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MONOTONE OPERATORS IN VECTOR LATTICES AND LATTICE-NORMED SPACES

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Abstract—We show that every monotone linear operator from a vector lattice to a lattice-normed space can be represented as the composition of a surjective lattice homomorphism and a linear isometry. We also give some applications to the theory of continuous and measurable bundles of Banach lattices.

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Let X be a vector lattice and let Y be a vector space endowed with a norm $\lvert\!\lvert\cdot\rvert\!\rvert\colon Y \to E$ taking values in a Dedekind complete vector lattice E. A linear operator $T\colon X \to Y$ is *monotone* whenever

$$|x_1| \leqslant |x_2| \ \text{ implies } \ \lvert\!\lvert Tx_1\rvert\!\rvert \leqslant \lvert\!\lvert Tx_2\rvert\!\rvert$$

for all $x_1, x_2 \in X$.

We show that a linear operator $T\colon X \to Y$ is monotone if and only if T can be represented as the composition

$$T = J \circ H, \quad H\colon X \twoheadrightarrow Z, \quad J\colon Z \hookrightarrow Y$$

of a surjective lattice homomorphism H from X onto an E-normed lattice Z and a linear isometry J.

We also establish a few corollaries for operators in lattice-normed spaces and give some applications to the theory of Banach bundles concerning a function representation of Banach–Kantorovich lattices and the notion of lifting in measurable bundles of Banach lattices.

Vector spaces under consideration are assumed real. Vector lattices are not assumed Archimedean.

Definition 1. Let X be a vector space and let E be an ordered vector space. A mapping $p\colon X \to E$ is a (*vector-valued* or *E-valued*) *seminorm* if

(1) $p(x_1 + x_2) \leqslant p(x_1) + p(x_2)$ for all $x_1, x_2 \in X$;

(2) $p(\alpha x) \leqslant |\alpha|\, p(x)$ for all $\alpha \in \mathbb{R}$ and $x \in X$.

It is easy to show that (1) and (2) imply

$$p(x) \geqslant 0, \quad p(0) = 0, \quad p(\alpha x) = |\alpha|\, p(x), \quad p(-x) = p(x)$$

for all $\alpha \in \mathbb{R}$ and $x \in X$. A seminorm $p\colon X \to E$ is a *norm* whenever

(3) $p(x) = 0$ implies $x = 0$ for each $x \in X$.

A vector space X endowed with an E-valued norm is called an *E-normed space*. The vector-valued norm on X is usually denoted by $\lvert\!\lvert\cdot\rvert\!\rvert$ or $\lvert\!\lvert\cdot\rvert\!\rvert_X$. (Obviously, for $E = \mathbb{R}$, the notion of E-valued norm coincides with the classical notion of norm, in which case the standard symbol $\|\cdot\|$ is employed.)

Given E-normed spaces X and Y, by a *linear isometry* from X into Y we mean a linear operator $J\colon X \to Y$ such that $\lvert\!\lvert Jx\rvert\!\rvert = \lvert\!\lvert x\rvert\!\rvert$ for all $x \in X$.

Say that a set $S \subseteq X$ is *closed* with respect to a seminorm $p\colon X \to E$ if

$$\inf_{s\in S} p(x - s) = 0 \ \text{ implies } \ x \in S$$

for each $x \in X$.

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The following statement is verified directly:

Proposition 1. *Let X be a vector space, let E be a Dedekind complete vector lattice, let $p\colon X \to E$ be a seminorm, and let Δ be a vector subspace of X. Consider the quotient vector space X/Δ and define the mapping $|\!\cdot\!|\colon X/\Delta \to E$ by putting*

$$|\mathbf{x}| := \inf p[\mathbf{x}] = \inf_{x\in\mathbf{x}} p(x), \quad \mathbf{x} \in X/\Delta,$$

where $p[\mathbf{x}] := \{p(x) : x \in \mathbf{x}\}$. In other words,

$$|x+\Delta| = \inf p[x+\Delta] = \inf_{\delta\in\Delta} p(x+\delta), \quad x \in X.$$

Then $|\!\cdot\!|$ is a seminorm on X/Δ. Moreover, $|\!\cdot\!|$ is a norm if and only if Δ is closed with respect to p.

Definition 2. The (semi)norm $|\!\cdot\!|$ in the statement of Proposition 1 is referred to as the *quotient (semi)norm* of p with respect to Δ.

In the sequel we need the following classical fact from the theory of vector lattices:

Theorem 1 [1, 18.7–18.9]. *If Δ is an order ideal in a vector lattice X, then the quotient vector space X/Δ is a vector lattice with respect to the order relation $\mathbf{x}_1 \leqslant \mathbf{x}_2$ defined for cosets $\mathbf{x}_1, \mathbf{x}_2 \in X/\Delta$ by the following pairwise equivalent conditions:*

(1) $(\exists\, x_1 \in \mathbf{x}_1)\,(\exists\, x_2 \in \mathbf{x}_2)\; x_1 \leqslant x_2$;
(2) $(\forall\, x_1 \in \mathbf{x}_1)\,(\exists\, x_2 \in \mathbf{x}_2)\; x_1 \leqslant x_2$;
(3) $(\forall\, x_2 \in \mathbf{x}_2)\,(\exists\, x_1 \in \mathbf{x}_1)\; x_1 \leqslant x_2$;
(4) $(\forall\, x_1 \in \mathbf{x}_1)\,(\forall\, x_2 \in \mathbf{x}_2)\,(\exists\, \delta \in \Delta)\; x_2 - x_1 \geqslant \delta$.

Moreover, the canonical mapping

$$(x \mapsto x+\Delta)\colon X \to X/\Delta$$

is a lattice homomorphism:

$$(x_1+\Delta) \vee (x_2+\Delta) = (x_1 \vee x_2)+\Delta,$$
$$(x_1+\Delta) \wedge (x_2+\Delta) = (x_1 \wedge x_2)+\Delta,$$
$$|x+\Delta| = |x|+\Delta$$

for all $x_1, x_2, x \in X$.

Definition 3. Let X be a vector lattice and let E be an ordered vector space. A seminorm $p\colon X \to E$ is said to be *monotone* and is called a *lattice seminorm* or a *Riesz seminorm* (cf. [1, Section 62; 2, 4.1; 3, 2.1.4]) whenever

$$|x_1| \leqslant |x_2| \text{ implies } p(x_1) \leqslant p(x_2)$$

for all $x_1, x_2 \in X$. It is clear that a seminorm p is monotone if and only if p enjoys the following two properties:

(1) $0 \leqslant x_1 \leqslant x_2$ implies $p(x_1) \leqslant p(x_2)$ for all $x_1, x_2 \in X$;
(2) $p(|x|) = p(x)$ for each $x \in X$.

A vector lattice X endowed with a monotone E-valued norm is called an *E-normed lattice.* (In case E is also a vector lattice, X is referred to as a *lattice-normed lattice.*)

The following is a vector-valued version of [1, Theorem 62.3]:

Theorem 2. *Let X be a vector lattice and let E be a Dedekind complete vector lattice. Consider an order ideal Δ in X and the quotient vector lattice X/Δ* (see Theorem 1). *If $p\colon X \to E$ is a monotone seminorm then the quotient seminorm $|\!\cdot\!|\colon X/\Delta \to E$ of p is also monotone. In particular, if the ideal $\Delta \subseteq X$ is closed with respect to p, then $(X/\Delta, |\!\cdot\!|)$ is an E-normed lattice.*

PROOF. We verify that $\lvert\!\lvert\cdot\rvert\!\rvert$ meets (1) and (2) in Definition 3.

(1): Suppose that $\mathbf{x}_1, \mathbf{x}_2 \in X/\Delta$ and $0 \leqslant \mathbf{x}_1 \leqslant \mathbf{x}_2$. Then according to Theorem 1 we have

$$|x_1| \wedge |x_2| \in \mathbf{x}_1 \wedge \mathbf{x}_2 = \mathbf{x}_1$$

for all $x_1 \in \mathbf{x}_1$ and $x_2 \in \mathbf{x}_2$. Moreover, since p is monotone, $|x_1| \wedge |x_2| \leqslant |x_2|$ implies $p(|x_1| \wedge |x_2|) \leqslant p(x_2)$. Consequently,

$$\lvert\!\lvert\mathbf{x}_1\rvert\!\rvert = \inf p[\mathbf{x}_1] \leqslant p(|x_1| \wedge |x_2|) \leqslant p(x_2)$$

for all $x_2 \in \mathbf{x}_2$, and so $\lvert\!\lvert\mathbf{x}_1\rvert\!\rvert \leqslant \inf p[\mathbf{x}_2] = \lvert\!\lvert x_2\rvert\!\rvert$.

(2): Take an arbitrary $x \in X$. To show the equality $\lvert\!\lvert |x| + \Delta\rvert\!\rvert = \lvert\!\lvert x + \Delta\rvert\!\rvert$, we proceed by analogy with the proof of [1, 62.3] and consider the corresponding two inequalities separately.

For all $\delta \in \Delta$ we have

$$\big||x+\delta| - |x|\big| \leqslant |\delta| \in \Delta \;\Rightarrow\; |x+\delta| - |x| \in \Delta \;\Rightarrow\; |x+\delta| \in |x| + \Delta;$$

therefore, $|x| + \Delta \supseteq \{|x+\delta| : \delta \in \Delta\}$, and so

$$\lvert\!\lvert |x| + \Delta\rvert\!\rvert = \inf p\big[|x| + \Delta\big] \leqslant \inf_{\delta\in\Delta} p(|x+\delta|) = \inf_{\delta\in\Delta} p(x+\delta) = \inf p[x+\Delta] = \lvert\!\lvert x+\Delta\rvert\!\rvert.$$

On the other hand, Theorem 1 implies

$$\big||x| + \delta\big| + \Delta = \big|(|x| + \delta) + \Delta\big| = \big||x| + \Delta\big| = |x| + \Delta$$

for each $\delta \in \Delta$. By applying [1, 59.1(v)] with $\pi\colon X \to X/\Delta$ the canonical homomorphism, we conclude that $|x + \delta'| \leqslant \big||x| + \delta\big|$ for some $\delta' \in \Delta$, and hence $\inf p[x+\Delta] \leqslant p(|x| + \delta)$. Since $\delta \in \Delta$ is arbitrary, we obtain

$$\lvert\!\lvert x+\Delta\rvert\!\rvert = \inf p[x+\Delta] \leqslant \inf p\big[|x| + \Delta\big] = \lvert\!\lvert |x| + \Delta\rvert\!\rvert. \quad \square$$

DEFINITION 4. Let X be a vector lattice, let E be an ordered vector space, let $(Y, \lvert\!\lvert\cdot\rvert\!\rvert)$ be an E-normed space, and let $T\colon X \to Y$ be a linear operator.

(1) Say that T is *monotone*, if $|x_1| \leqslant |x_2|$ implies $\lvert\!\lvert Tx_1\rvert\!\rvert \leqslant \lvert\!\lvert Tx_2\rvert\!\rvert$ for all $x_1, x_2 \in X$.

(2) Refer to a triple (Z, H, J) as a *lattice-isometric decomposition* of T whenever Z is an E-normed lattice, $H\colon X \to Z$ is a surjective lattice homomorphism, $J\colon Z \to Y$ is a linear isometry, and $T = J \circ H$.

We are now ready to state the main results of the article.

Theorem 3. *Let E be a Dedekind complete vector lattice. A linear operator $T : X \to Y$ acting from a vector lattice X to an E-normed space Y is monotone if and only if T admits a lattice-isometric decomposition.*

PROOF. Only necessity needs demonstration. Suppose that T is monotone. Note that $\Delta := \ker T$ is an order ideal in X. Indeed, if $x \in X$, $\delta \in \Delta$, and $|x| \leqslant |\delta|$, then $\lvert\!\lvert Tx\rvert\!\rvert \leqslant \lvert\!\lvert T\delta\rvert\!\rvert = 0$ and thus $x \in \Delta$. By Theorem 1, the quotient $Z := X/\Delta$ is a vector lattice with respect to the natural order, and the canonical mapping $H\colon X \to Z$ is a surjective lattice homomorphism. The monotonicity of T means that $\lvert\!\lvert\cdot\rvert\!\rvert \circ T$ is a monotone seminorm on X; moreover, Δ is closed with respect to this seminorm due to the obvious equivalences

$$\inf_{\delta\in\Delta} \lvert\!\lvert T(x-\delta)\rvert\!\rvert = 0 \;\Leftrightarrow\; \lvert\!\lvert Tx\rvert\!\rvert = 0 \;\Leftrightarrow\; x \in \Delta.$$

By Theorem 2, the quotient Z is an E-normed lattice with respect to the quotient norm $\lvert\!\lvert\cdot\rvert\!\rvert_Z$ of $\lvert\!\lvert\cdot\rvert\!\rvert \circ T$. It remains to note that the linear operator $J : Z \to Y$, correctly defined by the formula $J(Hx) = Tx$, is an isometry:

$$\lvert\!\lvert J(Hx)\rvert\!\rvert = \lvert\!\lvert Tx\rvert\!\rvert = \inf_{\delta\in\Delta} \lvert\!\lvert T(x+\delta)\rvert\!\rvert = \lvert\!\lvert Hx\rvert\!\rvert_Z. \quad \square$$

Corollary 1. *Let X be a vector lattice, let Y be an E-normed space, with E a Dedekind complete vector lattice, and let $T\colon X \to Y$ be a surjective monotone linear operator. Then Y can be endowed with an order so that Y becomes an E-normed lattice and T becomes a lattice homomorphism.*

Proof. Let (Z, H, J) be a lattice-isometric decomposition of T (see Theorem 3). Since T is surjective, so is the linear isometry $J\colon Z \to Y$. Introduce the following order on Y:

$$y_1 \leqslant y_2 \Leftrightarrow J^{-1}y_1 \leqslant J^{-1}y_2.$$

Then, as is easy to see, Y is an E-normed lattice, and J serves as an order and isometric isomorphism between Z and Y. In this case, T turns into a lattice homomorphism, being the composition of the lattice homomorphisms H and J. $\square$

Corollary 2. *Let X be a vector lattice, let Y be a Banach space, and let $T\colon X \to Y$ be a monotone linear operator whose image is dense in Y. Then Y can be endowed with an order so that Y becomes a Banach lattice and T becomes a lattice homomorphism.*

Proof. Using Corollary 1 with $E = \mathbb{R}$, equip the image Y_0 of T with an order that turns Y_0 into a normed lattice and $T\colon X \to Y_0$ into a lattice homomorphism. It remains to note that there is an order on the norm completion Y of Y_0 with respect to which Y is a Banach lattice that includes Y_0 as a normed sublattice (see [2, Theorem 4.2] for instance). $\square$

Lemma 1. *Suppose that E is a vector lattice, X and Y are E-normed lattices, $T\colon X \to Y$ is a linear isometry, $u, v \in X$, $|u| \wedge |v| = 0$, and $Tu \geqslant Tv \geqslant 0$. Then $\boldsymbol{|}u + nv\boldsymbol{|} \leqslant \boldsymbol{|}u + v\boldsymbol{|}$ for all $n \in \mathbb{N}$; and, if E is Archimedean, then $v = 0$.*

Proof. In the case of $n = 1$ the desired inequality is obvious. On assuming $\boldsymbol{|}u+nv\boldsymbol{|} \leqslant \boldsymbol{|}u+v\boldsymbol{|}$, show $\boldsymbol{|}u + (n+1)v\boldsymbol{|} \leqslant \boldsymbol{|}u + v\boldsymbol{|}$. From the relations $-Tu \leqslant 0 \leqslant Tu - Tv$ and $-(n+1)Tv \leqslant 0 \leqslant nTv$ we obtain

$$-Tu - nTv \leqslant Tu - (n+1)Tv \leqslant Tu + nTv.$$

Therefore, $|Tu - (n+1)Tv| \leqslant Tu + nTv = |Tu + nTv|$ and hence

$$\boldsymbol{|}u - (n+1)v\boldsymbol{|} = \boldsymbol{|}Tu - (n+1)Tv\boldsymbol{|} \leqslant \boldsymbol{|}Tu + nTv\boldsymbol{|} = \boldsymbol{|}u + nv\boldsymbol{|} \leqslant \boldsymbol{|}u + v\boldsymbol{|}.$$

Since $|u| \wedge |v| = 0$, we conclude that $|u+(n+1)v| = |u-(n+1)v|$, and so $\boldsymbol{|}u+(n+1)v\boldsymbol{|} = \boldsymbol{|}u-(n+1)v\boldsymbol{|}$. Finally, we have $n\boldsymbol{|}v\boldsymbol{|} = \boldsymbol{|}nv\boldsymbol{|} \leqslant \boldsymbol{|}u+nv\boldsymbol{|} + \boldsymbol{|}u\boldsymbol{|} \leqslant \boldsymbol{|}u+v\boldsymbol{|} + \boldsymbol{|}u\boldsymbol{|}$ for all $n \in \mathbb{N}$; and, if E is Archimedean, the latter implies $\boldsymbol{|}v\boldsymbol{|} = 0$. $\square$

Remark. The above reasoning conceptually repeats a fragment of the proof of Theorem 1 in [4].

Theorem 4. *Let E be an Archimedean vector lattice, let X and Y be E-normed lattices, and let $T\colon X \to Y$ be a positive linear isometry. Then T is an order isomorphism of X onto $\operatorname{im} T$.*

Proof. If $x \in X$ and $Tx \geqslant 0$, then $u := x^+$ and $v := x^-$ satisfy the conditions of Lemma 1 (since $Tu - Tv = Tx \geqslant 0$); therefore, $v = 0$ and thus $x \geqslant 0$. $\square$

Corollary 3. *If E is an Archimedean vector lattice, then every positive isometric isomorphism between E-normed lattices is an order isomorphism.*

In the case of $E = \mathbb{R}$ the latter statement coincides with [4, Theorem 1].

Theorem 5. *Let X be a vector lattice, let Y be an E-normed lattice, with E a Dedekind complete vector lattice, and let a linear operator $T : X \to Y$ be positive, surjective, and monotone. Then T is a lattice homomorphism.*

Proof. By Theorem 3, the operator T admits a lattice-isometric decomposition (Z, H, J). Show that $J : Z \to Y$ is positive. Indeed, if $x \in X$ and $Hx \geqslant 0$, then $Hx = (Hx)^+ = H(x^+)$, and so $J(Hx) = JH(x^+) = T(x^+) \geqslant 0$ as T is positive. Moreover, since T is surjective, so is J. Therefore, J serves as a positive isometric isomorphism between the E-normed lattices Z and Y. By Corollary 3, the operator J is an order isomorphism and, in particular, a lattice homomorphism. Consequently, the composition $T = J \circ H$ is also a lattice homomorphism. $\square$

Turning to applications of the above results, we first prove a technical lemma on introducing an order in each stalk of a Banach bundle whose set of sections has a given lattice structure. (See [5; 3, 2.4 and 2.5] for the corresponding definitions and notation.)

Lemma 2. *Let Q be an arbitrary set, let E be a vector sublattice of $\mathbb{R}^Q$, let $\mathcal{X}$ be a Banach bundle over Q, let $\mathcal{U}$ be a vector subspace of $S(Q,\mathcal{X})$ stalkwise dense in $\mathcal{X}$, and let $\leqslant$ be an order on $\mathcal{U}$ such that $(\mathcal{U}, |\!|\!|\cdot|\!|\!|, \leqslant)$ is an E-normed lattice, where $|\!|\!|\cdot|\!|\!| : \mathcal{U} \to E$ is the pointwise norm. Then each stalk $\mathcal{X}(q)$, $q \in Q$, can be endowed with an order so that $\mathcal{X}(q)$ becomes a Banach lattice and $(\mathcal{U}, \leqslant)$ turns into a sublattice of $S(Q,\mathcal{X})$ with respect to the pointwise order:*

(1) $u \leqslant v \Leftrightarrow (\forall\, q \in Q)\ u(q) \leqslant v(q)$;

(2) $(u \vee v)(q) = u(q) \vee v(q)$, $(u \wedge v)(q) = u(q) \wedge v(q)$, *and* $|u|(q) = |u(q)|$ *for all* $q \in Q$.

Proof. For each $q \in Q$, consider the linear operator $T_q : u \mapsto u(q)$ from $\mathcal{U}$ to $\mathcal{X}(q)$ whose image is dense in $\mathcal{X}(q)$. As is easy to see, T_q is monotone. Indeed, if $|u| \leqslant |v|$, then $|\!|\!|u|\!|\!| \leqslant |\!|\!|v|\!|\!|$, and so

$$\|T_q u\| = \|u(q)\| = |\!|\!|u|\!|\!|(q) \leqslant |\!|\!|v|\!|\!|(q) = \|v(q)\| = \|T_q v\|.$$

Using Corollary 2, equip $\mathcal{X}(q)$ with an order that turns $\mathcal{X}(q)$ into a Banach lattice and $T_q : \mathcal{U} \to \mathcal{X}(q)$ into a lattice homomorphism. The latter ensures (2); and (2) implies (1), since

$$u \leqslant v \;\Leftrightarrow\; u \vee v = v \;\Leftrightarrow\; (\forall\, q \in Q)\ u(q) \vee v(q) = v(q) \;\Leftrightarrow\; (\forall\, q \in Q)\ u(q) \leqslant v(q). \quad \square$$

Remark. Within the proof of Lemma 2, the coincidence of $\leqslant$ with the pointwise order $\preccurlyeq$ on $\mathcal{U}$ can also be deduced from Corollary 3. Indeed, the identity isometric isomorphism between the E-normed lattices $(\mathcal{U}, |\!|\!|\cdot|\!|\!|, \leqslant)$ and $(\mathcal{U}, |\!|\!|\cdot|\!|\!|, \preccurlyeq)$, being positive (due to the positivity of all T_q), is an order isomorphism.

Basing on Lemma 2, it is possible to implement the main part of the program outlined in [6] on developing the representation theory for Banach–Kantorovich lattices. As a simple example, we present the corresponding representation theorem for the case of a $C(Q)$-valued norm.

Theorem 6. *Let Q be an extremally disconnected compact space and let $E := C(Q)$. Every E-normed Banach–Kantorovich lattice is order and isometrically isomorphic to the space $C(Q,\mathcal{X})$ of continuous sections of some ample continuous bundle $\mathcal{X}$ of Banach lattices over Q.*

Proof. According to [5, 3.4.2] (see also [3, 2.4.10]), for every E-normed Banach–Kantorovich lattice X, there exists an ample continuous Banach bundle $\mathcal{X}$ over Q such that the E-normed space $(X, |\cdot|_X)$ is isometric to the space $\mathcal{U} := C(Q,\mathcal{X})$ equipped with the pointwise norm $|\!|\!|\cdot|\!|\!|$. Using the available isomorphism $J : X \to \mathcal{U}$, endow $\mathcal{U}$ with an order by putting $u_1 \leqslant u_2 \Leftrightarrow J^{-1}u_1 \leqslant_X J^{-1}u_2$. Next, employing Lemma 2, make the stalks of $\mathcal{X}$ into Banach lattices and the order on $\mathcal{U}$ into the pointwise order. As a result, $\mathcal{X}$ turns into a continuous bundle of Banach lattices with lattice continuous structure $\mathcal{U}$; and the E-normed lattice X becomes not only isometric but also order isomorphic to $C(Q,\mathcal{X})$. $\square$

The introduction of the notion of lifting $(\cdot)_\sim : L^\infty(\Omega,\mathcal{X}) \to \mathcal{L}^\infty(\Omega,\mathcal{X})$ on the classes of measurable sections of a measurable bundle $\mathcal{X}$ of Banach lattices over a measure space Ω is accompanied by the requirement that the lifting agrees with the lattice operations (see [7, Definition 2.2]):

$$(\mathbf{u} \vee \mathbf{v})_\sim = \mathbf{u}_\sim \vee \mathbf{v}_\sim \ \text{ for all } \mathbf{u}, \mathbf{v} \in L^\infty(\Omega,\mathcal{X}).$$

As another application of the above-established results, we show that the latter requirement is redundant, and it suffices to require positivity:

$$\mathbf{u}_\sim \geqslant 0 \ \text{ for positive } \mathbf{u} \in L^\infty(\Omega,\mathcal{X}).$$

Theorem 7. *Let $\mathcal{X}$ be a measurable bundle of Banach lattices over a measure space Ω and let $(\cdot)_\sim : L^\infty(\Omega,\mathcal{X}) \to \mathcal{L}^\infty(\Omega,\mathcal{X})$ be a lifting in $\mathcal{X}$ such that $\mathbf{u}_\sim \geqslant 0$ on Ω for positive $\mathbf{u} \in L^\infty(\Omega,\mathcal{X})$. Then $(\mathbf{u} \vee \mathbf{v})_\sim = \mathbf{u}_\sim \vee \mathbf{v}_\sim$, $(\mathbf{u} \wedge \mathbf{v})_\sim = \mathbf{u}_\sim \wedge \mathbf{v}_\sim$, and $|\mathbf{u}|_\sim = |\mathbf{u}_\sim|$ on Ω for all $\mathbf{u}, \mathbf{v} \in L^\infty(\Omega,\mathcal{X})$.*

Proof. It suffices to fix an arbitrary point $\omega \in \Omega$ and apply Theorem 5 to the vector lattices $X := L^\infty(\Omega,\mathcal{X})$ and $E = \mathbb{R}$, the normed lattice $Y := \mathcal{X}(\omega)$, and the operator $T : \mathbf{u} \in X \mapsto \mathbf{u}_\sim(\omega) \in Y$ whose surjectivity follows from [5, 4.4.1] (see also [3, 2.4.2(5) and 2.5.10]). $\square$

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CONFLICT OF INTEREST

The author of this work declares that he has no conflicts of interest.

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