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A sentence preservation theorem for Boolean algebras

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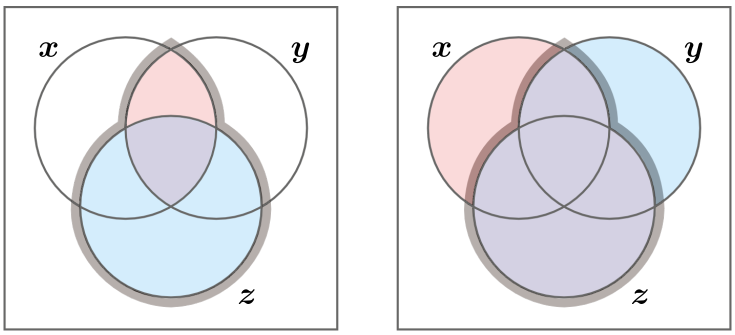
Abstract. At the initial stages of studying the theory of Boolean algebras, before trying to prove or disprove any simple sentence, students are usually asked to test their intuition using Venn diagrams or truth tables. A natural question arises: is it necessary to invent a proof after a positive check of this kind? Isn't such a check itself a rigorous proof of the verified sentence? And if this is not true in the general case, for which sentences is this true? We answer the question and prove an analog of the Jech Theorem for arbitrary (not necessarily complete) Boolean algebras.

Keywords: Boolean algebra, Venn diagram, truth table, Horn formula.

To Professor Anatoly G. Kusraev
on the occasion of his 70th birthday

Practical classes in the study of the theory of Boolean algebras usually begin with exercises that do not go far from elementary logic and simple set-theoretic identities. For instance, the axiomatics of Boolean algebras includes the distributivity law $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$. What about the dual statement $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$: is it provable, contradictory, or independent? When considering such hypotheses, students are traditionally asked to test their intuition with the help of Venn diagrams.

The basis of a Venn diagram is a set of figures depicted in a general position: so that the picture contains areas for all possible intersections and complements. Each figure corresponds to a variable, and if the hypothesis contains n variables, the diagram should contain 2^n elementary areas. To test an identity, we draw a diagram for each term, fill in the areas corresponding to subterms, and compare the results.



$(\boldsymbol{x} \wedge \boldsymbol{y}) \vee \boldsymbol{z}$ $\qquad$ $(\boldsymbol{x} \vee \boldsymbol{z}) \wedge (\boldsymbol{y} \vee \boldsymbol{z})$

If the results coincide, there is a chance that the identity is provable, and one can try to deduce it from the axioms.

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Venn diagrams are illustrative and effective at testing atomic identities, but they are not suitable for more complex statements that contain logical connectives. Venn diagrams are unlikely to help, for example, when testing the uniqueness of a complement:

$$x \wedge y = 0,\;\; x \vee y = 1,\;\; x \wedge z = 0,\;\; x \vee z = 1 \;\;\Rightarrow\;\; y = z. \tag{i}$$

In such cases, truth tables come to the rescue. When a Venn diagram transforms into a truth table, its figures become columns and elementary areas become rows. Each of the 2^n rows corresponds to a binary valuation of the n variables. To test an assertion, we add columns for the subformulas, check the cells for which the hypotheses are true, and test if the conclusion is true for the rows in which all the hypotheses are checked.

$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{z}$	$x\wedge y=0$	$x\vee y=1$	$x\wedge z=0$	$x\vee z=1$	$\boldsymbol{y=z}$
0	0	0	✓		✓		✓
1	0	0	✓	✓	✓	✓	✓
0	1	0	✓	✓	✓		
0	0	1	✓		✓	✓	
1	1	0		✓	✓	✓	
1	0	1	✓	✓		✓	
0	1	1	✓	✓	✓	✓	✓
1	1	1		✓		✓	✓

As it is seen from the above picture, the truth table confirms hypothesis (i). But does this kind of testing ensure the provability of the hypothesis? A positive test of a sentence φ using a truth table or Venn diagrams means exactly the validity $2 \vDash \varphi$ of the sentence in the simplest Boolean algebra $2 = \{0, 1\}$. On the other hand, according to the Soundness and Completeness theorems, the provability $\mathrm{BA} \vdash \varphi$ in the theory BA of Boolean algebras amounts to the validity $A \vDash \varphi$ in every Boolean algebra A. Therefore, it is clear that, in general, $2 \vDash \varphi$ does not imply $\mathrm{BA} \vdash \varphi$; and the sentence $(\forall x)(x = 0 \,\vee\, x = 1)$ is an obvious counterexample. In this regard, the following natural problem arises, the solution of which is the subject of this article:

Problem. *Describe a wide class of sentences φ for which $2 \vDash \varphi$ implies $\mathrm{BA} \vdash \varphi$.*

Results of this kind are usually called *sentence preservation theorems* in the literature. There are several classical papers that contain sentence preservation theorems related to Boolean algebras. The earliest are [1] and [2] in which term equality sentences of the form

$$\sigma_1 = \tau_1 \;\wedge\; \cdots \;\wedge\; \sigma_n = \tau_n \;\Rightarrow\; \sigma = \tau$$

are proven to be preserved in universally complete and Archimedean vector lattices (see IV.1.43, IV.3.11, XIII.3.21 in [1] and Theorem V.7.2 by A. I. Yudin presented in [2]). A significantly richer class of sentences is considered in [3], where a preservation theorem is proven for Horn formulas in universally complete vector lattices (see Theorem 5 therein). The most general transfer principle can be derived from the Jech Theorem [4, Theorem 2; 5, 4.3.9(1)], which justifies preservation of Horn sentences in the descents of algebraic systems from the Boolean-valued universe, i.e., in arbitrary universally complete algebraic Boolean-valued systems [5, 4.2].

Since every complete Boolean algebra A is isomorphic to the descent $2{\downarrow}$ of the two-element Boolean algebra $2 = \{0, 1\}$ from the A-valued universe $\mathbb{V}^{(A)}$ (see [5, 4.2.2]); the Jech Theorem ensures preservation of Horn sentences in complete Boolean algebras. However, this does not solve the above-stated problem, which concerns arbitrary, not necessarily complete, Boolean algebras. Moreover, as soon as the problem naturally arises at the very beginning of any course on the theory of Boolean algebras, we prefer an elementary approach that does not involve such advanced techniques as Boolean-valued analysis.

Any sentence preservation theorem is based on some representation of abstract algebraic systems, be it Boolean-valued, functional, topological, or set-theoretic representation. In the case under consideration, the basis is the Stone Theorem, according to which every Boolean algebra is isomorphic to an algebra of subsets. Our consideration is thus focused on algebras of that particular kind.

Throughout the article, B is the Boolean algebra $\mathcal{P}(S)$ of all subsets of a nonempty set S. We conventionally identify $\mathcal{P}(S)$ with 2^S by equating subsets of S and their characteristic functions. Therefore,

$$s \in b \ \Leftrightarrow \ b(s) = 1$$

whenever $b \in B$ and $s \in S$.

We consider the language of the theory of Boolean algebras to be the first-order language of the following signature:

$$\Sigma_{\mathrm{BA}} \ = \ \{=, \leqslant, \wedge, \vee, \neg, \Rightarrow, \Leftrightarrow, 0, 1\}.$$

When interpreted in any Boolean algebra, the symbol $\neg$ stands for the complement; the implication operation $a \Rightarrow b$ is defined as $\neg a \vee b$; and the equivalence operation $a \Leftrightarrow b$, as $(a \Rightarrow b) \wedge (b \Rightarrow a)$.

Symbols of the form $\ddot{x}$ are used to denote finite lists of variables $x_1, \dots, x_n$, with $n \geqslant 0$. When writing a formula φ initially as $\varphi(x_1, \dots, x_n)$ or $\varphi(\ddot{x})$, we assume that $x_1, \dots, x_n$ are pairwise different variables that include all the parameters of φ; i.e., all variables with free occurrences in $\varphi(\ddot{x})$ are supposed to be in the list $\ddot{x}$. The same convention is adopted when we write terms τ as $\tau(x_1, \dots, x_n)$ or $\tau(\ddot{x})$. The lengths of variable lists are usually not specified and are guessed from the context. For instance, if expressions $\varphi(\ddot{x})$, $\psi(\ddot{y})$, $\varphi(\ddot{a})$, $\psi(\ddot{b})$ occur in the same statement, the lists lengths are supposed to be coordinated as follows: $\ddot{x} = x_1, \dots, x_m$, $\ddot{y} = y_1, \dots, y_n$, $\ddot{a} = a_1, \dots, a_m$, $\ddot{b} = b_1, \dots, b_n$. When considering a list $\ddot{b} = b_1, \dots, b_n$, we abbreviate $\varphi(b_1(s), \dots, b_n(s))$ as $\varphi(\ddot{b}(s))$.

Definition 1. Let $\varphi(\ddot{x})$ be an arbitrary formula of signature Σ_{BA}. Given $\ddot{b} \in B$, define the element $[\![\varphi(\ddot{b})]\!] \in B$ by putting

$$[\![\varphi(\ddot{b})]\!] \ = \ \big\{ s \in S \ : \ 2 \vDash \varphi(\ddot{b}(s)) \big\}.$$

The following is obvious:

Lemma 1. *Let $\varphi(\ddot{x})$ and $\psi(\ddot{y})$ be formulas of signature Σ_{BA}. Then, for all $\ddot{a}, \ddot{b} \in B$, we have*

$$[\![\varphi(\ddot{a}) \wedge \psi(\ddot{b})]\!] \ = \ [\![\varphi(\ddot{a})]\!] \wedge [\![\psi(\ddot{b})]\!]; \quad [\![\varphi(\ddot{a}) \Rightarrow \psi(\ddot{b})]\!] \ = \ [\![\varphi(\ddot{a})]\!] \Rightarrow [\![\psi(\ddot{b})]\!];$$
$$[\![\varphi(\ddot{a}) \vee \psi(\ddot{b})]\!] \ = \ [\![\varphi(\ddot{a})]\!] \vee [\![\psi(\ddot{b})]\!]; \quad [\![\varphi(\ddot{a}) \Leftrightarrow \psi(\ddot{b})]\!] \ = \ [\![\varphi(\ddot{a})]\!] \Leftrightarrow [\![\psi(\ddot{b})]\!];$$
$$[\![\neg\varphi(\ddot{a})]\!] \ = \ \neg[\![\varphi(\ddot{a})]\!].$$

Lemma 2. *Let $\varphi(\ddot{x})$ and $\psi(\ddot{x})$ be formulas of signature Σ_{BA}. If $2 \vDash (\forall \ddot{x})(\varphi(\ddot{x}) \Leftrightarrow \psi(\ddot{x}))$, then $[\![\varphi(\ddot{b})]\!] = [\![\psi(\ddot{b})]\!]$ for all $\ddot{b} \in B$.*

$\lhd$ For all $\ddot{b} \in B$, we have $[\![\varphi(\ddot{b})]\!] = \big\{ s \in S : 2 \vDash \varphi(\ddot{b}(s)) \big\} = \big\{ s \in S : 2 \vDash \psi(\ddot{b}(s)) \big\} = [\![\psi(\ddot{b})]\!]$. $\rhd$

Lemma 3. *Let $\tau(\ddot{x})$ be a term of signature Σ_{BA}. Then $[\![\tau(\ddot{b}) = 1]\!] = \tau(\ddot{b})$ for all $\ddot{b} \in B$.*

$\lhd$ We proceed by induction on the complexity of τ.

Induction base. The simplest term is the constant 0, the constant 1, or a variable. The cases of 0 and 1 are trivial, while for the case in which $\tau(x)$ is x we have

$$[\![\tau(b) = 1]\!] \ = \ \big\{ s \in S : 2 \vDash \tau(b(s)) = 1 \big\} \ = \ \big\{ s \in S : \tau(b(s)) = 1 \big\}$$
$$= \ \big\{ s \in S : b(s) = 1 \big\} \ = \ \big\{ s \in S : s \in b \big\} \ = \ b \ = \ \tau(b)$$

for all $b \in B$.

Induction step. Consider the case in which $\tau(\ddot{x}, \ddot{y})$ is $\tau_1(\ddot{x}) \wedge \tau_2(\ddot{y})$, where τ_1 and τ_2 meet the equality under proof. Then, employing Lemmas 1 and 2, we deduce

$$[\![\tau(\ddot{a}, \ddot{b}) = 1]\!] \ = \ [\![\tau_1(\ddot{a}) \wedge \tau_2(\ddot{b}) = 1]\!] \ = \ [\![\tau_1(\ddot{a}) = 1 \ \wedge \ \tau_2(\ddot{b}) = 1]\!]$$
$$= \ [\![\tau_1(\ddot{a}) = 1]\!] \wedge [\![\tau_2(\ddot{b}) = 1]\!] \ = \ \tau_1(\ddot{a}) \wedge \tau_2(\ddot{b}) \ = \ \tau(\ddot{a}, \ddot{b})$$

for all $\ddot{a}, \ddot{b} \in B$. The rest of the cases are treated similarly. $\rhd$

Lemma 4. *Let $\sigma(\ddot{x})$ and $\tau(\ddot{y})$ be terms of signature Σ_{BA}. Then, for all $\ddot{a}, \ddot{b} \in B$, we have*

$$\begin{aligned} [\![\sigma(\ddot{a}) = \tau(\ddot{b})]\!] &= (\sigma(\ddot{a}) \Leftrightarrow \tau(\ddot{b})); \\ [\![\sigma(\ddot{a}) \leqslant \tau(\ddot{b})]\!] &= (\sigma(\ddot{a}) \Rightarrow \tau(\ddot{b})). \end{aligned}$$

$\lhd$ Owing to Lemma 3 we conclude that

$$[\![\sigma(\ddot{a}) = \tau(\ddot{b})]\!] = \{s \in S : 2 \vDash \sigma(\ddot{a}(s)) = \tau(\ddot{b}(s))\}$$
$$= \big\{s \in S : 2 \vDash \big(\sigma(\ddot{a}(s)) \Leftrightarrow \tau(\ddot{b}(s))\big) = 1\big\} = [\![(\sigma(\ddot{a}) \Leftrightarrow \tau(\ddot{b})) = 1]\!] = (\sigma(\ddot{a}) \Leftrightarrow \tau(\ddot{b})).$$

Similar arguments justify the second equality. $\rhd$

Definition 2. Given a subset $A \subseteq B$, denote by $B\langle A\rangle$ the Boolean subalgebra of B generated by A. If $b_1, \dots, b_n \in B$, we write $B\langle b_1, \dots, b_n\rangle$ instead of $B\langle \{b_1, \dots, b_n\}\rangle$.

The following is obvious:

Lemma 5. *Let $\tau(\ddot{x})$ be a term of signature Σ_{BA}. Then $\tau(\ddot{b}) \in B\langle \ddot{b}\rangle$ for all $\ddot{b} \in B$.*

Theorem 1. *Let $\varphi(x_1, \dots, x_n)$ be a formula of signature Σ_{BA}. Then*
(1) $[\![\varphi(b_1, \dots, b_n)]\!] \in B\langle b_1, \dots, b_n\rangle$ for all $b_1, \dots, b_n \in B$;
(2) for all $b_2, \dots, b_n \in B$ there exists $a \in B\langle b_2, \dots, b_n\rangle$ such that

$$[\![(\exists x)\, \varphi(x, b_2, \dots, b_n)]\!] = [\![\varphi(a, b_2, \dots, b_n)]\!].$$

Moreover, $[\![\varphi(1, b_2, \dots, b_n)]\!]$ is suitable for the role of a in (2).

$\lhd$ Prove first that (1) implies (2) for each φ. Consider $\ddot{c} = b_2, \dots, b_n \in B$ and put $a = [\![\varphi(1, \ddot{c})]\!]$. According to (1) we have $a \in B\langle 1, \ddot{c}\rangle = B\langle \ddot{c}\rangle$. Show that $[\![\varphi(a, \ddot{c})]\!] = [\![(\exists x)\, \varphi(x, \ddot{c})]\!]$. For all $s \in S$ we have

$$s \in [\![\varphi(a, \ddot{c})]\!] \;\Rightarrow\; 2 \vDash \varphi(a(s), \ddot{c}(s)) \;\Rightarrow\; (\exists\, t \in 2)\ 2 \vDash \varphi(t, \ddot{c}(s))$$
$$\Rightarrow\; 2 \vDash (\exists x)\, \varphi(x, \ddot{c}(s)) \;\Rightarrow\; s \in [\![(\exists x)\, \varphi(x, \ddot{c})]\!].$$

Conversely,

$$s \in [\![(\exists x)\, \varphi(x, \ddot{c})]\!] \;\Rightarrow\; 2 \vDash (\exists x)\, \varphi(x, \ddot{c}(s))$$
$$\Rightarrow\; (\exists\, t \in 2)\ 2 \vDash \varphi(t, \ddot{c}(s)) \;\Rightarrow\; 2 \vDash \varphi(0, \ddot{c}(s)) \,\vee\, 2 \vDash \varphi(1, \ddot{c}(s)).$$

If $2 \vDash \varphi(1, \ddot{c}(s))$, then $s \in [\![\varphi(1, \ddot{c})]\!] = a$, i.e., $a(s) = 1$; therefore, $2 \vDash \varphi(a(s), \ddot{c}(s))$ and so $s \in [\![\varphi(a, \ddot{c})]\!]$. Otherwise, if $2 \nvDash \varphi(1, \ddot{c}(s))$, then $s \notin [\![\varphi(1, \ddot{c})]\!] = a$, i.e., $a(s) = 0$; and in this case we have $2 \vDash \varphi(0, \ddot{c}(s))$, which implies $2 \vDash \varphi(a(s), \ddot{c}(s))$ and again $s \in [\![\varphi(a, \ddot{c})]\!]$.

Armed with the implication (1)$\Rightarrow$(2) proven above, we now proceed to prove (1) by induction on the complexity of φ.

Induction base. Every atomic formula has the form $\sigma(\ddot{x}) = \tau(\ddot{y})$ or $\sigma(\ddot{x}) \leqslant \tau(\ddot{y})$, where σ and τ are terms. According to Lemmas 4 and 5, for all $\ddot{a}, \ddot{b} \in B$, we have

$$\begin{aligned} [\![\sigma(\ddot{a}) = \tau(\ddot{b})]\!] &= (\sigma(\ddot{a}) \Leftrightarrow \tau(\ddot{b})) \in B\langle \ddot{a}, \ddot{b}\rangle; \\ [\![\sigma(\ddot{a}) \leqslant \tau(\ddot{b})]\!] &= (\sigma(\ddot{a}) \Rightarrow \tau(\ddot{b})) \in B\langle \ddot{a}, \ddot{b}\rangle. \end{aligned}$$

Induction step. Consider a nonatomic formula φ; assume that (1) holds for all formulas of lesser complexity; and show that (1) holds for φ.

Examine the case in which $\varphi(\ddot{x})$ has the form $\varphi_1(\ddot{x}) \wedge \varphi_2(\ddot{x})$. According to Lemma 1 we have

$$[\![\varphi(\ddot{b})]\!] = [\![\varphi_1(\ddot{b}) \wedge \varphi_2(\ddot{b})]\!] = [\![\varphi_1(\ddot{b})]\!] \wedge [\![\varphi_2(\ddot{b})]\!]$$

for all $\ddot{b} \in B$. By the induction hypothesis, $[\![\varphi_1(\ddot{b})]\!], [\![\varphi_2(\ddot{b})]\!] \in B\langle \ddot{b}\rangle$, and so $[\![\varphi(\ddot{b})]\!] \in B\langle \ddot{b}\rangle$. The rest of the propositional cases $\varphi_1 \vee \varphi_2$, $\neg\varphi_1$, $\varphi_1 \Rightarrow \varphi_2$, a.o., are treated similarly.

Consider the case in which $\varphi(\ddot{y})$ is $(\exists x)\,\psi(x,\ddot{y})$. Let $\ddot{b}\in B$. By the induction hypothesis, assertion (1) holds for ψ. Since (1) implies (2), there is $a\in B\langle\ddot{b}\rangle$ such that

$$[\![\varphi(\ddot{b})]\!] \;=\; [\![(\exists x)\,\psi(x,\ddot{b})]\!] \;=\; [\![\psi(a,\ddot{b})]\!].$$

The validity of (1) for ψ and the containment $a\in B\langle\ddot{b}\rangle$ yield

$$[\![\varphi(\ddot{b})]\!] \;=\; [\![\psi(a,\ddot{b})]\!] \;\in\; B\langle a,\ddot{b}\rangle \;=\; B\langle\ddot{b}\rangle.$$

Now, let $\varphi(\ddot{y})$ be of the form $(\forall x)\,\psi(x,\ddot{y})$. Consider $\ddot{b}\in B$. Since

$$2 \vDash (\forall\,\ddot{y})\big((\forall x)\,\psi(x,\ddot{y}) \Leftrightarrow \neg(\exists x)\,\neg\psi(x,\ddot{y})\big),$$

by Lemma 2 we have $[\![\varphi(\ddot{b})]\!]=[\![\neg(\exists x)\,\neg\psi(x,\ddot{b})]\!]$. By the induction hypothesis, (1) holds for ψ and hence for $\neg\psi$ due to Lemma 1. Since (1) implies (2), there is $a\in B\langle\ddot{b}\rangle$ such that $[\![(\exists x)\,\neg\psi(x,\ddot{b})]\!]=[\![\neg\psi(a,\ddot{b})]\!]$, and so

$$[\![(\exists x)\,\neg\psi(x,\ddot{b})]\!] \;\in\; B\langle a,\ddot{b}\rangle \;=\; B\langle\ddot{b}\rangle.$$

Employing Lemma 1, we conclude that

$$[\![\varphi(\ddot{b})]\!] \;=\; [\![\neg(\exists x)\,\neg\psi(x,\ddot{b})]\!] \;=\; \neg[\![(\exists x)\,\neg\psi(x,\ddot{b})]\!] \;\in\; B\langle\ddot{b}\rangle.\ \rhd$$

Definition 3. Given any signature, the notions of *syntactically strongly generic* and *syntactically generic formulas* are introduced by means of the following recursive definition [4, Section 4]:

(1) every atomic formula is syntactically strongly generic;
(2) if φ and ψ are syntactically strongly generic, then so are $\varphi\wedge\psi$, $(\exists x)\,\varphi$, and $(\forall x)\,\varphi$;
(3) every syntactically strongly generic formula is syntactically generic;
(4) if φ and ψ are syntactically generic, then so are $\varphi\wedge\psi$, $(\exists x)\,\varphi$, and $(\forall x)\,\varphi$;
(5) if φ is syntactically strongly generic, then $\neg\varphi$ is syntactically generic;
(6) if φ is syntactically strongly generic and ψ is syntactically generic, then $\varphi\Rightarrow\psi$ is syntactically generic.

A formula is (*strongly*) *generic* if it is logically equivalent to a syntactically (strongly) generic formula.

Definition 4. Given any signature, the notion of *syntactically Horn formula* is introduced by means of the following recursive definition [6, 6.2; 4, Section 4]:

(1) if $\alpha,\alpha_1,\dots,\alpha_n$ are atomic formulas, then α, $\neg(\alpha_1\wedge\dots\wedge\alpha_n)$, and $\alpha_1\wedge\dots\wedge\alpha_n\Rightarrow\alpha$ are syntactically Horn formulas;
(2) if φ and ψ are syntactically Horn formulas, then so are $\varphi\wedge\psi$, $(\exists x)\,\varphi$, and $(\forall x)\,\varphi$.

Horn formulas are those logically equivalent to syntactically Horn formulas.

The following can be proven by induction on formula complexity [4, Section 4]:

Proposition 1. *The class of generic formulas coincides with that of Horn formulas.*

The following theorem is analogous to the Jech Theorem on the relationship between validity in Boolean-valued and two-valued systems [4, Theorem 2; 5, 4.3.9(1)]:

Theorem 2. *Let $\varphi(x_1,\dots,x_n)$ be a formula of signature Σ_{BA}. Consider an arbitrary Boolean subalgebra $A\subseteq B$ and any elements $a_1,\dots,a_n\in A$.*

(1) *If φ is strongly generic, then* $[\![\varphi(a_1,\dots,a_n)]\!]=1 \Leftrightarrow A\vDash\varphi(a_1,\dots,a_n)$.
(2) *If φ is generic, then* $[\![\varphi(a_1,\dots,a_n)]\!]=1 \Rightarrow A\vDash\varphi(a_1,\dots,a_n)$.

$\lhd$ (1): Owing to Lemma 2 we may assume φ syntactically strongly generic and proceed by induction on the complexity of φ.

Induction base. Every atomic formula has the form $\sigma(\ddot{x})=\tau(\ddot{y})$ or $\sigma(\ddot{x})\leqslant\tau(\ddot{y})$, where σ and τ are terms. Employing Lemma 4, for all $\ddot{a},\ddot{b}\in A$, we derive

$$[\![\sigma(\ddot{a})=\tau(\ddot{b})]\!]=1 \Leftrightarrow (\sigma(\ddot{a})\Leftrightarrow\tau(\ddot{b}))=1 \Leftrightarrow \sigma(\ddot{a})=\tau(\ddot{b}) \Leftrightarrow A\vDash(\sigma(\ddot{a})=\tau(\ddot{b})),$$
$$[\![\sigma(\ddot{a})\leqslant\tau(\ddot{b})]\!]=1 \Leftrightarrow (\sigma(\ddot{a})\Rightarrow\tau(\ddot{b}))=1 \Leftrightarrow \sigma(\ddot{a})\leqslant\tau(\ddot{b}) \Leftrightarrow A\vDash(\sigma(\ddot{a})\leqslant\tau(\ddot{b})).$$

Induction step. Consider a nonatomic syntactically strongly generic formula φ; assume that

$$(\forall\, \ddot{b} \in A)\big(\llbracket \psi(\ddot{b})\rrbracket = 1 \;\Leftrightarrow\; A \vDash \psi(\ddot{b})\big) \tag{ii}$$

whenever $\psi(\ddot{y})$ is a proper subformula of φ; and show that (ii) holds for $\psi = \varphi$.

Let $\varphi(\ddot{x})$ be $\psi_1(\ddot{x}) \wedge \psi_2(\ddot{x})$. Given $\ddot{b} \in A$, by the induction hypothesis (ii), we have

$$\llbracket \psi_i(\ddot{b})\rrbracket = 1 \;\Leftrightarrow\; A \vDash \psi_i(\ddot{b}), \quad i = 1, 2.$$

Taking account of Lemma 1, we derive

$$\begin{gathered}\llbracket \psi_1(\ddot{b}) \wedge \psi_2(\ddot{b})\rrbracket = 1 \;\Leftrightarrow\; \llbracket \psi_1(\ddot{b})\rrbracket \wedge \llbracket \psi_2(\ddot{b})\rrbracket = 1 \;\Leftrightarrow\; \llbracket \psi_1(\ddot{b})\rrbracket = 1 \;\wedge\; \llbracket \psi_2(\ddot{b})\rrbracket = 1 \\ \Leftrightarrow\; A \vDash \psi_1(\ddot{b}) \;\wedge\; A \vDash \psi_2(\ddot{b}) \;\Leftrightarrow\; A \vDash \psi_1(\ddot{b}) \wedge \psi_2(\ddot{b}).\end{gathered}$$

Next, let $\varphi(\ddot{y})$ be $(\exists x)\, \psi(x, \ddot{y})$. Given $\ddot{b} \in A$, by the induction hypothesis (ii), we have

$$(\forall c \in A)\big(\llbracket \psi(c, \ddot{b})\rrbracket = 1 \;\Leftrightarrow\; A \vDash \psi(c, \ddot{b})\big) \tag{iii}$$

According to Theorem 1, there is $a \in B\langle \ddot{b}\rangle$ such that $\llbracket (\exists x)\, \psi(x, \ddot{b})\rrbracket = \llbracket \psi(a, \ddot{b})\rrbracket$. Moreover, since $\ddot{b} \in A$, we have $a \in B\langle \ddot{b}\rangle \subseteq A$. Consequently,

$$\llbracket (\exists x)\, \psi(x, \ddot{b})\rrbracket = 1 \;\Rightarrow\; \llbracket \psi(a, \ddot{b})\rrbracket = 1 \;\Rightarrow\; A \vDash \psi(a, \ddot{b}) \;\Rightarrow\; A \vDash (\exists x)\, \psi(x, \ddot{b}).$$

To justify the reverse implication, suppose that $A \vDash (\exists x)\, \psi(x, \ddot{b})$. Then $A \vDash \psi(c, \ddot{b})$ for some $c \in A$. By (iii) we have $\llbracket \psi(c, \ddot{b})\rrbracket = 1$, from which we consecutively derive

$$\begin{gathered}(\forall s \in S)\; 2 \vDash \psi(c(s), \ddot{b}(s)) \;\Rightarrow\; (\forall s \in S)(\exists t \in 2)\; 2 \vDash \psi(t, \ddot{b}(s)) \\ \Rightarrow\; (\forall s \in S)\; 2 \vDash (\exists x)\, \psi(x, \ddot{b}(s)) \;\Rightarrow\; \llbracket (\exists x)\, \psi(x, \ddot{b})\rrbracket = 1.\end{gathered}$$

Finally, let $\varphi(\ddot{y})$ be of the form $(\forall x)\, \psi(x, \ddot{y})$. As above, assertion (iii) holds for arbitrary $\ddot{b} \in A$, which allows us to conclude the following:

$$\begin{gathered}\llbracket (\forall x)\, \psi(x, \ddot{b})\rrbracket = 1 \;\Leftrightarrow\; (\forall s \in S)\; 2 \vDash (\forall x)\, \psi(x, \ddot{b}(s)) \;\Leftrightarrow \\ (\forall s \in S)(\forall t \in 2)\; 2 \vDash \psi(t, \ddot{b}(s)) \;\Rightarrow\; (\forall a \in A)(\forall s \in S)\; 2 \vDash \psi(a(s), \ddot{b}(s)) \qquad \text{(iv)} \\ \Leftrightarrow\; (\forall a \in A)\; \llbracket \psi(a, \ddot{b})\rrbracket = 1 \;\Leftrightarrow\; (\forall a \in A)\; A \vDash \psi(a, \ddot{b}) \;\Leftrightarrow\; A \vDash (\forall x)\, \psi(x, \ddot{b}).\end{gathered}$$

The reverse implication in line (iv) can be justified by taking $a = 0$ and $a = 1$:

$$\begin{gathered}(\forall a \in A)(\forall s \in S)\; 2 \vDash \psi(a(s), \ddot{b}(s)) \;\Rightarrow\; (\forall s \in S)\; 2 \vDash \psi(0, \ddot{b}(s)) \wedge (\forall s \in S)\; 2 \vDash \psi(1, \ddot{b}(s)) \\ \Rightarrow\; (\forall s \in S)\big(2 \vDash \psi(0, \ddot{b}(s)) \wedge 2 \vDash \psi(1, \ddot{b}(s))\big) \;\Rightarrow\; (\forall s \in S)(\forall t \in 2)\; 2 \vDash \psi(t, \ddot{b}(s)).\end{gathered}$$

(2): As in the proof of (1), by Lemma 2 we may assume φ syntactically generic and proceed by induction on the complexity of φ. The induction base corresponds to a syntactically strongly generic formula, in which case (2) follows from (1) proven above. Therefore, we are left to consider the induction step.

Consider a formula φ built according to one of the rules (4)–(6) of Definition 3; assume that

$$(\forall\, \ddot{b} \in A)\big(\llbracket \psi(\ddot{b})\rrbracket = 1 \;\Rightarrow\; A \vDash \psi(\ddot{b})\big) \tag{v}$$

whenever $\psi(\ddot{y})$ is a proper subformula of φ; and show that (v) holds for $\psi = \varphi$.

The cases in which φ has the form $\psi_1 \wedge \psi_2$, $(\exists x)\, \psi$, or $(\forall x)\, \psi$ are treated exactly in the same way as in the proof of (1), but shorter: with one-sided implications "$\Rightarrow$" instead of equivalences "$\Leftrightarrow$".

Let $\varphi(\ddot{x})$ be $\neg\psi(\ddot{x})$, with ψ a syntactically strongly generic formula. Then, taking account of Lemma 1, we conclude that, given arbitrary $\ddot{b} \in A$, the equality $\llbracket \varphi(\ddot{b})\rrbracket = 1$ implies

$$\llbracket \psi(\ddot{b})\rrbracket = \neg\neg\llbracket \psi(\ddot{b})\rrbracket = \neg\llbracket \neg\psi(\ddot{b})\rrbracket = \neg\llbracket \varphi(\ddot{b})\rrbracket = \neg 1 = 0 \neq 1.$$

Since ψ meets (1), we have $A \nvDash \psi(\ddot{b})$, i.e., $A \vDash \neg\psi(\ddot{b})$, and so $A \vDash \varphi(\ddot{b})$.

Now, let $\varphi(\ddot{x})$ be $\psi_1(\ddot{x}) \Rightarrow \psi_2(\ddot{x})$, where ψ_1 is syntactically strongly generic and ψ_2 is syntactically generic. Suppose that $\ddot{b} \in A$ satisfy $[\![\psi_1(\ddot{b}) \Rightarrow \psi_2(\ddot{b})]\!] = 1$ and show the validity $A \vDash (\psi_1(\ddot{b}) \Rightarrow \psi_2(\ddot{b}))$. Assume on the contrary that $A \vDash \psi_1(\ddot{b})$ and $A \nvDash \psi_2(\ddot{b})$. Since ψ_1 meets (1), and ψ_2 meets the induction hypothesis (v), we have $[\![\psi_1(\ddot{b})]\!] = 1$ and $[\![\psi_2(\ddot{b})]\!] \neq 1$. By taking account of Lemma 1, we derive

$$\begin{aligned}[\![\psi_1(\ddot{b}) \Rightarrow \psi_2(\ddot{b})]\!] &= \big([\![\psi_1(\ddot{b})]\!] \Rightarrow [\![\psi_2(\ddot{b})]\!]\big) = \neg[\![\psi_1(\ddot{b})]\!] \vee [\![\psi_2(\ddot{b})]\!] \\ &= \neg 1 \vee [\![\psi_2(\ddot{b})]\!] = 0 \vee [\![\psi_2(\ddot{b})]\!] = [\![\psi_2(\ddot{b})]\!] \neq 1,\end{aligned}$$

which is the desired contradiction. $\triangleright$

Theorem 3. *If φ is a Horn sentence of signature Σ_{BA}, then $2 \vDash \varphi$ implies $A \vDash \varphi$ for every Boolean algebra A.*

$\triangleleft$ Owing to the Stone Theorem [5, 1.2.4], we may assume that A is a Boolean subalgebra of $\mathcal{P}(S)$ for some nonempty set S. If $2 \vDash \varphi$, then $[\![\varphi]\!] = \{s \in S : 2 \vDash \varphi\} = 1$, and so by Theorem 2 we have $A \vDash \varphi$. $\triangleright$

Therefore, if a Horn sentence is true in the simplest Boolean algebra $\{0, 1\}$, then it is provable in the theory of Boolean algebras. This means that, within the theory of Boolean algebras, a successful verification with the help of Venn diagrams or truth tables serves as a rigorous proof for any hypothesis that can be expressed by a Horn sentence. So are, for instance, the dual distributivity law

$$(\forall x)(\forall y)\big((x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)\big)$$

and the uniqueness of a complement

$$(\forall x)(\forall y)(\forall z)\big(x \wedge y = 0 \ \wedge \ x \vee y = 1 \ \wedge \ x \wedge z = 0 \ \wedge \ x \vee z = 1 \ \Rightarrow \ y = z\big)$$

considered at the beginning of the article.

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