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LOCALLY CONVEX SPACES WITH ALL ARCHIMEDEAN CONES CLOSED

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Abstract: We provide an exhaustive description of the class of locally convex spaces in which all Archimedean cones are closed. We introduce the notion of quasidense set and prove that the above class consists of all finite-dimensional and countable-dimensional spaces X whose topological dual X' is quasidense in the algebraic dual $X^{\#}$ of X.

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§ 1. Basic Notation and Terminology

Let $\mathbb{N}$ stand for the naturals $\{1, 2, \dots\}$. The sets of integers, rationals, and reals are denoted by $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$. By $\mathbb{R}^+$ we denote the collection $\{\lambda \in \mathbb{R} : \lambda \geqslant 0\}$ of positive reals. The set $\mathbb{R}$ is endowed with the standard operations and topology that make it a field and a locally convex space. We designate as $\mathbb{R}_{\mathrm{D}}$ the set of reals endowed with the discrete topology.

The sign "$\subset$" denotes the nonstrict inclusion of sets. The assignment symbol "$:=$" introduces notation and indicates the equalities valid by definition.

In what follows, by a *vector space* we mean a vector space over $\mathbb{R}$. The term *subspace* means a vector subspace.

Closed, open, and semiopen numerical intervals are denoted by $[\alpha, \beta]$, $]\alpha, \beta[$, and $[\alpha, \beta[$. If X is a vector space, $x, y \in X$, and $x \neq y$; then $[x, y] := x + [0, 1]\,(y - x)$ and $[x, y[\;:= x + [0, 1[\,(y - x)$. For convenience, we put $[x, x] := [x, x[\;:= \{x\}$. A set $S \subset X$ is *absorbing* if for each $x \in X$ there exists a real $\varepsilon > 0$ such that $[0, \varepsilon x[\;\subset S$. Let $\operatorname{core} S$ denote the *core* of S, i.e., the set of elements $s \in S$ for which $S - s$ is absorbing. By $\operatorname{lin} S$ and $\operatorname{co} S$ we designate the linear span and the convex hull of S. A subset $W \subset X$ is a *wedge* if $W \neq \varnothing$, $W + W \subset W$, and $\mathbb{R}^+ W \subset W$. A wedge W is a *cone* if $W \cap -W = \{0\}$.

Given a set I, let $\mathbb{R}^I_{\mathrm{fin}}$ denote the subspace of finitely supported functions in $\mathbb{R}^I$, i.e., of functions $x \colon I \to \mathbb{R}$ whose support $\operatorname{supp} x := \{i \in I : x(i) \neq 0\}$ is finite. Tuples $x = \big(x(1), \dots, x(n)\big) \in \mathbb{R}^n$, with $n \in \mathbb{N}$, are traditionally regarded as functions $x \colon \{1, \dots, n\} \to \mathbb{R}$. In what follows, we use the notation

$$e_n := \chi_{\{n\}} = (0, \dots, 0, \underset{(n)}{1}, 0, 0, \dots) \in \mathbb{R}^{\mathbb{N}};$$
$$\mathbb{R}^{\mathbb{N}}_n := \operatorname{lin}\{e_1, \dots, e_n\} = \big\{x \in \mathbb{R}^{\mathbb{N}}_{\mathrm{fin}} : \operatorname{supp} x \subset \{1, \dots, n\}\big\}.$$

The linear operator $\pi_n \colon \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^n$ is defined by the formula

$$\pi_n s := s\big|_{\{1,\dots,n\}} = \big(s(1), \dots, s(n)\big). \qquad (1)$$

We agree to use notation (1) not only for sequences $s \in \mathbb{R}^{\mathbb{N}}$ but also for tuples $s \in \mathbb{R}^m$, with $m \geqslant n$.

Given a vector space X, let $X^{\#}$ denote the *algebraic dual* of X, i.e., the space of all linear functionals $f \colon X \to \mathbb{R}$. If X is endowed with a vector topology; then X' denotes the *topological dual* of X, i.e., the subspace of $X^{\#}$ consisting of continuous functionals.

Let $\operatorname{cl} S$ and $\operatorname{int} S$ or, more precisely, $\operatorname{cl}_X S$ and $\operatorname{int}_X S$ denote the closure and the interior of a set S in a topological space X.

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§ 2. Introduction

2.1. The concept of a cone is closely related to that of an *ordered vector space*, a real vector space X equipped with an order relation $\leqslant$ such that the inequality $x \leqslant y$ implies $x + z \leqslant y + z$ and $\lambda x \leqslant \lambda y$ for all $x, y, z \in X$ and $\lambda \in \mathbb{R}^+$. Namely, if $(X, \leqslant)$ is an ordered vector space then $X^+ := \{x \in X : x \geqslant 0\}$ is a cone; and vice versa: if $K \subset X$ is a cone and $x \leqslant_K y \Leftrightarrow y - x \in K$, then $(X, \leqslant_K)$ is an ordered vector space and $X^+ = K$ (see, for example, [1, 3.2; 2]).

2.2. An ordered vector space $(X, \leqslant)$ is *Archimedean* provided that

$$\text{if } x, y \in X,\ y \geqslant 0, \text{ and } x \leqslant \tfrac{1}{n} y \text{ for all } n \in \mathbb{N}; \text{ then } x \leqslant 0.$$

A cone K in a vector space X is *Archimedean* whenever the corresponding ordered vector space $(X, \leqslant_K)$ is Archimedean.

2.3. Proposition [3, 3.1]. *The following properties of a cone K in a vector space X are equivalent:*
(a) *K is Archimedean;*
(b) $X \backslash K = \operatorname{core}(X \backslash K)$;
(c) $[x, y[\subset K$ *implies* $y \in K$ *for all* $x, y \in X$;
(d) $\{\lambda \in \mathbb{R} : x + \lambda y \in K\}$ *is closed for all* $x, y \in X$;
(e) *the intersection of K with every subspace of X of dimension $\leqslant 2$ is closed;*
(f) *the intersection of K with every finite-dimensional subspace of X is closed;*
(g) *K is sequentially closed in some Hausdorff vector topology on X;*
(h) *K is sequentially closed in the strongest locally convex topology on X.*

$\triangleleft$ The implications (a)$\Leftarrow$(b)$\Leftarrow$(c)$\Leftarrow$(d)$\Leftarrow$(e)$\Leftarrow$(f)$\Leftarrow$(g)$\Leftarrow$(h) are obvious.

(a)$\Rightarrow$(f): Consider a finite-dimensional subspace $X_0 \subset X$ and put $K_0 := K \cap X_0$. Since the cone K_0 has nonempty interior in $\operatorname{lin} K_0 \subset X_0$, it is closed in $\operatorname{lin} K_0$ (see [2, 2.4]) and hence in X_0.

(f)$\Rightarrow$(h): Let τ be the strongest locally convex topology on X. Owing to (f), it suffices to show that the linear span of every τ-convergent sequence $x_n \to x$ has finite dimension. Otherwise, there would exist a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ such that $x \notin \operatorname{lin}\{x_{n_k} : k \in \mathbb{N}\}$, and then the convergence $x_{n_k} \to x$ in τ would contradict the existence of a (continuous) linear functional equal to 0 on $\{x_{n_k} : k \in \mathbb{N}\}$ and 1 at x. $\triangleright$

2.4. In 2.3, we trace some connection between the algebraic property of being Archimedean and the topological property of closedness. In the case when an ordered vector space X is endowed with some Hausdorff vector topology, the Archimedean property of X^+ and its closedness can be reformulated in terms of passage to the limit in linear inequalities. From 2.3(f) it is obvious that X^+ is Archimedean whenever we can pass to the limit in linear inequalities with fixed vectors and variable coefficients: If $x_1, \dots, x_n$ are elements of X and $(\lambda_{ij})_{j \in \mathbb{N}}$ $(i = 1, \dots, n)$ are convergent numeric sequences; then

$$\sum_{i=1}^{n} \lambda_{ij} x_i \geqslant 0 \ (j \in \mathbb{N}) \ \Rightarrow \ \sum_{i=1}^{n} \lim_{j \to \infty} \lambda_{ij} x_i \geqslant 0. \tag{2}$$

If X^+ is closed then passage to the limit in linear inequalities is admissible without any restrictions: For all convergent nets $(x_{ij})_{j \in J}$ in X and $(\lambda_{ij})_{j \in J}$ in $\mathbb{R}$ $(i = 1, \dots, n)$, we have

$$\sum_{i=1}^{n} \lambda_{ij} x_{ij} \geqslant 0 \ (j \in J) \ \Rightarrow \ \sum_{i=1}^{n} \lim_{j \in J} \lambda_{ij} \lim_{j \in J} x_{ij} \geqslant 0. \tag{3}$$

The above considerations justify the advisability of describing the class of topological vector spaces with all Archimedean cones closed or, which is the same, with (2) implying (3).

2.5. Every closed cone is obviously Archimedean, whereas the converse is not true in general. The set $(\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}})^+$ of positive finitely supported sequences in each of the classical Banach spaces ℓ^p, with $1 \leqslant p \leqslant \infty$, serves as a basic example of a nonclosed Archimedean cone. As is known, in the finite-dimensional case, a cone is Archimedean if and only if it is closed. Until recently, this simple observation actually exhausted the information about the class of spaces with all Archimedean cones closed. The corresponding study was initiated in [3], where the problem was solved for locally convex spaces of uncountable dimension.

Theorem [3, 4.2 and 4.3]**.** *Let E be a Hamel basis for a Hausdorff locally convex space X, and let $e_0 \in E$. If $|E| > |\mathbb{N}|$; then $\mathbb{R}^+C$ is a nonclosed Archimedean cone in X, where*

$$C = \left\{ \sum_{e\in E} x(e)e \,:\, x \in \left(\mathbb{R}^E_{\mathrm{fin}}\right)^+,\ x(e_0) = 1,\ \sum_{e\neq e_0} x(e) \leqslant 1,\ \sum_{e\neq e_0} \sqrt{x(e)} \geqslant 1 \right\}.$$

Therefore, in the case of uncountable dimension, nonclosed Archimedean cones exist even in the strongest locally convex topology. On the other hand, in finite-dimensional spaces, all Archimedean cones are closed. The case of countable dimension turned out more complicated, and any exhaustive description of locally convex spaces with all Archimedean cones closed remained an open problem until now.

2.6. Since the closed convex sets are the same in all locally convex topologies compatible with a given duality (see, for example, [1, 10.4.9; 4, 8-3-6]), the property under study of a locally convex space X is completely determined by the topological dual X' or, more precisely, by the location of X' in $X^{\#}$. Moreover, we may assume that X is endowed with the weak topology compatible with the duality between X and X'. Considering that every countable-dimensional vector space X is isomorphic to $\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}$, and the algebraic dual $X^{\#}$ of X is isomorphic to $\mathbb{R}^{\mathbb{N}}$; the problem under consideration can be reformulated as follows:

Problem [3, 4.5]. Describe the vector subspaces $Y \subset \mathbb{R}^{\mathbb{N}}$ for which all Archimedean cones in $\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}$ are closed in the weak topology induced on $\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}$ by Y under the duality $\langle x \,|\, y\rangle = \sum_{n\in\mathbb{N}} x(n)y(n)$.

The article is aimed at solving the above problem. Section 3 contains some available and auxiliary information on duality and polars. Sections 4–7 study the concept of quasi-interior and the related new concepts of quasilocal boundedness, quasidenseness, and projectivity that play a key role on the path to a solution. In Section 8, we establish Theorem 8.5, the main result of the article, which contains a solution of the problem. The conclusion is given in Theorem 8.6, where we provide some exhaustive description of the class of locally convex spaces with all Archimedean cones closed: the class consists of finite-dimensional and countable-dimensional spaces X whose dual X' is quasidense in $X^{\#}$. In the same section, we give two corollaries, one confirming a conjecture of [3], and the other offering a short justification of the main result of [5]. In the final Section 9, we formulate a few questions that remain open by now.

§ 3. Duality

We present the basic information, notation, and conventions that are related to duality between vector spaces and weak locally convex topologies.

3.1. A triple $(X, Y, \langle\cdot\,|\,\cdot\rangle)$ is a *duality space* if X and Y are vector spaces and $\langle\cdot\,|\,\cdot\rangle$ is a duality between X and Y, i.e., a bilinear functional $\langle\cdot\,|\,\cdot\rangle \colon X \times Y \to \mathbb{R}$ such that $\ker\langle\cdot| = \{0\}$ and $\ker|\cdot\rangle = \{0\}$, where $\langle x| = \langle x\,|\,\cdot\rangle$ and $|y\rangle = \langle\cdot\,|\,y\rangle$ for $x \in X$ and $y \in Y$. A duality space is also referred to as a *dual pair* (see [4, 8-2-1]). Instead of $(X, Y, \langle\cdot\,|\,\cdot\rangle)$, we write $X|Y$ or simply X, if Y and $\langle\cdot\,|\,\cdot\rangle$ are clear from the context. For $x \in X$, the functional $\langle x| \in Y^{\#}$ is denoted by $\hat{x}$.

When considering a duality space $(X, Y, \langle\cdot\,|\,\cdot\rangle)$, we assume Y endowed with the *conjugate duality*: $Y = Y|X = (Y, X, \langle\cdot\,|\,\cdot\rangle^*)$, where $\langle y\,|\,x\rangle^* = \langle x\,|\,y\rangle$. In particular, if $y \in Y$ then $\hat{y} = |y\rangle \in X^{\#}$.

3.2. Every Hausdorff locally convex space X is regarded as the duality space $X|X' = (X, X', \langle\cdot\,|\,\cdot\rangle)$ with $\langle x\,|\,f\rangle = f(x)$. Conversely, every duality space $X = (X, Y, \langle\cdot\,|\,\cdot\rangle)$ is endowed with the corresponding weak topology $\sigma(X|Y)$, turning into a Hausdorff locally convex space. Moreover, the mapping $|\cdot\rangle \colon y \in Y \mapsto \hat{y} \in X^{\#}$ is a linear and topological isomorphism between the locally convex spaces $Y = Y|X$ and $X' = X'|X$.

3.3. Let X be a vector space with a Hausdorff locally convex topology τ. The weak topology $\sigma := \sigma(X|X')$ implied on the duality space $X|X'$ need not coincide with τ. Nevertheless, owing to the equality $(X,\sigma)' = (X,\tau)'$, the locally convex spaces (X,σ) and (X,τ) have the same closed convex sets, closures of convex sets, and bounded sets (see [4, 8-1-3, 8-3-6, 8-3-7, 8-4-1]). Therefore, if the use of the topology in a statement on a locally convex space X remains within the above-mentioned concepts, we may assume that $X = X|X'$ in the proof of the statement.

3.4. Given a set I, the vector space $\mathbb{R}^I_{\text{fin}}$ is regarded as the duality space $\mathbb{R}^I_{\text{fin}}|\mathbb{R}^I = \big(\mathbb{R}^I_{\text{fin}}, \mathbb{R}^I, \langle\cdot\,|\,\cdot\rangle\big)$ with $\langle x\,|\,y\rangle = \sum_{i\in I} x(i)y(i)$. The same duality is implied when considering pairs of the form $\mathbb{R}^I_{\text{fin}}|Y$, where Y is a subspace of $\mathbb{R}^I$. It is easy to see that $\{\hat{y} : y \in \mathbb{R}^I\} = (\mathbb{R}^I_{\text{fin}})^\#$.

The space $\mathbb{R}^I$ is endowed with the conjugate duality: $\mathbb{R}^I = \mathbb{R}^I\big|\mathbb{R}^I_{\text{fin}}$. The default weak topology $\sigma\big(\mathbb{R}^I\big|\mathbb{R}^I_{\text{fin}}\big)$ of this space coincides with the Tikhonoff topology of the product $\mathbb{R}^I = \prod_{i\in I}\mathbb{R}$ that is also called the pointwise convergence topology. This, in particular, implies the well-known compactness criterion for $\mathbb{R}^I$: A subset of $\mathbb{R}^I$ is compact if and only if it is closed and bounded. (This is so, for example, because every bounded subset of $\mathbb{R}^I$ lies in a product of the form $\prod_{i\in I}[\alpha_i,\beta_i]$ which is compact by the Tikhonoff Theorem.)

If $n \in \mathbb{N}$ then $\mathbb{R}^n = \mathbb{R}^I = \mathbb{R}^I_{\text{fin}}$, where $I = \{1,\dots,n\}$, which means that all the conventions and notation introduced above for $\mathbb{R}^I$ and $\mathbb{R}^I_{\text{fin}}$ are applicable to $\mathbb{R}^n$. In particular, $\mathbb{R}^n$ is assumed to be the duality space $\mathbb{R}^n|\mathbb{R}^n$ with $\langle x\,|\,y\rangle = \sum_{i=1}^n x(i)y(i)$.

3.5. Let X be a vector space. Given an arbitrary subspace $Y \subset X^\#$, the bilinear functional $\langle\cdot\,|\,\cdot\rangle\colon X\times Y \to \mathbb{R}$ acting as $\langle x\,|\,y\rangle = y(x)$ need not be a duality: only the second of the conditions $\ker\langle\cdot| = \{0\}$ and $\ker|\cdot\rangle = \{0\}$ is guaranteed.

Proposition. *The following properties of a subspace $Y \subset X^\#$ are equivalent:*

(a) *$(x,y) \mapsto y(x)$ is a duality between X and Y;*

(b) *the weak topology $\sigma(X|Y)$ is Hausdorff;*

(c) *if $x_1, x_2 \in X$ and $x_1 \neq x_2$, then $y(x_1) \neq y(x_2)$ for some $y \in Y$;*

(d) *for every nonzero $x \in X$ there is $y \in Y$ such that $y(x) \neq 0$;*

(e) *for all linearly independent $x_1,\dots,x_n \in X$ and all $\lambda_1,\dots,\lambda_n \in \mathbb{R}$, there exists $y \in Y$ such that $y(x_1) = \lambda_1,\ \dots,\ y(x_n) = \lambda_n$;*

(f) *Y is dense in $X^\#|X$;*

(g) *$\operatorname{cl} Y = X^\#$ in the topological space $\mathbb{R}^X_{\mathrm{D}}$.*

In the literature, the dense subspaces of $X^\#|X$ are also called *total over* X, *separating*, or *fundamental*. If Y is a subspace of X', with X a Hausdorff locally convex space; then the denseness of Y in X' is equivalent to the denseness of Y in $X^\#$ (see [4, 8-3-9]). For this reason, the terms "dense subspace of X'" and "dense subspace of $X^\#$" are interchangeable, and the use of the term "dense subspace" leads to no ambiguity.

Given a vector space X and a dense subspace $Y \subset X^\#$, we imply the duality $\langle x\,|\,y\rangle = y(x)$ between X and Y and introduce the corresponding dual pairs $X|Y$ and $Y|X$ into consideration.

3.6. The description below of the dense subspaces of $\mathbb{R}^\mathbb{N}$ can be easily deduced from 3.5:

Proposition. *The following properties of a subspace $Y \subset \mathbb{R}^\mathbb{N}$ are equivalent:*

(a) *Y is a dense subspace of $\mathbb{R}^\mathbb{N}$;*

(b) *$\pi_n Y = \mathbb{R}^n$ for all $n \in \mathbb{N}$;*

(c) *$\pi_n e_n \in \pi_n Y$ for all $n \in \mathbb{N}$;*

(d) *Y is dense in $\mathbb{R}^\mathbb{N}_{\mathrm{D}}$.*

3.7. Let $X|Y$ be a dual pair, let $R \subset \mathbb{R}$, and let $S \subset X$. Agree to call the set

$$S^{\langle R\rangle} := \{y \in Y : \langle s\,|\,y\rangle \in R \text{ for all } s \in S\}$$

the *polar of S with respect to R.*

We list the classical special cases of polars:

$S^{\oplus} := S^{\langle \mathbb{R}^+ \rangle}$ is the *dual wedge* to S, often denoted in the literature by S';

$S^{\ominus} := S^{\langle -\mathbb{R}^+ \rangle} = -S^{\oplus}$ is the *normal wedge* to S; the polar $(S-x)^{\ominus}$ is called the normal wedge to S at a point $x \in X$;

$S^{\boxplus} := (S\backslash\{0\})^{\langle]0,\infty[\rangle} = \{y \in Y : \langle s \,|\, y\rangle > 0 \text{ for all } s \in S\backslash\{0\}\}$ is the collection of *strictly positive* elements of the dual space with respect to S;

$S^{\odot} := S^{\langle]-\infty,1] \rangle}$ is the *one-sided polar* of S;

$S^{\circ} := S^{\langle [-1,1] \rangle}$ is the *absolute polar* of S;

$S^{\perp} := S^{\langle \{0\} \rangle}$ is the *annihilator* of S.

Recall that the space Y is endowed by default with the conjugate duality: $Y = Y|X$. In particular, $T^{\langle R \rangle} \subset X$ for $T \subset Y$ (see 3.8(a) below).

3.8. Proposition. *Let $X|Y$ be a dual pair. Given $R \subset \mathbb{R}$ and $S \subset X$, the following hold:*

(a) *$S \subset S^{\langle R \rangle\langle R \rangle}$;*

(b) *if $R_0 \subset R$ and $S_0 \subset S$ then $S_0^{\langle R \rangle} \supset S^{\langle R_0 \rangle}$;*

(c) *if $0 \in R$ then $0 \in S^{\langle R \rangle}$;*

(d) *if R is a convex, or absolutely convex, or closed subset of $\mathbb{R}$, then $S^{\langle R \rangle}$ is convex, or absolutely convex, or closed subset of Y;*

(e) *if R is absorbing (i.e., $0 \in \operatorname{int} R$) and S is bounded, then $S^{\langle R \rangle}$ is absorbing;*

(f) *if $R \cap -R$ is bounded and S is absorbing, then $S^{\langle R \rangle}$ is bounded;*

(g) *if $R \cap -R$ is bounded and $S^{\langle R \rangle}$ is absorbing, then S is bounded.*

3.9. Corollary [6, 5.102]**.** *Given a subset S of a Hausdorff locally convex space, the following hold:*

(a) *$S^{\oplus}$, $S^{\ominus}$, $S^{\boxplus}$, $S^{\odot}$ are convex;*

(b) *S°, $S^{\perp}$ are absolutely convex;*

(c) *$S^{\oplus}$, $S^{\ominus}$, $S^{\odot}$, S°, $S^{\perp}$ are closed and contain 0;*

(d) *if S is absorbing, then $S^{\odot}$ and S° are bounded;*

(e) *S is bounded $\Leftrightarrow$ $S^{\odot}$ is absorbing $\Leftrightarrow$ S° is absorbing.*

3.10. Bipolar Theorem [6, 5.103]**.** *For every nonempty set S in a Hausdorff locally convex space,*

$$S^{\odot\odot} = \operatorname{cl}\operatorname{co}(S \cup \{0\}), \quad S^{\circ\circ} = \operatorname{cl}\operatorname{co}(S \cup -S).$$

3.11. Theorem [2, 2.13]**.** *If $X|Y$ is a dual pair and W is a wedge in X, then $W^{\oplus}$ is a closed wedge in Y and $W^{\oplus\oplus} = \operatorname{cl} W$.*

§ 4. Quasi-Interior

Throughout this section, X is a Hausdorff locally convex space. According to 3.1 and 3.2, the spaces X and X' are endowed with the natural duality, and so $X = X|X'$ and $X' = X'|X$. (In particular, if $S \subset X'$ then $S^{\odot} \subset X$.)

An element x of $S \subset X$ is a *relatively quasi-interior point* of S, in symbols $x \in \operatorname{qri} S$, whenever $\operatorname{cl} \mathbb{R}^+(S-x)$ is a vector subspace of X. This concept was introduced and studied in [7] and was further expanded in convex analysis and optimization methods. In the case under consideration, a slightly different but very close notion of quasi-interior turns out to be useful (see, for example, [8, 2.1.2]). Most of the facts about quasi-interior presented in this section have analogs established in [7] for relative quasi-interior. Each of these statements is provided with a corresponding reference.

4.1. The *quasi-interior* $\operatorname{qi} S$ of a set $S \subset X$ is defined as follows:

$$\operatorname{qi} S := \{x \in S : \operatorname{cl} \mathbb{R}^+(S-x) = X\}.$$

(Note that in this definition we cannot weaken the containment $x \in S$ to $x \in X$ even for a convex set S. For example, if $X \neq \{0\}$ and $S = K\backslash\{0\}$, where K is a dense cone in X; then $0 \notin S$, while $\operatorname{cl} \mathbb{R}^+(S-0) = \operatorname{cl} K = X$.) The elements of $\operatorname{qi} S$ are *quasi-interior points* of S. The set S is *quasiopen* whenever $\operatorname{qi} S = S$.

4.2. If need be, the symbol qi, as well as cl, is provided with a qualifying index: $\mathrm{qi}_X S = \{x \in S : \mathrm{cl}_X \mathbb{R}^+(S - x) = X\}$ for $S \subset X$. If Y is a dense subspace of X then $\mathrm{qi}_Y S = \mathrm{qi}_X S$ for all $S \subset Y$. (Indeed, by the denseness of Y in X, the relations $\mathrm{cl}_Y \mathbb{R}^+(S - y) = Y$ and $\mathrm{cl}_X \mathbb{R}^+(S - y) = X$ are equivalent for each $y \in S$.) In this context, we agree to use the notation $\mathrm{qi}\, S$ without specifying the ambient space of the quasi-interior of S.

4.3. Lemma. *If C is a nonempty convex subset of X, then*

$$C^{\ominus} = (\mathrm{cl}\, \mathbb{R}^+ C)^{\odot}, \quad C^{\ominus\odot} = \mathrm{cl}\, \mathbb{R}^+ C.$$

◁ Given an arbitrary $f \in X'$, the inclusion $\mathbb{R}^+(\mathrm{cl}\, \mathbb{R}^+ C) \subset \mathrm{cl}\, \mathbb{R}^+ C$ implies

$$f \in C^{\ominus} \;\Leftrightarrow\; f \leqslant 0 \text{ on } \mathrm{cl}\, \mathbb{R}^+ C \;\Leftrightarrow\; f \leqslant 1 \text{ on } \mathrm{cl}\, \mathbb{R}^+ C \;\Leftrightarrow\; f \in (\mathrm{cl}\, \mathbb{R}^+ C)^{\odot};$$

whence, by the Bipolar Theorem, it follows that

$$C^{\ominus\odot} = (\mathrm{cl}\, \mathbb{R}^+ C)^{\odot\odot} = \mathrm{cl}\, \mathrm{co}\, (\mathrm{cl}\, \mathbb{R}^+ C \cup \{0\}) = \mathrm{cl}\, \mathbb{R}^+ C. \;▷$$

4.4. Proposition [7, 2.8]. *For all convex $C \subset X$ and $x \in X$, we have*

$$x \in \mathrm{qi}\, C \;\Leftrightarrow\; x \in C \text{ and } (C - x)^{\oplus} = \{0\}.$$

◁ We may assume that $x = 0$. If $0 \in \mathrm{qi}\, C$ then $\mathrm{cl}\, \mathbb{R}^+ C = X$; whence, by 4.3, it follows that

$$-C^{\oplus} = C^{\ominus} = (\mathrm{cl}\, \mathbb{R}^+ C)^{\odot} = X^{\odot} = \{0\}.$$

Conversely, if $C^{\oplus} = \{0\}$; then, on appealing again to 4.3, we conclude that

$$\mathrm{cl}\, \mathbb{R}^+ C = C^{\ominus\odot} = (-C^{\oplus})^{\odot} = \{0\}^{\odot} = X. \;▷$$

In what follows, we repeatedly use 4.4 without explicit references.

4.5. Proposition [7, 2.9]. *If C is a convex set, $x \in \mathrm{qi}\, C$, and $y \in C$; then $[x, y[\subset \mathrm{qi}\, C$.*

◁ We may assume that $x = 0$. Let $0 \in \mathrm{qi}\, C$, $y \in C$, and $0 < \lambda < 1$. To show that $\lambda y \in \mathrm{qi}\, C$, consider $f \in (C - \lambda y)^{\oplus}$ and establish that $f = 0$. Inserting the values $c = 0$ and $c = y$ into the inequality $f(c - \lambda y) \geqslant 0$, we obtain $f(y) \leqslant 0$ and $f(y) \geqslant 0$, respectively. Hence $f(c) = f(c - \lambda y) \geqslant 0$ for all $c \in C$, i.e., $f \in C^{\oplus}$. On the other hand, $0 \in \mathrm{qi}\, C$ implies $C^{\oplus} = \{0\}$. ▷

4.6. Corollary [7, 2.11]. *The quasi-interior of a convex set is convex.*

4.7. Proposition [7, 2.12]. *If C is a convex set with nonempty quasi-interior, then $\mathrm{cl}\, \mathrm{qi}\, C = \mathrm{cl}\, C$.*

◁ It suffices to show that $C \subset \mathrm{cl}\, \mathrm{qi}\, C$. Let $y \in C$. By hypothesis, there is $x \in \mathrm{qi}\, C$. Owing to 4.5, $[x, y[\subset \mathrm{qi}\, C$. It remains to note that $y \in \mathrm{cl}\, [x, y[$. ▷

4.8. Proposition [7, 2.13]. *Let C and D be convex subsets of X, with $\mathrm{int}\, D \neq \varnothing$. Then*

$$\mathrm{qi}\, C \cap \mathrm{int}\, D = \mathrm{qi}(C \cap D).$$

◁ By shifting, the claim reduces to the equivalence

$$0 \in \mathrm{qi}\, C \cap \mathrm{int}\, D \;\Leftrightarrow\; 0 \in \mathrm{qi}(C \cap D).$$

Let $0 \in \mathrm{qi}\, C \cap \mathrm{int}\, D$. Consider $f \in (C \cap D)^{\oplus}$ and show that $f = 0$. Since $0 \in C \cap \mathrm{int}\, D$; for every $c \in C$ there is a real $\varepsilon > 0$ such that $\varepsilon c \in C \cap D$, and hence $f(c) = \frac{1}{\varepsilon} f(\varepsilon c) \geqslant 0$. Consequently, $f \in C^{\oplus}$. On the other hand, $0 \in \mathrm{qi}\, C$ implies $C^{\oplus} = \{0\}$.

Conversely, let $(C \cap D)^{\oplus} = \{0\}$. Assume to the contrary that $0 \notin \mathrm{int}\, D$. Since $\mathrm{int}\, D \neq \varnothing$; by the Separation Theorem, there exists a nonzero $f \in X'$ such that $f \geqslant 0$ on D. But then, in particular, $f \in (C \cap D)^{\oplus}$; and, hence, $f = 0$. ▷

4.9. Proposition. *Let C be a convex subset of X.*

(a) $\operatorname{int} C \subset \operatorname{core} C \subset \operatorname{qi} C$.

(b) *The inclusions in* (a) *can be strict simultaneously. For example, if $f \in X^{\#}\backslash X'$ and $C = \{x \in X : f(x) \geqslant 0\}$; then $\operatorname{int} C = \varnothing$, $\operatorname{core} C = \{x \in X : f(x) > 0\}$, and $\operatorname{qi} C = C$.*

(c) *If X is finite-dimensional then $\operatorname{int} C = \operatorname{qi} C$.*

(d) *If $\operatorname{int} C \neq \varnothing$ then $\operatorname{int} C = \operatorname{qi} C$* (see [7, 2.14]).

(e) *The absolutely convex compact set $C := \{x \in \mathbb{R}^{\mathbb{N}} : |x| \leqslant 1\}$ in $\mathbb{R}^{\mathbb{N}}$ has empty interior and nonempty quasi-interior.*

◁ (d): If $\operatorname{int} C \neq \varnothing$; then, using 4.8, we see that

$$\operatorname{int} C = \operatorname{qi} X \cap \operatorname{int} C = \operatorname{qi}(X \cap C) = \operatorname{qi} C.$$

(e): For every $x \in C$ we have $\mathbb{R}^{+}(C - x) \subset \ell^{\infty} \neq \mathbb{R}^{\mathbb{N}}$ and so $x \notin \operatorname{int} C$. On the other hand, $0 \in \operatorname{qi} C$, since $\operatorname{cl} \mathbb{R}^{+}C = \operatorname{cl} \ell^{\infty} = \mathbb{R}^{\mathbb{N}}$. ▷

4.10. Proposition. *If C is a convex subset of X then $\operatorname{qi} C$ is quasiopen, i.e., $\operatorname{qi}\operatorname{qi} C = \operatorname{qi} C$.*

◁ It suffices to assume $0 \in \operatorname{qi} C$ and justify that $0 \in \operatorname{qi}\operatorname{qi} C$. Consider $f \in (\operatorname{qi} C)^{\oplus}$ and show that $f = 0$. Since $\operatorname{qi} C \neq \varnothing$, by 4.7, we have $\operatorname{cl}\operatorname{qi} C = \operatorname{cl} C$; therefore, the positivity of f on $\operatorname{qi} C$ implies the positivity of f on C. Consequently, $f = 0$, since $C^{\oplus} = \{0\}$. ▷

4.11. The operation $\operatorname{qi} : \mathscr{P}(X) \to \mathscr{P}(X)$ is not in general the interior with respect to any topology on X, since the equality

$$\operatorname{qi}(S_1 \cap S_2) = \operatorname{qi} S_1 \cap \operatorname{qi} S_2$$

can be violated even for convex sets S_1 and S_2. For example, if $X \neq \{0\}$ and K is a dense cone in X, then $0 \in \operatorname{qi}(K) \cap \operatorname{qi}(-K)$, while $K \cap -K = \{0\}$ and so $0 \notin \operatorname{qi}(K \cap -K)$.

4.12. The following is a slightly strengthened version of Proposition [8, 2.1.1]:

Proposition. (a) $\operatorname{qi} C^{\oplus} \subset C^{\boxplus}$ *for every convex set $C \subset X$.*

(b) *If W is a closed wedge in X then $\operatorname{qi} W^{\oplus} = W^{\boxplus}$.*

◁ (a): Assume that $f \in \operatorname{qi} C^{\oplus}$, but $f(x) = 0$ for some $x \in C\backslash\{0\}$. Then $\langle x \mid g - f\rangle = \langle x \mid g\rangle \geqslant 0$ for all $g \in C^{\oplus}$; i.e., $x \in (C^{\oplus} - f)^{\oplus}$, which is impossible, since $(C^{\oplus} - f)^{\oplus} = \{0\}$.

(b): Assume that $f > 0$ on $W\backslash\{0\}$, but $f \notin \operatorname{qi} W^{\oplus}$. Then $(W^{\oplus} - f)^{\oplus} \neq \{0\}$ and so there exists a nonzero $x \in X$ such that $\langle x \mid g\rangle \geqslant \langle x \mid f\rangle$ for all $g \in W^{\oplus}$. Since $W^{\oplus}$ is a wedge, this implies $\langle x \mid g\rangle \geqslant 0$ for all $g \in W^{\oplus}$; therefore, $x \in W^{\oplus\oplus} = \operatorname{cl} W = W$ (see 3.11) and, hence, $\langle x \mid f\rangle > 0$. But then $\langle x \mid g\rangle \geqslant \langle x \mid f\rangle > 0$ for all $g \in W^{\oplus}$, which is impossible as $0 \in W^{\oplus}$. ▷

4.13. Theorem. *For every convex set $C \subset \mathbb{R}^{\mathbb{N}}$, we have*

$$\operatorname{qi} C = \{c \in C : \pi_n c \in \operatorname{int} \pi_n C \text{ for all } n \in \mathbb{N}\}.$$

In particular, a convex set C is quasiopen in $\mathbb{R}^{\mathbb{N}}$ if and only if every projection $\pi_n C$ is open in $\mathbb{R}^n$.

◁ It suffices to establish the equivalence

$$0 \in \operatorname{qi} C \ \Leftrightarrow\ 0 \in C \text{ and } 0 \in \operatorname{int} \pi_n C \text{ for all } n \in \mathbb{N},$$

which, by 4.4 and 4.9(c), reduces to the equivalence

$$C^{\oplus} = \{0\} \ \Leftrightarrow\ (\pi_n C)^{\oplus} = \{0\} \text{ for all } n \in \mathbb{N}.$$

Suppose that $C^{\oplus} = \{0\}$. If $n \in \mathbb{N}$, $x \in \mathbb{R}^n$, and $\hat{x} \geqslant 0$ on $\pi_n C$; then the sequence

$$y := \big(x(1), \dots, x(n), 0, 0, \dots\big) \in \mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}$$

meets the inequality $\hat{y} \geqslant 0$ on C, whence $y = 0$ and, in particular, $x = \pi_n y = 0$.

Conversely, suppose that $(\pi_n C)^{\oplus} = \{0\}$ for all $n \in \mathbb{N}$. If $n \in \mathbb{N}$, $x \in \mathbb{R}^{\mathbb{N}}_n$, and $\hat{x} \geqslant 0$ on C; then $(\pi_n x)^{\wedge} \geqslant 0$ on $\pi_n C$, whence $\pi_n x = 0$ and so $x = 0$. ▷

§ 5. Quasilocal Boundedness

This section studies the new concept of quasilocally bounded locally convex space. As it turns out later (see § 6), these spaces have a few useful properties related to the closedness of cones in the respective weak topologies. The main result of this section is Theorem 5.10 claiming that $\mathbb{R}^{\mathbb{N}}$ is quasilocally bounded.

In what follows, X is a Hausdorff locally convex space.

5.1. A convex set $C \subset X$ is *quasilocally bounded at* $x \in \operatorname{qi} C$ if $x \in \operatorname{qi} B$ for some bounded subset $B \subset C$. The space X is *quasilocally bounded* if every convex set C in X is quasilocally bounded at every $x \in \operatorname{qi} C$.

It is easy to see that every normed space is quasilocally bounded. Some wider class of examples of quasilocally bounded spaces is provided by 5.9.

5.2. Proposition. *If X is infinite-dimensional and $X' = X^{\#}$ then every bounded subset of X has empty quasi-interior. In particular, X is not quasilocally bounded in this case.*

◁ Let X be infinite-dimensional, let $X' = X^{\#}$, and let S be a subset of X with nonempty quasi-interior. We may assume that $0 \in \operatorname{qi} S$ and thus $\operatorname{cl} \mathbb{R}^+ S = X$. Consider a maximal linearly independent subset $E \subset S$. Then $S \subset \operatorname{lin} E$, and the set E is infinite, because otherwise $X = \operatorname{cl} \mathbb{R}^+ S \subset \operatorname{cl} \operatorname{lin} E = \operatorname{lin} E$ despite X being infinite-dimensional. Therefore, there exists an unbounded function $f : E \to \mathbb{R}$ that extends to a functional $\overline{f} \in X^{\#} = X'$ unbounded on S. ▷

5.3. Lemma. *Let C be a convex subset of X and let $x \in \operatorname{qi} C$. Put $D := \frac{1}{2}(C + x) = \frac{1}{2}(C - x) + x$. Then $x \in \operatorname{qi} D$ and $D \subset \operatorname{qi} C$. In particular, if $0 \in \operatorname{qi} C$ then $\frac{1}{2} C \subset \operatorname{qi} C$.*

◁ The containment $x \in \operatorname{qi} D$ follows from $x \in D$ and $(D - x)^{\oplus} = \frac{1}{2}(C - x)^{\oplus} = \{0\}$. Moreover, by 4.5, for all $c \in C$ we have $\frac{1}{2}(c + x) \in [x, c[\subset \operatorname{qi} C$ and so $D \subset \operatorname{qi} C$. ▷

5.4. Proposition. *Let a convex set $C \subset X$ be quasilocally bounded at $x \in \operatorname{qi} C$. Then*
(a) *there is a bounded convex subset $B \subset C$ such that $x \in \operatorname{qi} B$, $B \subset \operatorname{qi} C$, and $\operatorname{cl} B \subset \operatorname{qi} \operatorname{cl} C$;*
(b) *if C is closed then there exists a closed bounded convex subset $B \subset C$ such that*

$$x \in \operatorname{qi} B \subset \operatorname{cl} \operatorname{qi} B = B \subset \operatorname{qi} C.$$

◁ We may assume that $x = 0$.

(a): Let $0 \in \operatorname{qi} B_0$, where B_0 is a bounded subset of C. Then $B := \frac{1}{2} \operatorname{co} B_0$ is the desired set. Indeed, the containment $0 \in \operatorname{qi} B$ is obvious and, by 5.3, we have $B \subset \frac{1}{2} C \subset \operatorname{qi} C$ and $\operatorname{cl} B \subset \frac{1}{2} \operatorname{cl} C \subset \operatorname{qi} \operatorname{cl} C$.

(b): According to (a), there is a bounded convex subset $B_0 \subset C$ such that $0 \in \operatorname{qi} B_0$ and $\operatorname{cl} B_0 \subset \operatorname{qi} \operatorname{cl} C = \operatorname{qi} C$. Since $\operatorname{qi} \operatorname{cl} B_0 \neq \varnothing$, by 4.7, we have $\operatorname{cl} \operatorname{qi} \operatorname{cl} B_0 = \operatorname{cl} B_0$; and, hence,

$$0 \in \operatorname{qi} \operatorname{cl} B_0 \subset \operatorname{cl} \operatorname{qi} \operatorname{cl} B_0 = \operatorname{cl} B_0 \subset \operatorname{qi} C.$$

Therefore, $B := \operatorname{cl} B_0$ is the desired set. ▷

5.5. Proposition. *The space X is quasilocally bounded if and only if every dense wedge $W \subset X$ includes a bounded subset $B \subset W$ such that $0 \in \operatorname{qi} B$.*

◁ Only sufficiency needs some demonstration. Let C be a convex subset of X and let $0 \in \operatorname{qi} C$. Show that $0 \in \operatorname{qi} B$ for some bounded subset $B \subset C$. By hypothesis, the dense wedge $\mathbb{R}^+ C$ includes a bounded subset $\overline{B} \subset \mathbb{R}^+ C$ such that $0 \in \operatorname{qi} \overline{B}$. We may assume that $\overline{B}$ is convex. Then $B := \overline{B} \cap C$ is the desired set. Since $\operatorname{cl} \mathbb{R}^+ \overline{B} = X$, to establish the required relation $\operatorname{cl} \mathbb{R}^+ B = X$ it suffices to consider an arbitrary $\bar{b} \in \overline{B}$ and note that $\bar{b} \in \mathbb{R}^+ B$. Indeed, $\overline{B} \subset \mathbb{R}^+ C$ implies $\bar{b} = \lambda c$ for some $\lambda \geqslant 0$ and $c \in C$. If $\lambda \leqslant 1$ then $\bar{b} = \lambda c \in [0, c] \subset C$; so $\bar{b} \in \overline{B} \cap C = B$ and, in particular, $\bar{b} \in \mathbb{R}^+ B$. If $\lambda > 1$ then $c \in [0, \lambda c] = [0, \bar{b}] \subset \overline{B}$, whence $c \in \overline{B} \cap C = B$ and again $\bar{b} = \lambda c \in \mathbb{R}^+ B$. ▷

5.6. Say that a locally convex space is *subnormable* if its topology is weaker than some normed topology.

Obviously, every normable space is subnormable. Metrizability, in general, does not imply subnormability. For example, $\mathbb{R}^{\mathbb{N}}$ is metrizable, but has no bounded absorbing sets and therefore is not subnormable (see 5.7).

5.7. Proposition. *The following properties of a locally convex space X are equivalent:*
(a) *X is subnormable;*
(b) *there exists a norm on X whose unit ball is bounded in X;*
(c) *there is a bounded absorbing set in X;*
(d) *X admits a bounded set with nonempty core;*
(e) *X has a bounded Hamel basis.*

$\lhd$ (a)$\Rightarrow$(b): If B is the unit ball in a norm whose topology is stronger than that of X, then B is bounded in X. Indeed, every neighborhood U of the origin in X contains εB for some $\varepsilon > 0$, and hence $B \subset \frac{1}{\varepsilon}U$.

The implications (b)$\Rightarrow$(c)$\Leftrightarrow$(d) are trivial.

(c)$\Rightarrow$(e): If B is a bounded absorbing set and E is a Hamel basis, then there is a numerical family $\lambda_e > 0$ such that $\lambda_e e \in B$ for all $e \in E$, and then $\{\lambda_e e : e \in E\}$ is a bounded Hamel basis.

(e)$\Rightarrow$(a): Let E be a bounded Hamel basis for X. Define a norm on X by putting

$$\|x\| := \sum_{e \in E} |x_e|,$$

where x_e is the coefficient at e in the expansion of x with respect to E. Show that the topology of the norm $\|\cdot\|$ is stronger than that of X. Every neighborhood of the origin in X contains an absolutely convex subneighborhood U. The boundedness of E implies $E \subset \lambda U$ for some real $\lambda > 0$. Since the unit ball B in the norm $\|\cdot\|$ is the absolutely convex hull of E, we have $B \subset \lambda U$. Consequently, $\frac{1}{\lambda}B \subset U$ and, hence, U is a neighborhood of the origin in the topology of the norm $\|\cdot\|$. $\rhd$

5.8. Proposition. *X is subnormable if and only if X' is subnormable.*

$\lhd$ If B is a bounded absorbing subset of X; then, by 3.9(d),(e), its polar $B^{\odot}$ is a bounded absorbing subset of X'. For the same reason, the existence of a bounded absorbing set in X' implies the existence of such set in X. $\rhd$

5.9. Proposition. *Every subnormable space is quasilocally bounded.*

$\lhd$ Let B be a bounded absorbing subset of X (see 5.7), and let W be a dense wedge in X. Then $\mathbb{R}^+(W \cap B) = W$ and so $\operatorname{cl} \mathbb{R}^+(W \cap B) = X$, i.e., $0 \in \operatorname{qi}(W \cap B)$. It remains to refer to 5.5. $\rhd$

Note that the converse does not hold: for example, $\mathbb{R}^{\mathbb{N}}$ is quasilocally bounded (see 5.10) but not subnormable.

5.10. Theorem. *$\mathbb{R}^{\mathbb{N}}$ is quasilocally bounded.*

$\lhd$ Using 5.5, consider an arbitrary dense wedge $W \subset \mathbb{R}^{\mathbb{N}}$ and show that $0 \in \operatorname{qi} B$ for some bounded subset $B \subset W$. By 4.13, for each $n \in \mathbb{N}$ we have $0 \in \operatorname{int} \pi_n W$. Since the projection $\pi_n W$ is a wedge in $\mathbb{R}^n$; therefore, $\pi_n W = \mathbb{R}^n$. Therefore, for all $n \in \mathbb{N}$ and $m \in \{1, \dots, n\}$, there are $w^+_{nm}, w^-_{nm} \in W$ such that $\pi_n w^+_{nm} = \pi_n e_m$ and $\pi_n w^-_{nm} = -\pi_n e_m$. Put

$$B := \operatorname{co}\Bigl(\bigcup_{n \in \mathbb{N}} \{w^+_{n1}, \dots, w^+_{nn}, w^-_{n1}, \dots, w^-_{nn}\} \cup \{0\}\Bigr).$$

Obviously, $B \subset W$. For each $n \in \mathbb{N}$ the set $\pi_n B$ contains the vectors

$$\pi_n w^{\pm}_{n1} = \pm\pi_n e_1, \ \dots, \ \pi_n w^{\pm}_{nn} = \pm\pi_n e_n$$

whose convex hull is the unit ball of the ℓ^1-norm in $\mathbb{R}^n$. Hence, $0 \in \operatorname{int} \pi_n B$ for all $n \in \mathbb{N}$ and so $0 \in \operatorname{qi} B$ by 4.13. To prove that B is bounded, consider an arbitrary $k \in \mathbb{N}$ and show that the numerical set $B(k) := \{b(k) : b \in B\}$ is bounded. Indeed, for all $n \geqslant k$ and $m \in \{1, \dots, n\}$, we have

$$w^{\pm}_{nm}(k) = \bigl(\pi_n w^{\pm}_{nm}\bigr)(k) = (\pm\pi_n e_m)(k) \in \{-1, 0, 1\},$$

and so $B(k)$ lies in the convex hull of the finite set

$$\bigcup_{n=1}^{k-1} \bigl\{w^{\pm}_{n1}(k), \dots, w^{\pm}_{nn}(k)\bigr\} \cup \{-1, 0, 1\}. \ \rhd$$

§ 6. Quasidenseness

In this section, the connection is studied between two new concepts: those of quasidense subspace and strictly closed cone. The main results here are Theorem 6.11 and Corollary 6.12, according to which the strict closedness of cones is inherited by the topologies induced by quasidense subspaces of quasilocally bounded spaces.

Throughout the section, X is a Hausdorff locally convex space. We let $\mathscr{P}_{\text{cbc}}(X)$ denote the collection of all closed bounded convex subsets of X.

6.1. Lemma. *Let Z be a subspace of X. The following properties of a subset $S \subset X$ are equivalent:*

(a) *if $B \in \mathscr{P}_{\text{cbc}}(X)$ and $\operatorname{qi}(Z \cap B) \neq \varnothing$ then $S \cap B \neq \varnothing$;*

(b) *if $B \in \mathscr{P}_{\text{cbc}}(X)$ and $\operatorname{qi}(Z \cap B) \neq \varnothing$ then $S \cap \operatorname{qi} B \neq \varnothing$;*

(c) *if $B \in \mathscr{P}_{\text{cbc}}(Z)$ and $\operatorname{qi} B \neq \varnothing$ then $S \cap \operatorname{cl} B \neq \varnothing$;*

(d) *if $B \in \mathscr{P}_{\text{cbc}}(Z)$ and $\operatorname{qi} B \neq \varnothing$ then $S \cap \operatorname{qi} \operatorname{cl} B \neq \varnothing$,*

where the operations cl *and* qi *are performed in X.*

◁ (a)⇒(b): Let $B \in \mathscr{P}_{\text{cbc}}(X)$ and $z \in \operatorname{qi}(Z \cap B)$. Put $C := \frac{1}{2}(B - z) + z$. Obviously, $C \in \mathscr{P}_{\text{cbc}}(X)$. From the equalities $Z \cap C - z = \frac{1}{2}(Z \cap B - z)$ and $(Z \cap B - z)^{\oplus} = \{0\}$ it follows that $(Z \cap C - z)^{\oplus} = \{0\}$, i.e., $z \in \operatorname{qi}(Z \cap C)$. Then $S \cap C \neq \varnothing$ according to (a). Moreover, $C \subset \operatorname{qi} B$ by 5.3. Hence, $S \cap \operatorname{qi} B \neq \varnothing$.

(b)⇒(d): Let $B \in \mathscr{P}_{\text{cbc}}(Z)$ and $\operatorname{qi} B \neq \varnothing$. It is clear that $\operatorname{cl} B \in \mathscr{P}_{\text{cbc}}(X)$. The inclusion $Z \cap \operatorname{cl} B \supset B$ implies $\operatorname{qi}(Z \cap \operatorname{cl} B) \supset \operatorname{qi} B \neq \varnothing$, and so $S \cap \operatorname{qi} \operatorname{cl} B \neq \varnothing$ according to (b).

The implication (d)⇒(c) is trivial.

(c)⇒(a): Let $B \in \mathscr{P}_{\text{cbc}}(X)$ and $\operatorname{qi}(Z \cap B) \neq \varnothing$. Since $Z \cap B \in \mathscr{P}_{\text{cbc}}(Z)$, condition (c) implies $S \cap \operatorname{cl}(Z \cap B) \neq \varnothing$. In this case, $\operatorname{cl}(Z \cap B) \subset \operatorname{cl} B = B$. ▷

6.2. A subset $S \subset X$ satisfying the equivalent conditions (a)–(d) in 6.1 is *quasidense in X with respect to Z*. A quasidense set in X with respect to X is *quasidense* in X. Therefore, the following are equivalent:

(a) S is quasidense in X;

(b) if $B \in \mathscr{P}_{\text{cbc}}(X)$ and $\operatorname{qi} B \neq \varnothing$ then $S \cap B \neq \varnothing$;

(c) if $B \in \mathscr{P}_{\text{cbc}}(X)$ and $\operatorname{qi} B \neq \varnothing$ then $S \cap \operatorname{qi} B \neq \varnothing$.

6.3. Proposition. *If there exists a bounded set in X with nonempty quasi-interior, then every quasidense subset of X is dense in X.*

◁ Let B be a bounded subset of X with nonempty quasi-interior. We may assume that $0 \in \operatorname{qi} B$. Suppose that X has a quasidense set S that is not dense. Then there are U and u such that U is a closed convex subset of X, $u \in \operatorname{int} U$, and $U \subset X \backslash S$. Put $\overline{B} := \operatorname{cl} \operatorname{co} B$ and $C := (\overline{B} + u) \cap U$. It is clear that $C \in \mathscr{P}_{\text{cbc}}(X)$ and $\operatorname{qi} C \neq \varnothing$. Indeed, since $u \in \overline{B} + u$ and $\big((\overline{B} + u) - u\big)^{\oplus} = \overline{B}^{\oplus} \subset B^{\oplus} = \{0\}$, we have $u \in \operatorname{qi}(\overline{B} + u)$, whence, by 4.8, it follows that $u \in \operatorname{qi}(\overline{B} + u) \cap \operatorname{int} U = \operatorname{qi}\big((\overline{B} + u) \cap U\big) = \operatorname{qi} C$. Then the quasidenseness of S in X implies that $S \cap C$ is nonempty, which contradicts the inclusions $C \subset U \subset X \backslash S$. ▷

On the other hand, if all bounded sets in X have empty quasi-interior (see, for example, 5.2); then $\mathscr{P}_{\text{cbc}}(X) = \varnothing$, in which case, by 6.2, every subset of X is quasidense.

6.4. Corollary. *All quasidense subsets of $\mathbb{R}^{\mathbb{N}}$ are dense.*

◁ Owing to 6.3, it suffices to note that $\{x \in \mathbb{R}^{\mathbb{N}} : |x| \leqslant 1\}$ is a bounded subset of $\mathbb{R}^{\mathbb{N}}$ with nonempty quasi-interior (see 4.9(e)). ▷

The converse is not true: $\mathbb{R}^{\mathbb{N}}_{\text{fin}}$ is an example of a dense but not quasidense subset of $\mathbb{R}^{\mathbb{N}}$. Indeed,

$$B := \big\{ s \in \mathbb{R}^{\mathbb{N}} : |s(n) - 1| \leqslant \tfrac{1}{2} \text{ for all } n \in \mathbb{N} \big\}$$

belongs to $\mathscr{P}_{\text{cbc}}(\mathbb{R}^{\mathbb{N}})$, and $(1, 1, \dots) \in \operatorname{qi} B$, while $\mathbb{R}^{\mathbb{N}}_{\text{fin}} \cap B = \varnothing$.

6.5. A *pseudo-base* (respectively, a *base*) of a wedge K in X is a convex subset $B \subset K$ such that $0 \notin B$ and for every nonzero $x \in K$ there exists a real (respectively, a unique real) $\lambda > 0$ for which $\lambda x \in B$. Obviously, every wedge with a pseudo-base is a cone. As is known (see, for example, [9, 3.6]), the existence of a base for a cone K is equivalent to the existence of a linear functional strictly positive on $K\backslash\{0\}$.

6.6. Proposition. *The following properties of a wedge $K \subset X$ are equivalent:*
(a) $K^{\boxplus} \neq \varnothing$, *i.e., there is a functional $f \in X'$ such that $f > 0$ on $K\backslash\{0\}$;*
(b) *K has a pseudo-base B such that $0 \notin \operatorname{cl} B$;*
(c) *K has a base B such that $0 \notin \operatorname{cl} B$;*
(d) $K\backslash\{0\} \subset \mathbb{R}^+U$ *for some open convex set $U \subset X$ such that $0 \notin U$;*
(e) $K\backslash\{0\} \subset \operatorname{int} \overline{K}$ *for some cone $\overline{K} \subset X$.*

6.7. A wedge $K \subset X$ is *strictly closed* if every element $x \in X\backslash K$ is strictly separated from $K\backslash\{0\}$; i.e., there is $f \in X'$ such that $f > 0$ on $K\backslash\{0\}$ and $f(x) < 0$.

Obviously, a strictly closed wedge is a cone. In a finite-dimensional space, every closed cone is strictly closed. The sets of positive sequences $(\mathbb{R}^{\mathbb{N}})^+$ and $(\mathbb{R}^{\mathbb{N}}_{\text{fin}})^+$ are examples of closed cones in $\mathbb{R}^{\mathbb{N}}|\mathbb{R}^{\mathbb{N}}_{\text{fin}}$ and $\mathbb{R}^{\mathbb{N}}_{\text{fin}}|\mathbb{R}^{\mathbb{N}}_{\text{fin}}$ that are not strictly closed. On the other hand, all closed cones in $\mathbb{R}^{\mathbb{N}}_{\text{fin}}|\mathbb{R}^{\mathbb{N}}$ are strictly closed (see 8.4).

6.8. Proposition. *The following properties of a wedge $K \subset X$ are equivalent:*
(a) *K is a strictly closed cone;*
(b) *K is closed and $K^{\boxplus} \neq \varnothing$;*
(c) *K is closed and has a closed pseudo-base;*
(d) *K is closed and has a closed base.*

◁ By 6.6, only the implication (b)⇒(a) needs demonstration. Suppose that K is closed, $g \in K^{\boxplus}$, and $x \in X\backslash K$. By the Strict Separation Theorem, there exists a functional $h \in X'$ such that $h \geqslant 0$ on K and $h(x) < 0$. Let $\lambda > 0$ be a real such that $\lambda h(x) < -g(x)$. Then the functional $f := g + \lambda h$ strictly separates x from $K\backslash\{0\}$. ▷

6.9. Lemma. *If $C \in \mathscr{P}_{\text{cbc}}(X)$, $C \neq \varnothing$, and $0 \notin C$; then $\mathbb{R}^+C$ is a closed cone in X.*

◁ The set $\mathbb{R}^+C$ is obviously a cone. To prove its closedness, consider nets $\lambda_\alpha \in \mathbb{R}^+$ and $c_\alpha \in C$, suppose that $\lambda_\alpha c_\alpha \to x \in X$, and demonstrate that $x \in \mathbb{R}^+C$. We may assume that $x \neq 0$. Since C is closed and does not contain 0, by the Strict Separation Theorem, there exists $f \in X'$ such that $f \geqslant 1$ on C. The boundedness of C implies the boundedness of the net $f(c_\alpha)$. Passing to a subnet, we may assume that $f(c_\alpha) \to \mu$ for some real $\mu \geqslant 1$. Put $\lambda := \frac{f(x)}{\mu}$. The relations $\frac{1}{f(c_\alpha)} \to \frac{1}{\mu}$ and $f(\lambda_\alpha c_\alpha) \to f(x)$ imply

$$\lambda_\alpha = \tfrac{1}{f(c_\alpha)} f(\lambda_\alpha c_\alpha) \to \tfrac{1}{\mu} f(x) = \lambda.$$

Since C is bounded and $\lambda_\alpha c_\alpha \to x \neq 0$, the net λ_α cannot converge to zero; hence, $\lambda > 0$. Therefore, we may assume that $\lambda_\alpha > 0$ for all α. Since $\frac{1}{\lambda_\alpha} \to \frac{1}{\lambda}$ and $\lambda_\alpha c_\alpha \to x$, we have $c_\alpha = \frac{1}{\lambda_\alpha}\lambda_\alpha c_\alpha \to \frac{1}{\lambda}x$. Appealing to the closedness of C, we conclude that $\frac{1}{\lambda}x \in C$ and so $x \in \mathbb{R}^+C$. ▷

6.10. Lemma. *Suppose that Y and Z are dense subspaces of X', $B \in \mathscr{P}_{\text{cbc}}(X')$, $\operatorname{qi}\operatorname{cl}(Z \cap B) \neq \varnothing$, and $Y \cap B = \varnothing$. Then $(Z \cap B)^{\oplus} \subset X$ is a cone that is strictly closed in $X|Z$ and dense* (*in particular, not closed*) *in $X|Y$.*

◁ By hypothesis, $C := Z \cap B$ and $D := \operatorname{cl} C$ satisfy the relations $D \in \mathscr{P}_{\text{cbc}}(X')$, $0 \notin D \subset B$, and $\operatorname{qi} D \neq \varnothing$. Lemma 6.9 implies that $\mathbb{R}^+D$ is a closed cone in X'. Put

$$K := C^{\oplus} = D^{\oplus} = (\mathbb{R}^+D)^{\oplus}.$$

Owing to 3.11, we have $K^{\oplus} = (\mathbb{R}^+D)^{\oplus\oplus} = \mathbb{R}^+D$. Since $C \subset Z$, the wedge $K = C^{\oplus}$ is closed in $X|Z$. Moreover, according to 4.12(a), $Z \cap K^{\boxplus} \supset Z \cap \operatorname{qi} K^{\oplus} \supset Z \cap \operatorname{qi} D = \operatorname{qi} D \neq \varnothing$ hold and, hence, K is a strictly closed cone in $X|Z$ (see 6.8(b)). The condition $Y \cap B = \varnothing$ implies $Y \cap K^{\oplus} = Y \cap \mathbb{R}^+D \subset Y \cap \mathbb{R}^+B = \{0\}$. Using 4.4, we conclude that $0 \in \operatorname{qi}_{X|Y} K$, i.e., $\operatorname{cl}_{X|Y} K = X$. ▷

6.11. Theorem. *Let Y and Z be dense subspaces of X'. Consider the following properties:*

(a) *every strictly closed cone in $X|Z$ is strictly closed in $X|Y$;*

(b) *every strictly closed cone in $X|Z$ is closed in $X|Y$;*

(c) *Y is quasidense in X' with respect to Z.*

Then the implications (a)⇒(b)⇒(c) *hold; and if Z is quasilocally bounded, then* (a)–(c) *are equivalent.*

◁ The implication (a)⇒(b) is trivial; and (b)⇒(c) is immediate from 6.10 in view of 6.1(a). Assume that Z is quasilocally bounded and establish the implication (c)⇒(a).

Let Y be quasidense in X' with respect to Z, and let K be a strictly closed cone in $X|Z$. To prove the strict closedness of K in $X|Y$, consider an arbitrary $x \in X\backslash K$ and demonstrate the existence of $y \in Y$ such that $y > 0$ on $K\backslash\{0\}$ and $y(x) < 0$. In terms of $K^{\boxplus}$ and

$$H_0 := \{f \in X' : f(x) < 0\},$$

the current goal is formulated as follows:

$$Y \cap K^{\boxplus} \cap H_0 \neq \varnothing. \tag{4}$$

Owing to the strict closedness of K in $X|Z$, we have $Z \cap K^{\boxplus} \cap H_0 \neq \varnothing$, which, by 4.12(b), means the existence of $z \in \operatorname{qi}(Z \cap K^{\oplus}) \cap H_0$ (see 4.2). Since Z is quasilocally bounded, there is $B \in \mathscr{P}_{\mathrm{cbc}}(Z)$ such that $z \in \operatorname{qi} B$ and $B \subset Z \cap K^{\oplus}$ (see 5.4(b)). Put

$$H := \{f \in X' : f(x) \leqslant 0\}, \quad C := \operatorname{cl} B \cap H.$$

It is clear that $C \in \mathscr{P}_{\mathrm{cbc}}(X')$. Moreover, since $\operatorname{int} H = H_0 \neq \varnothing$; according to 4.8, we have

$$z \in \operatorname{qi} B \cap H_0 = \operatorname{qi}(B \cap H) = \operatorname{qi}(Z \cap B \cap H) \subset \operatorname{qi}(Z \cap C).$$

Invoking the quasidenseness of Y in X' with respect to Z, we conclude that $Y \cap \operatorname{qi} C \neq \varnothing$. Using 4.8 again, we have

$$\operatorname{qi} C = \operatorname{qi}(\operatorname{cl} B \cap H) = \operatorname{qi}\operatorname{cl} B \cap H_0 \subset H_0,$$

which means $Y \cap \operatorname{qi} C \cap H_0 \neq \varnothing$. To justify (4), it remains to note that

$$\operatorname{qi} C \subset \operatorname{qi}\operatorname{cl} B \subset \operatorname{qi}\operatorname{cl} K^{\oplus} = \operatorname{qi} K^{\oplus} \subset K^{\boxplus}. \ \triangleright$$

6.12. Corollary. *Let Y be a dense subspace of X', and let X' be quasilocally bounded. Then the following are equivalent:*

(a) *every strictly closed cone in X is strictly closed in $X|Y$;*

(b) *every strictly closed cone in X is closed in $X|Y$;*

(c) *Y is quasidense in X'.*

If, contrary to (c), *there exists $B \in \mathscr{P}_{\mathrm{cbc}}(X')$ such that $\operatorname{qi} B \neq \varnothing$ and $Y \cap B = \varnothing$; then $B^{\oplus}$ is an example of a strictly closed cone in X that is not closed* (*moreover, is dense*) *in $X|Y$.*

§ 7. Projective Sets

The space $\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}$ equipped with the strongest locally convex topology is the direct limit (colimit) $\varinjlim \mathbb{R}^n$ of the sequence $(\mathbb{R}^n)_{n\in\mathbb{N}}$ within the category of locally convex spaces with respect to the embeddings $\mathbb{R}^n \leftrightarrow \mathbb{R}^n \times \{(0,\dots,0)\} \subset \mathbb{R}^m$ $(n \leqslant m)$ and coincides with the strict inductive limit of the sequence of subspaces $\mathbb{R}^{\mathbb{N}}_n = \operatorname{lin}\{e_1,\dots,e_n\} \subset \mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}$ (see [4, 13-3-3; 10, 19.4]). The corresponding dual space $\mathbb{R}^{\mathbb{N}}\big|\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}$ is the inverse limit $\varprojlim \mathbb{R}^n$ in the same category with respect to the projections $(x \mapsto \pi_n x)\colon \mathbb{R}^m \to \mathbb{R}^n$ $(m \geqslant n)$ and coincides with the topological projective limit of the sequence of locally convex spaces $\mathbb{R}^n$ under the projections $\pi_n\colon \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^n$ (see [10, 19.8]).

The above considerations serve as a conceptual premise for introducing the notions of projective set, projective sequence, and projective limit whose study this section is devoted to. The main goal is to link the concepts with that of quasi-interior.

As before, $\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}$ and $\mathbb{R}^{\mathbb{N}}$ are equipped with the natural duality and endowed with the corresponding weak topologies (see 3.4).

7.1. Let $(S_n)_{n\in\mathbb{N}}$ be a sequence of subsets $S_n \subset \mathbb{R}^n$. A sequence $(x_n)_{n\in\mathbb{N}}$ is a *chain* in $(S_n)_{n\in\mathbb{N}}$ if $x_n \in S_n$ and $x_n = \pi_n x_{n+1}$ for all $n \in \mathbb{N}$. In this case, $\bigcup_{n\in\mathbb{N}} x_n$ is the element $s \in \mathbb{R}^{\mathbb{N}}$ such that $\pi_n s = x_n$ for all $n \in \mathbb{N}$.

Lemma. *The following properties of $S \subset \mathbb{R}^{\mathbb{N}}$ are equivalent:*
(a) *if $s \in \mathbb{R}^{\mathbb{N}}$ and $\pi_n s \in \pi_n S$ for all $n \in \mathbb{N}$ then $s \in S$;*
(b) *if $(x_n)_{n\in\mathbb{N}}$ is a chain in $(\pi_n S)_{n\in\mathbb{N}}$ then $\bigcup_{n\in\mathbb{N}} x_n \in S$;*
(c) *S is closed in $\mathbb{R}^{\mathbb{N}}_{\mathrm{D}}$.*

A set $S \subset \mathbb{R}^{\mathbb{N}}$ with the equivalent properties (a)–(c) is *projective*. Arbitrary Cartesian products $\prod_{n\in\mathbb{N}} \Lambda_n$, where $\Lambda_n \subset \mathbb{R}$, serve as examples of projective sets. Moreover, every closed subset of $\mathbb{R}^{\mathbb{N}}$ is also closed in $\mathbb{R}^{\mathbb{N}}_{\mathrm{D}}$ and is thus projective.

7.2. Lemma 7.1 implies that, for every $S \subset \mathbb{R}^{\mathbb{N}}$, the following sets coincide:
(a) $\{\bar{s} \in \mathbb{R}^{\mathbb{N}} : \pi_n \bar{s} \in \pi_n S$ for all $n \in \mathbb{N}\}$;
(b) $\{\bigcup_{n\in\mathbb{N}} x_n : (x_n)_{n\in\mathbb{N}}$ is a chain in $(\pi_n S)_{n\in\mathbb{N}}\}$;
(c) the largest subset $\overline{S} \subset \mathbb{R}^{\mathbb{N}}$ such that $\pi_n \overline{S} = \pi_n S$ for all $n \in \mathbb{N}$;
(d) the projective subset $\overline{S} \subset \mathbb{R}^{\mathbb{N}}$ such that $\pi_n \overline{S} = \pi_n S$ for all $n \in \mathbb{N}$;
(e) the smallest projective subset of $\mathbb{R}^{\mathbb{N}}$ including S;
(f) the closure of S in the topological space $\mathbb{R}^{\mathbb{N}}_{\mathrm{D}}$.

The set described by one of the equivalent ways (a)–(f) is the *projective closure* of S.

7.3. Lemma. *The following properties of a sequence of sets $S_n \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ are equivalent:*
(a) *there is $S \subset \mathbb{R}^{\mathbb{N}}$ such that $S_n = \pi_n S$ for all $n \in \mathbb{N}$;*
(b) *$S_n = \pi_n S_m$ for $n \leqslant m$;*
(c) *$S_n = \pi_n S_{n+1}$ for all $n \in \mathbb{N}$.*

Moreover,

$$\begin{aligned}\bigcap_{n\in\mathbb{N}} \pi_n^{-1}(S_n) &= \{s \in \mathbb{R}^{\mathbb{N}} : \pi_n s \in S_n \text{ for all } n \in \mathbb{N}\} \\ &= \Big\{ \bigcup_{n\in\mathbb{N}} x_n : (x_n)_{n\in\mathbb{N}} \text{ is a chain in } (S_n)_{n\in\mathbb{N}} \Big\}\end{aligned} \tag{5}$$

is the largest among the sets S satisfying (a), *presenting the only projective set among them, and coincides with the projective closure of each of them.*

A sequence $(S_n)_{n\in\mathbb{N}}$ satisfying the conditions of the above lemma is a *projective sequence*. The set (5) is denoted by $\varprojlim (S_n)_{n\in\mathbb{N}}$ or, in short, by $\varprojlim S_n$ and is the *projective limit* of the sequence $(S_n)_{n\in\mathbb{N}}$ or the projective limit of S_n.

Therefore, given a projective sequence $(S_n)_{n\in\mathbb{N}}$ and $S \subset \mathbb{R}^{\mathbb{N}}$, the following hold:

$$\varprojlim S_n = \{s \in \mathbb{R}^{\mathbb{N}} : \pi_n s \in S_n \text{ for all } n \in \mathbb{N}\}; \tag{6}$$

$$S = \varprojlim S_n \ \Leftrightarrow\ S \text{ is projective and } \pi_n S = S_n \text{ for all } n \in \mathbb{N}; \tag{7}$$

$$S \text{ is projective } \Leftrightarrow\ S = \varprojlim \pi_n S. \tag{8}$$

7.4. Lemma. *If $(C_n)_{n\in\mathbb{N}}$ is a projective sequence of convex sets with nonempty interiors, then $(\operatorname{int} C_n)_{n\in\mathbb{N}}$ is a projective sequence and*

$$\varprojlim \operatorname{int} C_n = \operatorname{qi} \varprojlim C_n.$$

◁ Fix an arbitrary $n \in \mathbb{N}$ and show that $\operatorname{int} C_n = \pi_n(\operatorname{int} C_{n+1})$.

The inclusion $\operatorname{int} C_n \supset \pi_n(\operatorname{int} C_{n+1})$ is obvious: if $x \in \mathbb{R}^{n+1}$, $\varepsilon > 0$, and the ball $B(x,\varepsilon)$ of the uniform norm lies in C_{n+1}; then $B(\pi_n x, \varepsilon) = \pi_n\big(B(x,\varepsilon)\big) \subset \pi_n C_{n+1} = C_n$.

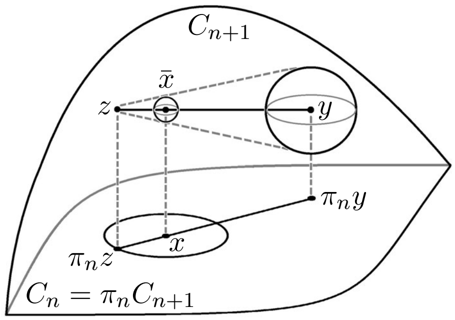

Fig. 1

Let $x \in \operatorname{int} C_n$ (see Fig. 1). Show that $x = \pi_n \bar{x}$ for some $\bar{x} \in \operatorname{int} C_{n+1}$. Consider an arbitrary $y \in \operatorname{int} C_{n+1}$. If $\pi_n y = x$ then there is nothing to prove. Let $\pi_n y \neq x$. Since $x \in \operatorname{int} C_n$, there is a real $\lambda > 0$ such that $x + \lambda(x - \pi_n y) \in C_n$. Owing to the equality $C_n = \pi_n C_{n+1}$, there exists $z \in C_{n+1}$ for which $x + \lambda(x - \pi_n y) = \pi_n z$. Put $\bar{x} := \frac{\lambda}{\lambda+1} y + \frac{1}{\lambda+1} z$. The containment $y \in \operatorname{int} C_{n+1}$ implies $\bar{x} \in [y, z[\subset \operatorname{int} C_{n+1}$. Moreover,

$$\pi_n \bar{x} = \tfrac{\lambda}{\lambda+1} \pi_n y + \tfrac{1}{\lambda+1} \pi_n z = \tfrac{\lambda}{\lambda+1} \pi_n y + \tfrac{1}{\lambda+1} \big(x + \lambda(x - \pi_n y) \big) = x.$$

The equality $\varprojlim \operatorname{int} C_n = \operatorname{qi} \varprojlim C_n$ follows from (6) and 4.13, because

$$s \in \varprojlim \operatorname{int} C_n \ \Leftrightarrow \ s \in \varprojlim C_n \text{ and } \pi_n s \in \operatorname{int} C_n \text{ for all } n \in \mathbb{N} \ \Leftrightarrow \ s \in \operatorname{qi} \varprojlim C_n$$

for all $s \in \mathbb{R}^{\mathbb{N}}$. ▷

7.5. In the statement of Lemma 7.4, the convexity and nonemptiness of interiors are both essential.

For example, if $S_1 = [0, 2]$, $S_2 = [0, 1]^2 \cup ([1, 2] \times \{0\})$, and $S_n = S_2 \times \mathbb{R}^{n-2}$ for $n > 2$; then $(S_n)_{n \in \mathbb{N}}$ is a projective sequence and $\operatorname{int} S_n \neq \varnothing$ for all $n \in \mathbb{N}$, but the sequence $(\operatorname{int} S_n)_{n \in \mathbb{N}}$ is not projective because $\operatorname{int} S_1 =]0, 2[$, while $\pi_1(\operatorname{int} S_2) = \pi_1(]0, 1[^2) =]0, 1[$.

If $S_1 = [0, 1]$ and $S_n = [0, 1] \times \{(0, \dots, 0)\}$ for $n > 1$; then $(S_n)_{n \in \mathbb{N}}$ is a projective sequence and all sets S_n are convex, but the sequence $(\operatorname{int} S_n)_{n \in \mathbb{N}}$ is not projective, since $\operatorname{int} S_1 =]0, 1[$ and $\pi_1(\operatorname{int} S_2) = \varnothing$.

7.6. The following is a consequence of 4.13, 7.4, and (8):

Corollary. *If C is a projective convex subset of $\mathbb{R}^{\mathbb{N}}$, then $\operatorname{qi} C$ is projective. Moreover, if $\operatorname{qi} C \neq \varnothing$ then $(\operatorname{int} \pi_n C)_{n \in \mathbb{N}}$ is a projective sequence, $\operatorname{qi} C = \varprojlim \operatorname{int} \pi_n C$; and, in particular, $\pi_n \operatorname{qi} C = \operatorname{qi} \pi_n C = \operatorname{int} \pi_n C$ for all $n \in \mathbb{N}$.*

7.7. Let P be a property of subsets of a locally convex space. Call P *projectively invariant* whenever the following is true for every projective set $S \subset \mathbb{R}^{\mathbb{N}}$: S has property P if and only if $\pi_n S$ have property P for all $n \in \mathbb{N}$; i.e., if S is the projective limit of sets with property P.

Proposition. *The following properties are projectively invariant:*

(a) *to be nonempty;*

(b) *to be convex;*

(c) *to be bounded;*

(d) *to be compact* (*closed and bounded*)*;*

(e) *to be convex and quasiopen;*

(f) *to be convex and have nonempty quasi-interior.*

◁ The projective invariance of (a) and (c) is obvious. The projective invariance of (b) and (d) is provided by the linearity and continuity of the mappings $\pi_n : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^n$ and by the representation $S = \bigcap_{n \in \mathbb{N}} \pi_n^{-1}(\pi_n S)$ of a projective set S; see (5). The projective invariance of (e) and (f) follows from 4.13 and 7.4. ▷

It is easy to see that the projective limit of closed sets is closed. Nevertheless, closedness is not projectively invariant even under the additional requirement of convexity. For example, the convex set

$$S = \{s \in \mathbb{R}^{\mathbb{N}} : s(1) > 0,\ s(1) \cdot s(2) \geqslant 1\}$$

is closed (and thus projective), but the projection $\pi_1 S =]0, \infty[$ is not closed.

7.8. A subset of $\mathbb{R}^{\mathbb{N}}$ of the form $\prod_{n\in\mathbb{N}} \Lambda_n$, where Λ_n are open subsets of $\mathbb{R}$, is called an *open box*. The topology on $\mathbb{R}^{\mathbb{N}}$ for which open boxes serve as basic open sets is the *box topology*. It is easy to see that, for every $s \in \mathbb{R}^{\mathbb{N}}$, the convex open boxes

$$\prod_{n\in\mathbb{N}}]s(n) - \varepsilon_n, s(n) + \varepsilon_n[, \quad \varepsilon_n > 0, \tag{9}$$

form a base of neighborhoods of s in the box topology.

Proposition. (a) *An arbitrary convex open box in* $\mathbb{R}^{\mathbb{N}}$ *is an example of a projective quasiopen set. Moreover, all subsets of* $\mathbb{R}^{\mathbb{N}}$ *open in the box topology are quasiopen.*

(b) *The converse of* (a) *fails. For example, the projective convex set*

$$C := \left\{s \in \mathbb{R}^{\mathbb{N}} : |s(1) - s(n)| < \tfrac{1}{n}\right\}$$

is quasiopen, but $\operatorname{int} C = \varnothing$ *in the box topology.*

◁ (a): The first assertion follows from 7.7(e), and the second is a consequence of the first, since each box neighborhood includes a subneighborhood of the form (9).

(b): The projectivity of C is obvious. Its convexity and quasiopenness follow from 7.7(e), since the projections of $\pi_n C$ are convex open subsets of $\mathbb{R}^n$. The equality $\operatorname{int} C = \varnothing$ in the box topology is due to the fact that all elements $c \in C$ satisfy the condition $\lim_{n\to\infty} c(n) = c(1)$, while every nonempty open box obviously includes a sequence that violates this condition. ▷

§ 8. Criterion for Closedness of Archimedean Cones

In this section, after characterizing the Archimedean cones in $\mathbb{R}^{\mathbb{N}}_{\text{fin}}$ (Theorem 8.4), we use the above auxiliary results for solving the target problem describing the class of subspaces $Y \subset \mathbb{R}^{\mathbb{N}}$ for which all Archimedean cones are closed in $\mathbb{R}^{\mathbb{N}}_{\text{fin}}|Y$. The main result is Theorem 8.5 asserting that the class consists of quasidense subspaces. Propositions 8.8 and 8.9 describe the class in more detail by proposing a few necessary and sufficient conditions for quasidenseness of a set in $\mathbb{R}^{\mathbb{N}}$.

As a corollary, we obtain some exhaustive description of locally convex spaces with all Archimedean cones closed (Theorem 8.6). Moreover, the answers are given to questions on the so-called "subtle" spaces which are the dense subspaces $Y \subset \mathbb{R}^{\mathbb{N}}$ for which $\mathbb{R}^{\mathbb{N}}_{\text{fin}}|Y$ includes nonclosed Archimedean cones. Namely, Corollary 8.10 confirms the conjecture of [3] that $\operatorname{lin} \mathbb{Q}^{\mathbb{N}}$ is not subtle, and, in particular, gives a negative answer to the question in [3] and [5] of whether all proper subspaces of $\mathbb{R}^{\mathbb{N}}$ are subtle. Moreover, using Theorem 8.5, it is possible to obtain a short justification of the result of [5] on the existence of a subtle hyperspace (Corollary 8.11).

8.1. A sequence of subsets $S_n \subset \mathbb{R}^{\mathbb{N}}_n$ ($n \in \mathbb{N}$) is *inductive* if it possesses each of the following equivalent properties:

(a) there exists $S \subset \mathbb{R}^{\mathbb{N}}_{\text{fin}}$ such that $S_n = S \cap \mathbb{R}^{\mathbb{N}}_n$ for all $n \in \mathbb{N}$;

(b) $S_n = S_m \cap \mathbb{R}^{\mathbb{N}}_n$ for $n \leqslant m$;

(c) $S_n = S_{n+1} \cap \mathbb{R}^{\mathbb{N}}_n$ for all $n \in \mathbb{N}$.

Moreover, S satisfying (a) is unique and equal to $\bigcup_{n\in\mathbb{N}} S_n$.

8.2. Lemma [11, 3.3]**.** *Let K be a closed locally quasicompact cone in a locally convex space X, let X_0 be a closed subspace of X, and let $f_0 \in X_0'$. If $f_0 \in (K \cap X_0)^{\boxplus}$ then f_0 is extendible onto X to $f \in K^{\boxplus}$.*

8.3. Lemma. *If $K_n \subset \mathbb{R}^{\mathbb{N}}_n$ $(n \in \mathbb{N})$ is an inductive sequence of closed cones, then $(\pi_n K_n)^{\boxplus}$ $(n \in \mathbb{N})$ is a projective sequence and*

$$\varprojlim (\pi_n K_n)^{\boxplus} = \Big(\bigcup_{n\in\mathbb{N}} K_n \Big)^{\boxplus}. \tag{10}$$

◁ Consider an arbitrary $n \in \mathbb{N}$ and establish the inclusion $(\pi_n K_n)^{\boxplus} \subset \pi_n(\pi_{n+1}K_{n+1})^{\boxplus}$. (The reverse inclusion and equality (10) are easily verified.) Let $y \in (\pi_n K_n)^{\boxplus}$. Put

$$X := \mathbb{R}^{n+1}, \quad X_0 := \mathbb{R}^n \times \{0\} \subset X, \quad K := \pi_{n+1}K_{n+1}.$$

Then K is a closed cone in X and

$$\hat{y} \circ \pi_n \in (\pi_n K_n \times \{0\})^{\boxplus} = (K \cap X_0)^{\boxplus}.$$

By 8.2, there is an element $z \in K^{\boxplus}$ such that $\hat{z} = \hat{y} \circ \pi_n$ on X_0. In this case, $y = \pi_n z$ and hence $y \in \pi_n(\pi_{n+1}K_{n+1})^{\boxplus}$. ▷

8.4. Theorem. *The following properties of $K \subset \mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}$ are equivalent:*

(a) *K is an Archimedean cone;*
(b) *K is a closed cone;*
(c) *K is a strictly closed cone;*
(d) *$K \cap \mathbb{R}^{\mathbb{N}}_n$ is a closed cone in $\mathbb{R}^{\mathbb{N}}_n$ for all $n \in \mathbb{N}$;*
(e) *$K = \bigcup_{n\in\mathbb{N}} K_n$ for some inductive sequence of closed cones $K_n \subset \mathbb{R}^{\mathbb{N}}_n$.*

◁ The implications (c)⇒(b)⇒(a)⇒(d)⇒(e) are evident (see 2.3(f)). Show (e)⇒(c). In (e), the union $K = \bigcup_{n\in\mathbb{N}} K_n$ is obviously an Archimedean cone. As is known, the strongest locally convex topology τ on $\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}$ is sequential (see [4, Prob. 12-3-113]). From 2.3(h) it follows that the Archimedean cone K is closed in τ and hence in the weak topology $\sigma(\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}} | \mathbb{R}^{\mathbb{N}})$. Further, the cones $\pi_n K_n$ are closed in $\mathbb{R}^n$ and thus they are strictly closed. From 6.8 it follows that $(\pi_n K_n)^{\boxplus} \neq \varnothing$ for all $n \in \mathbb{N}$. Invoking 8.3, we conclude that $K^{\boxplus} = \varprojlim (\pi_n K_n)^{\boxplus} \neq \varnothing$. ▷

8.5. The criterion below provides a solution to the problem this article is devoted to.

Theorem. *The following properties of a dense subspace $Y \subset \mathbb{R}^{\mathbb{N}}$ are equivalent:*

(a) *all Archimedean cones in $\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}} | Y$ are closed;*
(b) *all Archimedean cones in $\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}} | Y$ are strictly closed;*
(c) *Y is quasidense in $\mathbb{R}^{\mathbb{N}}$.*

If, contrary to (c)*, there exists a compact convex set $C \subset \mathbb{R}^{\mathbb{N}}$ such that $\mathrm{qi}\, C \neq \varnothing$ and $Y \cap C = \varnothing$* (see 8.8(b))*; then $C^{\oplus}$ is an example of an Archimedean but not closed* (moreover, dense) *cone in $\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}} | Y$.*

◁ According to 5.10, the space $\mathbb{R}^{\mathbb{N}}$ dual to $\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}$ is quasilocally bounded. Moreover, by 8.4, the classes of strictly closed and Archimedean cones in $\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}$ coincide. It remains to use 6.12 with $\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}$ as X. ▷

8.6. Owing to Theorems 2.5 and 8.5, we can now give some exhaustive description of the spaces in which all Archimedean cones are closed.

Theorem. *Given a Hausdorff locally convex space X, all Archimedean cones in X are closed if and only if X has finite or countable dimension and the dual space X' is quasidense in $X^{\#}$ with respect to the weak* topology.*

8.7. Lemma. *Let C be a projective convex subset of $\mathbb{R}^{\mathbb{N}}$ and let $x \in \mathrm{qi}\, C$. Put $D := \frac{1}{2}(C + x)$. Then $x \in \mathrm{qi}\, D$ and $\mathrm{cl}\, D \subset \mathrm{qi}\, C$.*

◁ We may assume that $x = 0$. The containment $0 \in \mathrm{qi}\, \frac{1}{2}C$ is obvious. Since $\mathrm{qi}\, C = \varprojlim \mathrm{int}\, \pi_n C$ (see 7.6), to prove that $\mathrm{cl}\, \frac{1}{2}C \subset \mathrm{qi}\, C$ it suffices to fix an arbitrary $n \in \mathbb{N}$ and show that $\pi_n\, \mathrm{cl}\, \frac{1}{2}C \subset \mathrm{int}\, \pi_n C$. According to 4.13, $0 \in \mathrm{qi}\, C$ implies $0 \in \mathrm{int}\, \pi_n C$. Hence, $\mathrm{cl}\, \frac{1}{2}\pi_n C \subset \mathrm{int}\, \pi_n C$ (see, for example, [1, 7.1.1(1)]). It remains to note that, by the continuity of the linear operator $\pi_n : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^n$, we have $\pi_n\, \mathrm{cl}\, \frac{1}{2}C \subset \mathrm{cl}\, \frac{1}{2}\pi_n C$. ▷

8.8. Proposition. *The following properties of $S \subset \mathbb{R}^{\mathbb{N}}$ are equivalent:*
(a) *S is quasidense in $\mathbb{R}^{\mathbb{N}}$;*
(b) *if C is a compact convex subset of $\mathbb{R}^{\mathbb{N}}$ and $\operatorname{qi} C \neq \varnothing$, then $S \cap C \neq \varnothing$;*
(c) *if B is a nonempty projective bounded quasiopen convex subset of $\mathbb{R}^{\mathbb{N}}$, then $S \cap B \neq \varnothing$;*
(d) *if $(B_n)_{n\in\mathbb{N}}$ is a projective sequence of nonempty bounded open convex sets, then $S \cap \varprojlim B_n \neq \varnothing$;*
(e) *if C is a projective convex subset of $\mathbb{R}^{\mathbb{N}}$ and $\operatorname{qi} C \neq \varnothing$, then $S \cap C \neq \varnothing$.*

◃ Using the compactness criterion in $\mathbb{R}^{\mathbb{N}}$ (see 3.4), we see that (a)⇔(b) is valid by definition 6.2(b).

(b)⇒(c): Let $B \subset \mathbb{R}^{\mathbb{N}}$ possess the properties listed in (c) and let $b \in B$. Put $C := \operatorname{cl} \frac{1}{2}(B + b)$. By 8.7, we have $\operatorname{qi} C \neq \varnothing$ and $C \subset B$. Since C is compact (see 3.4); (b) implies $S \cap C \neq \varnothing$ and, in particular, $S \cap B \neq \varnothing$.

The equivalence (c) ⇔ (d) is ensured by the projective invariance of the combination of the properties of B listed in (c) (see 7.7).

(c)⇒(e): Let C be a projective convex subset of $\mathbb{R}^{\mathbb{N}}$ and let $\operatorname{qi} C \neq \varnothing$. By 8.7, there is a closed convex set $D \subset \mathbb{R}^{\mathbb{N}}$ such that $\operatorname{qi} D \neq \varnothing$ and $D \subset C$. Further, owing to 5.10 and 5.4(b), there exists a compact convex subset $B \subset D$ with $\operatorname{qi} B \neq \varnothing$. Being closed, B is projective and, hence, according to 7.6, $\operatorname{qi} B$ is also projective. Moreover, $\operatorname{qi} B$ is quasiopen (see 4.10). Then (c) implies $S \cap B \neq \varnothing$, whence $S \cap C \neq \varnothing$ since $B \subset D \subset C$.

The implication (e)⇒(b) is obvious, since compactness implies closedness, and closedness implies projectivity. ▹

8.9. Proposition. *A set $S \subset \mathbb{R}^{\mathbb{N}}$ is quasidense in each of the cases listed below:*
(a) *S includes $\Lambda^{\mathbb{N}}$, where Λ is a dense subset of $\mathbb{R}$;*
(b) *S includes $\prod_{n\in\mathbb{N}} \Lambda_n$, where Λ_n are dense subsets of $\mathbb{R}$;*
(c) *S includes a projective subset P satisfying the following:*

$$\operatorname{cl} \{p(1) : p \in P\} = \mathbb{R}; \tag{11}$$

$$\operatorname{cl} \{p(n+1) : p \in P,\ \pi_n p = x\} = \mathbb{R} \quad \textit{for all } n \in \mathbb{N} \textit{ and } x \in \pi_n P. \tag{12}$$

◃ Conditions (a) and (b) are particular cases of (c), while (c) can easily be verified using 8.8(d). Indeed, let $P \subset \mathbb{R}^{\mathbb{N}}$ satisfy (11) and (12) and let $(B_n)_{n\in\mathbb{N}}$ be a projective sequence of nonempty open convex sets. Proceeding by recursion on $n \in \mathbb{N}$, construct a sequence of $b_n \in B_n \cap \pi_n P$ as follows: According to (11), there is $b_1 \in B_1 \cap \pi_1 P$. Assume that $b_n \in B_n \cap \pi_n P$ is defined. Since $B_n = \pi_n B_{n+1}$, there is a real $\lambda \in \mathbb{R}$ such that $\big(b_n(1), \dots, b_n(n), \lambda\big) \in B_{n+1}$. Owing to the openness of B_{n+1}, there exists a real $\varepsilon > 0$ such that

$$\big(b_n(1), \dots, b_n(n), \mu\big) \in B_{n+1} \quad \text{for all } \mu \in \left]\lambda - \varepsilon, \lambda + \varepsilon\right[.$$

From (12) it follows that there is $p \in P$ satisfying the conditions $\pi_n p = b_n$ and $p(n+1) \in \left]\lambda - \varepsilon, \lambda + \varepsilon\right[$. Put $b_{n+1} := \pi_{n+1} p$. Then $(b_n)_{n\in\mathbb{N}}$ is a chain both in $(B_n)_{n\in\mathbb{N}}$ and in $(\pi_n P)_{n\in\mathbb{N}}$, which means that $\bigcup_{n\in\mathbb{N}} b_n$ belongs to both $\varprojlim B_n$ and P. ▹

8.10. In [3], the term *subtle spaces* is used for the dense subspaces $Y \subset \mathbb{R}^{\mathbb{N}}$ that admit existence of nonclosed Archimedean cones in $\mathbb{R}^{\mathbb{N}}_{\text{fin}} | Y$. (Owing to 8.5, we know now that the subtle spaces are exactly the dense subspaces that are not quasidense.) The following fact confirms conjecture [3, 4.9] and also gives a (negative) answer to the question in [3] and [5] of whether all proper subspaces of $\mathbb{R}^{\mathbb{N}}$ are subtle:

Corollary. *All Archimedean cones in $\mathbb{R}^{\mathbb{N}}_{\text{fin}} | \operatorname{lin} \mathbb{Q}^{\mathbb{N}}$ and $\mathbb{R}^{\mathbb{N}}_{\text{fin}} | \operatorname{lin} \mathbb{N}^{\mathbb{N}}$ are closed.*

◃ Obviously, $\operatorname{lin} \mathbb{Q}^{\mathbb{N}} \subset \mathbb{R}^{\mathbb{N}}$ meets 8.9(a). The same is true of $\operatorname{lin} \mathbb{N}^{\mathbb{N}}$. Indeed, by Kronecker's Approximation Theorem, $\Lambda := \{m\sqrt{2} + n \,:\, m, n \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. It remains to note that

$$\operatorname{lin} \mathbb{N}^{\mathbb{N}} = \operatorname{lin} (\mathbb{N}^{\mathbb{N}} - \mathbb{N}^{\mathbb{N}}) = \operatorname{lin} (\mathbb{N} - \mathbb{N})^{\mathbb{N}} = \operatorname{lin} \mathbb{Z}^{\mathbb{N}} \supset \sqrt{2}\, \mathbb{Z}^{\mathbb{N}} + \mathbb{Z}^{\mathbb{N}} = (\sqrt{2}\, \mathbb{Z} + \mathbb{Z})^{\mathbb{N}} = \Lambda^{\mathbb{N}}. \ ▹$$

8.11. Theorem 8.5 makes it possible to obtain a short demonstration for the main result of [5].

Corollary. *If $L \in (\mathbb{R}^{\mathbb{N}})^{\#}$ and $L(s) = \lim_{n\to\infty} s(n)$ for convergent sequences $s \in \mathbb{R}^{\mathbb{N}}$; then $\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}\big|\ker L$ contains a nonclosed Archimedean cone.*

◁ Put $C := \prod_{n\in\mathbb{N}} \left[1 - \frac{1}{n}, 1 + \frac{1}{n}\right]$. Then $\ker L \cap C = \varnothing$. By 8.5, 7.8, and 8.8, the latter implies that $\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}\big|\ker L$ contains a nonclosed Archimedean cone and, moreover, $C^{\oplus}$ is an example of such cone. ▷

§ 9. Open Questions

In conclusion, we formulate several natural questions that remain open by now.

9.1. Quasilocal boundedness. In Section 5, we only undertake an initial study of the concept of quasilocal boundedness. Within a more detailed study, it would be appropriate to clarify relationship between the following properties of a Hausdorff locally convex space X and, in particular, to find out which of them are equivalent:

(a) X is quasilocally bounded;

(b) every convex subset of X with nonempty quasi-interior contains a bounded subset with nonempty quasi-interior;

(c) there exists a bounded set in X with nonempty quasi-interior;

(d) there exists a dense subnormable subspace in X;

(e) for every dense wedge $W \subset X$, there exists a subnormable subspace $Z \subset X$ such that $Z \cap W$ is dense in X;

(f) every dense subspace of X contains a dense subnormable subspace.

The following implications seem obvious: (a)⇒(b)⇒(c)⇔(d), (e)⇒(f)⇒(d), (e)⇒(a).

9.2. Limits of quasilocally bounded spaces. The space $\mathbb{R}^{\mathbb{N}}\big|\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}$ is the inverse limit $\varprojlim \mathbb{R}^n$ of the sequence $(\mathbb{R}^n)_{n\in\mathbb{N}}$ within the category of locally convex spaces under the natural projections $\pi_n : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^n$ and $\pi_{nm} : \mathbb{R}^m \to \mathbb{R}^n$ $(n \leqslant m)$. Analysis of the proofs of Theorems 4.13 and 5.10 makes it possible to propose the conjecture that the statements of those theorems can be generalized to the case of arbitrary inverse limits. Namely, let $\big(X, (\pi_i)_{i\in I}\big)$ be the inverse limit of a net $(X_i)_{i\in I}$ of locally convex spaces. Given a convex set C in X, does the following equality hold:

$$\operatorname{qi} C = \{c \in C : \pi_i c \in \operatorname{qi} \pi_i C \text{ in } X_i \text{ for all } i \in I\}?$$

If X_i are quasilocally bounded, does their inverse limit $\varprojlim X_i$ have the same property?

9.3. Closed total wedges. Theorem 6.11 and Corollary 6.12 offer criteria for the inheritance of strict closedness of cones in a space with quasilocally bounded dual. From the proof of 6.11, it is clear that the claim remains valid if the quasilocal boundedness is weakened to the requirement that the wedges of the form $K^{\oplus}$ be quasilocally bounded at their quasi-interior points, where K is a closed cone in the dual space. Presented below is a description of such wedges which follows from [2, 2.13].

A subset S of a Hausdorff locally convex space X is *total* if $\operatorname{cl}\operatorname{lin} S = X$ (see [4, 2-3-12]).

Proposition. *The following properties of $W \subset X$ are equivalent:*

(a) *W is a closed total wedge;*

(b) *W is a closed wedge and $W^{\oplus}$ is a cone;*

(c) *$W^{\oplus}$ is a cone and $W^{\oplus\oplus} = W$;*

(d) *$W = K^{\oplus}$ for some closed cone $K \subset X'$.*

Therefore, the statements of 6.11 and 6.12 remain valid for the respective spaces Z and X' in which all closed total wedges are quasilocally bounded at their quasi-interior points. Is the latter condition equivalent to the quasilocal boundedness of the space?

9.4. Strictly closed cones. Denote by $\triangledown(X)$ and $\triangledown_{\rm s}(X)$ the sets of closed and strictly closed cones in a locally convex Hausdorff space X. If X' is quasilocally bounded then, according to 6.12, for each subspace $Y \subset X'$ we have

$$Y \text{ is quasidense} \;\Leftrightarrow\; \triangledown_{\rm s}(X) \subset \triangledown(X|Y) \;\Leftrightarrow\; \triangledown_{\rm s}(X) = \triangledown_{\rm s}(X|Y).$$

How do the equalities $\triangledown(X) = \triangledown(X|Y)$ and $\triangledown(X|Y) = \triangledown_{\rm s}(X|Y)$ relate to the above statements?

9.5. Strictly closed cones in $\mathbb{R}^{\mathbb{N}}$. A similar question arises in connection with Theorem 8.5. Let Y be a subspace of $\mathbb{R}^{\mathbb{N}}$. Since closed cones are Archimedean, according to Theorem 8.5, the following implication holds:

$$Y \text{ is quasidense} \;\Rightarrow\; \triangledown\big(\mathbb{R}^{\mathbb{N}}_{\rm fin}|Y\big) = \triangledown_{\rm s}\big(\mathbb{R}^{\mathbb{N}}_{\rm fin}|Y\big).$$

Is the converse true? In other words, is it possible to supplement the list 8.5(a)–(c) with the claim that all closed cones in $\mathbb{R}^{\mathbb{N}}_{\rm fin}|Y$ are strictly closed?

9.6. Polars of cones. If $K_n \subset \mathbb{R}^{\mathbb{N}}_n$ $(n \in \mathbb{N})$ is an inductive sequence of closed cones then, by 8.3, the sequence of the polars $(\pi_n K_n)^{\boxplus}$ is projective. Is the sequence of the dual wedges $(\pi_n K_n)^{\oplus}$ projective in this situation?

9.7. Quasidenseness and projectivity. According to 8.8, the quasidenseness of a set $S \subset \mathbb{R}^{\mathbb{N}}$ is equivalent to either of the following conditions:

(a) S intersects every nonempty convex set $B \subset \mathbb{R}^{\mathbb{N}}$ that is bounded, quasiopen, and *projective*;

(b) S intersects every convex set $C \subset \mathbb{R}^{\mathbb{N}}$ that has nonempty quasi-interior and is *projective*.

Is the requirement that B and C be projective essential under the above two conditions? If we remove the projectivity, do the conditions remain equivalent to the quasidenseness of S provided that S is a dense vector subspace of $\mathbb{R}^{\mathbb{N}}$?

9.8. Examples of quasidense spaces. Call $S \subset \mathbb{R}^{\mathbb{N}}$ *exponentially dense*, or *Cartesian dense*, or *recursively dense* whenever S possesses the respective property (a), or (b), or (c) in 8.9. The three properties are connected by the obvious implications (a)⇒(b)⇒(c) and, according to 8.9, imply quasidenseness. Currently, the short list of available examples of quasidense subspaces of $\mathbb{R}^{\mathbb{N}}$ includes only spaces that are exponentially dense in $\mathbb{R}^{\mathbb{N}}$ (see 8.10). In this connection, three natural questions arise about the existence of quasidense subspaces $Y \subset \mathbb{R}^{\mathbb{N}}$ such that

(a) Y is Cartesian dense, but not exponentially dense;

(b) Y is recursively dense, but not Cartesian dense;

(c) Y is not recursively dense.

9.9. Box density. By 8.8, a subset of $\mathbb{R}^{\mathbb{N}}$ is quasidense if and only if it intersects every nonempty projective quasiopen convex set B. Since the totality of such sets B includes the open boxes (9), that form a base of the box topology on $\mathbb{R}^{\mathbb{N}}$; it follows that every quasidense set is box-dense. Is the converse true? Does there exist a dense subspace $Y \subset \mathbb{R}^{\mathbb{N}}$ that is box-dense but not quasidense?

9.10. Topological nature of quasidenseness. If the answer to the previous question turns out negative, and box-denseness is not equivalent to quasidenseness; then is there any other topology τ on $\mathbb{R}^{\mathbb{N}}$ such that τ-denseness is equivalent to quasidenseness? Is quasidenseness topological for dense subspaces of $\mathbb{R}^{\mathbb{N}}$?

9.11. Cones and linearly independent sets. It is easy to show that existence of a nonclosed linearly independent set in a Hausdorff locally convex space implies existence of a nonclosed Archimedean cone in the space (see [3, 4.7]). Is the converse assertion true? Does there exist a dense subspace $Y \subset \mathbb{R}^{\mathbb{N}}$ such that all linearly independent sets in $\mathbb{R}^{\mathbb{N}}_{\rm fin}|Y$ are closed, but $\mathbb{R}^{\mathbb{N}}_{\rm fin}|Y$ contains nonclosed Archimedean cones?

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