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LATERAL CONVERGENCE AND HOMOMORPHISMS OF BANACH BUNDLES

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Abstract—We introduce and study the concepts of injective and lateral convergence in a topological space and obtain some results on the existence of homomorphisms of continuous Banach bundles, as well as continuous and weakly continuous vector-valued functions and sections that take preassigned values at the points of injectively and laterally convergent sequences.

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While speaking about continuous Banach bundles (CBBs), we use the terminology and notation of [1; 2, 2.4]. In general topology, we follow the monographs [3, 4]. In particular, the concepts of regular and completely regular topological space include separation. By a neighborhood of a point we mean a set that includes some open subset containing the point. The symbol "$\subset$" denotes the nonstrict inclusion.

If $\mathscr{X}$ and $\mathscr{Y}$ are CBBs over a topological space Q, the symbol $Hom(\mathscr{X},\mathscr{Y})$ denotes the set of all homomorphisms from $\mathscr{X}$ to $\mathscr{Y}$. (In [1, 2] this set is denoted by $Hom_Q(\mathscr{X},\mathscr{Y})$, and its elements are called Q-homomorphisms in [1].) According to Theorem [1, 2.4.4], $Hom(\mathscr{X},\mathscr{Y})$ coincides with the set of all mappings H that assign bounded linear operators $H(q)\in B\big(\mathscr{X}(q),\mathscr{Y}(q)\big)$ to points $q\in Q$, have locally bounded pointwise norm $\|\!|H|\!\| : q\in Q\mapsto \|H(q)\|$, and satisfy the condition $H\otimes u\in C(Q,\mathscr{Y})$ for all $u\in C(Q,\mathscr{X})$, where $H\otimes u : q\in Q\mapsto H(q)u(q)\in\mathscr{Y}(q)$. The symbol $\mathscr{X}^*$ denotes the set $Hom(\mathscr{X},Q\times\{\mathbb{R}\})$ of all homomorphisms acting from $\mathscr{X}$ to the constant CBB over Q with stalk $\mathbb{R}$.

For each point $q\in Q$, consider the subspace $\mathscr{X}^*(q):=\{H(q) : H\in\mathscr{X}^*\}$ of the dual Banach space $\mathscr{X}(q)'=B\big(\mathscr{X}(q),\mathbb{R}\big)$. In the theory of CBBs, the question remains open of how representative is $\mathscr{X}^*(q)$ in $\mathscr{X}(q)'$. For instance, there are various wide classes of bundles $\mathscr{X}$ (see, e.g., [5, 3.4.4]) for which at each point $q\in Q$ the space $\mathscr{X}^*(q)$ is norming, i.e., satisfies the condition

$$\|x\| \;=\; \sup\{|\langle x\,|\,y\rangle| : y\in\mathscr{X}^*(q),\ \|y\|\leqslant 1\},\quad x\in\mathscr{X}(q);$$

but not a single case of violation of this condition has been found so far. Moreover, the possibility of the equality $\mathscr{X}^*=\{0\}$ for a nonzero bundle $\mathscr{X}$ is questionable, while for all bundles known by now we have

$$\|x\| \;=\; \sup\{|H(q)x| : H\in\mathscr{X}^*,\ \|H\|_\infty\leqslant 1\},\quad x\in\mathscr{X}(q).$$

In this regard, general methods for constructing homomorphisms that have certain approximating properties remain relevant. These include the results obtained in this article on the existence of homomorphisms H that take preassigned values $H(q_n)$ and $H(q)$ for a given convergent sequence $q_n\to q$.

A sequence of points $q_n\in Q$ *converges injectively* to a point $q\in Q$ if $q_n\to q$, $q_n\neq q_m$ for $n\neq m$, and $q_n\neq q$ for all $n\in\mathbb{N}$. A *covering of a sequence* $(q_n)_{n\in\mathbb{N}}$ is an arbitrary sequence of neighborhoods U_n of q_n. A covering $(U_n)_{n\in\mathbb{N}}$ of a convergent sequence $q_n\to q$ is *laterally tangent* to q if

$$\operatorname{cl}U_m\cap\operatorname{cl}\bigcup_{n\neq m}U_n=\varnothing\quad\text{for all } m\in\mathbb{N} \tag{1}$$

and, moreover, q is a proper limit point of the union $\bigcup_{n\in\mathbb{N}}\operatorname{cl}U_n$.

Dedicated to Professor Alexander A. Tolstonogov on the occasion of his 85th birthday.

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In a regular space, every injectively convergent sequence $q_n \to q$ admits a covering that is laterally tangent to q (see Proposition 1.5). If the space is completely regular, then the injective convergence $q_n \to q$ allows us to construct continuous vector-valued functions, sections, and homomorphisms that take preassigned values at q_n and q. One of the results in this direction, Proposition 2.6, asserts the existence of a homomorphism $H \in \mathit{Hom}(\mathscr{X}, \mathscr{Y})$ that takes the values $H(q_n) = H_n(q_n)$ for any predetermined sequence of homomorphisms $H_n \in \mathit{Hom}(\mathscr{X}, \mathscr{Y})$ satisfying the condition $\|H_n\|_\infty \to 0$.

The uniform convergence $\|H_n\|_\infty \to 0$ is a rather restrictive requirement. In the context under consideration, a condition similar to pointwise convergence looks more natural: $\|H_n(q_n)u(q_n)\| \to 0$ for a sufficiently representative set of sections $u \in C(Q, \mathscr{X})$. The existence of the desired homomorphism H in this case can be ensured due to a more accurate covering of the sequence $(q_n)_{n\in\mathbb{N}}$ (see Theorem 2.10); namely, it suffices to have a covering $(U_n)_{n\in\mathbb{N}}$ that satisfies (1) and is such that q is the *unique* proper limit point of the union $\bigcup_{n\in\mathbb{N}} \operatorname{cl} U_n$. A covering with such a property is called *laterally converging* to q.

In the above-mentioned Theorem 2.10, the existence of a laterally convergent covering is an essential requirement. Example 2.11 shows that the injective convergence $q_n \to q$ alone is not sufficient for the existence of the desired homomorphism H, even in the case of a compact space Q.

Section 1 is devoted to the study of lateral coverings of convergent sequences; and in Section 2 the obtained results are applied to construct continuous vector-valued functions, sections, and homomorphisms that take preassigned values on sequences that admit lateral coverings.

§ 1. Lateral Convergence

In what follows, Q is an arbitrary topological space.

1.1. Definition. Agree to call sets $U, V \subset Q$ *disjoint* and write $U \perp V$ if $\operatorname{cl} U \cap \operatorname{cl} V = \varnothing$.

Say that a sequence of sets $U_n \subset Q$

(a) *is disjoint* if its members are pairwise disjoint: $U_m \perp U_n$ for $m \neq n$;

(b) *is lateral* if $U_m \perp \bigcup_{n\neq m} U_n$ for all $m \in \mathbb{N}$ or, which is the same, $U_m \perp \bigcup_{n>m} U_n$ for all $m \in \mathbb{N}$;

(c) *is laterally tangent* to a point $q \in Q$ if the sequence is lateral and, moreover, q is a proper limit point of the union $\bigcup_{n\in\mathbb{N}} \operatorname{cl} U_n$, i.e.,

$$q \in \Big(\operatorname{cl} \bigcup_{n\in\mathbb{N}} U_n\Big) \setminus \bigcup_{n\in\mathbb{N}} \operatorname{cl} U_n;$$

(d) *converges laterally* to a point $q \in Q$ if the sequence is lateral and, moreover, q is the unique proper limit point of the union $\bigcup_{n\in\mathbb{N}} \operatorname{cl} U_n$, i.e.,

$$\Big(\operatorname{cl} \bigcup_{n\in\mathbb{N}} U_n\Big) \setminus \bigcup_{n\in\mathbb{N}} \operatorname{cl} U_n = \{q\};$$

(e) *contracts* to a point $q \in Q$ if for every neighborhood V of q there exists $m \in \mathbb{N}$ such that $U_n \subset V$ for all $n \geqslant m$ (i.e., the filter generated by the tails $\bigcup_{n\geqslant m} U_n$, $m \in \mathbb{N}$, converges to q).

1.2. Lemma. *Consider $U_n \subset Q$ $(n \in \mathbb{N})$ and $q \in Q$. Put $\mathscr{U}_m := \bigcup_{n\geqslant m} \operatorname{cl} U_n$ $(m \in \mathbb{N})$. The following are equivalent:*

(a) *(U_n) converges laterally to q, i.e., (U_n) is lateral and $\operatorname{cl} \mathscr{U}_1 \setminus \mathscr{U}_1 = \{q\}$;*

(b) *(U_n) is disjoint and $\operatorname{cl} \mathscr{U}_m \setminus \mathscr{U}_m = \{q\}$ for all $m \in \mathbb{N}$;*

(c) *(U_n) is disjoint, $q \in \operatorname{cl} \mathscr{U}_1 \setminus \mathscr{U}_1$, and $\big(\forall\, p \in Q\setminus\{q\}\big)\big(\exists\, m \in \mathbb{N}\big)\big(p \notin \operatorname{cl} \mathscr{U}_m\big)$;*

(d) *$q \in \operatorname{cl} \mathscr{U}_1 \setminus \mathscr{U}_1$ and $\big(\forall\, p \in Q\setminus\{q\}\big)\big(\exists\, m \in \mathbb{N}\big)\big(p \notin \operatorname{cl} \bigcup_{n\neq m} U_n\big)$.*

◁ (a)⇒(b): It suffices to show the equality $\operatorname{cl} \mathscr{U}_m \setminus \mathscr{U}_m = \{q\}$ for $m \geqslant 2$. For all $i \in \{1, \dots, m-1\}$ we have

$$\operatorname{cl} U_i \cap \operatorname{cl} \mathscr{U}_m \subset \operatorname{cl} U_i \cap \operatorname{cl} \bigcup_{n\neq i} U_n = \varnothing;$$

therefore,

$$\begin{aligned}\operatorname{cl}\mathscr{U}_m\setminus\mathscr{U}_m&=\big(\operatorname{cl}\mathscr{U}_m\setminus(\operatorname{cl}U_1\cup\dots\cup\operatorname{cl}U_{m-1})\big)\setminus\mathscr{U}_m\\&=\big((\operatorname{cl}U_1\cup\dots\cup\operatorname{cl}U_{m-1}\cup\operatorname{cl}\mathscr{U}_m)\setminus(\operatorname{cl}U_1\cup\dots\cup\operatorname{cl}U_{m-1})\big)\setminus\mathscr{U}_m\\&=\big(\operatorname{cl}\mathscr{U}_1\setminus(\operatorname{cl}U_1\cup\dots\cup\operatorname{cl}U_{m-1})\big)\setminus\mathscr{U}_m\\&=\operatorname{cl}\mathscr{U}_1\setminus\mathscr{U}_1=\{q\}.\end{aligned}$$

(b)⇒(c): Let $p\in Q\setminus\{q\}$. Since the sets $\operatorname{cl}U_n$ are pairwise disjoint, there exists m such that $p\notin\mathscr{U}_m$. Then from the equality $\operatorname{cl}\mathscr{U}_m\setminus\mathscr{U}_m=\{q\}$ it follows that $p\notin\operatorname{cl}\mathscr{U}_m$.

(c)⇒(d): Let $p\in Q\setminus\{q\}$. Consider the smallest $k\in\mathbb{N}$ for which $p\notin\operatorname{cl}\mathscr{U}_k$. If $k\leqslant 2$ then there is nothing to prove. Let $k>2$. Then $p\in\operatorname{cl}\mathscr{U}_{k-1}=\operatorname{cl}U_{k-1}\cup\operatorname{cl}\mathscr{U}_k$, and hence, $p\in\operatorname{cl}U_{k-1}$. Since the sets $\operatorname{cl}U_1,\dots,\operatorname{cl}U_{k-2}$ do not intersect $\operatorname{cl}U_{k-1}$, we obtain

$$p\notin\operatorname{cl}U_1\cup\dots\cup\operatorname{cl}U_{k-2}\cup\operatorname{cl}\mathscr{U}_k=\operatorname{cl}\bigcup_{n\neq k-1}U_n.$$

(d)⇒(a): According to (d) for each point $p\in\operatorname{cl}\mathscr{U}_1\setminus\{q\}$ there is m such that

$$p\in\operatorname{cl}\mathscr{U}_1\setminus\operatorname{cl}\bigcup_{n\neq m}U_n=\operatorname{cl}U_m\cup\operatorname{cl}\bigcup_{n\neq m}U_n\setminus\operatorname{cl}\bigcup_{n\neq m}U_n=\operatorname{cl}U_m\subset\mathscr{U}_1,$$

which means $\operatorname{cl}\mathscr{U}_1\setminus\mathscr{U}_1\subset\{q\}$.

To justify the equality

$$\operatorname{cl}U_k\cap\operatorname{cl}\bigcup_{n\neq k}U_n=\varnothing,\quad k\in\mathbb{N},$$

it is sufficient to note that for each $p\in\operatorname{cl}U_k$ the natural m in (d) cannot differ from k. ▷

1.3. Definition. A *covering of a sequence* of points $q_n\in Q$ is an arbitrary sequence of sets $U_n\subset Q$, each of which is a neighborhood of the corresponding point q_n. A covering consisting of open (closed) neighborhoods is called an *open* (*closed*) *covering*.

1.4. Definition. Say that a sequence of points $q_n\in Q$

(a) *is injective* if $q_n\neq q_m$ for $n\neq m$;

(b) *converges injectively* to a point $q\in Q$ if the sequence converges to q, is injective, and, moreover, $q_n\neq q$ for all $n\in\mathbb{N}$;

(c) *converges laterally* to a point $q\in Q$ if the sequence converges to q and admits a covering that converges laterally to q.

1.5. Proposition. *In a regular topological space Q, every injectively convergent sequence $q_n\to q$ admits a covering that is laterally tangent to q.*

◁ From the injective convergence $q_n\to q$ and the Hausdorff property of Q it follows that

$$q_n\notin\operatorname{cl}\{q_m:m>n\},\quad n\in\mathbb{N}.$$

Define open sets U_n and V_n ($n\in\mathbb{N}$) by the following recursive procedure: put $V_0:=Q$ and, using the regularity of Q, for each $n\in\mathbb{N}$ choose open subsets $U_n,V_n\subset V_{n-1}$ so that

$$q_n\in U_n,\quad\operatorname{cl}\{q_m:m>n\}\subset V_n,\quad U_n\perp V_n.$$

Then $(U_n)_{n\in\mathbb{N}}$ is the desired covering, since

$$\operatorname{cl}U_n\cap\operatorname{cl}\bigcup_{m>n}U_n\subset\operatorname{cl}U_n\cap\operatorname{cl}V_n=\varnothing\ \text{ for all } n\in\mathbb{N};$$

$$q\in\operatorname{cl}\{q_n:n\in\mathbb{N}\}\subset\operatorname{cl}\bigcup_{n\in\mathbb{N}}U_n;$$

$$q\in\bigcap_{n\in\mathbb{N}}\operatorname{cl}\{q_m:m>n\}\subset\bigcap_{n\in\mathbb{N}}V_n\subset\bigcap_{n\in\mathbb{N}}(Q\setminus\operatorname{cl}U_n)=Q\setminus\bigcup_{n\in\mathbb{N}}\operatorname{cl}U_n.\ \triangleright$$

1.6. Proposition. (a) *Let $(U_n)_{n\in\mathbb{N}}$ and $(V_n)_{n\in\mathbb{N}}$ be two coverings of a sequence $(q_n)_{n\in\mathbb{N}}$ such that $V_n \subset \operatorname{cl} U_n$ for all $n \in \mathbb{N}$. If $(U_n)_{n\in\mathbb{N}}$ is disjoint (lateral, laterally tangent to q, laterally convergent to q), then so is $(V_n)_{n\in\mathbb{N}}$.*

(b) *If a sequence of points admits a disjoint (lateral, laterally tangent to q, laterally convergent to q) covering, then it admits an open covering and a closed covering that have the same property.*

(c) *If a sequence of points converges laterally to q, then it converges injectively to q.*

(d) *If a sequence of points converges laterally to q, then q is the unique limit of the sequence.*

1.7. Example. In a Hausdorff space, every injectively convergent sequence of isolated points q_n converges laterally. As a laterally convergent covering serves the sequence of the singletons $\{q_n\}$. In particular, in the compact ordinal $\omega + 1$, with ω the first infinite ordinal, the sequence $(n)_{n\in\mathbb{N}}$ converges laterally to ω.

1.8. Proposition. (a) *In a metrizable space, the classes of injectively convergent and of laterally convergent sequences coincide.*

(b) *If a sequence $(q_n)_{n\in\mathbb{N}}$ converges injectively to a point q in a metric space (Q, d), then any sequence of closed balls $B(q_n, \varepsilon_n)$, $\varepsilon_n > 0$, that are pairwise disjoint and do not contain q, is a laterally convergent covering of $(q_n)_{n\in\mathbb{N}}$. The above condition is satisfied, for instance, in the case when*

$$0 < \varepsilon_n < \frac{1}{2} \inf_{m\neq n} d(q_m, q_n) \quad \text{for all } n \in \mathbb{N}. \tag{2}$$

◁ Suppose that $q_n \to q$ in a metric space (Q, d) and let $B(q_n, \varepsilon_n)$ be pairwise disjoint closed balls that do not contain q. By 1.2(c) it suffices to take a point $p \in Q\backslash\{q\}$ and find $m \in \mathbb{N}$ for which $p \notin \operatorname{cl}\bigcup_{n\geqslant m} B(q_n, \varepsilon_n)$. Put $\varepsilon := d(q, p)$ and consider m such that $q_n \in B(q, \frac{1}{3}\varepsilon)$ for $n \geqslant m$. Since $q \notin B(q_n, \varepsilon_n)$, for $n \geqslant m$ we have $\varepsilon_n < \frac{1}{3}\varepsilon$ and so $B(q_n, \varepsilon_n) \subset B(q, \frac{2}{3}\varepsilon)$, as $d(r, q_n) \leqslant \varepsilon_n$ implies

$$d(r, q) \leqslant d(r, q_n) + d(q_n, q) \leqslant \varepsilon_n + \frac{1}{3}\varepsilon < \frac{2}{3}\varepsilon.$$

Therefore, $\operatorname{cl}\bigcup_{n\geqslant m} B(q_n, \varepsilon_n) \subset B(q, \frac{2}{3}\varepsilon)$, while $p \notin B(q, \frac{2}{3}\varepsilon)$.

Next, in the case of (2) the balls $B(q_n, \varepsilon_n)$ are pairwise disjoint, since the existence of a point $p \in B(q_m, \varepsilon_m) \cap B(q_n, \varepsilon_n)$ with $m \neq n$ implies the contradictory relations

$$d(q_m, q_n) \leqslant d(q_m, p) + d(p, q_n) \leqslant \varepsilon_m + \varepsilon_n < \frac{1}{2}d(q_m, q_n) + \frac{1}{2}d(q_m, q_n) = d(q_m, q_n).$$

Moreover, it follows from (2) that $q \notin B(q_n, \varepsilon_n)$ for all $n \in \mathbb{N}$, since $d(q_m, q_n) > 2\varepsilon_n$ for $m > n$ and so

$$d(q, q_n) = \lim_{m\to\infty} d(q_m, q_n) \geqslant 2\varepsilon_n > \varepsilon_n.$$

It remains to note that the injective convergence of $(q_n)_{n\in\mathbb{N}}$ implies the inequality $\inf_{m\neq n} d(q_m, q_n) > 0$ for all $n \in \mathbb{N}$. ▷

1.9. Example. There is a compact Hausdorff space containing a sequence that converges injectively but not laterally.

◁ Consider the Cartesian product $Q = (\omega + 1) \times (\omega_1 + 1)$ of the compact ordinals $\omega + 1$ and $\omega_1 + 1$, with ω the first infinite ordinal and ω_1 the first uncountable ordinal. Then in Q the sequence $\big((n, \omega_1)\big)_{n\in\mathbb{N}}$ converges injectively to (ω, ω_1), but does not converge laterally.

Indeed, consider an arbitrary covering $(U_n)_{n\in\mathbb{N}}$ of $\big((n, \omega_1)\big)_{n\in\mathbb{N}}$. For each $n \in \mathbb{N}$, in the space $\omega_1 + 1$ there is a neighborhood V_n of ω_1 such that $\{n\} \times V_n \subset U_n$. Since the intersection $V = \bigcap_{n\in\mathbb{N}} V_n$ is a neighborhood of ω_1 in $\omega_1 + 1$, there exists an element $\alpha \in V$ distinct from ω_1. Then for all $m \in \mathbb{N}$

$$(\omega, \omega_1) \neq (\omega, \alpha) = \lim_{n\to\infty} (n, \alpha) \in \operatorname{cl} \bigcup_{n\geqslant m} \{n\}\times V \subset \operatorname{cl} \bigcup_{n\geqslant m} U_n,$$

and hence by 1.2(c) the covering $(U_n)_{n\in\mathbb{N}}$ does not converge to (ω, ω_1). ▷

1.10. Let us note a few more simple properties of laterally convergent sequences.

Proposition. *Let $(q_n)_{n\in\mathbb{N}}$ converge laterally to q in Q.*

(a) *Every subsequence $(q_{n_m})_{m\in\mathbb{N}}$ converges laterally to q.*

(b) *If Q_0 is a topological subspace of Q containing $(q_n)_{n\in\mathbb{N}}$ and q, then $(q_n)_{n\in\mathbb{N}}$ converges laterally to q in Q_0.*

(c) *If τ is a stronger topology on Q and $(q_n)_{n\in\mathbb{N}}$ converges in τ, then it converges laterally to q in τ.*

(d) *If $(U_n)_{n\in\mathbb{N}}$ is a laterally convergent covering of $(q_n)_{n\in\mathbb{N}}$ and $p_n \in \operatorname{cl} U_n$ for all $n\in\mathbb{N}$, then $\operatorname{cl}\{p_n : n\in\mathbb{N}\}\setminus\{p_n : n\in\mathbb{N}\}\subset\{q\}$. In particular, if $(p_n)_{n\in\mathbb{N}}$ is convergent, then $p_n\to q$.*

1.11. Theorem. *For the lateral convergence $q_n\to q$ to be valid in Q it is necessary, and if Q is regular, is sufficient, that the following two conditions hold:*

(a) *$q_n\to q$ injectively;*

(b) *there exist an open set U containing $(q_n)_{n\in\mathbb{N}}$, and a sequence $(V_n)_{n\in\mathbb{N}}$ of neighborhoods of q such that*

$$\operatorname{cl} U\cap\bigcap_{n\in\mathbb{N}} V_n=\{q\}. \tag{3}$$

◁ Necessity. Let $(U_n)_{n\in\mathbb{N}}$ be an open covering of $(q_n)_{n\in\mathbb{N}}$ that converges laterally to q (see 1.6(b)). The injective convergence $q_n\to q$ is noted in 1.6(c). Furthermore, since $q\notin\bigcup_{n\in\mathbb{N}}\operatorname{cl} U_n$, for every $n\in\mathbb{N}$ the set $V_n:=Q\setminus\operatorname{cl} U_n$ is a neighborhood of q. Putting $U:=\bigcup_{n\in\mathbb{N}}U_n$, we conclude that

$$\operatorname{cl} U\cap\bigcap_{n\in\mathbb{N}} V_n=\operatorname{cl} U\cap\bigcap_{n\in\mathbb{N}}(Q\setminus\operatorname{cl} U_n)=\operatorname{cl} U\cap\Big(Q\setminus\bigcup_{n\in\mathbb{N}}\operatorname{cl} U_n\Big)=\Big(\operatorname{cl}\bigcup_{n\in\mathbb{N}}U_n\Big)\setminus\bigcup_{n\in\mathbb{N}}\operatorname{cl} U_n=\{q\}.$$

Sufficiency. Let $(q_n)_{n\in\mathbb{N}}$ converge injectively to q, and let U and $(V_n)_{n\in\mathbb{N}}$ satisfy (b). Without loss of generality, we require $V_1=Q$. Since Q is regular, for each $n\in\mathbb{N}$ there exists an open set G_n such that $q\in G_n$ and $\operatorname{cl} G_n\subset V_n$, with $G_1=Q$. Replacing G_n by the intersection $G_1\cap\dots\cap G_n$, we assume that $G_1\supset G_2\supset\dots$.

Since $U\cap\bigcap_{n\in\mathbb{N}}G_n\subset\operatorname{cl} U\cap\bigcap_{n\in\mathbb{N}}V_n=\{q\}$, each element of U other than q belongs to only finitely many sets G_m. In particular, for each $n\in\mathbb{N}$ we may consider the natural

$$m(n)=\max\{m\in\mathbb{N} : q_n\in G_m\}. \tag{4}$$

Using the regularity of Q, which guarantees the separation of each point q_n from the closed set $\{q_{n+1},q_{n+2},\dots\}\cup\{q\}$, and applying recursion, it is easy to define two sequences of open sets U_n and W_n satisfying the following conditions for all $n\in\mathbb{N}$:

$$\begin{aligned}&\operatorname{cl} U_n\cap\operatorname{cl} W_n=\varnothing;\\ &q_n\in U_n\subset U\cap G_{m(n)}\cap W_{n-1};\\ &\{q_{n+1},q_{n+2},\dots\}\cup\{q\}\subset W_n\subset W_{n-1},\end{aligned} \tag{5}$$

where $W_0=Q$. We show that the sequence $(U_n)_{n\in\mathbb{N}}$ converges laterally to q by checking 1.2(c). From (5) and the convergence $q_n\to q$, we see that the sequence $(U_n)_{n\in\mathbb{N}}$ is disjoint and

$$q\in\operatorname{cl}\bigcup_{n\in\mathbb{N}}U_n\setminus\bigcup_{n\in\mathbb{N}}\operatorname{cl} U_n.$$

Next, consider an arbitrary point $p\neq q$. If $p\notin\operatorname{cl} U$, then $p\notin\operatorname{cl}\bigcup_{n\geqslant1}U_n$, since $\bigcup_{n\geqslant1}U_n\subset U$. Now let $p\in\operatorname{cl} U\setminus\{q\}$. From (3) and the inclusions $\operatorname{cl} G_n\subset V_n$ it follows that $p\notin\operatorname{cl} G_k$ for some $k\in\mathbb{N}$. Since $q_n\to q\in G_k$, there is $m\in\mathbb{N}$ such that $g_n\in G_k$ for $n\geqslant m$. According to (4), from $g_n\in G_k$ it follows that $m(n)\geqslant k$, which means $G_k\supset G_{m(n)}\supset U_n$ for $n\geqslant m$. Thus, $p\notin\operatorname{cl} G_k\supset\operatorname{cl}\bigcup_{n\geqslant m}U_n$. ▷

1.12. Corollary. *For the lateral convergence $q_n \to q$ to be valid in Q it is necessary, and if Q is regular, is sufficient, that the following two conditions hold:*

(a) *the sequence $(q_n)_{n\in\mathbb{N}}$ is injective and $q_n \to q$;*

(b) *$(q_n)_{n\in\mathbb{N}}$ admits a covering $(U_n)_{n\in\mathbb{N}}$ such that*

$$\Big(\operatorname{cl}\bigcup_{n\in\mathbb{N}} U_n\Big) \setminus \bigcup_{n\in\mathbb{N}} \operatorname{cl} U_n = \{q\}.$$

◁ Only sufficiency needs some clarification. Let $(q_n)_{n\in\mathbb{N}}$ be an injective sequence converging to q and let $(U_n)_{n\in\mathbb{N}}$ be a covering of $(q_n)_{n\in\mathbb{N}}$ subject to (b). Then for every $n \in \mathbb{N}$ we have $q \notin \operatorname{cl} U_n$, whence $q_n \neq q$ (and thus the convergence $q_n \to q$ is injective) and $V_n := Q \setminus \operatorname{cl} U_n$ is a neighborhood of q. Moreover, the open set $U := \bigcup_{n\in\mathbb{N}} \operatorname{int} U_n$ contains $(q_n)_{n\in\mathbb{N}}$ and satisfies the following conditions:

$$q \in \operatorname{cl} U \cap \bigcap_{n\in\mathbb{N}} V_n = \Big(\operatorname{cl}\bigcup_{n\in\mathbb{N}} \operatorname{int} U_n\Big) \cap \bigcap_{n\in\mathbb{N}} (Q \setminus \operatorname{cl} U_n)$$
$$\subset \Big(\operatorname{cl}\bigcup_{n\in\mathbb{N}} U_n\Big) \cap \Big(Q \setminus \bigcup_{n\in\mathbb{N}} \operatorname{cl} U_n\Big) = \Big(\operatorname{cl}\bigcup_{n\in\mathbb{N}} U_n\Big) \setminus \bigcup_{n\in\mathbb{N}} \operatorname{cl} U_n = \{q\}.$$

It remains to apply Theorem 1.11. ▷

1.13. Proposition. *Suppose that $(q_n)_{n\in\mathbb{N}}$ is an injective sequence in a regular topological space, which converges to q and admits a covering contracting to q. Then $(q_n)_{n\in\mathbb{N}}$ converges laterally to q.*

◁ Consider a covering $(U_n)_{n\in\mathbb{N}}$ of $(q_n)_{n\in\mathbb{N}}$ which contracts to q. Without loss of generality, we may assume that $q \notin \operatorname{cl} U_n$ for all $n \in \mathbb{N}$. Then, on the one hand,

$$q \in \operatorname{cl}\{q_n : n \in \mathbb{N}\} \setminus \bigcup_{n\in\mathbb{N}} \operatorname{cl} U_n \subset \Big(\operatorname{cl}\bigcup_{n\in\mathbb{N}} U_n\Big) \setminus \bigcup_{n\in\mathbb{N}} \operatorname{cl} U_n.$$

On the other hand, if V is an arbitrary neighborhood of q and $U_n \subset V$ for $n > m$, then

$$\Big(\operatorname{cl}\bigcup_{n\in\mathbb{N}} U_n\Big) \setminus \bigcup_{n\in\mathbb{N}} \operatorname{cl} U_n \subset \Big(\bigcup_{n\leqslant m} \operatorname{cl} U_n \cup \operatorname{cl}\bigcup_{n>m} U_n\Big) \setminus \bigcup_{n\leqslant m} \operatorname{cl} U_n \subset \operatorname{cl}\bigcup_{n>m} U_n \subset \operatorname{cl} V.$$

Therefore, $\big(\operatorname{cl}\bigcup_{n\in\mathbb{N}} U_n\big) \setminus \bigcup_{n\in\mathbb{N}} \operatorname{cl} U_n = \{q\}$, and by 1.12 the sequence $(q_n)_{n\in\mathbb{N}}$ converges laterally to q. ▷

1.14. Theorem. *Let $(q_n)_{n\in\mathbb{N}}$ be a sequence laterally converging to a point q in a regular space Q. The following are equivalent:*

(a) *every covering of $(q_n)_{n\in\mathbb{N}}$ laterally converging to q contracts to q;*

(b) *Q is countably compact.*

◁ (a)⇒(b): Assume to the contrary that Q is not countably compact. Then there exists an infinite subset $P \subset Q$ that has no limit points (see, for instance, [3, Chapter III, Problem 189]). Undoubtedly, P contains only finitely many terms of $(q_n)_{n\in\mathbb{N}}$, and so we can find an injective sequence of elements $p_n \in P$ that do not belong to $\{q_n : n \in \mathbb{N}\} \cup \{q\}$. Note that each of the sets $P_m := \{p_n : n \geqslant m\}$ $(m \in \mathbb{N})$ has no limit points and is therefore closed.

By hypothesis, the sequence $(q_n)_{n\in\mathbb{N}}$ admits a covering $(W_n)_{n\in\mathbb{N}}$ that converges laterally to q. Owing to the regularity of Q, there exists a closed covering $(V_n)_{n\in\mathbb{N}}$ of $(q_n)_{n\in\mathbb{N}}$ such that $V_n \subset W_n \setminus P_1$ for all $n \in \mathbb{N}$. Put

$$U_n := V_n \cup \{p_n\}\ (n \in \mathbb{N}), \quad \mathscr{U}_m := \bigcup_{n\geqslant m} U_n,\ \mathscr{V}_m := \bigcup_{n\geqslant m} V_n\ (m \in \mathbb{N}).$$

We show that the closed covering $(U_n)_{n\in\mathbb{N}}$ of $(q_n)_{n\in\mathbb{N}}$ converges laterally to q by checking 1.2(b). Indeed, due to 1.6(a) the covering $(V_n)_{n\in\mathbb{N}}$ converges laterally to q. Therefore, $U_m \cap U_n = V_m \cap V_n = \varnothing$ for $m \neq n$ and, moreover,

$$\operatorname{cl}\mathscr{U}_m \setminus \mathscr{U}_m = \operatorname{cl}(\mathscr{V}_m \cup P_m) \setminus (\mathscr{V}_m \cup P_m) = (\operatorname{cl}\mathscr{V}_m \cup P_m) \setminus (\mathscr{V}_m \cup P_m)$$
$$= (\operatorname{cl}\mathscr{V}_m \setminus \mathscr{V}_m) \setminus P_m = \{q\} \setminus P_m = \{q\}$$

for all $m \in \mathbb{N}$. It remains to note that $(U_n)_{n\in\mathbb{N}}$ does not contract to q, since $p_n \in U_n$ and $p_n \nrightarrow q$.

(b)⇒(a): Suppose that $(U_n)_{n\in\mathbb{N}}$ is a covering of $(q_n)_{n\in\mathbb{N}}$ that is laterally converging but not contracting to q. Then there exist a neighborhood V of q and sequences $(n_m)_{m\in\mathbb{N}} \subset \mathbb{N}$ and $(p_m)_{m\in\mathbb{N}} \subset Q$ such that $n_m \to \infty$ and $p_m \in U_{n_m}\backslash V$ for all $m \in \mathbb{N}$. Since $(U_n)_{n\in\mathbb{N}}$ is disjoint, $P := \{p_m : m \in \mathbb{N}\}$ is infinite. Obviously, $q \notin \operatorname{cl} P$. Moreover, from 1.2(d) it is clear that the elements of $Q\backslash\{q\}$ also cannot be limit points for P. Thus, Q contains an infinite set that has no limit points, and is thus not countably compact (see [3, Chapter III, Problem 189]). ▷

1.15. Recall that the *pseudocharacter* $\psi(q, Q)$ of a T_1 space Q at a point $q \in Q$ is the smallest among the cardinalities $|\mathscr{V}|$ of sets $\mathscr{V}$ consisting of open subsets of Q and satisfying the equality $\cap\mathscr{V} = \{q\}$:

$$\psi(q, Q) := \min\bigl\{|\mathscr{V}| : \mathscr{V} \subset \mathit{Open}(Q),\ \cap\mathscr{V} = \{q\}\bigr\}.$$

For instance, the pseudocharacter of a T_1 space at a point possessing a countable neighborhood base is countable.

The following is a direct corollary to Theorem 1.11:

Proposition. *Let Q be a regular space and $q \in Q$. If the pseudocharacter $\psi(q, Q)$ is countable, then every sequence in Q injectively converging to q converges laterally.*

1.16. Example. The converse to Proposition 1.15 does not hold even under the additional assumption that Q is compact. Indeed, let $Q = D \cup \{q\}$ be a one-point compactification of an uncountable discrete space D. The pseudocharacter $\psi(q, Q)$ is uncountable, since for every sequence $(V_n)_{n\in\mathbb{N}}$ of open neighborhoods of q, the complement $Q \setminus \bigcap_{n\in\mathbb{N}} V_n = \bigcup_{n\in\mathbb{N}}(Q \setminus V_n)$ is countable. Moreover, each injective sequence of elements in D converges laterally to q (see Example 1.7).

1.17. Let us describe another illustrative case when a compact space has uncountable pseudocharacter at the limit of a laterally convergent sequence.

Example. Let P be a Hausdorff compact space that has uncountable pseudocharacter $\psi(p, P)$ at some point p (or, which is the same in this case, contains a point p that has no countable neighborhood base, see [3, Chapter III, Problem 68; 4, 3.1.F]). We will assume that P does not intersect $\mathbb{N}$. Equip $Q := P \cup \mathbb{N}$ with the topology in which the open sets are the unions $U \cup V$ with U an open subset of P and V a subset of $\mathbb{N}$ satisfying the condition

$$(\exists\, m \in \mathbb{N})(\forall\, n \geqslant m)(n \in V)$$

in the case of $p \in U$. (Such a topological space Q is homeomorphic to the join $P \cup_{\{(p,\omega)\}} (\omega + 1)$ in the sense of [4, 2.1.12] or to the quotient space of the sum $P \oplus (\omega + 1)$ by the equivalence relation identifying the points p and ω.) Then Q is a Hausdorff compact space (see, for instance, [4, 3.2.11]) with uncountable pseudocharacter $\psi(p, Q)$, whereas the sequence $(n)_{n\in\mathbb{N}}$ converges laterally to p (see Example 1.7).

§ 2. Homomorphisms of Banach Bundles

In this section, we establish several results on the existence of homomorphisms of CBBs, as well as continuous and weakly continuous vector-valued functions and sections taking preassigned values at the points of injectively and laterally convergent sequences. As before, Q is a topological space.

2.1. Definition. A *function covering of a sequence* of points $q_n \in Q$ is an arbitrary sequence of continuous functions $f_n : Q \to [0, 1]$ such that $f_n(q_n) = 1$ for all $n \in \mathbb{N}$. A function covering $(f_n)_{n\in\mathbb{N}}$ of a convergent sequence $q_n \to q$ is called *laterally tangent* (*laterally convergent*) to q, if so is the sequence of the supports $\operatorname{supp} f_n := \operatorname{cl}\{q \in Q : f_n(q) \neq 0\}$.

2.2. As is easy to see, for every covering $(U_n)_{n\in\mathbb{N}}$ of a sequence $(q_n)_{n\in\mathbb{N}}$ in a completely regular space there exists a function covering $(f_n)_{n\in\mathbb{N}}$ of the sequence such that $\operatorname{supp} f_n \subset U_n$ for all $n \in \mathbb{N}$. In particular, if a convergent sequence $q_n \to q$ in a completely regular space admits a covering that is laterally tangent (convergent) to q, then the sequence also admits a function covering that has the same property. With this observation taken into account, the following fact ensues from 1.5:

Proposition. *In a completely regular topological space, every injectively convergent sequence $q_n \to q$ admits a function covering that is laterally tangent to q.*

2.3. Definition. We describe the construction of the pointwise sum of a disjoint sequence of sections of a bundle of vector spaces, which will be repeatedly used below in various special cases (see 2.4, 2.6, 2.7, 2.9, 2.10).

Let $\mathscr{X} := \big(\mathscr{X}(q)\big)_{q\in Q}$ be a family of vector spaces, let $S(Q,\mathscr{X}) := \prod_{q\in Q}\mathscr{X}(q)$ be the vector space of sections of $\mathscr{X}$, and let $(s_n)_{n\in\mathbb{N}}$ be a sequence of sections $s_n \in S(Q,\mathscr{X})$ that have pairwise disjoint supports $\operatorname{supp} s_n := \operatorname{cl}\{q\in Q : s_n(q)\neq 0\}$. In this context, it becomes possible to consider the section $\sum_{n\in\mathbb{N}} s_n \in S(Q,\mathscr{X})$ that is the pointwise sum of s_n and takes the following values:

$$\Big(\sum_{n\in\mathbb{N}} s_n\Big)(q) = \begin{cases} s_n(q), & q\in\operatorname{supp} s_n,\\ 0, & q\in Q\setminus\bigcup_{n\in\mathbb{N}}\operatorname{supp} s_n.\end{cases}$$

In most of the cases considered below, the mappings s_n are continuous in one sense or another and belong to a subspace of $S(Q,\mathscr{X})$ with a natural vector topology, while the section $\sum_{n\in\mathbb{N}} s_n$ turns out to be the pointwise sum of the uniformly convergent series $\sum_{n=1}^{\infty} s_n$.

2.4. Lemma. *Let the following conditions hold:*
(a) *Q is a completely regular topological space;*
(b) *a sequence of points $q_n\in Q$ converges injectively to $q\in Q$;*
(c) *$\mathscr{X}$ is a CBB over Q;*
(d) $x_n\in\mathscr{X}(q_n)$, $\|x_n\|\to 0$.
Then there exists a bounded continuous section $s\in C(Q,\mathscr{X})$ such that

$$s(q_n) = x_n\ (n\in\mathbb{N}),\quad s(q) = 0.$$

◁ By Dupré's Theorem [6, 1.1], for every $n\in\mathbb{N}$ there is a continuous section $s_n\in C(Q,\mathscr{X})$ such that $s_n(q_n)=x_n$ and $|\!|\!|s_n|\!|\!| \leqslant \|x_n\|$. Let $(f_n)_{n\in\mathbb{N}}$ be a function covering of $(q_n)_{n\in\mathbb{N}}$ that is laterally tangent to q (see 2.2). Then the section $s := \sum_{n\in\mathbb{N}} f_n s_n \in S(Q,\mathscr{X})$ takes the required values and, moreover, is bounded and continuous, being the pointwise sum of the uniformly convergent series $\sum_{n=1}^{\infty} f_n s_n$ in the Banach space $C^b(Q,\mathscr{X})\subset\ell^\infty(Q,\mathscr{X})$ of bounded continuous sections (see [1, 2.3.6]). ▷

2.5. Proposition. *Let the following conditions hold:*
(a) *Q is a completely regular topological space;*
(b) *a sequence of points $q_n\in Q$ converges injectively to $q\in Q$;*
(c) *$\mathscr{X}$ is a CBB over Q;*
(d) $x_n\in\mathscr{X}(q_n)$, $x\in\mathscr{X}(q)$;
(e) $(q_n,x_n)\to(q,x)$ *in the topological space $Q\otimes\mathscr{X}$* (see [1, 2.1.4]).
Then there exists a bounded continuous section $s\in C(Q,\mathscr{X})$ taking the values

$$s(q_n) = x_n\ (n\in\mathbb{N}),\quad s(q) = x.$$

◁ By Dupré's Theorem [6, 1.1], there exists a section $s_x\in C^b(Q,\mathscr{X})$ such that $s_x(q)=x$. It follows from [1, 2.3.8] that $\lim_{n\to\infty}\|x_n - s_x(q_n)\| = 0$. By Lemma 2.4, there is a section $s_0\in C^b(Q,\mathscr{X})$ such that $s_0(q_n) = x_n - s_x(q_n)$ for all $n\in\mathbb{N}$ and $s_0(q)=0$. Then $s := s_0 + s_x$ is the desired section. ▷

2.6. Proposition. *Let the following conditions hold:*
(a) *Q is a completely regular topological space;*
(b) *a sequence of points $q_n\in Q$ converges injectively to $q\in Q$;*
(c) *$\mathscr{X}$ and $\mathscr{Y}$ are CBBs over Q;*
(d) $H_n\in \mathit{Hom}(\mathscr{X},\mathscr{Y})$, $\|H_n\|_\infty\to 0$.
Then there exists a bounded homomorphism $H\in \mathit{Hom}(\mathscr{X},\mathscr{Y})$ such that

$$H(q_n) = H_n(q_n)\ (n\in\mathbb{N}),\quad H(q) = 0.$$

◁ Denote by $\mathscr{Z}$ the Banach bundle over Q with stalks $B\big(\mathscr{X}(p),\mathscr{Y}(p)\big)$ at $p\in Q$. Let $(f_n)_{n\in\mathbb{N}}$ be a function covering of $(q_n)_{n\in\mathbb{N}}$ that is laterally tangent to q (see 2.2). Then the section $H := \sum_{n\in\mathbb{N}} f_n H_n \in S(Q,\mathscr{Z})$ takes the required values and, moreover, is a bounded homomorphism from $\mathscr{X}$ to $\mathscr{Y}$, being the pointwise sum of the uniformly convergent series $\sum_{n=1}^{\infty} f_n H_n$ in the Banach space $\mathit{Hom}^b(\mathscr{X},\mathscr{Y})\subset\ell^\infty(Q,\mathscr{Z})$ of bounded homomorphisms (see [1, 2.4.11]). ▷

2.7. Proposition. *Let the following conditions be satisfied:*
(a) *Q is a completely regular topological space;*
(b) *a sequence of points $q_n \in Q$ converges injectively to $q \in Q$;*
(c) *X is a topological vector space;*
(d) *a sequence of vectors $x_n \in X$ converges to $x \in X$.*
Then there exists a continuous function $u : Q \to X$ taking the values

$$u(q_n) = x_n \ (n \in \mathbb{N}), \quad u(q) = x. \tag{6}$$

If, moreover, the set $\{x_n : n \in \mathbb{N}\}$ is bounded, then there is a bounded function $u \in C(Q, X)$ satisfying (6).

◁ Let $(f_n)_{n\in\mathbb{N}}$ be a function covering of $(q_n)_{n\in\mathbb{N}}$ that is laterally tangent to q (see 2.2). Define the function $u_0 : Q \to X$ by setting $u_0 := \sum_{n\in\mathbb{N}} f_n \otimes (x_n - x)$. Then $u_0(q_n) = x_n - x$ for all $n \in \mathbb{N}$, $u_0(q) = 0$, and, moreover, u_0 is bounded and continuous, being the pointwise sum of the series $\sum_{n=1}^{\infty} f_n \otimes (x_n - x)$ that is uniformly convergent with respect to the uniformity of the space X. Therefore, $u := u_0 + x$ is the sought function. ▷

2.8. Proposition. *Let the following conditions hold:*
(a) *Q is a completely regular topological space;*
(b) *a sequence of points $q_n \in Q$ converges injectively to $q \in Q$;*
(c) *$\mathscr{X} = Q \times \{X\}$ is the constant CBB over Q with stalk X;*
(d) *a sequence of functionals $y_n \in X'$ converges weakly* to $y \in X'$.*
Then there exists a bounded homomorphism $H \in \mathscr{X}^*$ taking the values

$$H(q_n) = y_n \ (n \in \mathbb{N}), \quad H(q) = y.$$

◁ By 2.7, there is a weakly* bounded weakly* continuous mapping $H : Q \to X'$ taking the values $H(q_n) = y_n$ $(n \in \mathbb{N})$ and $H(q) = y$. It is clear that $H \otimes u \in C(Q)$ for all constant functions $u : Q \to X$. Moreover, since X is complete, H is bounded in norm. It remains to use [1, 2.4.9]. ▷

2.9. A sequence $(x_n)_{n\in\mathbb{N}}$ in a Banach space X *w-w*-converges* to $x \in X$ if for all $y_n, y \in X'$ the weak* convergence $y_n \to y$ implies $\langle x_n \,|\, y_n\rangle \to \langle x \,|\, y\rangle$ (see [5, 3.1.7]). Say that a section $w \in S(Q, \mathscr{X})$ of a CBB $\mathscr{X}$ over Q is *weakly continuous* and write $w \in C_{\mathrm{w}}(Q, \mathscr{X})$ if $H \otimes w \in C(Q)$ for all $H \in \mathscr{X}^*$ (see [5, 3.5]).

Proposition. *Let the following conditions hold:*
(a) *Q is a completely regular Fréchet–Urysohn space;*
(b) *a sequence of points $q_n \in Q$ converges injectively to $q \in Q$;*
(c) *$\mathscr{X} = Q \times \{X\}$ is the constant CBB over Q with stalk X;*
(d) *a sequence of vectors $x_n \in X$ w-w*-converges to $x \in X$.*
Then there exists a weakly continuous section $w \in C_{\mathrm{w}}(Q, \mathscr{X})$ taking the values

$$w(q_n) = x_n \ (n \in \mathbb{N}), \quad w(q) = x.$$

◁ As in the proof of Proposition 2.7, consider a function covering $(f_n)_{n\in\mathbb{N}}$ of $(q_n)_{n\in\mathbb{N}}$ that is laterally tangent to q (see 2.2), and put $u := \sum_{n\in\mathbb{N}} f_n \otimes (x_n - x) \in S(Q, \mathscr{X})$, i.e.,

$$u(p) = \begin{cases} f_n(p)(x_n - x), & p \in \operatorname{supp} f_n, \\ 0, & p \in Q \setminus S, \end{cases}$$

where $S := \bigcup_{n\in\mathbb{N}} \operatorname{supp} f_n$. Then the sum $w := u + x$ takes the required values at q_n and q, and it only remains to verify that $u \in C_{\mathrm{w}}(Q, \mathscr{X})$.

Consider an arbitrary homomorphism $H \in \mathscr{X}^*$ and show that $H \otimes u : Q \to \mathbb{R}$ is continuous. First of all, note that $H \otimes u$ is continuous on the complement $Q \setminus \operatorname{cl} S$, since it is identically zero on it. Next, $H \otimes u$ is continuous on S, as each of the supports $\operatorname{supp} f_n$ lies in the open set $Q \setminus \operatorname{cl} \bigcup_{m\neq n} \operatorname{supp} f_m$ on which

$$H(p)u(p) = f_n(p)H(p)(x_n - x).$$

It remains to show that $H \otimes u$ is continuous on $\operatorname{cl} S \setminus S$. Assume this function discontinuous at some point $p \in \operatorname{cl} S \setminus S$. Then there exist a real $\varepsilon > 0$, a sequence of points $p_m \in S$, and a strictly increasing sequence of naturals $n_m \in \mathbb{N}$ such that

$$p \in \operatorname{cl}\{p_m : m \in \mathbb{N}\}, \quad p_m \in \operatorname{supp} f_{n_m}, \quad |H(p_m)u(p_m)| > \varepsilon$$

for all $m \in \mathbb{N}$. Since Q is Fréchet–Urysohn, there is a subsequence $p_{m_k} \to p$. It is easy to verify that the sequence of vectors $u(p_m) = f_{n_m}(p_m)(x_{n_m} - x)$ w-w*-converges to zero, and hence this is true for its subsequence $u(p_{m_k})$. At the same time, $H(p_{m_k}) \to H(p)$ weakly*, which leads to the contradictory relations

$$\varepsilon < |H(p_{m_k})u(p_{m_k})| \to |H(p)u(p)| = 0. \ \triangleright$$

2.10. Theorem. *Let the following conditions be satisfied:*
(a) *Q is a completely regular topological space;*
(b) *a sequence of points $q_n \in Q$ converges laterally to $q \in Q$;*
(c) *$\mathscr{X}$ and $\mathscr{Y}$ are CBBs over Q;*
(d) *$H_n \in \mathit{Hom}(\mathscr{X}, \mathscr{Y})$, $\|H_n\|_\infty \leqslant 1$ $(n \in \mathbb{N})$;*
(e) *$\mathscr{U}$ is a countable subset of $C(Q, \mathscr{X})$;*
(f) *$\operatorname{cl} \mathscr{U}(q) = \mathscr{X}(q)$, where $\mathscr{U}(q) := \{u(q) : u \in \mathscr{U}\}$;*
(g) *$\|H_n(q_n)u(q_n)\| \to 0$ for all $u \in \mathscr{U}$.*
Then there exists a homomorphism $H \in \mathit{Hom}(\mathscr{X}, \mathscr{Y})$ such that

$$\|H\|_\infty \leqslant 1, \quad H(q_n) = H_n(q_n) \ (n \in \mathbb{N}), \quad H(q) = 0.$$

$\triangleleft$ Denote by $\mathscr{Z}$ the Banach bundle over Q with stalks $B\big(\mathscr{X}(p), \mathscr{Y}(p)\big)$ at $p \in Q$. By [1, 2.4.9], a bounded section $H \in S(Q, \mathscr{Z})$ belongs to $\mathit{Hom}(\mathscr{X}, \mathscr{Y})$ whenever there exists a stalkwise dense set $\mathscr{V} \subset C(Q, \mathscr{X})$ in $\mathscr{X}$ such that $H \otimes v \in C(Q, \mathscr{Y})$ for all $v \in \mathscr{V}$. We begin by introducing a set $\mathscr{V}$ that is convenient for further exposition.

Consider a covering $(W_n)_{n\in\mathbb{N}}$ of $(q_n)_{n\in\mathbb{N}}$ that converges laterally to q and put

$$\mathscr{W} := \Big\{w \in C(Q, \mathscr{X}) \,:\, w = 0 \text{ on } \operatorname{cl} \bigcup_{m\neq n} W_m \text{ for some } n \in \mathbb{N}\Big\}.$$

Note that $\mathscr{W}(p) = \mathscr{X}(p)$ for each point

$$p \in Q\setminus\{q\} = \Big(Q \setminus \operatorname{cl} \bigcup_{n\in\mathbb{N}} W_n\Big) \cup \bigcup_{n\in\mathbb{N}} \operatorname{cl} W_n.$$

Indeed, due to the complete regularity of Q, in each of the cases $p \notin \operatorname{cl}\bigcup_{n\in\mathbb{N}} W_n$ and $p \in \bigcup_{n\in\mathbb{N}} \operatorname{cl} W_n$ there exists a continuous function $f : Q \to [0, 1]$ such that $f(p) = 1$ and $f = 0$ on $\operatorname{cl}\bigcup_{m\neq n} W_m$ for some $n \in \mathbb{N}$. By Dupré's Theorem [6, 1.1], for each $x \in \mathscr{X}(p)$ there is a section $u \in C(Q, \mathscr{X})$ with the value $u(p) = x$, and hence x is the value at p of the section fu belonging to $\mathscr{W}$. Thus, $\mathscr{W}(p) = \mathscr{X}(p)$ for all $p \neq q$, and so the union

$$\mathscr{V} := \mathscr{W} \cup \{u_m : m \in \mathbb{N}\}$$

is stalkwise dense in $\mathscr{X}$, where $\{u_m : m \in \mathbb{N}\} = \mathscr{U}$.

Next, according to (g) there is a strictly increasing sequence of $n_m \in \mathbb{N}$ for which

$$\|H_n(q_n)u_1(q_n)\| < \frac{1}{m}, \ \ldots, \ \|H_n(q_n)u_m(q_n)\| < \frac{1}{m}, \quad m, n \in \mathbb{N}, \ n \geqslant n_m.$$

As is easy to see, there exists a covering of $(q_n)_{n\in\mathbb{N}}$ constituted by subneighborhoods $V_n \subset W_n$ such that

$$\|H_n \otimes u_1\| \leqslant \frac{1}{m}, \ \ldots, \ \|H_n \otimes u_m\| \leqslant \frac{1}{m} \ \text{ on } V_n \text{ for } n_m \leqslant n < n_{m+1}.$$

Consider a function covering $(f_n)_{n\in\mathbb{N}}$ of $(q_n)_{n\in\mathbb{N}}$ such that $\operatorname{supp} f_n \subset V_n$ for all $n \in \mathbb{N}$ (see 2.2). It is clear that $(f_n)_{n\in\mathbb{N}}$, like $(V_n)_{n\in\mathbb{N}}$, converges laterally to q (see 1.6(a)), and

$$\lim_{n\to\infty} \|f_n H_n \otimes u_m\|_\infty = 0 \quad \text{for all } m \in \mathbb{N}.$$

Then the section

$$H := \sum_{n\in\mathbb{N}} f_n H_n \in S(Q, \mathscr{Z})$$

takes the required values at q_n and q and, moreover, $\|H\|_\infty \leqslant 1$. It remains to show that $H \otimes v \in C(Q, \mathscr{Y})$ for all $v \in \mathscr{V}$.

If $v \in \mathscr{W}$ then there is $n \in \mathbb{N}$ for which $v = 0$ on the set

$$\operatorname{cl} \bigcup_{m\neq n} W_m \supset \bigcup_{m\neq n} V_m \supset \bigcup_{m\neq n} \operatorname{supp} f_m H_m,$$

and then $H \otimes v = f_n H_n \otimes v \in C(Q, \mathscr{Y})$. If, otherwise, $v = u_m$ for some $m \in \mathbb{N}$, then

$$\lim_{n\to\infty} \|f_n H_n \otimes v\|_\infty = \lim_{n\to\infty} \|f_n H_n \otimes u_m\|_\infty = 0,$$

and so the section $H \otimes v = \sum_{n\in\mathbb{N}} f_n H_n \otimes v$ is continuous as the pointwise sum of the uniformly convergent series $\sum_{n=1}^{\infty} (f_n H_n \otimes v)$ in the Banach space $C^b(Q, \mathscr{Y})$ of bounded continuous sections (see [1, 2.3.6]). ▷

2.11. The following example shows that the lateral convergence requirement (b) in Theorem 2.10 is essential and cannot be weakened to injective convergence.

Example. A situation is possible when the following conditions are satisfied:
(a) Q is a compact Hausdorff space;
(b) a sequence of points $q_n \in Q$ converges injectively to $q \in Q$;
(c) $\mathscr{X}$ and $\mathscr{Y}$ are CBBs over Q;
(d) $H_n \in \mathit{Hom}(\mathscr{X}, \mathscr{Y})$, $\|H_n\|_\infty \leqslant 1$ $(n \in \mathbb{N})$;
(e) $\mathscr{X}(q) = \{0\}$,
but there is no homomorphism $H \in \mathit{Hom}(\mathscr{X}, \mathscr{Y})$ such that $H(q_n) = H_n(q_n)$ for all $n \in \mathbb{N}$.

◁ Consider a compact Hausdorff space S that has uncountable pseudocharacter $\psi(t, S)$ at some point $t \in S$, and put

$$Q = (\omega + 1) \times S, \quad q_n = (n, t), \quad q = (\omega, t).$$

(For instance, in the case of $S = \omega_1 + 1$ and $t = \omega_1$, the space Q and the injectively convergent sequence $q_n \to q$ coincide with those considered in Example 1.9.) Let $\mathscr{Y}$ be the constant CBB over Q with stalk $\mathbb{R}$, and define $\mathscr{X}$ to be the subbundle of $\mathscr{Y}$ with stalks

$$\mathscr{X}(p) = \begin{cases} \mathbb{R}, & p \neq q; \\ \{0\}, & p = q \end{cases}$$

and continuous structure $\{u \in C(Q) : u(q) = 0\} = C(Q, \mathscr{X})$ (see [1, 2.2.1 and 2.2.2]). For each $n \in \mathbb{N}$, denote by χ_n the characteristic function of the subset $\{n\} \times S \subset Q$:

$$\chi_n(\alpha, s) = \begin{cases} 1, & \alpha = n; \\ 0, & \alpha \neq n \end{cases}$$

for all $(\alpha, s) \in Q$. Note that the functions $\chi_n : Q \to \mathbb{R}$ are continuous, and by virtue of the equality $\chi_n(q) = \chi_n(\omega, t) = 0$ they belong to $C(Q, \mathscr{X})$. Finally, we define the sequence of homomorphisms $H_n \in \mathit{Hom}(\mathscr{X}, \mathscr{Y})$ by setting

$$H_n(\alpha, s)x = \begin{cases} \chi_n(\alpha, s)x & \text{for even } n; \\ 0 & \text{for odd } n \end{cases}$$

for all $(\alpha, s) \in Q$ and $x \in \mathscr{X}(\alpha, s)$.

Assume to the contrary that there is a homomorphism $H \in \mathit{Hom}(\mathscr{X}, \mathscr{Y})$ taking the values $H(q_n) = H_n(q_n)$ for all $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$ we have $H(n,t)\chi_n(n,t) = 1$ for even n and $H(n,t)\chi_n(n,t) = 0$ for odd n, and hence, in the space S there exists a neighborhood V_n of t such that

$$H \otimes \chi_n > \tfrac{2}{3} \quad \text{on } \{n\} \times V_n \quad \text{for even } n;$$
$$H \otimes \chi_n < \tfrac{1}{3} \quad \text{on } \{n\} \times V_n \quad \text{for odd } n.$$

Since the pseudocharacter $\psi(t, S)$ is uncountable, there is a point $t' \in \bigcap_{n\in\mathbb{N}} V_n \setminus \{t\}$. Consider a function $f \in C(S)$ that takes the values $f(t') = 1$ and $f(t) = 0$, and define the section $u \in C(Q, \mathscr{X})$ by setting

$$u(\alpha, s) = f(s), \quad (\alpha, s) \in Q.$$

Then $(n, t') \to (\omega, t')$, whereas the sequence of the reals

$$(H \otimes u)(n, t') = H(n, t')f(t') = H(n, t')1 = H(n, t')\chi_n(n, t') = (H \otimes \chi_n)(n, t')$$

has no limit. ▷

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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