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QUASIDENSENESS IN $\mathbb{R}^{\mathbb{N}}$ AND PROJECTIVE PARALLELOTOPES

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Abstract—We establish two new criteria for the closedness of Archimedean cones in countable-dimensional locally convex spaces in terms of projective parallelotopes and projective automorphisms. We also answer some open questions about quasidenseness and quasi-interior.

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The paper [1] provides an exhaustive description of the class of the locally convex spaces whose all Archimedean cones are closed. Namely, there is introduced the concept of quasidense subset of a locally convex space and shown that the class consists of all finite-dimensional spaces and all countable-dimensional spaces X whose topological dual X' is quasidense in the algebraic dual $X^{\#}$ equipped with the weak topology $\sigma(X^{\#}, X)$. The use of the concept of quasidenseness made it possible to solve a few problems related to Archimedean cones, but the concept remains new and little studied, as evidenced in particular by the list of open questions at the end of [1]. In the case of $\dim X = |\mathbb{N}|$, the locally convex space $\big(X^{\#}, \sigma(X^{\#}, X)\big)$ is isomorphic to $\mathbb{R}^{\mathbb{N}}$; therefore, the primary task of the study is the characterization of quasidense subsets in $\mathbb{R}^{\mathbb{N}}$. The article is devoted to solving this problem.

In the auxiliary Sections 1 and 2, we give preliminary information and consider the automorphisms of sequence spaces. In the central Section 3, we introduce and study the concept of projective parallelotope and prove two new criteria for quasidenseness in $\mathbb{R}^{\mathbb{N}}$. Sections 4 and 5 answer the four open questions posed in [1], as well as the questions on the representativeness of parallelotopes in their connection with quasidenseness and quasi-interior.

§ 1. Preliminary Information

We start with clarifying the general notation and terms and also reproduce some of the definitions and facts in [1] so that the article be suitable for independent reading. A more complete set of relevant information, including proofs and examples, is available in [1] and the literature cited therein.

1.1. The symbol "$\subset$" denotes the nonstrict inclusion of sets. The assignment symbol ":=" introduces notation and indicates the equalities valid by definition.

Let $\mathbb{N}$ stand for the naturals $\{1, 2, \dots\}$. The rationals and reals are denoted by $\mathbb{Q}$ and $\mathbb{R}$. By $\mathbb{R}^+$ we denote the collection $\{\lambda \in \mathbb{R} : \lambda \geqslant 0\}$ of positive reals. The set $\mathbb{R}$ is endowed with the standard operations and topology that make $\mathbb{R}$ a field and a locally convex space. We designate as $\mathbb{R}_{\mathrm{D}}$ the set of reals endowed with the discrete topology. Closed and open numerical intervals are denoted by $[\alpha, \beta]$ and $]\alpha, \beta[$.

By $\operatorname{lin} S$, $\operatorname{co} S$, $\operatorname{cl} S$, and $\operatorname{int} S$ we designate the linear span, the convex hull, the closure, and the interior of a set S in a vector or topological space under consideration.

1.2. In what follows, by a *vector space* we mean a vector space over $\mathbb{R}$. The term *subspace* means a vector subspace. A subset K of a vector space is a *cone*, if $K + K \subset K$, $\mathbb{R}^+ K \subset K$, and $K \cap -K = \{0\}$. A cone K in a vector space X is *Archimedean* whenever the ordered vector space $(X, \leqslant_K)$ is Archimedean, where $x \leqslant_K y \Leftrightarrow y - x \in K$.

1.3. If X and Y are vector spaces, let $L(X, Y)$ denote the vector space of linear operators from X to Y. By $X^{\#}$ we denote the vector space $L(X, \mathbb{R})$ algebraically conjugate to X. If X and Y are topological

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vector spaces, then $\mathcal{L}(X,Y)$ stands for the vector space of continuous linear operators from X to Y, and X' is the space $\mathcal{L}(X,\mathbb{R})$ topologically conjugate to X. We use $L(X)$ and $\mathcal{L}(X)$ as abbreviations for $L(X,X)$ and $\mathcal{L}(X,X)$. Let $Aut(X)$ designate the set of automorphisms of a topological vector space X, i.e., the set of bijections $T\colon X\to X$ such that $T,T^{-1}\in\mathcal{L}(X)$.

1.4. By $\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}$ we denote the subspace of $\mathbb{R}^{\mathbb{N}}$ consisting of finitely supported sequences, i.e., of the functions $s\colon\mathbb{N}\to\mathbb{R}$ whose support $\{n\in\mathbb{N}:s(n)\neq 0\}$ is finite. Tuples $x=\big(x(1),\dots,x(n)\big)\in\mathbb{R}^n$, with $n\in\mathbb{N}$, are traditionally regarded as functions $x\colon\{1,\dots,n\}\to\mathbb{R}$. In what follows, we use the notation

$$e_n:=\chi_{\{n\}}=(0,\dots,0,\underset{(n)}{1},0,0,\dots)\in\mathbb{R}^{\mathbb{N}};$$
$$\mathbb{R}^{\mathbb{N}}_n:=\operatorname{lin}\{e_1,\dots,e_n\}=\{s\in\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}:s(m)=0\text{ for }m>n\}.$$

The linear operator $\pi_n\colon\mathbb{R}^{\mathbb{N}}\to\mathbb{R}^n$ is defined by

$$\pi_n s:=s\big|_{\{1,\dots,n\}}=\big(s(1),\dots,s(n)\big). \tag{1}$$

We will use (1) not only for sequences $s\in\mathbb{R}^{\mathbb{N}}$ but also for tuples $s\in\mathbb{R}^m$, with $m\geqslant n$. We also put $\mathbb{R}^0:=\{0\}$ and $\pi_0 s:=0\in\mathbb{R}^0$ for convenience.

1.5. Vector spaces X and Y form a *dual pair* with respect to duality $\langle\cdot\,|\,\cdot\rangle$, if $\langle\cdot\,|\,\cdot\rangle\colon X\times Y\to\mathbb{R}$ is a bilinear functional such that $\ker\langle\cdot|=\{0\}$ and $\ker|\cdot\rangle=\{0\}$, where $\langle x|=\langle x\,|\,\cdot\rangle\in Y^{\#}$ $(x\in X)$ and $|y\rangle=\langle\cdot\,|\,y\rangle\in X^{\#}$ $(y\in Y)$. These spaces X and Y are endowed by default with the corresponding weak topologies $\sigma(X,Y)$ and $\sigma(Y,X)$ and thereby become some Hausdorff locally convex spaces that we denote by $X|Y$ and $Y|X$. In this case, the mappings $\langle\cdot|\colon X\to Y'$ and $|\cdot\rangle\colon Y\to X'$ (or, more precisely, $\langle\cdot|\colon X|Y\to(Y|X)'$ and $|\cdot\rangle\colon Y|X\to(X|Y)'$) are linear and topological isomorphisms.

1.6. The space $\mathbb{R}^n$ with $n\in\mathbb{N}$ is considered in the dual pair with $\mathbb{R}^n$, where $\langle x\,|\,y\rangle=\sum_{i=1}^n x(i)y(i)$. The spaces $\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}$ and $\mathbb{R}^{\mathbb{N}}$ are regarded by default as a dual pair with respect to $\langle x\,|\,y\rangle=\sum_{n\in\mathbb{N}}x(n)y(n)$ and endowed with the corresponding weak topologies: $\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}:=\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}\big|\mathbb{R}^{\mathbb{N}}$ and $\mathbb{R}^{\mathbb{N}}:=\mathbb{R}^{\mathbb{N}}\big|\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}$. We imply the same duality when considering locally convex spaces of the form $\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}\big|Y$, where Y is a subspace of $\mathbb{R}^{\mathbb{N}}$ satisfying the following equivalent conditions (see [1, 3.5 and 3.6]):

(a) $\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}$ and Y form a dual pair with respect to $\langle x\,|\,y\rangle=\sum_{n\in\mathbb{N}}x(n)y(n)$;
(b) the weak topology $\sigma\big(\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}},Y\big)$ is Hausdorff;
(c) Y is dense in $\mathbb{R}^{\mathbb{N}}\big|\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}$;
(d) Y is dense in $\mathbb{R}^{\mathbb{N}}_{\mathrm{D}}$;
(e) $\pi_n Y=\mathbb{R}^n$ for all $n\in\mathbb{N}$;
(f) $\pi_n e_n\in\pi_n Y$ for all $n\in\mathbb{N}$.

1.7. A set $S\subset\mathbb{R}^{\mathbb{N}}$ is *projective* (see [1, 7.1]) if S is closed in $\mathbb{R}^{\mathbb{N}}_{\mathrm{D}}$ or, which is the same, S contains every $s\in\mathbb{R}^{\mathbb{N}}$ satisfying $\pi_n s\in\pi_n S$ for all $n\in\mathbb{N}$.

Arbitrary Cartesian products $\prod_{n\in\mathbb{N}}\Lambda_n$, with $\Lambda_n\subset\mathbb{R}$, are examples of projective sets. Moreover, every closed subset of $\mathbb{R}^{\mathbb{N}}$ is also closed in $\mathbb{R}^{\mathbb{N}}_{\mathrm{D}}$ and, therefore, projective.

A sequence of sets $S_n\subset\mathbb{R}^n$ $(n\in\mathbb{N})$ is *projective* (see [1, 7.1]) if it has the following equivalent properties:

(a) there is a set $S\subset\mathbb{R}^{\mathbb{N}}$ such that $S_n=\pi_n S$ for all $n\in\mathbb{N}$;
(b) $S_n=\pi_n S_m$ for $n\leqslant m$;
(c) $S_n=\pi_n S_{n+1}$ for all $n\in\mathbb{N}$.

In this case,

$$\bigcap_{n\in\mathbb{N}}\pi_n^{-1}(S_n)=\big\{s\in\mathbb{R}^{\mathbb{N}}:\pi_n s\in S_n\text{ for all }n\in\mathbb{N}\big\}$$

is the *projective limit* of $(S_n)_{n\in\mathbb{N}}$ which is denoted by $\varprojlim S_n$ (see [1, 7.3]). The projective limit $\varprojlim S_n$ is the largest among the sets S satisfying (a), presents the only projective set among them, and coincides with the closure of each of them in the topological space $\mathbb{R}^{\mathbb{N}}_{\mathrm{D}}$.

1.8. The *quasi-interior* $\operatorname{qi} S$ of a subset S in a Hausdorff locally convex space X is defined as follows:

$$\operatorname{qi} S := \{x \in S : \operatorname{cl} \mathbb{R}^+(S - x) = X\}.$$

The elements of $\operatorname{qi} S$ are the *quasi-interior points* of S. A set S is *quasiopen* whenever $\operatorname{qi} S = S$.

Theorem [1, 4.13]. *If $C \subset \mathbb{R}^{\mathbb{N}}$ is convex, then*

$$\operatorname{qi} C = \{c \in C : \pi_n c \in \operatorname{int} \pi_n C \text{ for all } n \in \mathbb{N}\}.$$

In particular, a convex set C is quasiopen in $\mathbb{R}^{\mathbb{N}}$ if and only if each projection $\pi_n C$ is open in $\mathbb{R}^n$.

1.9. An *open box* in $\mathbb{R}^{\mathbb{N}}$ is a set of the form $\prod_{n\in\mathbb{N}} \Lambda_n$, with Λ_n open subsets of $\mathbb{R}$. The *box topology* on $\mathbb{R}^{\mathbb{N}}$ is that for which the open boxes serve as basic open sets. For every $z \in \mathbb{R}^{\mathbb{N}}$, the open convex boxes

$$\prod_{n\in\mathbb{N}} \,]z(n) - r(n), z(n) + r(n)[, \quad r(n) > 0, \tag{2}$$

form a base of neighborhoods of z in the box topology.

Proposition [1, 7.8]. *An arbitrary open convex box in $\mathbb{R}^{\mathbb{N}}$ is an example of a projective quasiopen set. Moreover, all subsets of $\mathbb{R}^{\mathbb{N}}$ open in the box topology are quasiopen.*

1.10. A subset S of a locally convex space X is *quasidense* in X, if $S \cap B \neq \varnothing$ for every closed bounded convex set $B \subset X$ with nonempty quasi-interior (see [1, 6.2]).

Proposition [1, 6.4]. *All quasidense sets in $\mathbb{R}^{\mathbb{N}}$ are dense.*

1.11. Proposition [1, 8.8]. *The following properties of $S \subset \mathbb{R}^{\mathbb{N}}$ are equivalent:*

(a) *S is quasidense in $\mathbb{R}^{\mathbb{N}}$;*
(b) *if C is a compact convex subset of $\mathbb{R}^{\mathbb{N}}$ and $\operatorname{qi} C \neq \varnothing$, then $S \cap C \neq \varnothing$;*
(c) *if B is a nonempty projective bounded quasiopen convex subset of $\mathbb{R}^{\mathbb{N}}$, then $S \cap B \neq \varnothing$;*
(d) *if $(B_n)_{n\in\mathbb{N}}$ is a projective sequence of nonempty bounded open convex sets, then $S \cap \varprojlim B_n \neq \varnothing$;*
(e) *if C is a projective convex subset of $\mathbb{R}^{\mathbb{N}}$ and $\operatorname{qi} C \neq \varnothing$, then $S \cap C \neq \varnothing$.*

1.12. Theorem [1, 8.5]. *Let Y be a dense subspace of $\mathbb{R}^{\mathbb{N}}$.*

(a) *All Archimedean cones in $\mathbb{R}^{\mathbb{N}}_{\text{fin}}|Y$ are closed if and only if Y is quasidense in $\mathbb{R}^{\mathbb{N}}$.*
(b) *If C is a compact convex subset of $\mathbb{R}^{\mathbb{N}}$, $\operatorname{qi} C \neq \varnothing$, and $Y \cap C = \varnothing$* (see 1.11(b)), *then*

$$\{x \in \mathbb{R}^{\mathbb{N}}_{\text{fin}} : \langle x \,|\, c\rangle \geqslant 0 \text{ for } c \in C\}$$

is an example of an Archimedean cone that is not closed (*moreover, it is dense*) *in $\mathbb{R}^{\mathbb{N}}_{\text{fin}}|Y$.*

§ 2. Inductive and Projective Automorphisms

In this auxiliary section, we clarify the form of the numerical matrices that represent continuous linear operators in sequence spaces, and study the operators corresponding to upper triangular and lower triangular matrices.

2.1. Let X and Y be a dual pair of vector spaces and let $T \in \mathcal{L}(X)$. As is known, in this case there is a unique mapping $T' \colon Y \to Y$ satisfying $\langle Tx \,|\, y\rangle = \langle x \,|\, T'y\rangle$ for all $x \in X$ and $y \in Y$. The mapping T' belongs to $\mathcal{L}(Y)$ and is referred to as *dual* to T (see [2, 11-1]). Given $S \in \mathcal{L}(Y)$, the dual $S' \in \mathcal{L}(X)$ of S is defined similarly.

Every operator $T \in \mathcal{L}(X)$ is dual to $T' \in \mathcal{L}(Y)$, and every $S \in \mathcal{L}(Y)$ is dual to $S' \in \mathcal{L}(X)$; i.e., $T'' = T$ and $S'' = S$.

Given $T \in \mathcal{L}(X)$, the conditions $T \in \mathit{Aut}(X)$ and $T' \in \mathit{Aut}(Y)$ are equivalent. In this case, $(T^{-1})' = (T')^{-1}$.

In what follows, the role of X and Y will be played by the pair of $\mathbb{R}^{\mathbb{N}}_{\text{fin}}$ and $\mathbb{R}^{\mathbb{N}}$. Owing to the relation $(\mathbb{R}^{\mathbb{N}}_{\text{fin}})' = (\mathbb{R}^{\mathbb{N}}_{\text{fin}})^{\#}$, all linear operators $T \colon \mathbb{R}^{\mathbb{N}}_{\text{fin}} \to \mathbb{R}^{\mathbb{N}}_{\text{fin}}$ are continuous; i.e., $L(\mathbb{R}^{\mathbb{N}}_{\text{fin}}) = \mathcal{L}(\mathbb{R}^{\mathbb{N}}_{\text{fin}})$.

2.2. We use the term *matrix* for a numerical family $\mu \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ and call the sequences $\mu(\cdot,n)$ and $\mu(m,\cdot)$ $(m,n\in\mathbb{N})$ the *columns* and the *rows* of μ:

$$\left(\begin{array}{cccc|c|c} \mu(1,1) & \mu(1,2) & \mu(1,3) & \cdots & \mu(1,n) & \cdots \\ \mu(2,1) & \mu(2,2) & \mu(2,3) & \cdots & \mu(2,n) & \cdots \\ \mu(3,1) & \mu(3,2) & \mu(3,3) & \cdots & \mu(3,n) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline \mu(m,1) & \mu(m,2) & \mu(m,3) & \cdots & \mu(m,n) & \cdots \\ \hline \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right) \mu(m,\cdot)$$

$$\mu(\cdot,n)$$

$$\mu(\cdot,n) = \big(\mu(m,n)\big)_{m\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \text{ is the } n\text{th column } (n\in\mathbb{N}) \text{ of the matrix } \mu,$$
$$\mu(m,\cdot) = \big(\mu(m,n)\big)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \text{ is the } m\text{th row } (m\in\mathbb{N}) \text{ of the matrix } \mu.$$

The *transpose* μ^{T} of a matrix μ is defined by the formula $\mu^{\mathrm{T}}(m,n) := \mu(n,m)$ for all $m,n\in\mathbb{N}$.

A matrix μ is *column finitary* (*row finitary*) whenever $\mu(\cdot,n)\in\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}$ for all $n\in\mathbb{N}$ (respectively, $\mu(m,\cdot)\in\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}$ for all $m\in\mathbb{N}$). The special cases of column finitary and row finitary matrices are *upper triangular* and *lower triangular matrices*, i.e., such $\mu,\lambda\in\mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ that $\mu(m,n)=0$ for $m>n$ and $\lambda(m,n)=0$ for $n>m$:

$$\begin{pmatrix} \mu(1,1) & \mu(1,2) & \mu(1,3) & \cdots & \mu(1,n) & \cdots \\ 0 & \mu(2,2) & \mu(2,3) & \cdots & \mu(2,n) & \cdots \\ 0 & 0 & \mu(3,3) & \cdots & \mu(3,n) & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \mu(n,n) & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \quad \begin{pmatrix} \lambda(1,1) & 0 & 0 & 0 & \cdots & 0 & \cdots \\ \lambda(2,1) & \lambda(2,2) & 0 & 0 & \cdots & 0 & \cdots \\ \lambda(3,1) & \lambda(3,2) & \lambda(3,3) & 0 & \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda(m,1) & \lambda(m,2) & \lambda(m,3) & \cdots & \lambda(m,m) & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

2.3. Column finitary matrices μ and row finitary matrices λ are associated with the mappings $\mu^\wedge\colon \mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}\to\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}$ and $\lambda^\vee\colon\mathbb{R}^{\mathbb{N}}\to\mathbb{R}^{\mathbb{N}}$ defined as follows:

$$(\mu^\wedge x)(m) := \langle x\,|\,\mu(m,\cdot)\rangle = \sum_{n\in\mathbb{N}}\mu(m,n)x(n), \quad x\in\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}},\ m\in\mathbb{N},$$
$$(\lambda^\vee y)(m) := \langle\lambda(m,\cdot)\,|\,y\rangle = \sum_{n\in\mathbb{N}}\lambda(m,n)y(n), \quad y\in\mathbb{R}^{\mathbb{N}},\ m\in\mathbb{N}.$$

(Let us demonstrate why $\mu^\wedge x\in\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}$ for $x\in\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}$. Indeed, since the columns $\mu(\cdot,n)$ are finitely supported, there is a sequence of naturals m_n such that $\mu(m,n)=0$ for all $m\geqslant m_n$. Therefore, if $x(n)=0$ for $n>k$, then $(\mu^\wedge x)(m)=\sum_{n=1}^k\mu(m,n)x(n)=0$ for $m\geqslant\max\{m_1,\dots,m_k\}$.)

Given an operator T in $L(\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}})$ or $L(\mathbb{R}^{\mathbb{N}})$, define the matrix $[T]\in\mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ as follows:

$$[T](m,n) := (Te_n)(m), \quad m,n\in\mathbb{N}.$$

2.4. Proposition (cf. [2, 11-1-6, 11-1-10, and 11-1-11]).

(a) *If μ is a column finitary matrix, then $\mu^\wedge\in L(\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}})$ and $[\mu^\wedge]=\mu$.*

(b) *If λ is a row finitary matrix, then $\lambda^\vee\in\mathcal{L}(\mathbb{R}^{\mathbb{N}})$ and $[\lambda^\vee]=\lambda$.*

(c) *If $\nabla\in L(\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}})$, then $[\nabla]$ is a column finitary matrix, $[\nabla]^\wedge=\nabla$, $\nabla'\in\mathcal{L}(\mathbb{R}^{\mathbb{N}})$, and $[\nabla']=[\nabla]^{\mathrm{T}}$.*

(d) *If $\Delta\in\mathcal{L}(\mathbb{R}^{\mathbb{N}})$, then $[\Delta]$ is a row finitary matrix, $[\Delta]^\vee=\Delta$, $\Delta'\in L(\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}})$, and $[\Delta']=[\Delta]^{\mathrm{T}}$.*

◁ Only (d) needs demonstration. Let $\Delta\in\mathcal{L}(\mathbb{R}^{\mathbb{N}})$. Then

$$[\Delta](m,n) = (\Delta e_n)(m) = \langle e_m\,|\,\Delta e_n\rangle = \langle\Delta' e_m\,|\,e_n\rangle = (\Delta' e_m)(n)$$

for all $m,n\in\mathbb{N}$. This means $[\Delta](m,\cdot)=\Delta' e_m\in\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}$; i.e., $[\Delta]$ is row finitary. Next,

$$([\Delta]^\vee e_n)(m) = \langle[\Delta](m,\cdot)\,|\,e_n\rangle = ([\Delta](m,\cdot))(n) = [\Delta](m,n) = (\Delta e_n)(m)$$

for all $m,n\in\mathbb{N}$; whence, using the denseness of $\mathrm{lin}\{e_n : n\in\mathbb{N}\}$ in $\mathbb{R}^{\mathbb{N}}$ and the continuity of Δ and $[\Delta]^\vee$ (see (b)), we infer $[\Delta]^\vee=\Delta$. The equality $[\Delta']=[\Delta]^{\mathrm{T}}$ is obvious. ▷

2.5. Proposition. *The following properties of* $\nabla \in L(\mathbb{R}^{\mathbb{N}}_{\text{fin}})$ *are equivalent:*
(a) $[\nabla]$ *is an upper triangular matrix;*
(b) $\nabla(\mathbb{R}^{\mathbb{N}}_n) \subset \mathbb{R}^{\mathbb{N}}_n$ *for all* $n \in \mathbb{N}$*;*
(c) $\nabla e_n \in \mathbb{R}^{\mathbb{N}}_n$ *for all* $n \in \mathbb{N}$*.*

An operator $\nabla \in L(\mathbb{R}^{\mathbb{N}}_{\text{fin}})$ with the equivalent properties (a)–(c) is *inductive.*

2.6. Proposition. *The following properties of* $\Delta \in \mathcal{L}(\mathbb{R}^{\mathbb{N}})$ *are equivalent:*
(a) $[\Delta]$ *is a lower triangular matrix;*
(b) *if* $y \in \mathbb{R}^{\mathbb{N}}$*,* $n \in \mathbb{N}$*, and* $\pi_n y = 0$*; then* $\pi_n \Delta y = 0$*;*
(c) *if* $y, z \in \mathbb{R}^{\mathbb{N}}$*,* $n \in \mathbb{N}$*, and* $\pi_n y = \pi_n z$*; then* $\pi_n \Delta y = \pi_n \Delta z$*;*
(d) $\pi_n \Delta e_{n+1} = 0$ *for all* $n \in \mathbb{N}$*.*

An operator $\Delta \in \mathcal{L}(\mathbb{R}^{\mathbb{N}})$ with the equivalent properties (a)–(d) is *projective.*

2.7. Proposition. (a) $\nabla \in L(\mathbb{R}^{\mathbb{N}}_{\text{fin}})$ *is inductive if and only if* $\nabla' \in \mathcal{L}(\mathbb{R}^{\mathbb{N}})$ *is projective.*
(b) $\Delta \in \mathcal{L}(\mathbb{R}^{\mathbb{N}})$ *is projective if and only if* $\Delta' \in L(\mathbb{R}^{\mathbb{N}}_{\text{fin}})$ *is inductive.*

2.8. Proposition. (a) *For every sequence of* $x_n \in \mathbb{R}^{\mathbb{N}}_n$*, there is a unique operator* $\nabla \in L(\mathbb{R}^{\mathbb{N}}_{\text{fin}})$ *such that* $\nabla e_n = x_n$ *for all* $n \in \mathbb{N}$*. This operator* ∇ *is inductive.*

(b) *If a sequence of* $y_n \in \mathbb{R}^{\mathbb{N}}$ *satisfies the condition* $\pi_n y_{n+1} = 0$ *for all* $n \in \mathbb{N}$*, then there is a unique operator* $\Delta \in \mathcal{L}(\mathbb{R}^{\mathbb{N}})$ *such that* $\Delta e_n = y_n$ *for all* $n \in \mathbb{N}$*. This operator* Δ *is projective.*

◁ Item (a) does not need any demonstration.

(b): By hypothesis, the matrix λ defined by $\lambda(m,n) := y_n(m)$ is lower triangular, and so $\Delta := \lambda^{\vee}$ is the required operator (see 2.4(b)). The uniqueness of this operator follows from its continuity and the denseness of $\operatorname{lin}\{e_n : n \in \mathbb{N}\}$ in $\mathbb{R}^{\mathbb{N}}$; its projectivity is due to 2.6(d). ▷

2.9. Proposition. *Let* $\Delta \in \mathcal{L}(\mathbb{R}^{\mathbb{N}})$ *be projective.*
(a) Δ *is continuous as a mapping from* $\mathbb{R}^{\mathbb{N}}_{\text{D}}$ *into* $\mathbb{R}^{\mathbb{N}}_{\text{D}}$*.*
(b) *If* P *is a projective subset of* $\mathbb{R}^{\mathbb{N}}$*; then* $\Delta^{-1}(P)$ *is a projective subset of* $\mathbb{R}^{\mathbb{N}}$*.*
(c) *If* $\operatorname{im}\Delta = \mathbb{R}^{\mathbb{N}}$ *and* Y *is a dense subspace of* $\mathbb{R}^{\mathbb{N}}$*; then* $\Delta(Y)$ *is a dense subspace of* $\mathbb{R}^{\mathbb{N}}$*.*

◁ Item (a) is easy to prove using 2.6(c). Items (b) and (c) follow from (a) by 1.7 and 1.6(d) respectively. ▷

2.10. Proposition. *Let* $\nabla \in L(\mathbb{R}^{\mathbb{N}}_{\text{fin}})$ *be inductive. The following properties of* ∇ *are equivalent:*
(a) $\nabla \in Aut(\mathbb{R}^{\mathbb{N}}_{\text{fin}})$*;*
(b) ∇e_n $(n \in \mathbb{N})$ *are linearly independent;*
(c) $[\nabla](n,n) \neq 0$ *for all* $n \in \mathbb{N}$*.*

◁ The implication (a)⇒(b) is trivial.

(b)⇒(c): Since $\nabla e_1 \in \mathbb{R}^{\mathbb{N}}_1$ (see 2.5(c)) and $\nabla e_1 \neq 0$; therefore, $[\nabla](1,1) = (\nabla e_1)(1) \neq 0$. Now let $n > 1$. From (b) it follows that $\nabla e_1, \dots, \nabla e_{n-1}$ form a basis for $\mathbb{R}^{\mathbb{N}}_{n-1}$. If $[\nabla](n,n) = (\nabla e_n)(n)$ were equal to zero; then, as $\nabla e_n \in \mathbb{R}^{\mathbb{N}}_n$, the relations $\nabla e_n \in \mathbb{R}^{\mathbb{N}}_{n-1} = \operatorname{lin}\{\nabla e_1, \dots, \nabla e_{n-1}\}$ would be valid contrary to the linear independence of $\nabla e_1, \dots, \nabla e_n$.

(c)⇒(a): Since an upper triangular $n \times n$ matrix with nonzero diagonal entries is invertible; we conclude that $\nabla e_1, \dots, \nabla e_n$ form a basis for $\mathbb{R}^{\mathbb{N}}_n$ for each $n \in \mathbb{N}$, whence $\{\nabla e_n : n \in \mathbb{N}\}$ is a basis for $\mathbb{R}^{\mathbb{N}}_{\text{fin}}$. ▷

An inductive operator $\nabla \in L(\mathbb{R}^{\mathbb{N}}_{\text{fin}})$ with the equivalent properties (a)–(c) is an *inductive automorphism.* An inductive automorphism ∇ is *positive,* if $[\nabla](n,n) > 0$ for all $n \in \mathbb{N}$. Let $\nabla_{+}(\mathbb{R}^{\mathbb{N}}_{\text{fin}})$ stand for the set of all positive inductive automorphisms.

2.11. Proposition. *Let $\Delta \in \mathcal{L}(\mathbb{R}^{\mathbb{N}})$ be projective. The following properties of Δ are equivalent:*
(a) $\Delta \in Aut(\mathbb{R}^{\mathbb{N}})$;
(b) Δe_n $(n \in \mathbb{N})$ *are linearly independent and* $\operatorname{cl}\operatorname{im}\Delta = \mathbb{R}^{\mathbb{N}}$;
(c) $[\Delta](n,n) \neq 0$ *for all* $n \in \mathbb{N}$.

⊲ The implication (a)⇒(b) is trivial.

(b)⇒(c): Show that $\Delta' e_n$ $(n \in \mathbb{N})$ are linearly independent in $\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}$. Indeed, if $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ and $\sum_{i=1}^n \lambda_i \Delta' e_i = 0$, then

$$\left\langle \sum_{i=1}^n \lambda_i e_i \,\middle|\, \Delta y \right\rangle = \left\langle \Delta'\Big(\sum_{i=1}^n \lambda_i e_i\Big) \,\middle|\, y \right\rangle = \left\langle \sum_{i=1}^n \lambda_i \Delta' e_i \,\middle|\, y \right\rangle = 0$$

for all $y \in \mathbb{R}^{\mathbb{N}}$; whence, by the denseness of $\operatorname{im}\Delta$ in $\mathbb{R}^{\mathbb{N}}$, we infer $\sum_{i=1}^n \lambda_i e_i = 0$ and so $\lambda_1 = \dots = \lambda_n = 0$. Owing to 2.4(d), 2.7(b), and 2.10, we have $[\Delta](n,n) = [\Delta']^{\mathrm{T}}(n,n) = [\Delta'](n,n) \neq 0$ for all $n \in \mathbb{N}$.

The implication (c)⇒(a) is also easily deduced from 2.4(d), 2.7(b), and 2.10. ⊳

The requirement $\operatorname{cl}\operatorname{im}\Delta = \mathbb{R}^{\mathbb{N}}$ in (b) is essential. The shift operator $\Delta(y) = \big(0, y(1), y(2), y(3), \dots\big)$ serves as a counterexample.

A projective operator $\Delta \in \mathcal{L}(\mathbb{R}^{\mathbb{N}})$ with the equivalent properties (a)–(c) is a *projective automorphism*. A projective automorphism Δ is *positive* provided that $[\Delta](n,n) > 0$ for all $n \in \mathbb{N}$. Let $\triangle^{+}(\mathbb{R}^{\mathbb{N}})$ stand for the set of all positive projective automorphisms.

2.12. The following is an immediate consequence of the above:

Proposition. (a) *For all* $\nabla \in L\big(\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}\big)$ *and* $\Delta \in \mathcal{L}(\mathbb{R}^{\mathbb{N}})$,

$$\nabla \in \triangledown_{+}\big(\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}\big) \ \Leftrightarrow\ \nabla' \in \triangle^{+}(\mathbb{R}^{\mathbb{N}});$$
$$\Delta \in \triangle^{+}(\mathbb{R}^{\mathbb{N}}) \ \Leftrightarrow\ \Delta' \in \triangledown_{+}\big(\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}\big).$$

(b) $\triangledown_{+}\big(\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}\big)$ *and* $\triangle^{+}(\mathbb{R}^{\mathbb{N}})$ *are subgroups of* $Aut\big(\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}\big)$ *and* $Aut(\mathbb{R}^{\mathbb{N}})$ *under the composition of operators.*

§ 3. Projective Parallelotopes

In this section, we introduce the concept of projective parallelotope, establish the connection between parallelotopes and inductive and projective automorphisms (Theorem 3.4), and prove two new criteria for quasidenseness in $\mathbb{R}^{\mathbb{N}}$ in terms of parallelotopes and automorphisms (Theorem 3.8).

3.1. Given $\varkappa = (\varkappa_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathbb{R}^n$ and $r \in\,]0,\infty[^{\mathbb{N}}$, put

$$\Pi^r_{\varkappa} := \Big\{ y \in \mathbb{R}^{\mathbb{N}} : |y(1)| < r(1),\ \Big| y(n+1) - \sum_{i=1}^n \varkappa_n(i) y(i) \Big| < r(n+1) \text{ for all } n \in \mathbb{N} \Big\}.$$

In what follows, we assume $\varkappa_0 := 0 \in \mathbb{R}^0$ for convenience. (Recall that $\mathbb{R}^0 := \{0\}$ and $\pi_0 y := \pi_0 x := 0 \in \mathbb{R}^0$ for $y \in \mathbb{R}^{\mathbb{N}}$ and $x \in \mathbb{R}^n$.) With this convention adopted, we see that

$$\Pi^r_{\varkappa} = \big\{ y \in \mathbb{R}^{\mathbb{N}} : \big| y(n) - \langle \varkappa_{n-1} \,|\, \pi_{n-1} y \rangle \big| < r(n) \text{ for all } n \in \mathbb{N} \big\}.$$

The set $z + \Pi^r_{\varkappa}$, with $z \in \mathbb{R}^{\mathbb{N}}$, is the *parallelotope* (more precisely, the *projective parallelotope*) with *center* z, *tilt* $\varkappa$, and *radius* r.

Denote by Π^r_0 the parallelotope with zero center $0 \in \mathbb{R}^{\mathbb{N}}$, zero tilt $(0,0,\dots) \in \prod_{n\in\mathbb{N}} \mathbb{R}^n$, and radius $r \in\,]0,\infty[^{\mathbb{N}}$; and by Π^1_0, the parallelotope $\Pi^{(1,1,\dots)}_0$:

$$\Pi^r_0 = \big\{ y \in \mathbb{R}^{\mathbb{N}} : |y(n)| < r(n) \text{ for all } n \in \mathbb{N} \big\},$$
$$\Pi^1_0 = \big\{ y \in \mathbb{R}^{\mathbb{N}} : |y(n)| < 1 \text{ for all } n \in \mathbb{N} \big\}.$$

3.2. From the following lemma it is clear that each projection $\pi_n P$ of a parallelotope $P \subset \mathbb{R}^{\mathbb{N}}$ is an open parallelotope in $\mathbb{R}^n$, i.e., an open n-dimensional parallelepiped:

Lemma. *Let $\varkappa \in \prod_{n\in\mathbb{N}} \mathbb{R}^n$, $r \in]0,\infty[^{\mathbb{N}}$, $n \in \mathbb{N}$. Then*

$$\begin{aligned}\pi_n \Pi^r_\varkappa &= \big\{x \in \mathbb{R}^n : \big|x(m) - \langle \varkappa_{m-1} \,|\, \pi_{m-1}x\rangle\big| < r(m) \ \textit{for}\ m = 1,\dots,n\big\}\\ &= \big\{x \in \mathbb{R}^n : \pi_{n-1}x \in \pi_{n-1}\Pi^r_\varkappa \ \textit{and}\ \big|x(n) - \langle \varkappa_{n-1} \,|\, \pi_{n-1}x\rangle\big| < r(n)\big\}.\end{aligned}$$

◁ The inclusion "⊂" is obvious. Now suppose that $x \in \mathbb{R}^n$ and

$$\big|x(m) - \langle \varkappa_{m-1} \,|\, \pi_{m-1}x\rangle\big| < r(m), \quad m = 1,\dots,n.$$

Define $y \in \mathbb{R}^{\mathbb{N}}$ by putting $\pi_n y := x$ and $y(m) := \sum_{i=1}^{m-1} \varkappa_{m-1}(i)y(i)$ recursively for $m > n$. Then $y \in \Pi^r_\varkappa$ and so $x \in \pi_n\Pi^r_\varkappa$. ▷

3.3. Proposition. *Every parallelotope uniquely determines its center, tilt, and radius: if $z + \Pi^r_\varkappa = z' + \Pi^{r'}_{\varkappa'}$, then $z = z'$, $\varkappa = \varkappa'$, and $r = r'$.*

◁ Let $z + \Pi^r_\varkappa = z' + \Pi^{r'}_{\varkappa'}$. First of all, note that $z = z'$, since a nonempty bounded set cannot have two centers of symmetry. Therefore, we may assume that $z = z' = 0$.

Consider $n \in \mathbb{N}$ and show that $r(n) = r'(n)$ and $\varkappa_{n-1} = \varkappa'_{n-1}$. The case $n = 1$ is trivial. Let $n > 1$. For each $x \in \pi_{n-1}\Pi^r_\varkappa = \pi_{n-1}\Pi^{r'}_{\varkappa'}$, by 3.2 we see that

$$\big|\lambda - \langle \varkappa_{n-1} \,|\, x\rangle\big| < r(n) \ \Leftrightarrow\ (x,\lambda) \in \pi_n\Pi^r_\varkappa \ \Leftrightarrow\ (x,\lambda) \in \pi_n\Pi^{r'}_{\varkappa'} \ \Leftrightarrow\ \big|\lambda - \langle \varkappa'_{n-1} \,|\, x\rangle\big| < r'(n)$$

for all $\lambda \in \mathbb{R}$; whence $r(n) = r'(n)$ and $\langle \varkappa_{n-1} \,|\, x\rangle = \langle \varkappa'_{n-1} \,|\, x\rangle$. Since $x \in \pi_{n-1}\Pi^r_\varkappa$ is arbitrary, the last equality means that the functional $\langle \varkappa_{n-1} - \varkappa'_{n-1}|$ is constant on $\pi_{n-1}\Pi^r_\varkappa$. Consequently, $\varkappa_{n-1} = \varkappa'_{n-1}$, as $\pi_{n-1}\Pi^r_\varkappa$ is a neighborhood of the origin in $\mathbb{R}^{n-1}$ by 3.2. ▷

3.4. Theorem. *The following properties of $P \subset \mathbb{R}^{\mathbb{N}}$ are equivalent:*

(a) *P is a parallelotope centered at the origin; i.e., $P = \Pi^r_\varkappa$ for some $\varkappa \in \prod_{n\in\mathbb{N}} \mathbb{R}^n$ and $r \in]0,\infty[^{\mathbb{N}}$;*

(b) *$P = \big\{y \in \mathbb{R}^{\mathbb{N}} : \big|\langle \nabla e_n \,|\, y\rangle\big| < 1$ for all $n \in \mathbb{N}\big\}$ for some $\nabla \in \triangledown_{+}\big(\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}\big)$;*

(c) *$P = \Delta(\Pi^1_0)$ for some $\Delta \in \triangle^{+}(\mathbb{R}^{\mathbb{N}})$.*

Moreover, $\varkappa$, r, ∇, and Δ are uniquely determined by P and satisfy the following:

$$r(n) = \frac{1}{(\nabla e_n)(n)}; \quad \varkappa_{n-1} = -r(n)\,\pi_{n-1}\nabla e_n; \tag{3}$$

$$\nabla e_n = \frac{1}{r(n)}\big(-\varkappa_{n-1}(1),\dots,-\varkappa_{n-1}(n-1),1,0,0,\dots\big); \tag{4}$$

$$\Delta = (\nabla')^{-1} = (\nabla^{-1})'; \quad \nabla = (\Delta')^{-1} = (\Delta^{-1})'.$$

◁ (a)⇒(b): Let $\varkappa$ and r meet (a). According to 2.8(a), there is an inductive operator ∇ satisfying (4) for all $n \in \mathbb{N}$; moreover, $\nabla \in \triangledown_{+}\big(\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}\big)$, since $[\nabla](n,n) = (\nabla e_n)(n) = \frac{1}{r(n)} > 0$. It remains to note that

$$\big|y(n) - \langle \varkappa_{n-1} \,|\, \pi_{n-1}y\rangle\big| < r(n) \ \Leftrightarrow\ \left|\frac{1}{r(n)}y(n) - \left\langle \frac{1}{r(n)}\varkappa_{n-1} \,\middle|\, \pi_{n-1}y\right\rangle\right| < 1 \ \Leftrightarrow\ \big|\langle \nabla e_n \,|\, y\rangle\big| < 1$$

for $y \in \mathbb{R}^{\mathbb{N}}$ and $n \in \mathbb{N}$.

(b)⇒(c): Let P and ∇ satisfy (b). By 2.12, ∇' and $\Delta := (\nabla')^{-1}$ belong to the group $\triangle^{+}(\mathbb{R}^{\mathbb{N}})$. Moreover, for all $y \in \mathbb{R}^{\mathbb{N}}$ and $n \in \mathbb{N}$, the inequality $\big|\langle \nabla e_n \,|\, y\rangle\big| < 1$ is equivalent to $|(\nabla' y)(n)| < 1$, and so

$$y \in P \ \Leftrightarrow\ \nabla' y \in \Pi^1_0 \ \Leftrightarrow\ y \in (\nabla')^{-1}(\Pi^1_0) = \Delta(\Pi^1_0).$$

(c)⇒(a): Let $\Delta \in \triangle^{+}(\mathbb{R}^{\mathbb{N}})$ and $P = \Delta(\Pi_0^1)$. Owing to 2.12, the operator $\nabla := (\Delta^{-1})'$ belongs to $\triangledown_{+}(\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}})$. Define r and $\varkappa$ in accordance with (3) and show that $P = \Pi^r_\varkappa$. Indeed,

$$\begin{aligned}(\Delta^{-1}y)(n) &= \langle e_n \,|\, \Delta^{-1}y\rangle = \big\langle(\Delta^{-1})'e_n \,\big|\, y\big\rangle = \langle \nabla e_n \,|\, y\rangle \\ &= (\nabla e_n)(n)y(n) + \langle \pi_{n-1}\nabla e_n \,|\, \pi_{n-1}y\rangle = \frac{1}{r(n)}y(n) - \Big\langle \frac{1}{r(n)}\varkappa_{n-1} \,\Big|\, \pi_{n-1}y\Big\rangle\end{aligned}$$

for all $y \in \mathbb{R}^{\mathbb{N}}$ and $n \in \mathbb{N}$. Consequently, the inequalities $|(\Delta^{-1}y)(n)| < 1$ and $\big|y(n) - \langle\varkappa_{n-1} \,|\, \pi_{n-1}y\rangle\big| < r(n)$ are equivalent, and so

$$y \in P \;\Leftrightarrow\; y \in \Delta(\Pi_0^1) \;\Leftrightarrow\; \Delta^{-1}y \in \Pi_0^1 \;\Leftrightarrow\; y \in \Pi^r_\varkappa.$$

Demonstrate the uniqueness of $\varkappa$, r, ∇, and Δ in the claim. The uniqueness of $\varkappa$ and r is justified in 3.3. If ∇ meets (b); then, as is seen from the proof of the implication (c)⇒(a), the equality $P = \Pi^r_\varkappa$ holds, where r and $\varkappa$ are determined by (3). Then 3.3 implies that P uniquely determines the values ∇e_n with $n \in \mathbb{N}$ and hence the operator ∇. If Δ satisfies (c), then $\nabla := (\Delta^{-1})'$ satisfies (b), and so P uniquely determines ∇ and $\Delta = (\nabla')^{-1}$. ▷

3.5. The following is a consequence of 2.12(b) and 3.4:

Corollary. *If P is a parallelotope and $\Delta \in \triangle^{+}(\mathbb{R}^{\mathbb{N}})$, then $\Delta(P)$ and $\Delta^{-1}(P)$ are parallelotopes.*

3.6. Corollary. *Each parallelotope is a nonempty projective bounded quasiopen convex subset of $\mathbb{R}^{\mathbb{N}}$.*

◁ The properties listed are enjoyed by Π_0^1; therefore, according to 2.9(b) and 2.12(b), they are also enjoyed by every set of the form $z + \Delta(\Pi_0^1)$, with $z \in \mathbb{R}^{\mathbb{N}}$ and $\Delta \in \triangle^{+}(\mathbb{R}^{\mathbb{N}})$. It remains to refer to 3.4. ▷

3.7. Lemma. *Let C be a projective convex subset of $\mathbb{R}^{\mathbb{N}}$. Then $\operatorname{qi} C \neq \varnothing$ if and only if C includes some parallelotope.*

◁ Sufficiency follows from 3.6. Show necessity. Assume that $0 \in \operatorname{qi} C$. According to 1.8, the projection $\pi_n C$ is a neighborhood of the origin in $\mathbb{R}^n$ for each $n \in \mathbb{N}$; therefore, there are sequences of elements $c_n \in C$ and reals $\varepsilon_n > 0$ such that

$$\pi_n c_n = \varepsilon_n \pi_n e_n \quad \text{for all } n \in \mathbb{N}.$$

Since $\pi_n c_{n+1} = \pi_n \pi_{n+1} c_{n+1} = \pi_n(\varepsilon_{n+1}\pi_{n+1}e_{n+1}) = 0$; by 2.8(b), there is a projective operator $\Delta \in \mathcal{L}(\mathbb{R}^{\mathbb{N}})$ such that $\Delta e_n = \frac{1}{2^{n+1}} c_n$ for all $n \in \mathbb{N}$. Moreover,

$$[\Delta](n,n) = (\Delta e_n)(n) = \frac{1}{2^{n+1}} c_n(n) = \frac{1}{2^{n+1}}(\pi_n c_n)(n) = \frac{1}{2^{n+1}}(\varepsilon_n \pi_n e_n)(n) > 0,$$

and so $\Delta \in \triangle^{+}(\mathbb{R}^{\mathbb{N}})$. From 2.9(b) it follows that $D := \Delta^{-1}(C)$ is a projective convex subset of $\mathbb{R}^{\mathbb{N}}$, with $0 \in D$ and $2^{n+1}e_n \in D$ for all $n \in \mathbb{N}$. Put $z := (1, 1, \dots) \in \mathbb{R}^{\mathbb{N}}$ and show that $z + \Pi_0^1 \subset D$.

Let $y \in \Pi_0^1$. Since D is projective, it suffices to fix $n \in \mathbb{N}$ and establish that $x := \pi_n(z + y) \in \pi_n D$. Given $i \in \{1, \dots, n\}$, we have $x(i) = 1 + y(i)$, with $|y(i)| < 1$, and so $0 < x(i) < 2$. Consequently,

$$\begin{aligned} x &= x(1)\pi_n e_1 + x(2)\pi_n e_2 + \dots + x(n)\pi_n e_n \\ &= \frac{x(1)}{4}\pi_n(4e_1) + \frac{x(2)}{8}\pi_n(8e_2) + \dots + \frac{x(n)}{2^{n+1}}\pi_n\big(2^{n+1}e_n\big) + \Big(1 - \sum_{i=1}^{n} \frac{x(i)}{2^{i+1}}\Big)0, \end{aligned}$$

and hence x belongs to $\pi_n D$ as a convex combination of the elements

$$\pi_n(4e_1), \pi_n(8e_2), \dots, \pi_n\big(2^{n+1}e_n\big), 0 \in \pi_n D.$$

From $z + \Pi_0^1 \subset D$ it follows that $C = \Delta(D)$ includes the parallelotope $\Delta z + \Delta(\Pi_0^1)$ (see 3.4). ▷

In connection with 3.7, the question arises naturally as to whether the parallelotopes form a "quasi-interior base" in the following sense: If C is a projective convex subset of $\mathbb{R}^{\mathbb{N}}$ and $x \in \operatorname{qi} C$, then does C include a parallelotope centered at x or at least a parallelotope containing x? The answer is given below in 5.5 and 5.6.

3.8. Theorem. *The following properties of $S \subset \mathbb{R}^{\mathbb{N}}$ are equivalent:*
(a) *S is quasidense in $\mathbb{R}^{\mathbb{N}}$;*
(b) *S has nonempty intersection with every parallelotope;*
(c) *$\Delta(S)$ is box dense in $\mathbb{R}^{\mathbb{N}}$ for every $\Delta \in \triangle^{+}(\mathbb{R}^{\mathbb{N}})$.*

◁ The equivalence (a)⇔(b) follows from 1.11, 3.6, and 3.7.
(b)⇒(c): Let $\Delta \in \triangle^{+}(\mathbb{R}^{\mathbb{N}})$. Since every basic open box

$$B = \prod_{n\in\mathbb{N}} \,]z(n) - r(n), z(n) + r(n)[$$

(see (2)) is the parallelotope $z + \Pi_0^r$; by 3.5, the set $\Delta^{-1}(B)$ is also a parallelotope. Then (b) implies $S \cap \Delta^{-1}(B) \neq \varnothing$ and hence $\Delta(S) \cap B \neq \varnothing$.

(c)⇒(b): By 3.4, every parallelotope has the form $z+\Delta(\Pi_0^1)$ for some $z \in \mathbb{R}^{\mathbb{N}}$ and $\Delta \in \triangle^{+}(\mathbb{R}^{\mathbb{N}})$. Since $\Delta^{-1} \in \triangle^{+}(\mathbb{R}^{\mathbb{N}})$ (see 2.12(b)) and $\Delta^{-1}z + \Pi_0^1$ is a box; from (c) it follows that $\Delta^{-1}(S) \cap (\Delta^{-1}z + \Pi_0^1) \neq \varnothing$ and so $S \cap (z + \Delta(\Pi_0^1)) \neq \varnothing$. ▷

3.9. Corollary. *If S is a quasidense subset of $\mathbb{R}^{\mathbb{N}}$ and $\Delta \in \triangle^{+}(\mathbb{R}^{\mathbb{N}})$, then $\Delta(S)$ is quasidense in $\mathbb{R}^{\mathbb{N}}$.*

§ 4. Examples

In this section, we provide some counterexamples to the three conjectures in [1] on the polars of cones [1, 9.6], on connections between quasidenseness and projectivity [1, 9.7], and on the spaces with all linearly independent sets closed [1, 9.11].

4.1. If X and Y is a dual pair of vector spaces; then, given $S \subset X$, there are defined the polars (see [1, 3.7])

$$S^{\oplus} := \big\{y \in Y : \langle s\,|\,y\rangle \geqslant 0 \text{ for all } s \in S\big\};$$
$$S^{\boxplus} := \big\{y \in Y : \langle s\,|\,y\rangle > 0 \text{ for all } s \in S\backslash\{0\}\big\}.$$

A sequence of subsets $S_n \subset \mathbb{R}_n^{\mathbb{N}}$ $(n \in \mathbb{N})$ is *inductive* (see [1, 8.1]) if it has each of the following equivalent properties:
(a) there is $S \subset \mathbb{R}_{\text{fin}}^{\mathbb{N}}$ such that $S_n = S \cap \mathbb{R}_n^{\mathbb{N}}$ for all $n \in \mathbb{N}$;
(b) $S_n = S_m \cap \mathbb{R}_n^{\mathbb{N}}$ for $n \leqslant m$;
(c) $S_n = S_{n+1} \cap \mathbb{R}_n^{\mathbb{N}}$ for all $n \in \mathbb{N}$.
In this case, the set S satisfying (a) is unique and equal to $\bigcup_{n\in\mathbb{N}} S_n$.

Lemma [1, 8.3]**.** *If $K_n \subset \mathbb{R}_n^{\mathbb{N}}$ with $n \in \mathbb{N}$ is an inductive sequence of closed cones, then $(\pi_n K_n)^{\boxplus}$ with $n \in \mathbb{N}$ is a projective sequence and*

$$\varprojlim\, (\pi_n K_n)^{\boxplus} = \Big(\bigcup_{n\in\mathbb{N}} K_n\Big)^{\boxplus}.$$

The following example gives the negative answer to the question [1, 9.6] as to whether the sequence of polars $(\pi_n K_n)^{\oplus}$ is projective in this situation:

Example. Consider the closed cone $K_3 := \mathbb{R}^{+}(\{1\} \times D) \subset \mathbb{R}^3$, where $D = \{(y, z) : y^2 + (z-1)^2 \leqslant 1\}$ (see Fig. 1).

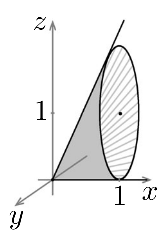

Fig. 1

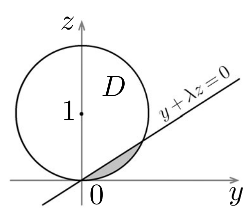

Fig. 2

Put $\mathbb{R}^3_2 := \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$, $K_2 := \pi_2(K_3 \cap \mathbb{R}^3_2) = \{(x, 0) : x \geqslant 0\}$, and show that $K_2^{\oplus} \neq \pi_2 K_3^{\oplus}$. Indeed, $(0,1) \in K_2^{\oplus}$, but no triple $(0,1,\lambda)$, with $\lambda \in \mathbb{R}$, belongs to $K_3^{\oplus}$; since the half-plane $\{(y,z) : y + \lambda z < 0\}$ intersects the circle D for every $\lambda \in \mathbb{R}$ (see Fig. 2), and, given a pair (y,z) in the intersection, we have $(1,y,z) \in K_3$ and $\langle (1,y,z) \,|\, (0,1,\lambda)\rangle = y + \lambda z < 0$. Therefore, for $S := K_3 \times \{(0,0,\dots)\} \subset \mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}$, the sequence of polars $\big(\pi_n(S \cap \mathbb{R}^{\mathbb{N}}_n)\big)^{\oplus}$ is not projective.

4.2. According to 1.11, the quasideneness of $S \subset \mathbb{R}^{\mathbb{N}}$ is equivalent to each of the two conditions:

(a) S intersects every nonempty convex set $B \subset \mathbb{R}^{\mathbb{N}}$ that is bounded, quasiopen, and *projective*;

(b) S intersects every convex set $C \subset \mathbb{R}^{\mathbb{N}}$ that has nonempty quasi-interior and is *projective*.

The example presented below answers the question [1, 9.7] and shows that the projectivity requirement for B and C in (a) and (b) is essential even for the case that S is a dense vector subspace of $\mathbb{R}^{\mathbb{N}}$.

Example. There exist a quasidense (and therefore dense) subspace $Y \subset \mathbb{R}^{\mathbb{N}}$ and a nonempty bounded quasiopen convex subset $B \subset \mathbb{R}^{\mathbb{N}}$ such that $Y \cap B = \varnothing$.

$\lhd$ Let T be a transcendence basis for $\mathbb{R}$ over $\mathbb{Q}$, i.e., a maximal algebraically independent subset of $\mathbb{R}$ over $\mathbb{Q}$ or, which is the same, an algebraically independent subset T of $\mathbb{R}$ over $\mathbb{Q}$ such that $\operatorname{alg}\mathbb{Q}(T) = \mathbb{R}$. (Here $\mathbb{Q}(T)$ is the subfield of $\mathbb{R}$ generated by T; and $\operatorname{alg} F$ is the subfield of $\mathbb{R}$ consisting of all reals that are algebraic over a subfield $F \subset \mathbb{R}$.) Consider a sequence $(t_n)_{n\in\mathbb{N}}$ of pairwise distinct elements in T and put $T_0 := T \backslash \{t_n : n \in \mathbb{N}\}$, $T_n := T_0 \cup \{t_1, \dots, t_n\}$, and $F_n := \operatorname{alg}\mathbb{Q}(T_n)$. Then $(F_n)_{n\in\mathbb{N}}$ is a sequence of subfields of $\mathbb{R}$ with the properties

(i) $F_n \subset F_{n+1}$ for all $n \in \mathbb{N}$;

(ii) $\bigcup_{n\in\mathbb{N}} F_n = \mathbb{R}$;

(iii) $\mathbb{R}$ is infinite-dimensional as a vector space over F_n for each $n \in \mathbb{N}$.

Indeed, owing to the equality $\operatorname{alg}\mathbb{Q}(T) = \mathbb{R}$, for every $\lambda \in \mathbb{R}$ there is a nonzero polynomial $p(x)$ with coefficients in $\mathbb{Q}(T)$ such that $p(\lambda) = 0$; whence, by the obvious relations $\mathbb{Q}(T_n) \subset \mathbb{Q}(T_{n+1})$ and $\mathbb{Q}(T) = \bigcup_{n\in\mathbb{N}} \mathbb{Q}(T_n)$, it follows that all coefficients of $p(x)$ belong to $\mathbb{Q}(T_n)$ for some $n \in \mathbb{N}$, and so $\lambda \in \operatorname{alg}\mathbb{Q}(T_n) = F_n$. Moreover, it is easy to see that $F_n \neq \mathbb{R}$ and, given $\lambda \in \mathbb{R}\backslash F_n$, the reals $\lambda, \lambda^2, \lambda^3, \dots$ are linearly independent over F_n.

According to (iii), there is a sequence of $x_n \in \mathbb{R}^{\mathbb{N}}_n$ such that, for each $n \in \mathbb{N}$, the reals $x_n(1), \dots, x_n(n)$ are linearly independent over F_n, and $x_n(n) > 0$. Consider the automorphism $\nabla \in \triangledown_{+}\big(\mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}\big)$ with values $\nabla e_n = x_n$ for all $n \in \mathbb{N}$ (see 2.8(a) and 2.10), and put

$$Y := \nabla'(\operatorname{lin}\mathbb{Q}^{\mathbb{N}}), \quad B := \big\{b \in \Pi^1_0 \cap \mathbb{R}^{\mathbb{N}}_{\mathrm{fin}} : b(1) > 0\big\}.$$

It is clear that B is nonempty, convex, bounded, and quasiopen (see 1.8). Using 1.11(d), it is easy to show that $\operatorname{lin}\mathbb{Q}^{\mathbb{N}}$ is a quasidense subspace of $\mathbb{R}^{\mathbb{N}}$. Next, $\nabla' \in \triangle^{+}(\mathbb{R}^{\mathbb{N}})$ (see 2.12(a)), and hence Y is quasidense in $\mathbb{R}^{\mathbb{N}}$ (see 3.9). The denseness of Y in $\mathbb{R}^{\mathbb{N}}$ follows from its quasidenseness according to 1.10 (see also 2.9(c)). To complete the proof, it suffices to establish the equality $Y \cap \mathbb{R}^{\mathbb{N}}_{\mathrm{fin}} = \{0\}$.

Show that $\nabla' z \notin \mathbb{R}^{\mathbb{N}}_{\mathrm{fin}}$ for every nonzero $z \in \operatorname{lin}\mathbb{Q}^{\mathbb{N}}$. Let

$$z = \sum_{j=1}^{k} \lambda_j q_j, \quad \lambda_j \in \mathbb{R},\ q_j \in \mathbb{Q}^{\mathbb{N}},$$

and assume that $z(l) \neq 0$ for some $l \in \mathbb{N}$. Owing to (i) and (ii), there is $m \geqslant l$ such that $\lambda_1, \dots, \lambda_k \in F_m$. Consider $n \geqslant m$ and show that $(\nabla' z)(n) \neq 0$. Indeed, $z(i) = \sum_{j=1}^{k} \lambda_j q_j(i) \in F_m \subset F_n$ for all $i \in \mathbb{N}$. Then $\langle x_n \,|\, z\rangle = \sum_{i=1}^{n} x_n(i) z(i)$ is a linear combination of the reals $x_n(1), \dots, x_n(n)$ that are linearly independent over F_n, with coefficients $z(1), \dots, z(n) \in F_n$; and the combination is nontrivial, since $n \geqslant l$ and $z(l) \neq 0$. Therefore, $\langle x_n \,|\, z\rangle \neq 0$. It remains to note that $\langle x_n \,|\, z\rangle = \langle \nabla e_n \,|\, z\rangle = \langle e_n \,|\, \nabla' z\rangle = (\nabla' z)(n)$. $\rhd$

4.3. In [3], some examples are provided of dense proper vector subspaces $Y \subset \mathbb{R}^{\mathbb{N}}$ for which all linearly independent sets are closed in $\mathbb{R}^{\mathbb{N}}_{\text{fin}}|Y$ (see [3, 4.8]); and it is shown that the existence of a nonclosed linearly independent set implies the existence of a nonclosed Archimedean cone (see [3, 4.7]). The question whether the converse holds was left open and formulated explicitly in [1, 9.11]. The example presented below gives the negative answer.

Given a numerical sequence $(\lambda_n)_{n\in\mathbb{N}}$ and a real λ, write $\lambda_n \twoheadrightarrow \lambda$ if $\lambda_n \to \lambda$ and there exists $\bar{n} \in \mathbb{N}$ such that $\lambda_n \neq \lambda$ for $n \geqslant \bar{n}$. Call $\Lambda \subset \mathbb{R}$ *sparse* if Λ has the equivalent properties

(a) Λ is closed and discrete;

(b) all bounded subsets of Λ are finite;

(c) there are no sequences $(\lambda_n)_{n\in\mathbb{N}}$ in Λ and reals $\alpha \in \mathbb{R}$ such that $\lambda_n \twoheadrightarrow \alpha$.

Lemma. *The set*

$$\Lambda(\alpha_1, \dots, \alpha_k) := \{\alpha_1 2^{n_1} + \dots + \alpha_k 2^{n_k} : n_1, \dots, n_k \in \mathbb{N}\}$$

is sparse for all $k \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_k \in \mathbb{R}$.

$\triangleleft$ Proceed by induction on k. The sparseness of $\Lambda(\alpha) = \{\alpha 2^n : n \in \mathbb{N}\}$ for every α is clear. Suppose that $\Lambda(\beta_1, \dots, \beta_{k-1})$ is sparse for all $\beta_1, \dots, \beta_{k-1}$, consider arbitrary $\alpha_1, \dots, \alpha_k$, and assume to the contrary that $\Lambda(\alpha_1, \dots, \alpha_k)$ is not sparse. Then there exist sequences $\nu_1, \dots, \nu_k \in \mathbb{N}^{\mathbb{N}}$ and a real α such that

$$\lambda(n) := \alpha_1 2^{\nu_1(n)} + \dots + \alpha_k 2^{\nu_k(n)} \twoheadrightarrow \alpha.$$

The current goal is to find a contradiction.

Observe that each of the sequences $\nu_1, \dots, \nu_k$ tends to infinity. Indeed, if, for example, $\nu_1 \not\twoheadrightarrow \infty$; then ν_1 has a constant subsequence $\nu_1(n_m) \equiv i$. In this case,

$$\lambda(n_m) = \alpha_1 2^i + \alpha_2 2^{\nu_2(n_m)} + \dots + \alpha_k 2^{\nu_k(n_m)} \twoheadrightarrow \alpha$$

and then

$$\alpha_2 2^{\nu_2(n_m)} + \dots + \alpha_k 2^{\nu_k(n_m)} \twoheadrightarrow \alpha - \alpha_1 2^i$$

despite the sparseness of $\Lambda(\alpha_2, \dots, \alpha_k)$.

Put $\mu(n) := \min\{\nu_1(n), \dots, \nu_k(n)\} - 1$ $(n \in \mathbb{N})$. Then

$$\lambda(n) = 2^{\mu(n)}\big(\alpha_1 2^{\mu_1(n)} + \dots + \alpha_k 2^{\mu_k(n)}\big),$$

with $\mu_i(n) = \nu_i(n) - \mu(n)$. Moreover, $\mu(n) \to \infty$ and, for each $n \in \mathbb{N}$, at least one of the naturals $\mu_1(n), \dots, \mu_k(n)$ is equal to 1. For definiteness, assume that μ_1 has a constant subsequence $\mu_1(n_m) \equiv 1$. In this case,

$$\lambda(n_m) = 2^{\mu(n_m)}\big(2\alpha_1 + \alpha_2 2^{\mu_2(n_m)} + \dots + \alpha_k 2^{\mu_k(n_m)}\big) \twoheadrightarrow \alpha;$$

whence, as $\mu(n_m) \to \infty$ it follows that $\alpha_2 2^{\mu_2(n_m)} + \dots + \alpha_k 2^{\mu_k(n_m)} \twoheadrightarrow -2\alpha_1$ contrary to the sparseness of $\Lambda(\alpha_2, \dots, \alpha_k)$. $\triangleright$

Example. Put $Y = \operatorname{lin} \prod_{n\in\mathbb{N}} \{2^{n+m} : m \in \mathbb{N}\}$. Then Y is a dense subspace of $\mathbb{R}^{\mathbb{N}}$, and all linearly independent sets in $\mathbb{R}^{\mathbb{N}}_{\text{fin}}|Y$ are closed, but there are nonclosed Archimedean cones.

$\triangleleft$ The space Y is dense in $\mathbb{R}^{\mathbb{N}}$; since, for each $m \in \mathbb{N}$,

$$e_m = \frac{1}{2^{m+1}}\big(d + 2^{m+1} e_m\big) - \frac{1}{2^{m+1}} d \in Y, \quad \text{where } d = (2^{n+1})_{n\in\mathbb{N}}.$$

The closedness of all linearly independent sets in $\mathbb{R}^{\mathbb{N}}_{\text{fin}}|Y$ can be proven in exactly the same way as in [3, 4.8]. According to 1.11(c), 1.9, and 1.12(a), in order to establish the existence of a nonclosed Archimedean cone in $\mathbb{R}^{\mathbb{N}}_{\text{fin}}|Y$, it suffices to show that $Y \cap\,]0,1[^{\mathbb{N}} = \varnothing$. Each $y \in Y$ has the form

$$y(n) = \alpha_1 2^{n+m_1(n)} + \dots + \alpha_k 2^{n+m_k(n)} = 2^n \lambda(n) \quad (n \in \mathbb{N}),$$

where $k \in \mathbb{N}$, $\alpha_1, \dots, \alpha_k \in \mathbb{R}$, $m_1(n), \dots, m_k(n) \in \mathbb{N}$, and

$$\lambda(n) = \alpha_1 2^{m_1(n)} + \dots + \alpha_k 2^{m_k(n)} \in \Lambda(\alpha_1, \dots, \alpha_k).$$

If $y \in\,]0,1[^{\mathbb{N}}$, i.e., $0 < 2^n \lambda(n) < 1$ for all $n \in \mathbb{N}$, then $\lambda(n) \twoheadrightarrow 0$, which contradicts the sparseness of $\Lambda(\alpha_1, \dots, \alpha_k)$. $\triangleright$

§ 5. Quasidenseness and Topological Denseness

In this section, we answer the question [1, 9.9] and establish that the quasidenseness in $\mathbb{R}^{\mathbb{N}}$ is not equivalent to the denseness with respect to the box topology. We also give some examples related to the question [1, 9.10] about the topological nature of quasidenseness and show that the parallelotopes do not form a base for any topology and therefore do not characterize the quasidenseness in $\mathbb{R}^{\mathbb{N}}$ as a topological denseness. Moreover, we give the negative answer to the question in 3.7 as to whether the parallelotopes form a base for quasi-interior.

5.1. Lemma. *There is a sequence $\varkappa \in \prod_{n\in\mathbb{N}} \mathbb{R}^n$ such that*

$$\Pi^1_\varkappa \cap \Pi^1_0 = \{0\}.$$

In particular, if $Y \subset \mathbb{R}^{\mathbb{N}}$ and $z \notin Y$, then $Y \cap (z + \Pi^1_\varkappa) \cap (z + \Pi^1_0) = \varnothing$.

◁ Let $(N_m)_{m\in\mathbb{N}}$ be a partition of $\mathbb{N}$ into infinite subsets $N_m \subset \mathbb{N}$. Define $\varkappa \in \prod_{n\in\mathbb{N}} \mathbb{R}^n$ by putting $\varkappa_n = n\pi_n e_m$ for $n \in N_m$. Consider $y \in \Pi^1_\varkappa$ with nonzero value $y(m) \neq 0$ for some $m \in \mathbb{N}$, and show that $y \notin \Pi^1_0$. Indeed, if $n \in N_m$ and $n \geqslant m$, then $\langle \varkappa_n \,|\, \pi_n y\rangle = \langle n\pi_n e_m \,|\, \pi_n y\rangle = ny(m)$ and so, as $y \in \Pi^1_\varkappa$,

$$|y(n+1) - ny(m)| = |y(n+1) - \langle \varkappa_n \,|\, \pi_n y\rangle| < 1$$

for these n; in particular, $|y(n+1)| > n|y(m)| - 1$. Therefore, $\sup_{n\in N_m} |y(n+1)| = \infty$. ▷

5.2. According to 3.8, every quasidense subset of $\mathbb{R}^{\mathbb{N}}$ is box dense (see also 1.9 and 1.11(c)). As the following example shows, the converse is not true even for dense vector subspaces of $\mathbb{R}^{\mathbb{N}}$, which gives the negative answer to the question [1, 9.9].

Example. There is a dense subspace of $\mathbb{R}^{\mathbb{N}}$ that is box dense but not quasidense in $\mathbb{R}^{\mathbb{N}}$.

◁ Let Z be a box dense subspace of $\mathbb{R}^{\mathbb{N}}$ that does not include ℓ^∞ and has dense intersection $Z \cap \ell^\infty$ in $\mathbb{R}^{\mathbb{N}}$. (For example, $\operatorname{lin} \mathbb{Q}^{\mathbb{N}}$ is suitable for the role of Z.) Owing to 5.1, there is a parallelotope P centered at the origin such that $P \cap \Pi^1_0 = \{0\}$. Using the absolute convexity of P, it is easy to show that $(\operatorname{lin} P) \cap \ell^\infty = \{0\}$. Put $Y_0 := Z \cap \ell^\infty + \operatorname{lin} P$ and consider $b \in \ell^\infty$ not belonging to Z. As is easy to see, $b \notin Y_0$ and so there is a subspace $Y \subset \mathbb{R}^{\mathbb{N}}$ of codimension 1 such that $Y_0 \subset Y$ and $b \notin Y$. Since $Z \cap \ell^\infty \subset Y$, the space Y is dense in $\mathbb{R}^{\mathbb{N}}$. Moreover, $P \subset Y$ and $b \notin Y$ imply $Y \cap (b + P) = \varnothing$; therefore, by 3.8, the space Y is not quasidense in $\mathbb{R}^{\mathbb{N}}$. It remains to show that Y is box dense in $\mathbb{R}^{\mathbb{N}}$.

Consider $s \in \mathbb{R}^{\mathbb{N}}$ and $r \in\,]0,\infty[^{\mathbb{N}}$ and prove that $Y \cap (s + \Pi^r_0) \neq \varnothing$. We may assume that $r \in \ell^\infty$, i.e., $\Pi^r_0 \subset \ell^\infty$. Since $\operatorname{codim} Y = 1$, there exists $\alpha \in \mathbb{R}$ such that $s - \alpha b \in Y$. Owing to the box denseness of Z in $\mathbb{R}^{\mathbb{N}}$, there is an element $z \in Z \cap (\alpha b + \Pi^r_0)$. The inclusion $\alpha b + \Pi^r_0 \subset \ell^\infty$ implies $z \in Z \cap \ell^\infty \subset Y$. Moreover, $z + s - \alpha b \in s + \Pi^r_0$. Therefore, $z + s - \alpha b \in Y \cap (s + \Pi^r_0)$. ▷

5.3. From 5.1 it follows that the parallelotopes do not form a base for any topology. Despite the fact that every quasidense subspace $Y \subset \mathbb{R}^{\mathbb{N}}$ intersects each parallelotope; if $Y \neq \mathbb{R}^{\mathbb{N}}$, there exist two parallelotopes P and Q with a common center such that

$$Y \cap P \cap Q = \varnothing.$$

In particular, criteria 1.11 and 3.8 do not characterize the quasidenseness in $\mathbb{R}^{\mathbb{N}}$ as a topological denseness even for vector subspaces. This observation, however, does not answer the question [1, 9.10] as to whether there is a topology τ on $\mathbb{R}^{\mathbb{N}}$ such that τ-denseness is equivalent to quasidenseness in $\mathbb{R}^{\mathbb{N}}$.

5.4. In what follows, we consider each of the spaces $\mathbb{R}^n$ $(n \in \mathbb{N})$ endowed with the Euclidean norm $\|\cdot\| := \|\cdot\|_2$. Given $S \subset \mathbb{R}^n$, put $\|S\| := \sup_{s\in S} \|s\|$. Denote by 0_n the origin $(0,\dots,0)$ of $\mathbb{R}^n$. If $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, we designate the tuple $\big(x(1),\dots,x(n),\lambda\big) \in \mathbb{R}^{n+1}$ as (x,λ). In particular, $(0_n,\lambda) = (0,\dots,0,\lambda) = \lambda\pi_{n+1}e_{n+1} \in \mathbb{R}^{n+1}$.

Lemma. *Suppose that $n \in \mathbb{N}$, C is a bounded convex subset of $\mathbb{R}^n$, $0 \in \operatorname{int} C$, and $\lambda > 0$. Define the subsets D and D^+ of $\mathbb{R}^{n+1}$ by putting*

$$D := \operatorname{co}\bigl(C\times\{-1\} \cup \{(0_n,\lambda)\}\bigr), \quad D^+ := \{d \in D : d(n+1) \geqslant 0\}.$$

Then

(a) $0 \in \operatorname{int} D$;
(b) $D \cap -D \subset D^+ \cup -D^+$;
(c) $\|D \cap -D\| \leqslant \|D^+\| = \max\bigl\{\lambda, \frac{\lambda}{\lambda+1}\|C\|\bigr\}$;
(d) $\pi_n D = C$.

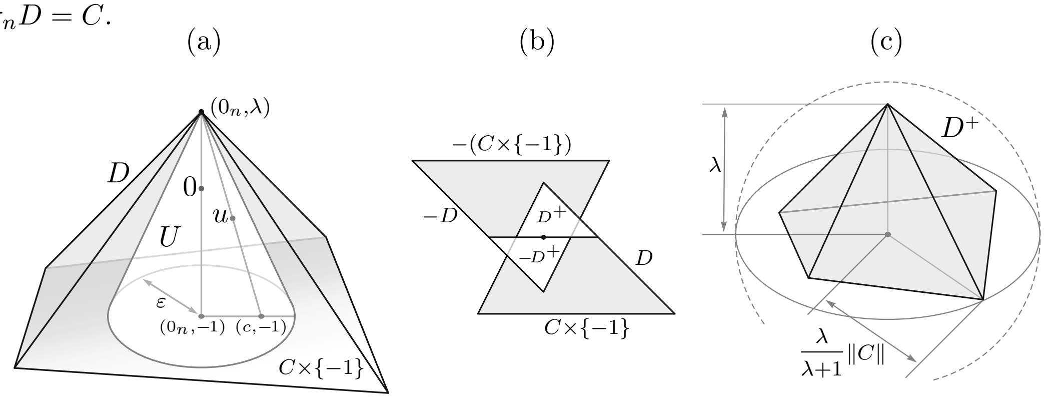

Fig. 3

◁ (a): By hypotheses, C includes some ball $\{x \in \mathbb{R}^n : \|x\| < \varepsilon\}$, $\varepsilon > 0$. An elementary check shows that the open neighborhood of the origin

$$U := \left\{u \in \mathbb{R}^{n+1} : \frac{\|\pi_n u\|}{\varepsilon} + \frac{u(n+1)+1}{\lambda+1} < 1,\ u(n+1) > -1\right\}$$

lies in D. Indeed, if $u \in U$ then, putting $c := \frac{\lambda+1}{\lambda-u(n+1)}\,\pi_n u$, we see that $\|c\| < \varepsilon$ and u belongs to the segment $[(c,-1),(0_n,\lambda)]$ (see Fig. 3(a)).

The remaining relations are trivial (see Fig. 3(b) and 3(c)). ▷

5.5. If x is a quasi-interior point of a projective convex set $C \subset \mathbb{R}^{\mathbb{N}}$; then, in view of Lemma 3.7, we might expect that C includes a parallelotope centered at x. Nevertheless, the proof of the lemma provides only the existence of a parallelotope $P \subset C$ whose center is different from x and, moreover, $x \notin P$. The following example shows that the above circumstance is significant:

Example. There is a projective convex subset $C \subset \mathbb{R}^{\mathbb{N}}$ such that $0 \in \operatorname{qi} C$, but C does not include any parallelotope centered at the origin.

◁ Consider the sequence of $C_n \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) defined recursively as follows (see Fig. 4):

$$C_1 := [-1,1],$$
$$C_{n+1} := \operatorname{co}\bigl(C_n\times\{-1\} \cup \{(0_n,\lambda_n)\}\bigr),$$

where the reals $\lambda_n > 0$ are chosen so that $\max\bigl\{\lambda_n, \frac{\lambda_n}{\lambda_n+1}\|C_n\|\bigr\} \leqslant \frac{1}{n}$.

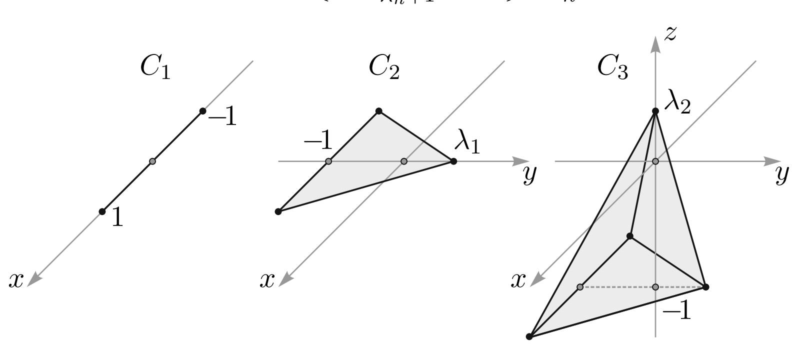

Fig. 4

By 5.4, the convex sets $(C_n)_{n\in\mathbb{N}}$ form a projective sequence, with $0 \in \operatorname{int} C_n$ and $\|C_n \cap -C_n\| \leqslant \frac{1}{n}$ for all $n \in \mathbb{N}$. Put $C := \varprojlim C_n$. From 1.8 it follows that $0 \in \operatorname{qi} C$. On the other hand, had C included a parallelotope centered at the origin, there would exist $c \in C \cap -C$ with $c(1) \neq 0$, and then the following contradictory relations would hold for all $n \in \mathbb{N}$:

$$|c(1)| \leqslant \|\pi_n c\| \leqslant \|\pi_n(C \cap -C)\| = \|C_n \cap -C_n\| \leqslant \frac{1}{n}. \ \triangleright$$

5.6. We strengthen the previous example and show that, in the case under consideration, the point $0 \in \operatorname{qi} C$ does not belong to any parallelotope included in C.

Lemma. *Let $P \subset \mathbb{R}^{\mathbb{N}}$ be a parallelotope and let $x \in P$. Then there is a parallelotope P_x centered at x such that $P_x \subset P$.*

$\triangleleft$ By 3.4, there exist $z \in \mathbb{R}^{\mathbb{N}}$ and $\Delta \in \triangle^{+}(\mathbb{R}^{\mathbb{N}})$ such that $P = z + \Delta(\Pi_0^1)$. As is easy to see, $y := \Delta^{-1}(x - z) \in \Pi_0^1$. Define $r \in \left]0, \infty\right[^{\mathbb{N}}$ by putting $r(n) := 1 - |y(n)|$. Then $y + \Pi_0^r \subset \Pi_0^1$ and so

$$P_x := x + \Delta(\Pi_0^r) = z + \Delta y + \Delta(\Pi_0^r) = z + \Delta(y + \Pi_0^r) \subset z + \Delta(\Pi_0^1) = P. \ \triangleright$$

Corollary. *If C does not include any parallelotope centered at x, then C does not include any parallelotope P such that $x \in P$.*

Therefore, every projective convex set $C \subset \mathbb{R}^{\mathbb{N}}$ with nonempty quasi-interior includes some parallelotope, but these parallelotopes do not always cover the quasi-interior of C.

5.7. A convex set C is *quasilocally bounded at $x \in \operatorname{qi} C$*, if $x \in \operatorname{qi} B$ for some bounded subset $B \subset C$. A space X is *quasilocally bounded*, if every convex set C in X is quasilocally bounded at each point $x \in \operatorname{qi} C$ (see [1, § 5]).

According to [1, 5.10], $\mathbb{R}^{\mathbb{N}}$ is quasilocally bounded, but 5.6 implies that the parallelotopes do not form a "base for quasilocal boundedness": if x is a quasi-interior point of a convex set $C \subset \mathbb{R}^{\mathbb{N}}$, then $x \in \operatorname{qi} B$ for some bounded subset $B \subset C$, but it may occur that there are no parallelotopes among these B.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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