnothing\!\uparrow\!|_{\neg b} \in \operatorname{mix} Z_{\delta(\beta)+1} \subset \operatorname{mix} \bigcup_{\gamma<\delta(\beta)+\omega} Z_\gamma = Z_{\delta(\beta)+\omega} = Z_{\delta(\alpha)}.$$

If $\alpha \in \operatorname{Lim}\mathrm{Ord}$ then

$$Y_\alpha = \operatorname{mix} \bigcup_{\beta<\alpha} Y_\beta \subset \operatorname{mix} \bigcup_{\beta<\alpha} Z_{\delta(\beta)} \subset \operatorname{mix} \bigcup_{\gamma<\delta(\alpha)} Z_\gamma = Z_{\delta(\alpha)}. \ \triangleright$$

12.8. Corollary. *If C is a cyclic subclass of $\mathbb{V}^{(B)}$ and $Y\!\uparrow \in C$ for all $Y \subset C$ then $C = \mathbb{V}^{(B)}$.*

$\triangleleft$ Suppose that $C \neq \mathbb{V}^{(B)}$. By Theorem 12.7, there exists a least ordinal α for which there is an element $x \in Z_\alpha \backslash C$. If $\alpha = \beta + 1$ then $x = Y\!\uparrow$ for some subset $Y \subset Z_\beta$, and so $Z_\beta \subset C$ implies $x \in C$. If $\alpha \in \operatorname{Lim}\mathrm{Ord}$ then $x = \sqcup P$ for some maximal antichain $P \subset {}^{\%}\big(\bigcup_{\beta<\alpha} Z_\beta\big)$, and then $\bigcup_{\beta<\alpha} Z_\beta \subset C$ implies $x \in \operatorname{mix} C = C$. $\triangleright$

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